Math 3010 § 1. **Second Midterm Exam** Name: Solutions
Treibergs March 28, 2018

1. *Multiple choice. The choices are given in the columns "Date / Place" and "Contribution." For each book in the table at the bottom, fill in the number of your choice for the place and date, and a letter for your choice of its mathematical contribution.*

Date/Place	**Contribution**
1. 150 BC Chang'an	A. 246 problems in proportions, measuring, 3×3 systems
2. 145 Alexandria	B. Astronomy and Trigonometry of Geocentric Universe
3. 650 Bhinmal	C. Chinese Remainder Theorem, polynomial equations
4. 830 Bagdad	D. Decimal arithmetic, linear and quadratic algebraic equations
5. 1150 Ujjian	E. First published solution of cubic equations.
6. 1202 Pisa	F. Introduced Hindu-Arabic notation and algebra to Europe
7. 1247 Hangzhou	G. Projective geometry
8. 1545 Bologna	H. Quadratic equations, composition formula for Pell's equation
9. 1637 Holland	I. Solved Pell's Equation using the cyclic process.
10. 1639 Paris	J. Used coordinates to relate curves to solutions of equations.

Book	Date / Place	Contribution
al-Khwārizmī's *Al-jabr wál mûqabalah*	4	D
Bhâskara II's *Bîjagaṇita*	5	I
Brahmagupta's *Brâhma-sphuṭa-siddhânta*	3	H
Cardano's *Ars magna*	8	E
Decartes' *La Géométrie*	9	J
Desargues' *Brouillon projet d'une attiente aux ...*	10	G
Leonardo's *Liber abaci*	6	F
Zhang Cang's *Nine Chapters on the Mathematical Art*	1	A
Ptolemy's *Almagest*	2	B
Qin Juishao's *Mathematical Treatise in Nine Sections*	7	C

2. *Find a number n that simultaneously satisfies the congruences. Check your answer.*

$$\begin{aligned} n &\equiv 3 \mod 4 \\ n &\equiv 4 \mod 5 \\ n &\equiv 5 \mod 7 \end{aligned}$$

Using Qin Jiushao's algorithm, the first congruence is satisfied if we put

$$n = 3 + 4k$$

for some integer k. Thus to satisfy the second we put

$$n = 3 + 4k = 4 - 5\ell$$

for some integer ℓ. This is equivalent to

$$4k + 5\ell = 1$$

Since $4(-1) + 5(1) = 1$, the general solution is

$$k = -1 + 5m, \qquad \ell = 1 - 4m$$

for some integer m. It follows that

$$n = 3 + 4k = 3 + 4(-1 + 5m) = -1 + 20m.$$

Finally, the last congruence implies

$$n = -1 + 20m = 5 - 7p$$

for some integer p. Equivalently

$$20m + 7p = 6.$$

Observing that $20(-1) + 7(3) = 1$, we get a particular solution $20(-6) + 3(18) = 6$. The general solution is thus

$$m = -6 + 7q, \qquad p = 18 - 20q$$

for some integer q. It follows that all n satisfying the congruences are

$$n = -1 + 20m = -1 + 20(-6 + 7q) = -121 + 140q$$

When $q = 1$, $n = 19$. Checking, we see that it satisfies all three congruences

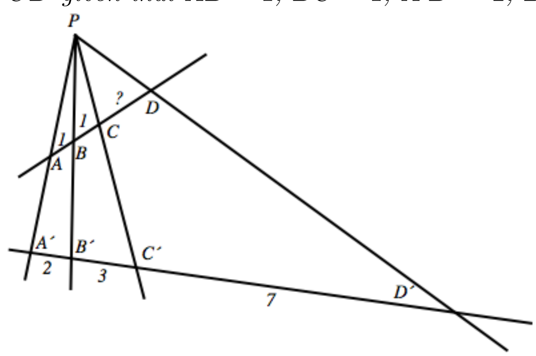
$$19 = 4 \cdot 4 + 3 = 3 \cdot 5 + 4 = 2 \cdot 7 + 5.$$

3. (a) *If integers x and y solve Pell's equation $x^2 - Ny^2 = 2$, show that $\tilde{x} = x^2 - 1$ and $\tilde{y} = xy$ is an integral solution of $\tilde{x}^2 - N\tilde{y}^2 = 1$.*

First if x and y are integers, then so are their squares and products so $\tilde{x} = x^2 - 1$ and $\tilde{y} = xy$ are integers. The equation says $Ny^2 = x^2 - 2$. Using this,

$$\begin{aligned}\tilde{x}^2 - N\tilde{y}^2 &= (x^2-1)^2 - Ny^2x^2 \\ &= x^4 - 2x^2 + 1 - (x^2-2)x^2 \\ &= 1.\end{aligned}$$

(b) *Suppose two lines cross four rays emanating from P in the plane. Find the distance CD given that $AB = 1$, $BC = 1$, $A'B' = 2$, $B'C' = 3$ and $C'D' = 7$.*



Let's use the cross ratio, which is equal for both crossing lines. Denote $x = CD$. The cross ratios are equal for both lines

$$\frac{AC \cdot BD}{AD \cdot BC} = \frac{A'C' \cdot B'D'}{A'D' \cdot B'C'}$$

Substituting given lengths, for example $BD = BC + CD = 1 + x$, we find

$$\frac{2(1+x)}{(2+x)\cdot 1} = \frac{5 \cdot 10}{12 \cdot 3}$$

which is equivalent to

$$2(1+x) = \frac{50}{36}(2+x)$$

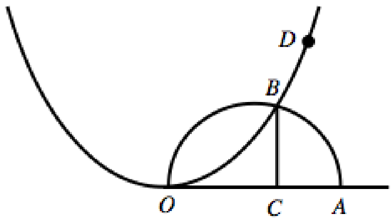
or

$$\frac{11}{18}x = \left(2 - \frac{25}{18}\right)x = \left(\frac{50}{18} - 2\right) = \frac{14}{18}$$

so

$$x = \frac{14}{11}.$$

4. (a) *Following Umar al-Khayyāmī, consider the cubic equation $x^3 + ax = b$ with $a, b > 0$. Let OBD be a vertical parabola whose vertex is O and which passes through the point $D = (\sqrt{a}, \sqrt{a})$. Let $A = \left(\frac{b}{a}, 0\right)$ and OBA is a circle whose diameter is OA. Show that $x = OC$ solves the cubic equation.*



The equations of the parabola and the circle centered on $\left(\frac{b}{2a}, 0\right)$ with radius $\frac{b}{2a}$ are

$$\sqrt{a}y = x^2$$
$$\left(x - \frac{b}{2a}\right) + y^2 = \left(\frac{b}{2a}\right)^2.$$

Squaring and simplifying,

$$y^2 = \frac{x^4}{a}$$
$$x^2 - \frac{b}{a}x + y^2 = 0.$$

Hence

$$x^2 - \frac{b}{a}x + \frac{x^4}{a} = 0.$$

Since we don't want the intersection point at $x = 0$ we may assume $x \neq 0$ so multiplying by $\frac{a}{x}$ gives the desired equation

$$ax - b + x^3 = 0.$$

(b) *Use the algorithm from Nine Chapters or Āryabhaṭīyah to find the square root of* $17,424$.

Since the number is less than 1000^2, we look for roots of the form $x = 100a + 10b + c$.

$17,424$	$200^2 = 40,000$ is too big so $a = 1$.
$-10,000$	Subtract $(100a)^2$.
$7,424$	Is greater than $2000b$ for $b = 3$ but not $b = 4$.
	Using $(100a + 10b)^2 = 10,000 + 2000b + 100b^2$
$-6,000$	we subtract $2000b$
$1,424$	
-900	and $100b^2$,
524	which is now $17,424 - (100a + 10b)^2$. It is greater than $260c$ for $c = 2$
	but not $c = 3$. Using $(130 + c)^2 = 130^2 + 260c + c^2$
-520	we subtract $260c$
4	
-4	and c^2,
0	Thus the square root is $x = 132$.

5. *Use Cardano's method to solve the equation* $x^3 = 9x + 28$.

Recall, to solve $x^3 = px + q$, *he put* $x = u + v$ *and solved the system*

$$3uv = p$$
$$u^3 + v^3 = q.$$

We have

$$3uv = 9$$
$$u^3 + v^3 = 28.$$

The first equation tells us $v = \dfrac{3}{u}$. Substituting into the second

$$u^3 + \left(\frac{3}{u}\right)^3 = 28$$

or

$$(u^3)^2 - 28(u^3) + 27 = 0.$$

This equation factors

$$(u^3 - 27)(u^3 - 1) = 0.$$

The two roots are therefore

$$u^3 = 27, \qquad v^3 = 1$$

or $u = 3$ and $v = 1$. Thus $x = 3 + 1 = 4$. Checking,

$$64 = (4)^3 \stackrel{?}{=} 9(4) + 28 = 64. \qquad \surd$$