Math 3010 § 1. **Second Midterm Exam** Name: Practice Problems
Treibergs March 7, 2018

Here are some problems soluble by methods encountered in the course. I have tried to select problems ranging over the topics we've encountered. Admittedly, they were chosen because they're fascinating to me. As such, they may have solutions that are longer than the questions you might expect on an exam. But some of them are samples of homework problems. Here are a few of my references.

References.

- Carl Boyer, *A History of Mathematics,* Princeton University Press, Princeton 1985
- Lucas Bunt, Phillip Jones, Jack Bedient, *The Historical Roots of Elementary mathematics,* Dover, Mineola 1988; orig. publ. Prentice- Hall, Englewood Cliffs 1976
- David Burton, *The History of Mathmatics An Introduction*, 7th ed., McGraw Hill, New York 2011
- Ronald Calinger, *Classics of Mathematics,* Prentice-Hall, Englewood Cliffs 1995; orig. publ. Moore Publ. Co. Inc., 1982
- Victor Katz, *A History of Mathematics An introduction,* 3rd. ed., Addison-Wesley, Boston 2009
- Morris Kline, *Mathematical Thought from Ancient to Modern Times,* Oxford University Press, New York 1972
- John Stillwell, *Mathematics and its History, 3rd ed.,* Springer, New York, 2010.
- Dirk Struik, *A Concise History of Mathematics,* Dover, New York 1967
- Harry N. Wright, *First Course in Theory of Numbers,* Dover, New York, 1971; orig. publ. John Wiley & Sons, Inc., New York, 1939.

---

1. *Find the least common multiple of* $3,130,512$ *and* $509,652$.

   By the factoring method, $3,130,512 = 2^4 \cdot 3 \cdot 7^2 \cdot 11^3$ and $509,652 = 2^2 \cdot 3^4 \cdot 11^2 \cdot 13$ so $\operatorname{lcm}(3,130,512, 509,652) = 2^4 \cdot 3^4 \cdot 7^2 \cdot 11^3 \cdot 13 = 1,098,809,712.$

   Or we may use the formula involving the greatest common denominator. $\gcd(3,130,512, 509,652) = 2^2 \cdot 3 \cdot 11^2 = 1452$ so

$$\operatorname{lcm}(3,130,512, 509,652) = \frac{3,130,512 \cdot 509,652}{1452} = 1,098,809,712.$$

2. *Use the Chinese square root algorithm to find* $\sqrt{226,576}$.

   Note that $1000^2 = 1,000,000$ is too large so seek the root in the form $100a + 10b + c$ where a, b and c are integers from 0 to 9. Discarding the lower terms, we need the largest a so that

$$226,576 \ge (100a)^2 = 10,000a^2$$

   For $a = 4$ this is $10,000a^2 = 160,000 \le 226,576$ which works, but for $a = 5$ this is $10,000a^2 = 250,000 > 226,576$ which is too large. Thus $a = 4$. Next we need the largest b so that

$$226,576 \ge (100a + 10b)^2 = (100a)^2 + 2(100a)(10b) + (10b)^2 = 160,000 + 8000b + 100b^2.$$

This implies that

$$226,576 - 160,000 = 66,576 \geq 2(100a)(10b) = 8000b$$

for which the largest integer is $b = 8$ since $66,576 \geq 8000b = 64,000$ but $b = 9$ is too large because then $66,576 < 8000b = 72,000$. Adding the last term we have to check that

$$66,576 \overset{?}{\geq} 2(100a)(10b) + (10b)^2.$$

But for $b = 8$ this is

$$2(100a)(10b) + (10b)^2 = 8000b + 100b^2 = 64,000 + 6,400 = 70,400$$

which is too big. This implies that $b = 7$. Now we seek the largest c so that

$$226,576 \geq (100a + 10b + c)^2 = (100a + 10b)2 + 2(100a + 10b)c + c^2 = 470^2 + 2 \cdot 470c + c^2.$$

This implies

$$226,576 - 220,900 = 5676 \geq 470c$$

for which the largest solution is $c = 6$ because then $5675 \geq 940c = 5640$ but for $c = 7$, $5675 < 940c = 6540$. Checking $c = 6$

$$5676 \overset{?}{\geq} 940c + c^2 = 5640 + 36 = 5675.$$

Thus $\boxed{476}$ is the square root of $226,576$ on the nose.

3. *Use the Chinese cube root algorithm to find* $\sqrt[3]{478,211,768}$.

Note that $1000^3 = 10^9$ is too large. Thus look for a solution of the form $100a + 10b + c$ where $a, b, c \in \{0, 1, \ldots, 9\}$. We want the largest a so that

$$478,211,768 \geq (100a)^3 = 1,000,000a^3$$

$a = 7$ works because $7^3 = 343$ but $8^3 = 512$ is too big. Now seek the largest b so that

$$478,211,768 \geq (100a + 10b)^3 = (100a)^3 + 3(100a)^2 10b + 3(100a)(10b)^2 + (10b)^3$$

or

$$\begin{aligned} 478,211,768 - (100a)^3 = 135,211,768 &\geq 3(100a)^2 10b + 3(100a)(10b)^2 + (10b)^3 \\ &= b(14700000 + 210000b + 1000b^2) \end{aligned}$$

The largest such $b = 8$ since

$$b(14700000 + 210000b + 1000b^2) = 131,552,000$$

but $b = 9$ is too large because then

$$b(14700000 + 210000b + 1000b^2) = 150,039,000.$$

Now seek the largest c so that

$$478,211,768 \geq (100a + 10b + c)^3 = (100a + 10b)^3 + 3(100a + 10b)^2 c + 3(100a + 10b)c^2 + c^3$$

or

$$\begin{aligned} 478,211,768 - (100a + 10b)^3 = 3,659,768 &\geq 3(100a + 10b)^2 c + 3(100a + 10b)c^2 + c^3 \\ &= c(1825200 + 2340c + c^2) \end{aligned}$$

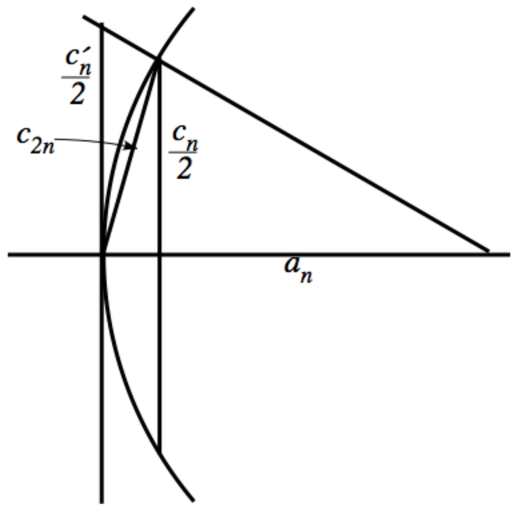
The largest such $c = 2$ when

$$c(1825200 + 2340c + c^2) = 3,659,768.$$

Thus $\boxed{782}$ is the cube root of $478,211,768$ on the nose.

4. *Let P_n and P'_n denote inscribed and circumcibed regular n-gons of a circle of radius r. Use Liu Hui's algorithm to compute inscribed areas $A(P_n)$ and $A(P'_n)$ for $n = 6, 12, 24, 48$ and 96. [Katz, A History of Mathematics, 2009, p. 227]*

   The algorithm starts with lengths c_n and c'_c of the sides of the inscribed and circumscribed $6 \cdot 2^{n-1}$-gon. Assuming $r = 1$ these are $c_6 = 1$ and $c'_6 = c_6/a_6 = 2/\sqrt{3}$. Then we proceed with Liu Hui's recursion.

   Using Pythagorean theorem, we can compute first a_n from r and $c_n/2$ and then c_{2n} from a_n and $c_n/2$.

$$a_n = \sqrt{r^2 - \left(\frac{c_n}{2}\right)^2};$$
$$c_{2n} = \sqrt{\left(\frac{c_n}{2}\right)^2 + (r - a_n)^2}.$$

   At the same time, we get the side of the circumscribing polygon by similar triangles

$$\frac{c'_n}{r} = \frac{c_n}{a_n}.$$

   From these the area of the next polygon may be computed. The number of triangles times base times height of a sector of P_{2n} yields

$$A(P_{2n}) = 2n \cdot \frac{1}{2}\frac{c_n}{2} r = \frac{1}{2} n c_n.$$

   We now do the computation in a little **R**©program. Of course, Liu Hui had to take square roots the long way by hand!

```
c=1;
cp=2/sqrt(3);
for(i in 0:10){k=3*2^i;
              print(c(2^i*6, c,k*c,cp,k*cp));
              a=sqrt(1-c^2/4);
              cn=sqrt(c^2/4+(1-a)^2);
              cp=c/a;
```

```
c=cn;
ap=sqrt(1-cp^2/4);
cq=sqrt(cp^2/4+(1-ap)^2);
cp=cq}
```

This produces the output, (where I have added labels and edited).

```
    n       c_n        A(P_2n)       c_n'       A(P_2n')
-----    ---------    ---------   -----------   ---------
    6    1.0000000    3.0000000   1.1547010     3.4641020
   12    0.5176381    3.1058285   0.6058109     3.6348654
   24    0.2610524    3.1326286   0.2704328     3.2451939
   48    0.1308063    3.1393502   0.1319399     3.1665579
   96    0.0654382    3.1410320   0.06557873    3.1477788
  192    0.0327235    3.1414525   0.03274100    3.1431358
  384    0.0163623    3.1415576   0.01636447    3.1419782
  768    0.0081812    3.141584    0.00818148    3.141689
 1536    0.0040906    3.141590    0.00409065    3.141617
 3072    0.0020453    3.141592    0.00204531    3.141599
 6144    0.0010226    3.141593    0.00102265    3.141594
```

5. *Problem 17 from Chapter VII of Nine Chapters came with a solution. Explain why the given solution works*

   "The price of one acre of good lane is 300 *pieces of gold; the price of 7 acres of bad land is* 500*. One has purchased altogether 100 acres; the price was* 10,000*. How much good land was bought and how much bad?"*

   Solution given: "Suppose there were 20 acres of good land and 80 acres of bad. Then the surplus is $s = 1714\frac{2}{7}$*. If there were 10 acres of good land and 90 acres of bad, the deficiency is* $d = 571\frac{3}{7}$*. Then the solution is*

   $$\frac{20d + 10s}{s + d} = 12\frac{1}{2} \text{ acres of good land and } 100 - 12\frac{1}{2} = 87\frac{1}{2} \text{ of bad.}$$

   [Katz, *A History of Mathematics,* 2009, p. 210]

   The system for x acres of good and y acres of bad being solved is

   $$\begin{aligned} x + y &= 100 \\ 300x + \frac{500}{7}y &= 10,000 \end{aligned}$$

   If one tries a couple of points that satisfy the first equation $x_1 + y_1 = 100$ and $x_1 + y_1 = 100$, the second yields the surplus s and deficit d by

   $$\begin{aligned} 300x_1 + \frac{500}{7}y_1 &= 10,000 + s \\ 300x_2 + \frac{500}{7}y_2 &= 10,000 - d \end{aligned}$$

   Let's look for solutions that are convex combinations of (x_1, y_1) and (x_2, y_2). That means, if for some t,

   $$(x, y) = [1 - t](x_1, y_1) + t(x_2, y_2)$$

satisfies $x + y = 100$. Substituting into the second equation,

$$300x + \frac{500}{7}y = 10,000 + [1 - t]s - td.$$

We seek t_0 such that $[1 - t_0]s - t_0 d = 0$ or $s = t_0(s + d)$. That t_0 is

$$t_0 = \frac{s}{s + d}.$$

It follows that

$$x = [1 - t_0]x_1 + t_0 x_2 = \frac{dx_1 + sx_2}{s + d}$$

as claimed.

6. *Solve Problem 3 from Chapter VII of Nine Chapters. There are 9 equal pieces of gold and 11 equal pieces of silver. The two lots weigh the same. If one piece is removed from each lot and put in the other, the lot containing mainly gold is found to contain 13 ounces less than the lot containing mainly silver. Find the weight of each piece of gold and silver. [Burton, The History of Mathematics 7th ed., 2007, p. 265]*

   Let x and y denote the weights of each piece of gold and of silver, respectively. The euqations to be solved are

$$\begin{aligned} 9x &= 11y \\ 8x + y &= 10y + x - 13 \end{aligned}$$

   Rearranging

$$\begin{aligned} 9x - 11y &= 0 \\ 7x - 9y &= -13 \end{aligned}$$

   Using the Chinese method of solving simultaneous equations, subtract 7 times the first eqution from 9 times the second

$$\begin{aligned} 9x - 11y &= 0 \\ -4y &= -117 \end{aligned}$$

   so that back-substituting,

$$\begin{aligned} y &= \frac{117}{4} = 29\frac{1}{4} \text{ oz.} \\ x = \frac{11}{9}y &= \frac{1287}{36} = 35\frac{3}{4} \text{ oz.} \end{aligned}$$

7. *This is problem 8 in Li Ye's 1259 book* Old Mathematics in Nine Expanded Sections.*There is a circular pond in the middle of a square, and the area of the square outside the pond is 3300 square* pu. *It is known only that the sum of the perimeters of the square and the circle is 300* pu. *Find the perimeters of the square and the circle. [Burton, The History of Mathematics 7th ed., 2007, p. 260]*

   Let x be the diameter of the circular pond. The perimeter of the pond is πx, making the perimeter of the square field $300 - \pi x$. Its area is $\frac{1}{16}(300 - \pi x)^2$. The area of the circular pond is $\frac{\pi}{4}x^2$. The difference is

$$\frac{1}{16}(300 - \pi x)^2 - \frac{\pi}{4}x^2 = 3300$$

which yields the equation

$$\pi(4-\pi)x^2 + 600\pi x - 37,200 = 0.$$

Thus, by the quadratic formula the diameter of the pond is

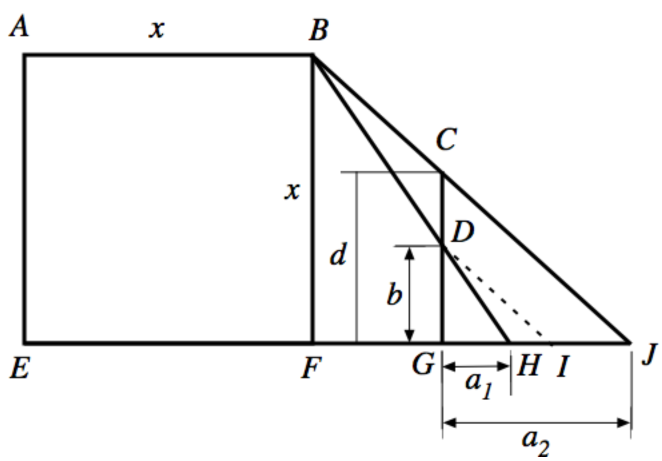
$$x = \frac{-600\pi + \sqrt{600^2\pi^2 + 4\pi(4-\pi)\cdot 37,200}}{2\pi(4-\pi)} = 19.2074 \ pu.$$

So the perimeters of the field and pond are $300 - \pi x = \boxed{239.6582}$ and $\pi x = \boxed{60.34183}$ *pu.* Using the Chinese approximation "$\pi = 3$," the equation is

$$3x^2 + 1800x - 37,200 = 0,$$

so $x = 20$ and the perimeters of the field and pond are $300 - 3x = \boxed{240}$ and $3x = \boxed{60}$ *pu.*

8. *This is a problem from Liu Hui's 264 book* Sea Island Mathematical Manual. *There is a square, walled city of unknown dimensions. A man erects two poles d feet apart in the north-south direction east of the city and joins them with with a string at eye-level. The southern pole is in a straight line with the southwestern and southeastern corners of the city. By moving eastward a_1 feet from the southern pole, the man's observation with the northeast corner of the city intersects the string at a point b feet from the southern end. He goes again a_2 feet from the pole until the northeastern corner is in line with the northern pole. What is the length of the side of the square city? [Burton, The History of Mathematics 7th ed., 2007, p. 265]*



In the diagram, the poles are located at C and G. Let I be the point on EJ such that DI is parallel to BJ. By the similarity of $\triangle(CGJ)$ and $\triangle(DGI)$ it follows that

$$\frac{GH + HI}{DG} = \frac{GJ}{CG}$$

The the similarity of the triangles $\triangle(BHJ)$ and $\triangle(DHI)$ and of $\triangle(BFH)$ and $\triangle(DGH)$,

$$\frac{HJ}{HI} = \frac{BH}{DH} = \frac{BF}{DG}.$$

Thus

$$x = BF = \frac{DG \cdot HJ}{HI} = \frac{DG \cdot (GJ - GH)}{\frac{GJ \cdot DG}{CG} - GH} = \frac{b \cdot (a_2 - a_1)}{\frac{a_2 \cdot b}{d} - a_1}.$$

9. *Does the equation have an integral solution? If so, find all solutions.*

$$132x + 378y = 30 \tag{1}$$

The equation has a solution if $d \mid 30$ where $d = \gcd(132, 378)$. Using the Euclidean algorithm,

$$\begin{aligned} 378 &= 2 \cdot 132 + 114 \\ 132 &= 1 \cdot 114 + 18 \\ 114 &= 6 \cdot 18 + 6 \\ 18 &= 3 \cdot 6 + 0 \end{aligned}$$

so $\gcd(132, 378) = 6$. Since $6 \mid 30$, the Diophantine equation is soluble. $30 = 5 \cdot 6$ so first solve (1) with RHS$= d$. Working backwards, we find

$$\begin{aligned} 6 &= 114 - 3 \cdot 18 \\ &= 114 - 6 \cdot (132 - 114) \qquad &&= 7 \cdot 114 - 6 \cdot 132 \\ &= 7 \cdot (378 - 2 \cdot 132) - 6 \cdot 132 \qquad &&= 7 \cdot 378 - 20 \cdot 132 \end{aligned}$$

Multiplying by 5 we see that

$$30 = 132 \cdot (-100) + 378 \cdot 35$$

so one solution is $x = -100$ and $y = 35$.

Let's divide (1) by $d = 6$.

$$22x + 63y = 5 \tag{2}$$

Note that any solution of (1) is a solution of (2) and vice versa. Note also that $\gcd(22, 63) = 1$. We know that $x = -100$ and $y = 35$ is one solution of (2) and so of (1). Suppose that $(\tilde{x}, \tilde{y})$ were another

$$22\tilde{x} + 63\tilde{y} = 5$$

Then, subtracting,

$$22(x - \tilde{x}) + 63(y - \tilde{y}) = 0$$

Since $\gcd(22, 63) = 1$, this says $22 \mid (y - \tilde{y})$ or

$$(y - \tilde{y}) = 22k$$

for some integer k. Substituting back, this says

$$22(x - \tilde{x}) + 63 \cdot 22k = 0$$

or

$$(x - \tilde{x}) + 63k = 0$$

We have shown that every solution of (1) has the form

$$\tilde{x} = x + 63k, \qquad \tilde{y} = y - 22k$$

where $(x, y) = (-100, 35)$ is one solution and k is any integer.

10. *Find the smallest positive solution.*

$$2360x \equiv 16 \mod 2244$$

This is equivalent to solving

$$2244x + 2360y = 16.$$

The Euclidean algorithm yields

$$\begin{aligned}
2360 &= 1 \cdot 2244 + 116 \\
2244 &= 19 \cdot 116 + 40 \\
116 &= 2 \cdot 40 + 36 \\
40 &= 1 \cdot 36 + 4 \\
36 &= 9 \cdot 4 + 0
\end{aligned}$$

so $d = \gcd(2244, 2360)$. Since $d \mid 16$ the equation is soluble. Working backwards,

$$\begin{aligned}
4 &= 40 - 36 \\
&= 40 - (116 - 2 \cdot 40) && = 3 \cdot 40 - 116 \\
&= 3 \cdot (2244 - 19 \cdot 116) - 116 && = 3 \cdot 2244 - 58 \cdot 116 \\
&= 3 \cdot 2244 - 58 \cdot (2360 - 2244) && = 61 \cdot 2244 - 58 \cdot 2360
\end{aligned}$$

Multiplying by 4 gives one solution

$$2360(-232) + 2244 \cdot 244 = 16.$$

so $x = -232$ and $y = 244$. Dividing by d gives an equivalent equation

$$590x + 561y = 4$$

where $\gcd(561, 590) = 1$. Thus all solutions of the Diophantine equations are

$$x = -232 + 561k, \qquad y = 244 - 590k$$

where k is an integer. Thus the smallest positive solution occurs when $k = 1$ and $x = 329$. To see that this is indeed a solution,

$$590 \cdot 329 = 561 \cdot 346 + 4.$$

11. *Solve the simultaneous congruences using Sun Zi's method.*

$$\begin{aligned}
x &\equiv 2 \mod 5 \\
x &\equiv 3 \mod 7 \\
x &\equiv 4 \mod 13
\end{aligned}$$

We seek x_1 x_2 and x_3 so that

$$\begin{array}{lll}
x_1 \equiv 1 \mod 5 & x_2 \equiv 0 \mod 5 & x_3 \equiv 0 \mod 5 \\
x_1 \equiv 0 \mod 7 & x_2 \equiv 1 \mod 7 & x_3 \equiv 0 \mod 7 \\
x_1 \equiv 0 \mod 13 & x_2 \equiv 0 \mod 13 & x_3 \equiv 1 \mod 13
\end{array}$$

Then a solution is $x = 2x_1 + 3x_2 + 4x_3$.

The second and third congruences imply $x_1 = 7 \cdot 13j = 91j$ for some integer j. Then the first congruence implies

$$91j + 5k = 1$$

which has a solution by inspection $j = 1$ and $k = -18$. Thus $x_1 = 91j = 91$.

The first and third congruences imply $x_2 = 5 \cdot 13j = 65\ell$ for some integer ℓ. Then the second congruence implies

$$65\ell + 7m = 1.$$

By the Euclidean algorithm

$$\begin{aligned} 65 &= 9 \cdot 7 + 2 \\ 7 &= 3 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

Working backwards

$$1 = 7 - 3 \cdot 2 = 7 - 3 \cdot (65 - 9 \cdot 7) = 28 \cdot 7 - 3 \cdot 65$$

so $\ell = -3$ and $x_2 = 65\ell = -195$.

The first and second and congruences imply $x_3 = 5 \cdot 7n = 35n$ for some integer n. Then the third congruence implies

$$35n + 13p = 1.$$

By the Euclidean algorithm

$$\begin{aligned} 35 &= 2 \cdot 13 + 9 \\ 13 &= 1 \cdot 9 + 4 \\ 9 &= 2 \cdot 4 + 1 \\ 4 &= 4 \cdot 1 + 0 \end{aligned}$$

Working backwards

$$1 = 9 - 2 \cdot 4 = 9 - 2 \cdot (13 - 9) = 3 \cdot 9 - 2 \cdot 13 = 3 \cdot (35 - 2 \cdot 13) - 2 \cdot 13 = 3 \cdot 35 - 8 \cdot 13$$

so $n = 3$ and $x_3 = 35n = 105$.

Thus an answer to the simultaneous congruences is

$$x = 2x_1 + 3x_2 + 4x_3 = 2 \cdot 91 + 3 \cdot (-195) + 4 \cdot 105 = 17.$$

To see that this is a solution, we observe that

$$17 = 3 \cdot 5 + 2 = 2 \cdot 7 + 3 = 1 \cdot 13 + 4.$$

12. *Find all solutions to the simultaneous congruences.*

$$\begin{aligned} x &\equiv 1 \mod 6 \\ x &\equiv 5 \mod 11 \\ x &\equiv 7 \mod 13 \end{aligned}$$

The first implies $x = 6j + 1$ for some integer j. The second then implies

$$6j + 1 \equiv 5 \mod 11$$

or

$$6j \equiv 4 \mod 11.$$

Hence

$$6j + 11k = 4$$

Because $6 \cdot 2 + 11 \cdot (-1) = 1$ we have $6 \cdot 8 + 11 \cdot (-4) = 4$. Hence, since $\gcd(6, 11) = 1$, all solutions have the form

$$j = 11\ell + 8, \qquad k = -6\ell - 4$$

where ℓ is an integer. Thus $x = 6j + 1 = 6(11\ell + 8) + 1 = 66\ell + 49$. The third congruence implies

$$66\ell + 49 \equiv 7 \mod 13$$

or

$$66\ell \equiv -42 \mod 13.$$

Hence

$$66\ell + 13m = -42.$$

Because $66 \cdot 1 + 13 \cdot (-5) = 1$ we have $66 \cdot (-42) + 13 \cdot (210) = -42$. Hence, since $\gcd(66, 13) = 1$, all solutions have the form

$$\ell = 13n - 42, \qquad k = -66m + 210$$

where n is an integer. It follows that all solutions are of the form

$$x = 66\ell + 49 = 66 \cdot (13n - 42) + 49 = 858n - 2723$$

Note that $6 \cdot 11 \cdot 13 = 858$ leaves zero remainder divided by 6, 11 and 13. If $n = 4$, $x = 709$. To see that it is a solution of the simultaneous congruences

$$709 = 118 \cdot 6 + 1 = 64 \cdot 11 + 5 = 54 \cdot 13 + 7.$$

13. *Problem 2 of Zhu Shijie's* Jade Mirror of Four Unknowns *is to find the sides of the right triangle* (a, b, c) *such that*

$$\begin{aligned} a^2 - (b + c - a) &= ab \\ b^2 + (a + c - b) &= bc \end{aligned}$$

Jade Mirror *suggests substituting* $x = a$ *and* $y = b + c$. *Using* $a^2 = c^2 - b^2$ *show that this implies*

$$\begin{aligned} b &= (y - x^2/y)/2 \\ c &= (y + x^2/y)/2. \end{aligned} \tag{3}$$

Deduce that the first two equations are equivalent to

$$(-2-x)y^2 + (2x+2x^2)y + x^3 = 0 \tag{4}$$
$$(2-x)y^2 + 2xy + x^3 = 0. \tag{5}$$

By subtracting one equation from the other, deduce that $y = x^2/2$. Substitute this back and obtain a quadratic equation for x. Then find x, a, b, c. [Stillwell, p. 91.]

Factoring

$$x^2 = a^2 = c^2 - b^2 = (c+b)(c-b) = y(c-b)$$

implies the 2×2 system

$$\begin{aligned} b - c &= -\frac{x^2}{y} \\ b + c &= y \end{aligned}$$

whose solution is (3)

$$b = \frac{1}{2}\left(y - \frac{x^2}{y}\right); \qquad c = \frac{1}{2}\left(y + \frac{x^2}{y}\right).$$

Substituting into the first equation

$$x^2 - (y - x) = \frac{x}{2}\left(y - \frac{x^2}{y}\right)$$

simplifies to (4)

$$(-2-x)y^2 + (2x+2x^2)y + x^3 = 0.$$

Substituting into the second,

$$\frac{1}{4}\left(y - \frac{x^2}{y}\right)^2 + \left(x + \frac{x^2}{y}\right) = \frac{1}{4}\left(y - \frac{x^2}{y}\right)\left(y + \frac{x^2}{y}\right).$$

Multiplying out

$$\frac{1}{4}\left(y^2 - 2x^2 + \frac{x^4}{y^2}\right) + \left(x + \frac{x^2}{y}\right) = \frac{1}{4}\left(y^2 - \frac{x^4}{y^2}\right)$$

which simplifies to

$$\frac{1}{4}\left(-2x^2 + \frac{x^4}{y^2}\right) + x + \frac{x^2}{y} = -\frac{x^4}{4y^2}.$$

Multiplying by $2y^2/x$ gives (5)

$$(2-x)y^2 + 2xy + x^3 = 0.$$

Sebtracting (4) from (5) yields

$$4y^2 - 2x^2y = 0$$

or

$$y = \frac{1}{2}x^2.$$

Substituting into (5) yields

$$\frac{1}{4}(2-x)x^4 + 2x^3 = 0$$

or

$$x^3(x^2 - 2x - 8)$$

whose roots are $x = 0, 4, -2$. Taking the only root that is a positive length $x = a = 4$,

$$y = \frac{1}{2}x^2 = 8; \qquad b = \frac{1}{2}\left(y - \frac{x^2}{y}\right) = 3, \qquad c = \frac{1}{2}\left(y + \frac{x^2}{y}\right) = 5.$$

14. *Use Qin Jiuhao's method to find a root of*

$$x^4 - 57981x^2 - 5808100 = 0.$$

The idea is to mimic the cube root algorithm. Let

$$f(x) = x^4 - 57981x^2 - 5808100.$$

We have

$$f(1000) = 10^{12} = 5.7981 \times 10^{10} - 5.8081 \times 10^6 \gg 0$$

so $x = 1000$ is too big. Look for solutions of the form $x = 100a + 10b + c$. We have

$$f(200) = -725048100, \qquad f(300) = 2875901900$$

so $a = 2$. The idea is to change variables to $x = 200 + y$ where $y = 10b$. This is done by dividing $f(x)$ by $y = x - 200$ to get a smaller polynomial in y. Using long division,

$$\begin{array}{r|cccccc}
 & x^3 & +200x^2 & -17981x & -3596200 & & \\
\cline{2-6}
x - 200 & x^4 & & -57981x^2 & & -5808100 \\
 & x^4 & -200x^3 & & & \\
\cline{2-3}
 & & 200x^3 & -57981x^2 & & \\
 & & 200x^3 & -40000x^2 & & \\
\cline{3-4}
 & & & -17981x^2 & & \\
 & & & -17981x^2 & +3596200\text{x} & \\
\cline{4-5}
 & & & & -3596200x & -5808100 \\
 & & & & -3596200x & +719240000 \\
\cline{5-6}
 & & & & & -725048100
\end{array}$$

Hence

$$f(x) = (x - 200)[x^3 + 200x^2 - 17981x - 3596200] - 725048100$$

Repeating,

$$
\begin{array}{rrrrrr}
 & x^2 & +400x & +62019 & & \\
\cline{2-5}
x-200 \;\big) & x^3 & +200x^2 & -17981x & -3596200 & \\
 & x^3 & -200x^2 & & & \\
\cline{2-3}
 & & 400x^2 & -17981x & & \\
 & & 400x^2 & -80000x & & \\
\cline{3-4}
 & & & 62019x & -3596200 & \\
 & & & 62019x & -12403800 & \\
\cline{4-5}
 & & & & 8807600 &
\end{array}
$$

Hence

$$f(x) = (x-200)\big[(x-200)[x^2+400x+62019]+8807600\big] - 725048100$$

Repeating,

$$
\begin{array}{rrrr}
 & x & +600 & \\
\cline{2-4}
x-200 \;\big) & x^2 & +400x & +62019 \\
 & x^2 & -200x & \\
\cline{2-3}
 & & 600x & +62019 \\
 & & 600x & -120000 \\
\cline{3-4}
 & & & 182019
\end{array}
$$

Hence

$$
\begin{aligned}
f(x) &= (x-200)\Big[(x-200)\big[(x-200)[x+600]+182019\big]+8807600\Big] - 725048100 \\
&= y\Big[y\big[y[y+800]+182019\big]+8807600\Big] - 725048100 \\
&= y^4 + 800y^3 + 182019y^2 + 8807600y - 725048100
\end{aligned}
$$

Let

$$g(y) = y^4 + 800y^3 + 182019y^2 + 8807600y - 725048100$$

Note that

$$g(100) = 2875901900$$

is too big. Now we look for $y = 10b$. Observe that

$$g(40) = -27753700, \qquad g(50) = 276629400$$

so that $b = 4$. Now let look for solutions of the form $y = 40 + z$. As before, we divide $g(y)$ by $z = y - 40$.

$$
\begin{array}{r|rrrrr}
 & y^3 & +840y^2 & +215619y & +17432360 & \\
\hline
y-40 & y^4 & +800y^3 & +182019y^2 & +8807600y & -725048100 \\
 & y^4 & -40y^3 & & & \\
\cline{2-3}
 & & 840y^3 & +182019y^2 & & \\
 & & 840y^3 & -33600y^2 & & \\
\cline{3-4}
 & & & 250819y^2 & +8807600y & \\
 & & & 215619y^2 & -8624760y & \\
\cline{4-5}
 & & & & 17432360y & -725048100 \\
 & & & & 17432360y & -697294400 \\
\cline{5-6}
 & & & & & -27753700
\end{array}
$$

Hence

$$g(y) = (y-40)[y^3 + 840y^2 + 215619y + 17432360] - 27753700$$

Repeating,

$$
\begin{array}{r|rrrr}
 & y^2 & +880y & +250819 & \\
\hline
y-40 & y^3 & +840y^2 & +215619y & +17432360 \\
 & y^3 & -40y^2 & & \\
\cline{2-3}
 & & 880y^2 & +215619y & \\
 & & 880y^2 & -35200y & \\
\cline{3-4}
 & & & 250819y & +17432360 \\
 & & & 250819y & -10032760 \\
\cline{4-5}
 & & & & 27465120
\end{array}
$$

Hence

$$g(y) = (y-40)\big[(y-40)[y^2 + 880y + 250819] + 27465120\big] - 27753700$$

Repeating,

$$
\begin{array}{r|llll}
 & y & +920 & \\
\hline
y-40 & y^2 & +880y & +250819 \\
 & y^2 & -40y & \\
\hline
 & & 920y & +250819 \\
 & & 920y & -36800 \\
\hline
 & & & 287619
\end{array}
$$

Hence

$$
\begin{aligned}
g(y) &= (y-40)\Big[(y-40)\big[(y-40)[y+920]+287619\big]+27465120\Big]-27753700 \\
&= z\Big[z\big[z[z+960]+287619\big]+27465120\Big]-27753700 \\
&= z^4+960z^3+287619z^2+27465120z-27753700
\end{aligned}
$$

Let

$$h(z) = z^4+960z^3+287619z^2+27465120z-27753700$$

We notice that $h(1)=0$. Thus a solution is $x = \boxed{241}$ since $f(241)=0$. Where we have used long division, Qin Jiushao used synthetic division which he executed on a counting board. For example, here is how the first long division above would be placed as a tableau. His tableau would have been vertically oriented bottom-to-top instead of the modern horizontal left-to-right orientation used for synthetic division.

$$
\begin{array}{cccccc}
-200 & 1 & 0 & -57981 & 0 & -5808100 \\
 & & -200 & -4000 & 3596200 & 719240000 \\
 & 1 & 200 & -17981 & -3596200 & -725048100
\end{array}
$$

The computation terminated with an integral solution in this case but, in general, the process can continued to produce as many decimal places for the solution as desired. [c.f. Franz, pp. 214–217.]

15. *Find a solution to Pell's equation using Bhâskara II's* chakravâla *(cyclic process).*

$$x^2-19y^2=1$$

Let $N=19$. If the three variables a, b and k satisfy $a^2-Nb^2=k$ we write the triple "(a,b,k)." Brahmagupta's formula relating two solutions of Pell's equation is

$$(x_1x_2+Ny_1y_2)^2-N(x_1y_2+x_2y_1)^2=(x_1^2-Ny_1^2)(x_2^2-Ny_2^2).$$

Thus if $x_1^2-Ny_1^2=k_1$ and $x_2^2-Ny_2^2=k_2$ then we get a formula that combines two solutions to make a third solution. It may be written

$$\mathcal{B}\Big((x_1,y_1,k_1),(x_2,y_2,k_2)\Big) = \Big(x_1x_2+Ny_1y_2, \quad x_1y_2+x_2y_1, \quad k_1k_2\Big).$$

By dividing by k^2 we get smaller solutions, but they may not be integral. The idea of the *chakravâla* is to start from any solution with a small k and then look for another solution whose combination can be divided. In this way we get a second solution with, hopefully, smaller k. The procedure is repeated until the original Pell's equation is solved. The Indian mathematicians didn't know if the procedure will ever be successful, but it has been proved to work in modern times.

Using the fact that $m^2 - N \cdot 1^2 = m^2 - N$ we have a family of solutions $(m, 1, m^2 - N)$. Bhâskara II chose m wisely so the combination can be divided. The combination gives

$$\mathcal{B}\Big((a,b,k),(m,1,m^2-N)\Big) = \Big(am+Nb,\ a+bm,\ k(m^2-N)\Big)$$

Dividing by k^2 gives the solution

$$\left(\frac{am+Nb}{k},\ \frac{a+bm}{k},\ \frac{m^2-N}{k}\right)$$

The number m is now chosen so that $\dfrac{a+bm}{k} = -n$ is an integer and so that $\dfrac{m^2-N}{k}$ is as small as possible. It turns out that $\dfrac{am+Nb}{k}$ and $\dfrac{m^2-N}{k}$ will be integral too [Stillwell, p. 80]. This means that m satisfies the Diophantine equation for m and n.

$$bm + kn = -a$$

If we find a solution m_0 and n_0, then if $\gcd(b,k) = 1$, all solutions have the form

$$m = m_0 + kt, \qquad n = n_0 - bt$$

for some integer t. Now t is chosen so that $|m^2 - N|$ is as small as possible.

Let us now show that a small solution can be improved by the *cakravâla*. By noodling around, we notice that $(a,b,k) = (13,3,-2)$ is a small solution. Solving the Diophantine equation

$$bm + kn = 3m - 2n = -13 = -a$$

we have $3 \cdot 1 - 2 \cdot 1 = 1$ so that $3 \cdot (-13) - 2 \cdot (-13) = -13$. Since $\gcd(3,2) = 1$, all solutions have the form

$$m = -13 + 2t, \qquad n = -13 + 3t$$

for t some integer. For the values $t = 2,4,5,6,7,8,9,10$ we have $m = -7,-5,-3,-1,1,3,5,7$ so $m^2 - N = 30,6,-10,-18,-18,-10,6,30$, resp., so that the minimum occurs when $t = 4,9$, $m = -5,5$ and $m^2 - N = 6,6$. Taking $m = 5$ we get the solution

$$\left(\frac{am+Nb}{k},\ \frac{a+bm}{k},\ \frac{m^2-N}{k}\right) = \left(\frac{13\cdot 5 + 19 \cdot 3}{-2},\ \frac{13 + 3 \cdot 5}{-2},\ \frac{5^2-19}{-2}\right) = (-61,-14,-3)$$

so $(61,14,-3)$ is a solution since the signs of a and b don't matter.

Cycling again with $(a,b,k) = (61,14,-3)$, solving the Diophantine equation

$$bm + kn = 14m - 3n = -61 = -a$$

we have $14 \cdot 2 - 3 \cdot 9 = 1$ so $14 \cdot (-122) - 3 \cdot (-549) = -61$. Since $\gcd(14,3) = 1$, all solutions have the form

$$m = -122 + 3t, \qquad n = -549 + 14t$$

for t some integer. For the values $t = 38, 39, 40, 41, 42, 43$ we have $m = -8, -5, -2, 1, 4, 7$ so $m^2 - N = 45, 6, -15, -18, -3, 30$, resp., so that the minimum occurs when $t = 42$, $m = 4$ and $m^2 - N = -3$. We get the solution

$$\left(\frac{am+Nb}{k}, \frac{a+bm}{k}, \frac{m^2-N}{k}\right) = \left(\frac{61\cdot 4 + 19\cdot 14}{-3}, \frac{61+14\cdot 4}{-3}, \frac{4^2-19}{-3}\right) = (-170, -39, 1)$$

so $\boxed{(170, 39, 1)}$ is a solution. Indeed

$$170^2 - 19\cdot 39^2 = 28900 - 19\cdot 1521 = 28900 - 28899 = 1.$$

Brahmagupta would have probably simply noticed that

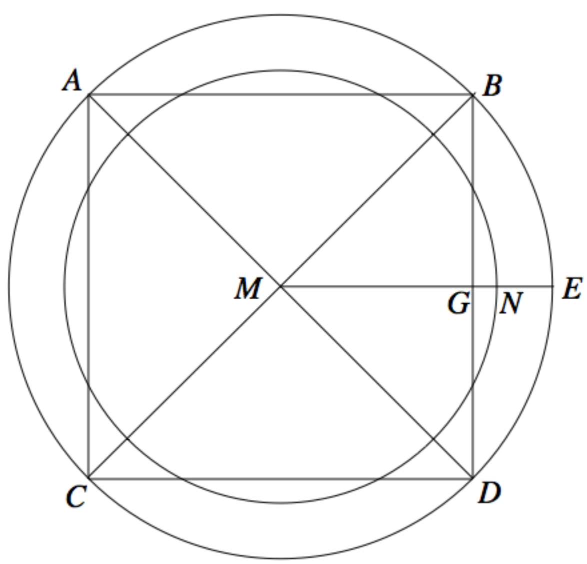
$$\mathcal{B}\Big((13, 3, -2),\ (13, 3, -2)\Big) = \Big(340, 78, 4\Big)$$

which can be divided by two to get the solution $(170, 39, 1)$.

16. *The* Baudhayana Sulbasutra *from India about 600* BC *tells how to circle the squqre and square the circle. Determine the implicit values of π that these recipes give. [Katz, A History of Mathematics, 2009, p. 238]*

    If it is desired to transform a square into a circle,
    a chord of length half the diagonal of the square
    is stretched from the center to the east,
    a part of it lying outside the eastern side of the square.
    With one third of the part lying outside
    added to the remainder of the half diagonal,
    the requisite circle is drawn.

    To transform a circle into a square
    the diameter is divided into eight parts;
    one such part, after being divided into twenty-nine parts,
    is reduced by twenty-eight of them
    and further by a sixth of the part left
    less the eight of the sixth part.
    (The remainder is then the side of the required square.)



For the first part, let $r = L(MN)$ be the radius of the desired circle. If the side of the original square is s, then a chord of half the length of the diagonal ME is $\dfrac{\sqrt{2}s}{2}$ and one third of the part lying outside GE has length $\dfrac{1}{3}\left(\dfrac{\sqrt{2}s - s}{2}\right)$. Adding to the remainder of the half the diagonal (MG) gives

$$r = \frac{s}{2} + \frac{1}{3}\left(\frac{\sqrt{2}s - s}{2}\right) = \frac{(\sqrt{2}+2)s}{6}$$

Comparing to the formula $s^2 = \pi r^2$,

$$\pi = \frac{s^2}{r^2} = \frac{36}{(\sqrt{2}+2)^2} \approx 3.088311755.$$

Let D be the diameter of the circle and s the side of the desired square. D is reduced twenty-eight twenty-ninths of one eighth $D - \dfrac{28D}{29 \cdot 8}$, and further reduced by a sixth of the part left less an eight of the sixth part $-\left(\dfrac{D}{6 \cdot 29 \cdot 8} - \dfrac{D}{8 \cdot 6 \cdot 29 \cdot 8}\right)$ resulting in

$$s = \left(1 - \frac{28}{29 \cdot 8} - \frac{1}{6 \cdot 29 \cdot 8} + \frac{1}{8 \cdot 6 \cdot 29 \cdot 8}\right) D$$

Comparing to the formula $4s^2 = \pi D^2$,

$$\pi = \frac{4s^2}{D^2} = 4\left(1 - \frac{28}{29 \cdot 8} - \frac{1}{6 \cdot 29 \cdot 8} + \frac{1}{8 \cdot 6 \cdot 29 \cdot 8}\right)^2 = \frac{95746225}{31002624} \approx 3.088326491.$$

17. *Show the following transformation of solutions of certain Pell's equations of Brahmagupta: if $u^2 - Nv^2 = -4$ then setting $x = (u^2+2)[\frac{1}{2}(u^2+1)(u^3+3)-1]$ and $y = \frac{1}{2}uv(u^2+1)(u^2+3)$ we have that x and y are integers and solve $x^2 - Ny^2 = 1$. Then find a solution to $x^2 - 13y^2 = 1$. [Katz, A History of Mathematics, 2009, p. 262]*

Note that if u is odd then $u^2 + 1$ is even and $\frac{1}{2}(u^2+1)$ is an integer so also is

$$x = (u^2+2)[\frac{1}{2}(u^2+1)(u^3+3) - 1].$$

If u is even then $u^2 + 2$ is even and $\frac{1}{2}(u^2+2)$ is an integer, as is x. If u or v is even, then $\frac{1}{2}uv$ is an integer, as is

$$y = \frac{1}{2}uv(u^2+1)(u^2+3).$$

If nether u nor v are even then $u^2 + 1$ is even and $\frac{1}{2}(u^2+1)$ is integral, as is y.

Using the equation $Nv^2 = u^2 + 4$,

$$\begin{aligned} Ny^2 &= \frac{1}{4}Nu^2v^2(u^2+1)^2(u^2+3)^2 \\ &= \frac{1}{4}u^2(u^2+4)(u^2+1)^2(u^2+3)^2 \\ &= \frac{1}{4}\left\{(u^2+2)^2 - 4\right\}(u^2+1)^2(u^2+3)^2 \\ &= \frac{1}{4}(u^2+2)^2(u^2+1)^2(u^2+3)^2 - (u^2+1)^2(u^2+3)^2 \end{aligned}$$

but

$$\begin{aligned}
x^2 &= (u^2+2)^2\left\{\frac{1}{2}(u^2+1)(u^2+3)-1\right\}^2 \\
&= (u^2+2)^2\left\{\frac{1}{4}(u^2+1)^2(u^2+3)^2-(u^2+1)(u^2+3)+1\right\} \\
&= \frac{1}{4}(u^2+2)^2(u^2+1)^2(u^2+3)^2-(u^2+2)^2\left\{(u^2+1)(u^2+3)-1\right\}
\end{aligned}$$

so

$$\begin{aligned}
x^2-Ny^2 &= -(u^2+2)^2\left\{(u^2+1)(u^2+3)-1\right\}+(u^2+1)^2(u^2+3)^2 \\
&= -(u^2+2)^2\left\{u^4+4u^2+2\right\}+(u^4+4u^2+3)^2 \\
&= -(u^2+2)^2\left\{(u^2+2)^2-2\right\}+\left[(u^2+2)^2-1\right]^2 \\
&= -(u^2+2)^4+\left\{2(u^2+2)^2+\left[(u^2+2)^2-1\right]^2\right\} \\
&= -(u^2+2)^4+\left\{(u^2+2)^4+1\right\} \\
&= 1.
\end{aligned}$$

We note that $3^2-13\cdot 1^2=-4$ so we may take $N=13$, $u=3$ and $v=1$. Then

$$\begin{aligned}
x &= (u^2+2)\left[\frac{1}{2}(u^2+1)(u^2+3)-1\right] = (3^2+2)\left[\frac{1}{2}(3^2+1)(3^3+3)-1\right] = 649, \\
y &= \frac{1}{2}uv(u^2+1)(u^2+3) = \frac{1}{2}\cdot 3\cdot 1\cdot(3^2+1)(3^2+3) = 180.
\end{aligned}$$

Indeed

$$649^2-13\cdot 180^2 = 421201-13\cdot 32400 = 421201-421200 = 1.$$

18. *Among the triangles whose sides are $a=12$, $b=5$ and an integer c, which are rational?*

The triangle inequalities, which say that the length of every side is less than the sum of the lengths of the other sides

$$c\le a+b, \qquad b\le a+c, \qquad a\le b+c$$

imply for $a=5$ and $b=12$ that

$$7=|a-b|=\max\{b-a,a-b\}\le c\le a+b=17.$$

If either equality holds, then the triangle degenerates to a line: the two shorter sides lie on the longest side. Their area is zero so they are rational.

Thus the non-degenerate triangles correspond to those whose third side $c=8,9,\ldots,16$. For example if $c=13$ then the triangle is right and its area is $\frac{1}{2}ab=30$ is rational.

To compute the area in general, let us employ Heron's formula.

$$A=\sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where } s=\frac{a+b+c}{2} \text{ is the semiperimeter.}$$

We find for $a = 5$ and $b = 12$

$$\begin{aligned} 16s(s-a)(s-b)(s-c) &= (a+b+c)(b+c-a)(a+c-b)(a+b-c) \\ &= (17+c)(c+7)(c-7)(17-c) = (289-c^2)(c^2-49) \end{aligned}$$

We obtain

c	$289 - c^2$	$c^2 - 49$	$(289 - c^2)(c^2 - 49)$	A
8	$225 = 3^2 \cdot 5^2$	$15 = 3 \cdot 5$	3375	14.52369
9	$208 = 2^2 \cdot 3 \cdot 17$	$32 = 2^5$	6656	20.39608
10	$189 = 3^3 \cdot 7$	$51 = 3 \cdot 17$	9639	24.54460
11	$168 = 2^3 \cdot 3 \cdot 7$	$72 = 2^3 \cdot 3^2$	12096	27.49545
12	$145 = 5 \cdot 29$	$95 = 5 \cdot 19$	13775	29.34174
13	120	120	14400	30.00000
14	$93 = 3 \cdot 31$	$147 = 3 \cdot 7^2$	13671	29.23076
15	$64 = 2^6$	$176 = 2^4 \cdot 11$	11264	26.53300
16	$33 = 3 \cdot 11$	$207 = 3^2 \cdot 23$	6831	20.66247

Note that, except for $c = 8$ and $c = 13$, each $(289 - c^2)(c^2 - 49)$ has at least one prime factor to the first power, which has an irrational square root, thus an irrational area. When $c = 8$, $A = \dfrac{15\sqrt{15}}{4}$ which is also irrational. Thus the only rational triangle is the right one with $c = 13$.

19. *Aryabhata (476–570) knew the following formula. Prove it.*

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

Let $S_n = 1^3 + 2^3 + \cdots + n^3$ and $T_n = 1 + 2 + \cdots + n$. We first prove

$$T_n = \frac{n(n+1)}{2}$$

Using induction, when $n = 1$ we have $T_1 = 1 = \dfrac{1 \cdot 2}{2}$ proving the base case. Assuming the induction hypothesis,

$$T_{n+1} = T_n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2},$$

proving the induction case.

For the original question, using induction, when $n = 1$ we have $S_1 = 1^3 = (1)^2$ proving the base case. Assume the formula holds for n. Then, using the induction hypothesis and the formula for T_n,

$$\begin{aligned} S_{n+1} &= S_n + (n+1)^3 \\ &= T_n^2 + (n+1)(n+1)^2 \\ &= T_n^2 + n(n+1)^2 + (n+1)^2 \\ &= T_n^2 + 2T_n(n+1) + (n+1)^2 \\ &= (T_n + (n+1))^2 \\ &= (T_{n+1})^2 . \end{aligned}$$

Combining formulas, we have also proved that

$$S_n = \frac{n^2(n+1)^2}{4}.$$

20. *Solve this problem from Mahavira's text (* AD *950) which was the first book devoted entirely to mathematics. [Katz, A History of Mathematics, 2009, p. 262]*

One-third of a herd of elephants
and three times the square root
of the remaining part of the herd
were seen on a mountain slope;
and in a lake was seen a male elephant
along with three female elephants
constituting the ultimate remainder.
How many were the elephants here?

The quadratic formula was well known among Indian mathematicians at this time. Let x denote the number of elephants in the herd. Then the verse tells us where parts of the herd reside

$$\frac{1}{3}x + 3\sqrt{x - \frac{1}{3}x} + 1 + 3 = x$$

so that

$$3\sqrt{\frac{2}{3}x} = \frac{2}{3}x - 4.$$

Squaring

$$9 \cdot \frac{2}{3}x = \left(\frac{2}{3}x - 4\right)^2 = \frac{4}{9}x^2 - \frac{16}{3}x + 16$$

or

$$\frac{4}{9}x^2 - \frac{34}{3}x + 16 = 0.$$

Solving for x by the quadratic formula

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{9}{8}\left(\frac{34}{3} \pm \sqrt{\frac{34^2}{3^2} - 4 \cdot \frac{4}{9} \cdot 16}\right) \\ &= \frac{9}{8}\left(\frac{34}{3} \pm \sqrt{100}\right) \\ &= \frac{9}{8}\left(\frac{34}{3} \pm \frac{30}{3}\right) = 3 \cdot 8, 3 \cdot \frac{1}{2} = 24, \frac{3}{2}. \end{aligned}$$

The number of elephants is a whole number so $x = \boxed{24}$ elephants.

21. *Use the half-angle formula, the addition formulas and Pythagorean theorem to compute the missing entries in the Indian table of sines. These were certainly known to Bhaskara I. (You do not need to use the Indian method to find square roots!) Then check using a calculator. [$Sin(\alpha) = R\sin\alpha$ where $R = 3438$ and α is in minutes.]*

Minutes	Sine	Sine Difference
0	0	* * *
900		
1800	1719	
2700		
3600		
4500		
5400	3438	

Recall that $\text{Sin}(\alpha) = R\sin\alpha$ is the Indian Sine. $\dfrac{900'}{60 \text{ min. per deg.}} = 15°$. The half angle formula is

$$\text{Sin}\left(\frac{\alpha}{2}\right) = R\sin\left(\frac{\alpha}{2}\right) = R\sqrt{\frac{1-\cos\alpha}{2}}$$

$2700' = 45°$, so $\text{Sin}(2700') = \frac{R}{\sqrt{2}} = 2431.033 \approx 2431$. Checking, $R\sin(45°) = 2431.033$.

$3600' = 60°$ so $\text{Cos}(60°) = R\cos 60° = 3438 \cdot \dfrac{\sqrt{3}}{2} = 2977.395 \approx 2977$. Checking $R\sin 60° = 2977.395$. Note $\text{Cos}(30°) = \text{Sin}(60°)$.

Now $\cos(1800') = \dfrac{\sqrt{3}}{2}$ so

$$\text{Sin}(900') = 3438\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = 889.8199 \approx 890$$

Checking $R\sin(15°) = 889.8199$.

Finally $4500' = 75°$. From the addition formula

$$\begin{aligned}\text{Sin}(4500') &= R\sin(45° + 30°)\\ &= R(\sin(45°)\cos(30°) + \cos(45°)\sin(30°))\\ &= 3438\left(\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right) = 3320.853 \approx 3321\end{aligned}$$

Checking, $R\sin(75°) = 3320.853$.

The differences are current sine minus the previous sine. The completed table is

Minutes	Sine	Sine Difference
0	0	* * *
900	890	890
1800	1719	829
2700	2431	712
3600	2977	546
4500	3321	344
5400	3438	117

22. *Use Brahmagupta's second order difference scheme to approximate the Indian* $\mathrm{Sin}(700')$. *Recall* $R = 3438$. *Use the partial table of Bhaskara.*

Minutes	Sine	Sine Difference
0	0	* * *
225	225	225
450	449	224
675	671	222
900	890	219
1025	1105	215
1250	1315	210

Let $700' = 675' + x$. In general, for $0 \le x \le 225$, Brahmagupta's formula amounts to finding the parabola $p(675' + x)$ that fits the points $p(450) = 449$, $p(675) = 671$ and $p(900) = 890$ and then evaluating $p(700)$. Let us give a modern derivation of Brahmagupta's interpolation formula. Because the step $d = 225'$ is constant, we may write

$$p(675 + x) = c_0 + c_1 x + c_2 x^2$$

We get a 3×3 system for the coefficients

$$\begin{aligned}
449 = y_1 = p(450) = p(675 - d) &= c_0 - c_1 d + c_2 d^2 \\
671 = y_2 = p(675) = p(675 - 0) &= c_0 \\
890 = y_3 = p(900) = p(675 + d) &= c_0 + c_1 d + c_2 d^2
\end{aligned}$$

Solving we get $c_0 = y_2$, $y_3 - y_1 = 2c_1 d$ and $y_3 + y_1 = 2c_0 + 2c_2 d^2$ so

$$\begin{aligned}
c_0 &= y_2, \\
c_1 &= \frac{y_3 - y_1}{2d} = \frac{(y_3 - y_2) + (y_2 - y_1)}{2d}, \\
c_2 &= \frac{y_3 - 2y_2 + y_1}{2d^2} = \frac{(y_3 - y_2) - (y_2 - y_1)}{2d^2}
\end{aligned}$$

Note that the coefficients are expressible in terms of the sine differences. Hence

$$p(675 + x) = y_2 + \frac{(y_3 - y_2) + (y_2 - y_1)}{2d} x + \frac{(y_3 - y_2) - (y_2 - y_1)}{2d^2} x^2$$

In the present case, $700' = 675' + 25'$ so

$$p(675 + 25) = 671 + \frac{219 + 222}{2 \cdot 225} \cdot 25 + \frac{119 - 222}{2(225)^2} \cdot 25^2 = 694.8642 \approx \boxed{695}.$$

In fact $R \sin 700' = 695.224$.

23. *Show that Bhaskara I's rational approximation formula approximates* $\mathrm{Sin}(\alpha)$ *with an error of no more than* 1%*. Find the values that are most in error. [Katz, A History of Mathematics, 2009, p. 263]*

$$\mathrm{Sin}(x) \approx \rho(\alpha) = \frac{4R\alpha(180 - \alpha)}{40,500 - \alpha(180 - \alpha)}, \qquad \text{where } R = 3438.$$

The historian R. C. Gupta has given the following suggestion on how this formula was derived [Ganita Bharati 8, (1986).] Both the functions

$$\begin{aligned}
P(\alpha) &= \frac{R\alpha(180 - \alpha)}{90 \cdot 90} \\
F(\alpha) &= \frac{\alpha(180 - \alpha)\,\mathrm{Sin}(\alpha)}{90 \cdot 90} = \frac{\mathrm{Sin}(\alpha) P(\alpha)}{R}
\end{aligned}$$

have the single-peaked, symmetric about $\alpha = 90$ shape of $\mathrm{Sin}(\alpha)$ and agree with it at the angles $\alpha = 0, 90, 180$. At $\alpha_0 = 30, 150$, $\mathrm{Sin}(\alpha_0) = \frac{1}{2} R$ whereas

$$P(\alpha_0) = \frac{R \cdot 30(180 - 30)}{90 \cdot 90} = \frac{5}{9} R; \qquad F(\alpha_0) = \frac{30(180 - 30)\,\mathrm{Sin}(\alpha_0)}{90 \cdot 90} = \frac{5}{18} R$$

Thus, requiring that the rational function of $P(\alpha)$ and $F(\alpha)$ give the correct values also at $\alpha = \alpha_0$ we have

$$\frac{P(\alpha_0) - \mathrm{Sin}(\alpha_0)}{F(\alpha_0) - \mathrm{Sin}(\alpha_0)} = \frac{\frac{5}{9} R - \frac{1}{2} R}{\frac{5}{18} R - \frac{1}{2} R} = -\frac{1}{4}$$

Solving for $\mathrm{Sin}(\alpha_0)$,

$$P(\alpha_0) - \mathrm{Sin}(\alpha_0) = -\frac{1}{4} \mathrm{Sin}(\alpha_0) \left(\frac{P(\alpha_0)}{R} - 1 \right)$$

or

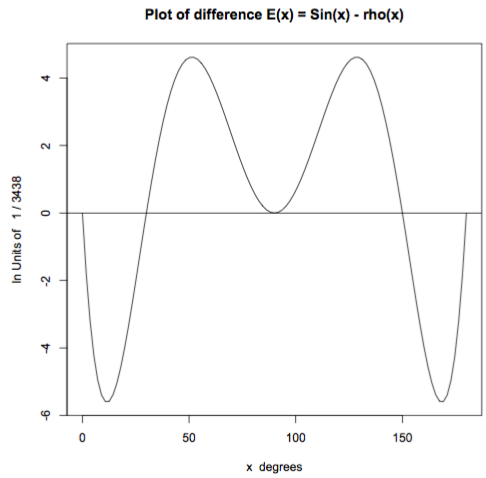
$$\mathrm{Sin}(\alpha_0) = \frac{4P(\alpha_0)}{5 - \dfrac{P(\alpha_0)}{R}} = \frac{4R\alpha_0(180 - \alpha_0)}{40500 - \alpha_0(180 - \alpha_0)} = \rho(\alpha_0)$$

At α_0 the rational function is also correct.

$$\rho(\alpha_0) = \frac{4R \cdot 30(180-30)}{40500 - 30(180-30)} = \frac{1}{2}R.$$

Let us look at the error

$$E(x) = \mathrm{Sin}(x) - \rho(x).$$

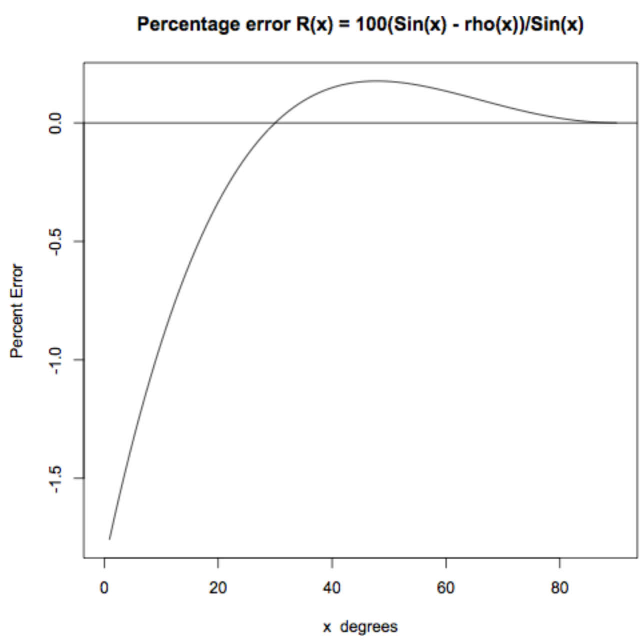


Its derivative is

$$E'(x) = \frac{R\pi}{180}\cos\left(\frac{\pi x}{180}\right) - \frac{16200R(180-2x)}{[40500 - x(180-x)]^2}$$

It equals zero at $x = 11.543829$ and $x = 51.345846$ at the minimum and maximum of $E(x)$. At these points $E(11.54382879) = -5.610008$ and $E(51.34584599) = 4.619629$, the points were the maximum discrepancy occurs. At worst, $|E(x) - \rho(x)| \le 5.61$. To get the percentage error, we consider the ratio

$$R(x) = 100\frac{\mathrm{Sin}(x) - \rho(x)}{\mathrm{Sin}(x)}.$$



The worst percentage error occurs at $z = 0°$ and $x = 180°$. We find the value using L'Hopital's rule.

$$\lim_{x\to 0+} R(x) = \lim_{x\to 0+} \frac{100E(x)}{\mathrm{Sin}(x)} \lim_{x\to 0+} = \lim_{x\to 0+} \frac{100\frac{d}{dx}E(x)}{\frac{\mathrm{d}}{\mathrm{dx}}\mathrm{Sin}(x)}$$

$$= 100 \lim_{x\to 0+} \frac{\frac{R\pi}{180}\cos\left(\frac{\pi x}{180}\right) - \frac{16200R(180-2x)}{[40500 - x(180-x)]^2}}{\frac{R\pi}{180}\cos\left(\frac{\pi x}{180}\right)}$$

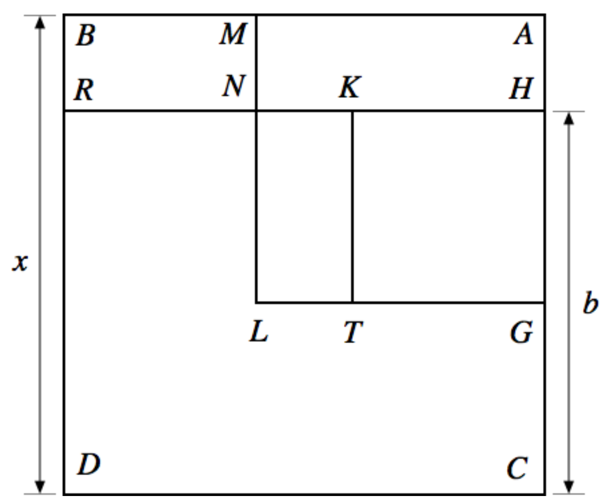
$$= 100 \frac{\frac{\pi}{180} - \frac{16200 \cdot 180}{40500^2}}{\frac{\pi}{180}} = -1.859164$$

We observe that $R(10) = -0.9269054$. Thus Bhaskara I's approximation makes less than 2% error for values close to 0° and 180° and less than 1% error if $10° \le x \le 170°$.

24. *Al-Kwarizmi gives the following rule for solving $bx + c = x^2$:*

    Halve the number of roots.
    Multiply this by itself.
    Add this square to the number.
    Extract the square root.
    Add this to half the number of roots.
    That is the solution.

    Translate this rule into a formula. Give a geometric argument for its validity using the figure, where $x = AB$, $b = HC$, c is represented by the rectangle $ABRH$, and G is the midpoint of HC. [Katz, A History of Mathematics, 2009, p. 318]



The number of roots x is b. The number is c. Thus the verse tells us

$$x = \sqrt{\frac{b}{2} \cdot \frac{b}{2} + c} + \frac{b}{2}.$$

Of course this is just the quadratic formula applied to $x^2 - bx - c = 0$, namely,

$$x = \frac{-(-b) \pm \sqrt{(-b)^2 - 4a(-c)}}{2a} = \frac{b \pm \sqrt{b^2 + 4c}}{2} = \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c}$$

where we take "+" since negative numbers were not recognized.

In the figure, $ABDC$ is a square of side x. On the one hand, the total square has area x^2 which equals the sum of the areas of rectangle $HRDC$, which is bx, and the rectangle $ABRH$ which is c, giving the equation.

On the other hand, let us express x in terms of lengths. Since G is the midpoint of HC, the lengths of HG and GC are $\frac{b}{2}$. On the segment RH, mark off distances RN and KH also equal to $\frac{b}{2}$. Thus the length NK, which is $x - \frac{b}{2} - \frac{b}{2} = x - b$ is the same as the length of AH. Thus the rectangles $MBRN$ and $KNLT$ have the same area. It follows that the square $AMLB$ has area equal to the area of the square $HKTG$, which is $\frac{b}{2} \cdot \frac{b}{2}$ plus the sum of the areas of the two rectangles $AMNH$ and $KNLT$ which equals the sum of the areas of the rectangles $AMNH$ and $MBRN$ which add to c. In other words, the area of the square $AMLG$ is $\frac{b}{2} \cdot \frac{b}{2} + c$. The length of a side of this square, AG is thus $\sqrt{\frac{b}{2} \cdot \frac{b}{2} + c}$. Finally, the total length of AC, namely x, is the sum of the lengths of AG and GC, namely $x = \sqrt{\frac{b}{2} \cdot \frac{b}{2} + c} + \frac{b}{2}$, as to be proved.

25. *Solve the following problem of Al-Kwarizmi:*

> *I have 10 divided into two parts,*
> *and have divided the first by the second,*
> *and the second by the first*
> *and the sum of the quotients is* $2\frac{1}{6}$.
> *Find the solution.*

[Katz, A History of Mathematics, 2009, p. 318]

Let x and $10 - x$ be the parts. The problem tells us that

$$\frac{x}{10 - x} + \frac{10 - x}{x} = \frac{13}{6}.$$

Multiplying the equation by $x(10 - x)$ yields

$$x^2 + (10 - x)^2 = \frac{13}{6}x(10 - x)$$

which simplifies to

$$6x^2 + 600 - 120x + 6x^2 = 130x - 13x^2$$

or

$$25x^2 - 250x + 600 = 0.$$

Dividing by 25,

$$x^2 - 10x + 24 = 0.$$

By the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{10 \pm \sqrt{100 - 96}}{2} = 5 \pm 1.$$

Thus the parts are 4 and 6.

26. *Here is a problem from the Egyption mathematician Abu Kamil ibn Aslam (850–930):*

$$x < y < z, \qquad x^2 + y^2 = z^2, \qquad xz = y^2, \qquad xy = 10$$

Put $y = \dfrac{10}{x}$ and $z = \dfrac{100}{x^3}$ and substitute

$$x^2 + \frac{100}{x^2} = \frac{100^2}{x^6}$$

Clearing fractions

$$x^8 + 100x^4 - 100^2 = 0$$

Solving for x^4,

$$x^4 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-100 \pm \sqrt{100^2 + 4 \cdot 100^2}}{2} = -50 \pm 50\sqrt{5}.$$

The positive root corresponds to

$$x^4 = -50 + 50\sqrt{5}$$

so

$$x = \sqrt{\sqrt{-50 + 50\sqrt{5}},} \quad .$$

27. *Give a combinatorial argument for Ibn al-Banna's (1256–13210) formula for combinations C_k^n, the number of subsets of size k taken from n things, where order is not important*

$$C_k^n = \frac{n - (k-1)}{k} C_{k-1}^n.$$

To each subset of $k - 1$ elements, pick one of the remaining $n - (k - 1)$ elements not in the subset to make up a k element subset. There are $[n - (k - 1)]C_{k-1}^n$ ways to do this. However, this scheme results in some duplications. The resulting k element subset could have occurred with any $k - 1$ of these k elements taken first, and the remaining element chosen last, in other words $C_{k-1}^k = k$ duplications. Hence we must divide by the number of duplications to yield

$$C_k^n = \frac{n - (k-1)}{C_{k-1}^k} C_{k-1}^n = \frac{n - (k-1)}{k} C_{k-1}^n.$$

For example

$$C_2^n = \frac{n-1}{2} C_1^n = \frac{n(n-1)}{2}, \quad C_3^n = \frac{n-2}{3} C_2^n = \frac{n(n-1)(n-2)}{3 \cdot 2}.$$

Continuing in this fashion (or by induction)

$$C_k^n = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

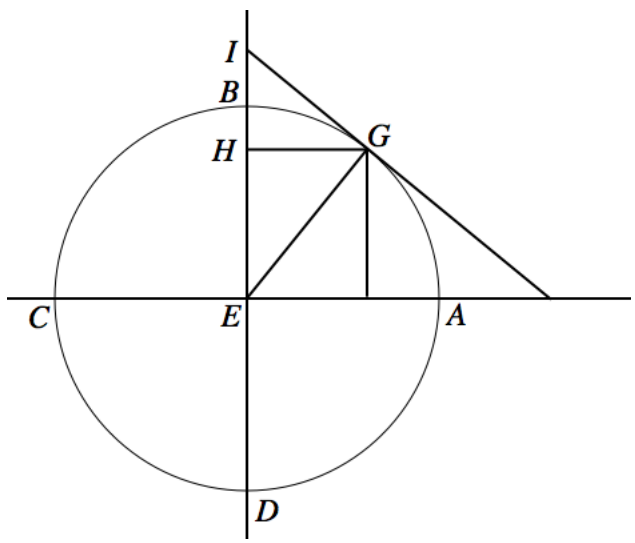
28. *Here is a question like those considered by Ahmad al-Abdari ibn Munim (1199–1213) of Marrakech. How many* 11 *letter words can be made from an alphabet of* 28 *letters such that three letters occur once, one letter occurs twice and two letters occur three times?*

There are $C^{28}_2 = 378$ choices for the two letters that appears three times. The lesser of these letters may occur in several places in the word. The number of places corresponds to the number of subsets of size three taken from 11, which is $C^{11}_3 = 11 \cdot 10 \cdot 9/3! = 165$. Then greater of these letters may occur in any three places among the remaining 8 letters, of $C^8_3 = 8 \cdot 7 \cdot 6/3! = 56$ times. Note that we didn't pick the first letter occurring three times and its positions and then the second and its positions because the same configuration occurs twice, once when a letter is chosen first with its positions, or when the letter is chosen second, with the same positions. There are 26 remaining letter choices for the letter that occurs twice. It may occur in any of two places among the remaining five places, or $C^5_2 = 5 \cdot 4/2 = 10$ times. There are 25 remaining choices for the first letter that occurs once, 24 then are left for the second letter that occurs once, and 23 unpicked letters for the last letter that occurs once. The product is the number of words

$$n = 378 \cdot 165 \cdot 56 \cdot 26 \cdot 10 \cdot 25 \cdot 24 \cdot 23 = 12,531,879,360,000.$$

29. *Here is a problem published by Umar ibn Ibrahim al-Khayyami (1048–1131). Find a point G on the circle $ABCD$ such that the tangent to the circle at G meets the vertical through the center E at the point I in such a way that*

$$\frac{AE}{GH} = \frac{EH}{BH}$$



Let us assign some dimensions to the circle and get an equation for the point G. al-Khayyami chose $EH = 10$ and $GH = x$. hence $GE^2 = x^2 + 100$.

Now since the line IG is tangent to the circle, $\angle EGI = 90°$ is a right angle. Thus the triangles $\triangle EGI$ is similar to $\triangle IHG$ is similar to $\triangle GHE$. It follows that

$$\frac{EG}{EI} = \frac{EH}{EG}$$

of $EG^2 = EH \cdot EI$. Dividing by 10,

$$\frac{GE^2}{10} = \frac{x^2}{10} + 10 = \frac{EH \cdot EI}{10} = EI.$$

Using the assumption $AE \cdot BH = EH \cdot GH$ we see from $AE = EG$, $HB = EB - EH = EG - EH$, the Pythagorean Theorem that

$$\frac{GH}{EG} = \frac{BH}{EH} = \frac{EG - EH}{EH} = \frac{EG^2 - EH^2}{EH(EG + EH)} = \frac{GH^2}{EH(EG + EH)}$$

which implies

$$\frac{EG + EH}{EG} = \frac{GH}{EH}. \tag{6}$$

Now using similarith of $\triangle IHG$ and $\triangle GHE$ so $HI/GH = GH/EH$, the assumption $EG/GH = EH/BH$ and (6)

$$\begin{aligned}
EI &= EB + BI \\
&= EG + (HI - BH) \\
&= EG + \left(\frac{GH^2}{EH} - \frac{EH \cdot GH}{EG}\right) \\
&= EG + GH\left(\frac{GH}{EH} - \frac{EH}{EG}\right) \\
&= EG + GH.
\end{aligned}$$

Returning to the equation

$$\frac{x^2}{10} + 10 = \frac{EH \cdot EI}{10} = EG + GH = EG + x.$$

Therefore

$$\begin{aligned}
100 + x^2 = EG^2 &= \left(\frac{x^2}{10} - x + 10\right)^2 \\
&= \frac{x^4}{100} + x^2 + 100 - \frac{x^3}{5} + 2x^2 - 20x
\end{aligned}$$

which simplifies to

$$x^3 + 200x = 20x^2 + 2000 \tag{7}$$

The solution is realized as the intersection of the hyperbola and the semicircle

$$\begin{gathered}
xy = \sqrt{20,000} \\
x^2 - 30x + y^2 - \sqrt{800}y + 400 = 0
\end{gathered}$$

The latter is equivalent to

$$(x - 15)^2 + (y - \sqrt{200})2 = x^2 - 30x + 225 + y^2 - \sqrt{800}y + 200 = -400 + 225 + 200 = 25,$$

a circle of radius 5 and center $(10\sqrt{2}, 15)$. The hyperbola intersects the circle in two points. If (x, y) is a solution then from the first equation

$$y = \frac{\sqrt{20,000}}{x}.$$

Substituting into the second

$$x^2 - 30x + \frac{20,000}{x^2} - \frac{4000}{x} + 400 = 0$$

or

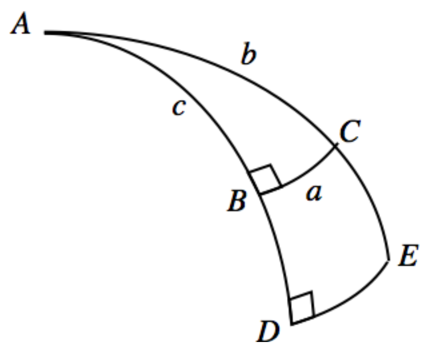
$$0 = x^4 - 30x^3 - 400x^2 - 4000x + 20,000 = (x - 10)(x^3 - 20x^2 + 200x - 2000)$$

One of the solutions of the geometric problem is $x = 10$ and $y = \sqrt{200}$. However this x does not satisfy the desired cubic equation (7). Hence the other intersection point solves al-Khayyami's problem.

30. *This is an exercise in spherical trigonometry, following Abu al-Wafa (940–998). On the unit sphere, suppose $\triangle ABC$ and $\triangle ADE$ are two spherical triangles with right angles at B and D. Then (writing modern sines),*

$$\frac{\sin BC}{\sin CA} = \frac{\sin DE}{\sin EA} \tag{8}$$

which is called the the rule of four quantities. *[Katz, A History of Mathematics, 2009, p. 311]*



We give a modern proof using spherical coordinates and a rotation of space. To start, suppose that $A = (0, 0, 1)$ is the north pole and AB is on the great circle $y = 0$. If c and b measure the spherical distance from the north pole and $\beta = \angle BAC$ then the $\mathbf{R}^3$ coordinates of B and C are

$$B = \begin{pmatrix} \sin c \\ 0 \\ \cos c \end{pmatrix}, \qquad C = \begin{pmatrix} \sin b\, \cos\beta \\ \sin b \sin\beta \\ \cos b \end{pmatrix}.$$

Let us rotate space so that the great circle $y = 0$ stays fixed but the point B moves to A. This is accomplished by multiplying by the rotations matrix which preserves angles, distances in space and spherical distances.

$$R = \begin{pmatrix} \cos c & 0 & -\sin c \\ 0 & 1 & 0 \\ \sin c & 0 & \cos c \end{pmatrix}.$$

The points A, B and C get moved to

$$RA = \begin{pmatrix} -\sin c \\ 0 \\ \cos c \end{pmatrix}, \qquad RB = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad RC = \begin{pmatrix} \sin b \, \sin c \, \cos \beta - \cos b \, \sin c \\ \sin b \, \sin \beta \\ \sin b \, \sin c \, \cos \beta + \cos b \, \cos c \end{pmatrix}$$

Since $\angle ABC = 90°$ is a right angle, the rotated point is in the $x = 0$ great circle, namely

$$RC = \begin{pmatrix} 0 \\ \sin a \\ \cos a \end{pmatrix}.$$

Equating the components of the two ways we computed RC we deduce

$$\sin a = \sin b \, \sin \beta$$
$$\cos a = \sin b \, \sin c \, \cos \beta + \cos b \, \cos c$$

These are special cases of the *sine law* and the *cosine law* for spherical triangles. If we now consider two other points D on the ray AB and E on the ray AC where c' is the length of AD, b' is the length AE and $\angle ADE = 90°$ is also a right angle. Then the sine law could have just as well been applied. As the triangles $\triangle BAC$ and $\triangle DAE$ share an angle at the vertex, we deduce from the sine law

$$\frac{\sin b}{\sin a} = \frac{1}{\sin \beta} = \frac{\sin b'}{\sin a'} \tag{9}$$

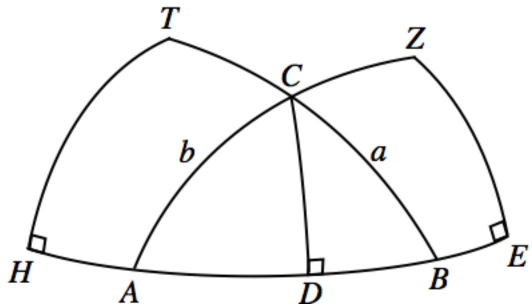
which is (8), as desired.

31. *Following Abu al-Wafa, using the Rule of Four Quantities, deduce the spherical sine law. Let $\triangle ABC$ be a triangle in the unit sphere. Let a, b and c denote the lengths of the sides opposite the corners, and A, B and C denote the angles at the corners, then*

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

[Katz, A History of Mathematics, 2009, p. 311]

The argument in the previous problem could have been used, with a general angle in place of the right angles at RB. Starting from the triangle $\triangle ABC$, Abu al-Wafa drew the great circle CD where D is on the great circle through AB such that $\angle ADC = 90°$ is a right angle. Extend the arcs AB to include E and H, AC to include Z and BC to include T such that the distances AE and AZ are half the distance form A to its antipodal point. Thus EZ is on the equator if A were the north pole and the angles $\angle AEZ$ and $\angle AZE$ are right angles. Similarly, suppose that H and T are on the equator for the pole B such that $\angle BHT$ and $\angle BTH$ are right angles.



The triangles $\triangle ADC$ and $\triangle AEZ$ are spherical right triangles with common angle at A and triangles $\triangle BDC$ and $\triangle BHT$ are spherical right triangles with common angle at B. By the rule of four quantities,

$$\frac{\sin DC}{\sin CA} = \frac{\sin EZ}{\sin ZA}$$
$$\frac{\sin DC}{\sin CB} = \frac{\sin HT}{\sin TB}$$

Because the arc EZ is on the equator for the pole A, its length equals the angle $\angle BAC = EZ$. Similarly, because the arc HT is on the equator for the pole b, its length equals the angle $\angle ABC = HT$. Moreover, the distance from pole to equator AZ and TB is $\frac{\pi}{2}$ radians. Using $a = BC$ and $b = CA$ we obtain

$$\frac{\sin DC}{\sin b} = \frac{\sin A}{1}$$
$$\frac{\sin DC}{\sin a} = \frac{\sin B}{1}$$

from which $\sin A \sin b = \sin B \sin a$ follows, the first equation in (11). The second equation using follows by the same argument replacing the points ABC by BCA.

32. *The* qibla *is the direction toward Mecca for prayers. If you are at point P, Mecca is at point M and the north pole is at point N, then the* qibla *is the angle at P of the spherical triangle $\triangle NPM$. Use the cosine law and the sine law to find the* qibla *for Rome. (Rome has latitude $41°53'$ N, longitude $12°30'$ E. Mecca has latitude $21°45'$ N, longitude $39°49'$ E.) [Katz,* A History of Mathematics, *2009, p. 320]*

The side NP has length $90° - 41°53' = 48°07'$. The side NM has length $90° - 21°45' = 68°15'$. The difference in longitudes is the angle $N = 39°49' - 12°30' = 27°19'$. The cosine law gives the length of the side PM

$$\begin{aligned}\cos PM &= \cos NP \, \cos NM + \sin NP \, \sin NM \, \cos N \\ &= \cos(48°07') \, \cos(68°15') + \sin(48°07') \, \sin(68°15') \, \cos(27°19') \\ &= 0.8617802\end{aligned}$$

so $PM = 30°29'$. By the sine law

$$\sin P = \frac{\sin NM \, \sin N}{\sin PM} = \frac{\sin(68°15') \, \sin(27°19')}{\sin(30°29')} = 0.8402392.$$

Since Mecca is southeast of Rome, we must take arcsine in the range $90° < P < 180°$ so that the *qibla* is $P = 122°50'$.

33. *Show that if t is a root of $x^3 = cx + d$ then*

$$r = -t, \quad \frac{t}{2} \pm \sqrt{c - \frac{3t^2}{4}}$$

are roots of $x^3 + d = cx$. Use this to solve $x^3 + 3 = 8x$. [Katz, A History of Mathematics, *2009, p. 419]*

$x = -t$ solves $-x^3 = -cx + d$. Cubing r and using $t^3 = ct + d$,

$$\begin{aligned}
r^3 &= \frac{t^3}{8} \pm \frac{3t^2}{4}\sqrt{c - \frac{3t^2}{4}} + \frac{3t}{2}\left(c - \frac{3t^2}{4}\right) \pm \left(c - \frac{3t^2}{4}\right)^{\frac{3}{2}} \\
&= -t^3 + \frac{3ct}{2} \pm \left\{\frac{3t^2}{4} + c - \frac{3t^2}{4}\right\}\sqrt{c - \frac{3t^2}{4}} \\
&= -ct - d + \frac{3ct}{2} \pm c\sqrt{c - \frac{3t^2}{4}} \\
&= -d + c\left(\frac{t}{2} \pm \sqrt{c - \frac{3t^2}{4}}\right) \\
&= -d + cr.
\end{aligned}$$

The equation $x^3 = 8x + 3$ has the root $t = 3$. Thus

$$r = \frac{t}{2} \pm \sqrt{c - \frac{3t^2}{4}} = \frac{3}{2} \pm \sqrt{8 - \frac{3 \cdot 3^2}{4}} = \frac{3 \pm \sqrt{5}}{2}$$

are roots of $x^3 + 3 = 8x$. Indeed

$$r^3 = 9 \pm 4\sqrt{5} = 8\left(\frac{3 \pm \sqrt{5}}{2}\right) - 3 = 8r - 3.$$

34. *Use Cardano's method to solve the cubic equation and check your answer.*

$$y^3 = 5y + 8$$

Cardano's trick is to assume $y = u + v$. Then substituting in the equation

$$u^3 + v^3 + 3uvy = u^3 + 3uv(u + v) + v^3 = u^3 + 3u^2v + 3uv^2 + v^3 = (u + v)^3 = y^3 = 5y + 8$$

one solves the system

$$\begin{aligned}
3uv &= 5 \\
u^3 + v^3 &= 8.
\end{aligned}$$

Letting $v = \dfrac{5}{3u}$ we get

$$u^3 + \left(\frac{5}{3u}\right)^3 = 8$$

or

$$(u^3)^2 - 8u^3 + \frac{5^3}{3^3} = 0$$

By the quadratic formula

$$u^3, v^3 = \frac{8}{2} \pm \sqrt{\frac{8^2}{4} - \frac{5^3}{3^3}} = 4 \pm \sqrt{16 - \frac{125}{27}} = 4 \pm \sqrt{\frac{307}{27}}$$

Thus one solution is

$$y = \sqrt[3]{4 + \sqrt{\frac{307}{27}}} + \sqrt[3]{4 - \sqrt{\frac{307}{27}}} = 2.802589$$

since $u^3 + v^3 = 8$ and

$$u^3 v^3 = 16 - \frac{307}{27} = \frac{125}{27} = \left(\frac{5}{3}\right)^3.$$

35. *Using Cardano's method, find a solution. Then find all solutions.*

$$x^3 - 2x^2 - 5x + 10 = 0.$$

The first step is to make a change of variables $x = y - a$ to get rid of the squared term. Substituting

$$(y^3 - 3ay^2 + 3a^2 y - a^3) - 2(y^2 - 2ay + a^2) - 5(y - a) + 10 = 0$$

We choose a to eliminate the y^2 term

$$-3a - 2 = 0$$

or $a = -\frac{2}{3}$. The equation becomes

$$y^3 + \left(\frac{4}{3} - \frac{8}{3} - 5\right) y + \left(\frac{8}{27} - \frac{8}{9} - \frac{10}{3} + 10\right) = 0$$

which may be written

$$y^3 = \frac{19}{3} y - \frac{164}{27}.$$

Cardano's trick is to write $y = u + v$. Then substituting,

$$u^3 + v^3 + 3uvy = u^3 + 3uv(u+v) + v^3 = u^3 + 3u^2 v + 3uv^2 + v^3 = (u+v)^3 = y^3 = \frac{19}{3} y - \frac{164}{27},$$

we see it solves the equation if u and v solve the system

$$3uv = \frac{19}{3}$$

$$u^3 + v^3 = -\frac{164}{27}.$$

Letting $v = \frac{19}{9u}$ we get

$$u^3 + \left(\frac{19}{9u}\right)^3 = -\frac{164}{27}$$

or

$$(u^3)^2 + \frac{164}{27} u^3 + \frac{19^3}{9^3} = 0$$

By the quadratic formula

$$u^3, v^3 = -\frac{82}{27} \pm \sqrt{\frac{164^2}{4 \cdot 27^2} - \frac{19^3}{9^3}} = -\frac{82}{27} \pm \sqrt{\frac{82^2}{3^6} - \frac{19^3}{3^6}} = \frac{-82 \pm \sqrt{135}i}{27} \tag{10}$$

The formula asks for the square root of a negative number, so we use complex numbers where $i^2 = -1$. Thus we have to find cube roots of complex numbers to get to u and v. Writing a complex number in polar coordinates $a + ib = re^{i\theta} = r\cos\theta + ir\sin\theta$ where $r = |a + ib| = \sqrt{a^2 + b^2}$ and $\theta = \operatorname{Atan}\left(\frac{b}{a}\right)$. The norms are

$$|u|^6 = |v|^6 = \frac{82^2 + 135}{27^2} = \left(\frac{19}{9}\right)^3$$

The directions are

$$\frac{u^3}{|u|^3}, \frac{v^3}{|v|^3} = e^{\pm i\theta} = \frac{-82 \pm \sqrt{135}i}{\sqrt{82^2 + 135}}$$

so $\theta = 3.000835$. It follows that

$$u, v = \frac{\sqrt{19}}{3} e^{\pm \frac{1}{3}\theta i} = \frac{\sqrt{19}}{3}\left(\cos\left(\frac{\theta}{3}\right) \pm \sin\left(\frac{\theta}{3}\right) i\right) = 0.7847007 \pm 1.222847\, i$$

Thus one solution is

$$y = u + v = \sqrt[3]{\frac{-82 + \sqrt{135}\, i}{27}} + \sqrt[3]{\frac{-82 - \sqrt{135}\, i}{27}} = 1.569401.$$

It follows that

$$x = y - a = 1.569401 + \frac{2}{3} = 2.236068 = \sqrt{5}$$

One checks that $x = \sqrt{5}$ solves the equation.

It is usually impossible to see what surd a given decimal fraction equals. Let us check that we can explicitly write the cube roots by figuring out a good candidates for u and v, and checking that their cubes are the ones given by (10). We know that $y = \sqrt{5} - \frac{2}{3}$ so the real part of $u, v = a \pm bi$ is $a = \frac{\sqrt{5}}{2} - \frac{1}{3}$. We also know that $|u| = \frac{\sqrt{19}}{3}$. Solving for the other b leg in a right triangle whose base is $a = \frac{\sqrt{5}}{2} - \frac{1}{3}$ and whose hypotenuse is $\frac{\sqrt{19}}{3}$ gives the complex numbers,

$$a \pm ib = \frac{\sqrt{5}}{2} - \frac{1}{3} \pm i\sqrt{\frac{3}{4} + \frac{\sqrt{5}}{3}}$$

whose sum is $\sqrt{5}$ is real. Its cube is the given by (10) since

$$\begin{aligned}
(a + ib)^3 &= a^3 - 3ab^2 + i(3a^2 b - b^3) \\
&= \frac{-82}{27} + \frac{(15 - 6\sqrt{5})\sqrt{27 + 12\sqrt{5}}}{27} i \\
&= -\frac{-82}{27} + \frac{\sqrt{135}}{27} i
\end{aligned}$$

because

$$(15 - 6\sqrt{5})^2(27 + 12\sqrt{5}) = 15 \cdot (27 - 12\sqrt{5})(27 + 12\sqrt{5}) = 15(27^2 - 5 \cdot 12^2) = 135.$$

Now that we know that $x = \sqrt{5}$ is a root, by long division we see that

$$x^3 - 2x^2 - 5x + 10 = (x - \sqrt{5})\Big(x^2 + (\sqrt{5} - 2)x - 2\sqrt{5}\Big)$$

The other roots may be found using the quadratic formula on the quadratic factor

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2} = \frac{-\sqrt{5} + 2 \pm \sqrt{9 - 4\sqrt{5} + 8\sqrt{5}}}{2} = \frac{-\sqrt{5} + 2 \pm (\sqrt{5} + 2)}{2} = 2, -\sqrt{5}.$$

36. *Solve using Cardano's formula. [Burton,* The History of Mathematics, *2011, p. 326]*

$$x^3 + 24x = 16.$$

Using Cardano's trick, assume $x = u - v$. Then

$$u^3 - v^3 - 3uvx = u^3 - 3u^2v + 3uv^3 - v^3 = (u - v)^3 = -24x + 16$$

The equation is satisfied if

$$\begin{aligned} 3uv &= 24 \\ u^3 - v^3 &= 16 \end{aligned}$$

Solve these equtions slightly differently than in the text. Cube the first and square the second.

$$\begin{aligned} 4u^3v^3 &= 4 \cdot 8^3 \\ u^6 - 2u^3v^3 + v^6 = (u^3 - v^3)^2 &= 16^2 \end{aligned}$$

Adding, we get

$$(u^3 + v^3)^2 = u^6 + 2u^3v^3 + v^6 = 16^2 + 4 \cdot 8^3 = 2^8 + 2^{11} = 2^8 \cdot 9$$

whose square root gives the second equation of

$$\begin{aligned} u^3 - v^3 &= 16 \\ u^3 + v^3 &= 48 \end{aligned}$$

whose solution is

$$\begin{aligned} u^3 &= 32 \\ v^3 &= 16. \end{aligned}$$

Hence a root of the equation is

$$x = u - v = \sqrt[3]{32} - \sqrt[3]{16} = 0.65496$$

which is a root of $x^3 + 24x = 16$.

37. *Use Viète's method to find a root of the equation.*

$$x^3 - 7x + 2 = 0$$

The idea is to change variables so that the left side becomes $4y^3 - 3y$ and use trig substitution. Thus, if $x = ky$, the equation becomes

$$k^3y^3 - 7ky = -2.$$

We let $k = \sqrt{\dfrac{4 \cdot 7}{3}}$ so that

$$\frac{4 \cdot 7}{3} y^3 - 7y = -2 \cdot \sqrt{\frac{3}{4 \cdot 7}}$$

so, multiplying by $\dfrac{3}{7}$,

$$4y^3 - 3y = -\frac{2 \cdot 3}{7} \sqrt{\frac{3}{4 \cdot 7}} = -\frac{3\sqrt{3}}{7\sqrt{7}}.$$

Substituting $y = \cos\theta$ we find

$$\begin{aligned}
\cos 3\theta = \cos(\theta + 2\theta) &= \cos\theta \, \cos 2\theta - \sin\theta \, \sin 2\theta \\
&= \cos\theta \, \left(\cos^2\theta - \sin^2\theta\right) - 2\sin^2\theta \, \cos\theta \\
&= \cos\theta \, \left(2\cos^2\theta - 1\right) - 2\left(1 - \cos^2\theta\right)\cos\theta \\
&= 4\cos^3\theta - 3\cos\theta
\end{aligned}$$

which is why Viète made the first substitution. Hence

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta = -\frac{3\sqrt{3}}{7\sqrt{7}}.$$

It follows that

$$\theta = \frac{1}{3} \arccos\left(-\frac{3\sqrt{3}}{7\sqrt{7}}\right)$$

so

$$x = ky = \frac{2\sqrt{7}}{\sqrt{3}} \cos\left\{\frac{1}{3} \arccos\left(-\frac{3\sqrt{3}}{7\sqrt{7}}\right)\right\} = 2.489289$$

One checks that this is a zero of $x^3 - 7x + 2$.

38. *Using Ludovico Ferrari's method, find one solution.*

$$x^4 - 4x^3 + 3x^2 + 4x - 5 = 0$$

The first step is to eliminate the cubic term. Substitute $x = y + a$ gives

$$\begin{gathered}
y^4 + 4ay^3 + 6a^2y^2 + 4a^3y + a^4 - 4(y^3 + 3ay^2 + 3a^2y + a^3) \\
+3(y^2 + 2ay + a^2) + 4(y + a) - 5 = 0
\end{gathered}$$

so

$$4a - 4 = 0$$

or $a = 1$. The resulting equation is

$$y^4 + (6 - 12 + 3)y^2 + (4 - 12 + 6 + 4)y + (1 - 4 + 3 + 4 - 5) = y^4 - 3y^2 + 2y - 1 = 0$$

Next Ferrari tries to make the right side a perfect square by choosing good constant b.

$$(y^2 + b)^2 = y^4 + 2by^2 + b^2 = 3y^2 - 2y + 1 + 2by^2 + b^2 = (3 + 2b)y^2 - 2y + (1 + b^2)$$

The right side is a square provided it has a single double root, or $0 = B^2 - 4AC$, which is

$$0 = 4 - 4(3 + 2b)(1 + b^2) = -8b^3 - 12b^2 - 8b - 8$$

or

$$b^3 + \frac{3}{2}b^2 + b + 1 = 0.$$

Substituting $b = z + c$,

$$z^3 + 3cz^2 + 3c^2z + c^3 + \frac{3}{2}(z^2 + 2cz + c^2) + (z + c) + 1 = 0$$

we get the z^2 term to vanish if $3c + \frac{3}{2} = 0$ or $c = -\frac{1}{2}$. With this c we get

$$z^3 + \left(\frac{3}{4} - \frac{3}{2} + 1\right) z - \frac{1}{8} + \frac{3}{8} - \frac{1}{2} + 1 = z^3 + \frac{1}{4}z + \frac{3}{4} = 0$$

Solving by Cardano's method, setting $z = u + v$ we get

$$u^3 + v^3 + 3uvz = (u + v)^3 = -\frac{1}{4}z - \frac{3}{4}.$$

Cardano's trick is x is a solution if u and v satisfy

$$\begin{aligned} u^3 + v^3 &= -\frac{3}{4} \\ 3uv &= -\frac{1}{4}. \end{aligned}$$

Substituting $v = -\frac{1}{12u}$ we find

$$u^3 - \left(\frac{1}{12u}\right)^3 = -\frac{3}{4}$$

which is quadratic in u^3

$$(u^3)^2 + \frac{3}{4}u^3 - \frac{1}{12^3} = 0$$

Thus, the quadratic formula gives

$$u^3, v^3 = -\frac{3}{8} \pm \sqrt{\frac{3^2}{8^2} + \frac{1}{12^3}} = \frac{-81 \pm 6\sqrt{183}}{216}$$

Hence

$$z = \frac{\sqrt[3]{-81 + 6\sqrt{183}} - \sqrt[3]{81 + 6\sqrt{183}}}{6}$$

so

$$b = z + c = \frac{\sqrt[3]{-81 + 6\sqrt{183}} - \sqrt[3]{81 + 6\sqrt{183}}}{6} - \frac{1}{2} = -1.317183$$

One checks that b is a zero of $b^3 + \frac{3}{2}b^2 + b + 1$.

It follows that

$$(y^2 + b)^2 = (3 + 2b)y^2 - 2y + (1 + b^2) = \left(\sqrt{3 + 2b}y - \sqrt{1 + b^2}\right)^2$$

which implies

$$y^2 + b = \pm\left(\sqrt{3 + 2b}y - \sqrt{1 + b^2}\right).$$

The "+" equation

$$y^2 - \sqrt{3+2b}y + b + \sqrt{1+b^2} = 0$$

has negative discriminant, so has complex roots. This is a manifestation of the fact that the graph of $y^4 - 3y^2 + 2y - 1$ crosses the y-axis at only two, not four points. The "–" equation

$$y^2 + \sqrt{3+2b}y + b - \sqrt{1+b^2} = 0$$

has the roots

$$y = \frac{-\sqrt{3+2b} \pm \sqrt{3+2b-4\left(b-\sqrt{1+b^2}\right)}}{2} = 1.447623, -2.052300$$

One checks that these are zeros of $y^4 - 3y^2 + 2y - 1 = 0$. Thus

$$x = y + 1 = 2.447623, -1.052300$$

which check as roots of $x^4 - 4x^3 + 3x^2 + 4x - 5 = 0$.

39. *Use Nicole Oresme's trick to sum the following series.*

$$S = 1 + \frac{3}{2^1} + \frac{5}{2^2} + \frac{7}{2^3} + \frac{9}{2^4} + \cdots = \sum_{k=1}^{\infty} \frac{2k-1}{2^{k-1}}$$

Note that the numerators are odd numbers. Using Zeno's sum

$$2 = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

we get

$$\begin{aligned} S + 2\cdot 2 = & \quad 1 + \frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + \frac{9}{2^4} + \cdots \\ & + 2\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots\right) \\ = & \quad \left(3 + \frac{5}{2} + \frac{7}{2^2} + \frac{9}{2^3} + \frac{11}{2^4} + \cdots\right) \\ = & \quad 2\left(\frac{3}{2} + \frac{5}{2^2} + \frac{7}{2^3} + \frac{9}{2^4} + \cdots\right) \\ = & \quad 2(S-1) \end{aligned}$$

so $S = 2\cdot 2 + 2 = 6$.

40. *Here is a problem in* Arithmetica Integra *by the German Cossist, "master of the unknown," Michael Stifel (1487–1567) of Wittenberg. In the sequence of odd numbers the first odd number is* 1^5*. After skipping one number, the sum of the next four numbers* $5+7+9+11 = 2^5$*. After skipping the next three numbers, the sum of the following nine numbers* $19+21+23+25+27+29+31+33+35 = 3^5$*. At each successive stage, one skips the next triangular number of odd integers. Formulate this power rule in modern notation and prove it by induction.*

The triangular numbers are given by the recursion $T_1 = 1$ and $T_n = T_{n-1} + n$ for $n \ge 1$. It follows that $T_n = \frac{(n+1)n}{2}$, as can be seen by induction. For $n = 1$, $\frac{(1+1)\cdot 1}{2} = 1$ which is T_1. Assuming it's true for n, by the recursion

$$T_{n+1} = T_n + (n+1) = \frac{(n+1)n}{2} + (n+1) = \frac{(n+1)n}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

proving the induction.

At the nth stage, n^2 odd numbers are taken starting from the S_nth odd number. After which, T_n odd numbers are skipped, which means we have he recursion

$$S_1 = 1; \qquad S_{n+1} = S_n + n^2 + T_n$$

So the second stage starts with the $S_2 = S_1 + 1^2 + T_1 = 1 + 1 + 1 = 3$rd odd number which is 5. Then third stage starts at the $S_3 = S_2 + 2^2 + T_2 = 3 + 4 + 3 = 10$th odd number, which is 19. The fourth stage starts with the $S_4 = S_3 + 3^2 + T_3 = 10 + 9 + 5 = 25$th odd number which is 49. The formula is for $n \geq 1$

$$S_n = 1 + \sum_{k=0}^{n-1} \left(n^2 + \frac{n(n+1)}{2} \right) = 1 + \sum_{k=0}^{n-1} \left(\frac{3}{2} n^2 + \frac{1}{2} n \right)$$

since at $n = 1$ the sum is $1 = S_1$ and for $n \geq 1$,

$$S_{n+1} - S_n = n^2 + \frac{n(n+1)}{2} = n^2 + T_n.$$

We use the fact that

$$\sum_{k=0}^{n} k = \frac{n(n+1)}{2}; \qquad \sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

so

$$S_n = 1 + \frac{(n-1)n(2n-1)}{4} + \frac{(n-1)n}{4} = 1 + \frac{(n-1)n^2}{2}$$

Finally, the sum of the first m odd numbers is

$$\sum_{k=1}^{m} (2k-1) = m^2.$$

Thus the sum of n^2 odd numbers starting at S_n is

$$\begin{aligned}
\sum_{k=S_n}^{S_n+n^2-1} (2k-1) &= \sum_{k=1}^{S_n+n^2-1} (2k-1) - \sum_{k=1}^{S_n-1} (2k-1) \\
&= (S_n + n^2 - 1)^2 - (S_n - 1)^2 \\
&= (S_n + n^2 - 1 + S_n - 1)(S_n + n^2 - 1 - S_n + 1) \\
&= (2S_n + n^2 - 2)n^2 \\
&= (2 + (n-1)n^2 + n^2 - 2)n^2 \\
&= n^3 \cdot n^2 \\
&= n^5
\end{aligned}$$

where we have used $A^2 - B^2 = (A + B)(A - B)$. The proof is complete.

41. *The Fibonacci numbers $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 5$ and so on with $F_{n+1} = F_n + F_{n-1}$ for $n \geq 2$ have some interesting identities. Show that*

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

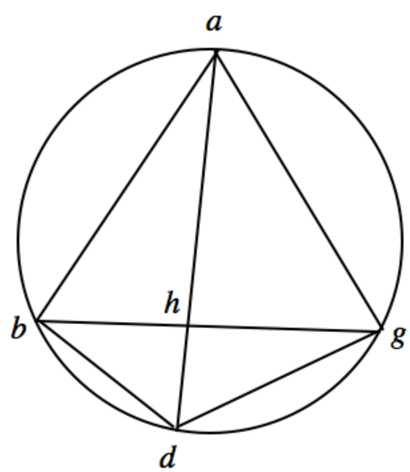
Write the recursion relation for each n.

$$\begin{aligned}
F_1 &= F_3 - F_2 \\
F_2 &= F_4 - F_3 \\
F_3 &= F_5 - F_4 \\
&\vdots \qquad \vdots \\
F_{n-1} &= F_{n+1} - F_n \\
F_n &= F_{n+2} - F_{n+1}.
\end{aligned}$$

Adding, the left side gives the sum and the right side telescopes.

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - F_2 = F_{n+2} - 1.$$

42. *From Leonardo of Pisa's* Practica geometriae*: Given a quadrilateral inscribed in a circle with $ab = ag = 10$ and $bg = 12$, find the diameter ad of the circle. [Katz,* A History of Mathematics, *2009, p. 359]*



As triangle $\triangle abg$ is iscoceles, the diameter through a bisects the angle at a, hence $\triangle abd$ and $\triangle agd$ are congruent right triangles. It follows that $bh = hg = 6$. By the Pythagorean theorem $ah^2 = ab^2 - bh^2 = 10^2 - 6^2 = 8^2$. Also $\triangle ahb$ is similar to $\triangle bhd$ so that

$$\frac{dh}{bh} = \frac{bh}{ah}$$

so

$$dh = \frac{bh^2}{ah} = \frac{6^2}{8} = \frac{9}{2}$$

Thus the diameter is $ad = ah + dh = 8 + \dfrac{9}{2} = \dfrac{25}{2}$.

43. *Solve the problem from Regiomontanus* De Triangulis Omnimodis*: In triangle $\triangle ABC$ suppose that the ratio $\dfrac{\angle A}{\angle B} = \dfrac{10}{7}$ and the ratio $\dfrac{\angle B}{\angle C} = \dfrac{7}{3}$. Find the three angles and the ratio of the sides. [Katz,* A History of Mathematics, *2009, p. 462]*

We know that the angles add up to $180°$. Writing $\alpha = \angle A$ and $\beta = \angle B$ so that $180° - \alpha - \beta = \angle C$, the equations on angles are

$$\begin{aligned}
7\alpha - 10\beta &= 0 \\
3\beta - 7(180 - \alpha - \beta) &= 0.
\end{aligned}$$

In other words

$$\begin{aligned} 7\alpha - 10\beta &= 0 \\ 7\alpha + 10\beta &= 7 \cdot 180, \end{aligned}$$

whose solution is $\alpha = 90°$ and $\beta = 63°$ so $\gamma = 180° - \alpha - \beta = 27°$. By the sine law,

$$\begin{aligned} \frac{AB}{BC} &= \frac{\sin\gamma}{\sin\alpha} = \frac{\sin 27°}{\sin 90°} = 0.4539905 \\ \frac{BC}{AC} &= \frac{\sin\alpha}{\sin\beta} = \frac{\sin 90°}{\sin 63°} = 1.122326 \\ \frac{AC}{AB} &= \frac{\sin\beta}{\sin\gamma} = \frac{\sin 63°}{\sin 27°} = 1.962611. \end{aligned}$$

44. *Solve another problem from* De Triangulis Omnimodis*: In the unit-spherical triangle $\triangle ABC$, suppose the angles are $\angle A = 90°$, $\angle B = 70°$ and $\angle C = 50°$. Find the lengths of the sides. [Katz,* A History of Mathematics, *2009, p. 462]*

Observe that the sum of the angles is more than 180°. This is because in the sphere, the sides of a triangle bulge out compared to the plane. Let angles $\alpha = \angle A$, $\beta = \angle B$ and $\gamma = \angle C$, and let x, y and z denote the lengths of the sides BC, AC and AB, respectively. By the spherical sine law

$$\frac{\sin x}{\sin\alpha} = \frac{\sin y}{\sin\beta} = \frac{\sin z}{\sin\gamma}.$$

Thus

$$\sin y = \sin\beta\,\sin x, \qquad \sin z = \sin\gamma\,\sin x \tag{11}$$

The spherical cosine law is

$$\cos x = \cos y\,\cos z + \sin y\,\sin z\,\cos\alpha$$

Taking advantage of $\alpha = 90°$ so $\cos\alpha = 0$ we find by squaring

$$1 - \sin^2 x = \cos^2 x = \cos^2 y\,\cos^2 z = (1 - \sin^2 y)(1 - \sin^2 z) = 1 - \sin^2 y - \sin^2 z + \sin^2 y\,\sin^2 z$$

which gives an equation for $\sin^2 x$ using (11)

$$\sin^2 x = (\sin^2\beta + \sin^2\gamma)\sin^2 x - \sin^2\beta\,\sin^2\gamma\,\sin^4 x$$

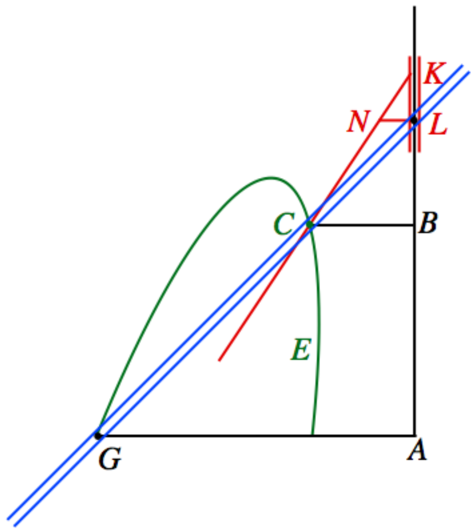
or, since $\sin^2 x = 0$ is not a solution,

$$\sin^2 x = \frac{\sin^2\beta + \sin^2\gamma - 1}{\sin^2\beta\,\sin^2\gamma} = \frac{\sin^2 70° + \sin^2 50° - 1}{\sin^2 70°\,\sin^2 50°}$$

Hence, by (11)

$$\begin{aligned} x &= \arcsin\left(\sqrt{\frac{\sin^2 70° + \sin^2 50° - 1}{\sin^2 70°\,\sin^2 50°}}\right) = 1.260430 \\ y &= \arcsin\left(\sqrt{\frac{\sin^2 70° + \sin^2 50° - 1}{\sin^2 50°}}\right) = 1.107974 \\ z &= \arcsin\left(\sqrt{\frac{\sin^2 70° + \sin^2 50° - 1}{\sin^2 70°}}\right) = 0.8175091. \end{aligned}$$

45. *This problem illustrates on of Decartes' machines. GL is a (blue) ruler pivoting at G. It is linked at L to a (red) device $CKLN$ that slides up and down AB, always keeping KN parralel to itself. The intersection C of the two moving lines GL and KN determine a (green) curve E. Decartes stated, without proof, that the curve is a hyperbola. Show that he was correct. [Katz,* A History of Mathematics, *2009, p. 503.]*



Let $y = CB$ and $x = AB$, and the constants $a = GA$, $b = KL$, $c = NL$. First we shall find BK, BL and AL in terms of x, y, a, b, and c. Triangles $\triangle KBC$ is similar to $\triangle KLN$ so

$$\frac{BK}{CB} = \frac{KL}{NL} \qquad \Longrightarrow \qquad BK = \frac{CB \cdot KL}{NL} = \frac{by}{c}.$$

Hence

$$BL = BK - KL = \frac{by}{c} - b$$

and

$$AL = AB + BL = x + \frac{by}{c} - b.$$

Second, triangles $\triangle LAG$ is similar to $\triangle LBC$ so

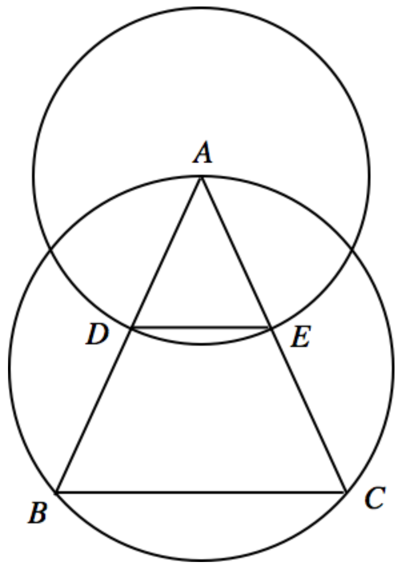
$$\frac{CB}{BL} = \frac{GA}{AL} \qquad \Longrightarrow \qquad CB \cdot AL = BL \cdot GA \qquad \Longrightarrow \qquad xy + \frac{by^2}{c} - by = \frac{aby}{c} - ab$$

or

$$by^2 + cxy - b(a+c)y = -abc.$$

The quadratic form $Ax^2 + Bxy + Cy^2$ has $A = 0$, $B = c$ and $C = b$ with discriminant $D = B^2 - 4AC = c^2 > 0$ so that the conic section is a hyperbola. Or you can see that the quadratic form has two null directions $y = 0$ and $by + cx = 0$ which are the asymptote directions. This only happens for the hyperbolas.

46. *Here is a problem from Copernicus* De revolutionibus. *Given three sides of an isosceles triangle, to find the angles. Circumscribe a circle about the triangle and draw another circle with center A and radius $AD = \frac{1}{2}AB$. Then show that each of the equal sides is to the base as the radius is to the chord subtending the vertex angle. All three angles are then determined. Perform the calculation with $AB = AC = 10$ and $BC = 6$.*



The sides $AD = AE = \frac{1}{2}AB$ are equal so that $\triangle ADE$ is also isosceles. Its vertex A has the same angle for both triangles, so the angles at D and E equal the angles at B and C. It follows that $\triangle ADE$ is similar to $\triangle ABC$ so that

$$\frac{AB}{BC} = \frac{AC}{BC} = \frac{AD}{DE}.$$

If we call 2α the angle at A then

$$\frac{DE}{2} = AD \sin\alpha.$$

Thus we get

$$2\alpha = 2\arcsin\left(\frac{DE}{2\,AD}\right) = 2\arcsin\left(\frac{BC}{2\,AB}\right)$$

Hence, using the given lengths,

$$2\alpha = 2\arcsin\left(\frac{BC}{2\,AB}\right) = 2\arcsin\left(\frac{6}{2\cdot 10}\right) = 34.91521^\circ$$

Using the sum of the angles of a triangle is 180°, the other two angles are

$$\frac{1}{2}(180^\circ - 2\alpha) = 72.54239^\circ.$$