Math 3010 § 1. **Second Midterm Exam** Name:____________________

Treibergs November 2, 2022

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1. *Multiple choice. The choices are given in the columns "Date / Place" and "Contribution." For each book in the table at the bottom, fill in the number of your choice for the place and date, and a letter for your choice of its mathematical contribution.*

Date/Place	**Contribution**
1. 263 Shansi Province	A. Decimal arithmetic, linear and quadratic algebraic equations
2. 628 Bhillamala	B. First published solution of cubic equations.
3. 830 Bagdad	C. Introduced Hindu-Arabic notation and algebra to Europe
4. 1150 Ujjian	D. Planet orbits around sun are generated by epicycles.
5. 1202 Pisa	E. Planets orbit around the sun on elliptical paths of varying speed.
6. 1543 Frauenberg	F. Rational triangles and a composition law for Pell's equation.
7. 1545 Bologna	G. Solved cubic equations using trigonometric identities.
8. 1591 Paris	H. Solved Pell's Equation using the cyclic process.
9. 1609 Prague	I. Tables of logarithms and instruction for their use.
10. 1614 Merchison	J. Was the standard text on systems of equations and measuring.

Book	Date / Place	Contribution
al-Khwārizmī's *Al-jabr wál mûqabalah*	3	A
Bhâskara II's *Lilavati*	4	H
Brahmagupta's *Brâhma-sphuṭa-siddhânta*	2	F
Cardano's *Ars Magna*	7	B
Copernicus's *De Revolutionibus Orbitum Coelestum*	6	D
Kepler's *Astronomia Nova*	9	E
Leonardo's *Liber Abaci*	5	C
Lui Hui's *Commentaries on the Nine Chapters*	1	J
Napier's *Mirifici Logarithmorum Canonis Descriptio*	10	I
Viete's *On the Review and Correction of Equations*	8	G

2. *Find a number n that simultaneously satisfies the congruences. Check your answer.*

$$\begin{aligned} n &\equiv 1 \mod 3 \\ n &\equiv 3 \mod 4 \\ n &\equiv 2 \mod 5 \end{aligned}$$

First find n_1 that satisfies $n \equiv 0 \mod 4$ and $n \equiv 0 \mod 5$. Such an $n_1 = 4 \cdot 5\ell$. We have $20 = 4 \cdot 5 \equiv 2 \mod 3$. Hence $n_1 = 40 = 2 \cdot 20 \equiv 2 \cdot 2 \mod 3 \equiv 4 \mod 3 \equiv 1 \mod 3$.

Second find n_2 that satisfies $n \equiv 0 \mod 3$ and $n \equiv 0 \mod 5$. Such an $n_2 = 3 \cdot 5\ell$. We have $3 \cdot 5 = 15 \equiv 3 \mod 4$ as desired, so take $n_2 = 15$.

Third find n_3 that satisfies $n \equiv 0 \mod 3$ and $n \equiv 0 \mod 4$. Such an $n_3 = 3 \cdot 4\ell$. We have $3 \cdot 4 = 12 \equiv 2 \mod 5$ as desired, so take $n_3 = 12$.

Thus a solution of the simultaneous congruences is $n = n_1 + n_2 + n_3 = 40 + 15 + 12 = 67$. Check:

$$67 = 22 \cdot 3 + 1 = 16 \cdot 4 + 3 = 13 \cdot 5 + 2.$$

In fact, since 3, 4 and 5 are pairwise relatively prime, all solutions of the simultaneous congruences are

$$n = 67 + 3 \cdot 4 \cdot 5p = 67 + 60p$$

where p is an integer. By taking $p = -1$ we find another smaller solution of the simultaneous congruences $n = 7$.

Another solution is as follows. The solution of the first congruence is $n \equiv 1 \mod 3$ says $n = 3k + 1$ for some integer k. To be equivalent to the second we need $3k + 1 = 4\ell + 3$, for an integer ℓ. Thus we solve the Diophantine equation

$$3k - 4\ell = 3 - 1 = 2.$$

Now $3 \cdot 1 - 4 \cdot 1 = -1$ so one solution is given by $3 \cdot (-2) - 4 \cdot (-2) = 2$. Thus all solutions are

$$k = -2 + 4m, \qquad \ell = -2 + 3m$$

where m is an integer. It follows that $n = 3k + 1 = 3(-2 + 4m) + 1 = 12m - 5$. To satisfy the third congruence requires $12m - 5 \equiv 2 \mod 5$ or $12m - 5 = 5q + 2$ for some integer q. This results in the Diophantine equation

$$12m - 5q = 5 + 2 = 7.$$

Now $12 \cdot 2 - 5 \cdot 5 = -1$ so one solution is $12 \cdot (-14) - 5 \cdot (-35) = 7$. Thus all solutions are

$$m = -14 + 5r, \qquad q = -35 + 12r.$$

where r is an integer. This implies all solutions are of the form

$$n = 12m - 5 = 12(-14 + 5r) - 5 = -173 + 60r.$$

Thus for $r = 4$ or $r = 3$ we get $n = 67$ or $n = 7$ as before.

3. (a) *Recall Brahmagupta's theorem that for fixed N, if (x_1, y_1, k_1) and (x_2, y_2, k_2) satisfy $x^2 - Ny^2 = k$ then so does $(x_1x_2 + Ny_1y_2,\ x_1y_2 + y_1x_2,\ k_1k_2)$. Starting from $(3, 1, 2)$, use Baskara II's methodto find an integer solution to Pell's Equation*

$$x^2 - 7y^2 = 1.$$

For an integer n we have the composition

$$(3, 1, 2) * (n, 1, n^2 - 7) = (3n + 7, 3 + n, 2(n^2 - 7)).$$

Thus

$$\left(\frac{3n+7}{2}, \frac{3+n}{2}, \frac{n^2-7}{2}\right)$$

Choosing n so that $2|(3 + n)$ and $(n^2 - 7)/2$ is small, we take $n = 3$. This yields

$$\left(\frac{3 \cdot 3 + 7}{2}, \frac{3+3}{2}, \frac{3^2 - 7}{2}\right) = \left(\frac{16}{2}, \frac{6}{2}, \frac{2}{2}\right) = (8, 3, 1).$$

This is a solution because

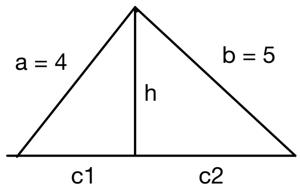
$$8^2 - 7 \cdot 3^2 = 64 - 63 = 1.$$

(b) *How many five letter words can be made in an alphabet of 26 letters where one of the letters is repeated three times and the other two are noty repeated, it e.g., AAABC.*

There are 26 choices for the first letter which may be placed in all three spots in any one of the $\binom{5}{3}$ subsets of three among the five letters. There are 25 choices for the leftmost unchosen space and 24 for the rightmost. All together, the number of such words is

$$26 \cdot \binom{5}{3} \cdot 25 \cdot 24 = 26 \cdot 10 \cdot 25 \cdot 24 = 156,000.$$

4. (a) *Consider a triangle with sides $a = 4$, $b = 5$ and $c = 6$. The altitude h splits c into segments c_1 and c_2. Find c_1 and c_2 and determine whether the triangle is rational.*



A triangle is rational if in addition to its sides, its area is rational. Since area is base times height, it is equivalent to show that the height h is rational. Let us rederive the formulas for c_1 and c_2. The height may be recovered from Pythagorean theorem for the left and right triangle.

$$\begin{aligned} a^2 &= h^2 + c_1^2 \\ b^2 &= h^2 + c_2^2. \end{aligned}$$

Subtracting we get

$$a^2 - b^2 = c_1^2 - c_2^2 = (c_1 - c_2)(c_1 + c_2).$$

Thus using $c = c_1 + c_2$,

$$\begin{aligned} c_1 - c_2 &= \frac{a^2 - b^2}{c} \\ c_1 + c_2 &= c. \end{aligned}$$

Hence adding and subtracting yields

$$\begin{aligned} c_1 &= \frac{1}{2}\left\{c + \frac{a^2 - b^2}{c}\right\} \\ c_2 &= \frac{1}{2}\left\{c - \frac{a^2 - b^2}{c}\right\} \end{aligned}$$

Plugging in the values of the sides

$$\begin{aligned} c_1 &= \frac{1}{2}\left\{6 + \frac{4^2 - 5^2}{6}\right\} = \frac{9}{4} \\ c_2 &= \frac{1}{2}\left\{6 - \frac{4^2 - 5^2}{6}\right\} = \frac{15}{4} \end{aligned}$$

From this we compute the height.

$$h = \sqrt{a^2 - c_1^2} = \sqrt{16 - \frac{81}{16}} = \frac{\sqrt{256 - 81}}{4} = \frac{\sqrt{175}}{4} = \frac{5}{4}\sqrt{7}$$

which is irrational. Hence the triangle is not rational.

(b) *If $F_1 = F_2 = 1$, What is the tenth Fibonacci number F_{10}?*

The Fibonacci sequence is recursively defined by $F_1 = 1$, $F_2 = 1$ and then the following terms for $n \geq 2$ are the sums of the previous two terms

$$F_{n+1} = F_n + F_{n-1}.$$

Thus we get the sequence

$$1, \quad 1, \quad 2, \quad 3, \quad 5, \quad 8, \quad 13, \quad 21, \quad 34, \quad 55, \quad 89, \quad 144, \quad 233, \quad 377, \quad 610, \quad \ldots$$

thus $F_{10} = 55$.

5. *Use Cardano's method to solve the equation $x^3 + 6x = 20$.*

Recall, to solve $x^3 = px + q$, he put $x = u + v$ and solved the system

$$\begin{aligned} 3uv &= p \\ u^3 + v^3 &= q. \end{aligned}$$

Substituting $x = u + v$ we have

$$(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3 = u^3 + v^3 + 3uvx = -6x + 20$$

Thus equating the constant and x terms

$$\begin{aligned} 3uv &= -6 \\ u^3 + v^3 &= 20. \end{aligned}$$

By the first equation,

$$v = -\frac{2}{u}$$

and by the second

$$u^3 - \frac{8}{u^3} = 20.$$

Thus

$$u^6 - 20u^3 - 8 = 0.$$

This is quadratic in u^3. By the quadratic formulas

$$u^3, v^3 = \frac{20 \pm \sqrt{20^2 + 4 \cdot 8}}{2} = 10 \pm \sqrt{5 \cdot 20 + 8} = 10 \pm \sqrt{108} = 10 \pm 6\sqrt{3}.$$

It follows that a solution of the cubic equation is

$$x = u + v = \left(10 + 6\sqrt{3}\right)^{1/3} + \left(10 - 6\sqrt{3}\right)^{1/3}.$$

This works out to be $x = 2$.