Math 3010 § 1. **First Midterm Exam** Name: Practice Problems
Treibergs January 22, 2018

Here are some problems soluble by methods encountered in the course. I have tried to select problems ranging over the topics we've encountered. Admittedly, they were chosen because they're fascinating and have solutions that are longer than the questions you might expect on an exam. Here are a few of my references.

References.

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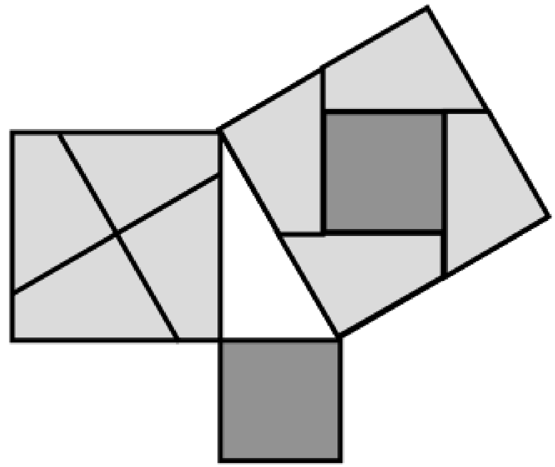
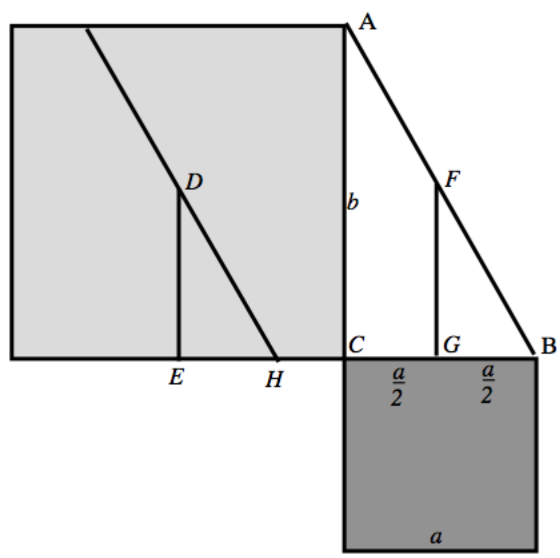
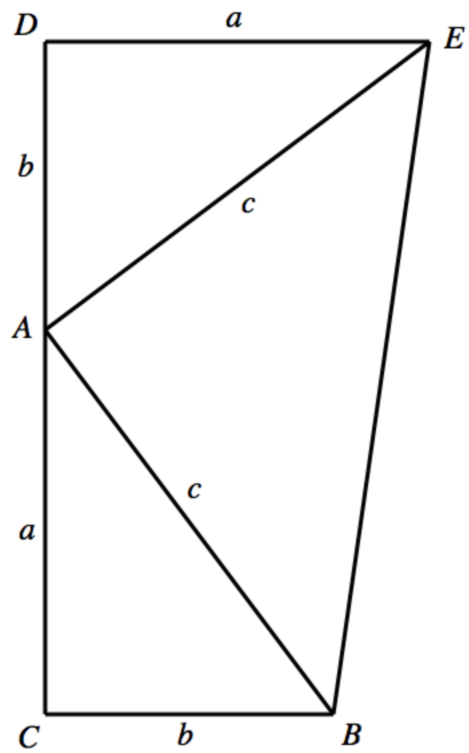
1. *Consider a right triangle with sides $a \le b$ and hypotenuse c. Through the center of the $b \times b$ square, draw two lines that are parallel and perpendicular to the hypotenuse. For each of the four quadrilaterals resulting by cutting the square, show sum of the lengths of two sides is c and the difference of the two other sides is a. Hence they may be placed in the $c \times c$ square without overlapping and leaving a hole of size $a \times a$, thus providing yet another proof of the Pythagorean Theorem.*



The line parallel to the hypotenuse has length c as it is a translate of the hypotenuse. The other line is a $90°$ rotation, thus has length c too. Let F be the midpoint of AB and FG a line parallel to AC. Then the triangles $\triangle(ABC)$ and $\triangle(FBG)$ are similar. Since F is the midpoint of AB we have $\frac{c}{2} = FB$ and so $\frac{a}{2} = GB$. Since $\triangle(FBG)$ and $\triangle(DEH)$ are congruent, $\frac{a}{2} = EH$. Thus the sides of the quadrilateral parallel to CB have lengths $\frac{b}{2} + \frac{a}{2}$ and $EC - EH = \frac{b}{2} - \frac{a}{2}$, whose difference is a. It follows that the four quadrilaterals may be placed nonoverlapping in the $c \times c$ square leaving a "hole" of size $a \times a$, proving the Pythagorean Theorem.



2. *Several years before James Garfield became president of the United States, he devised an original proof of the Pythagorean Theorem which was published in 1876 in the* New England Journal of Education. *Starting with right triangle $\triangle(ABC)$, place a congruent triangle $\triangle(EAD)$ so that AD extends CA. Then draw EB to form a quadrilateral $\diamond(EBCD)$. Prove that $a^2 + b^2 = c^2$ by relating the area of the quadrilateral to the area of the three triangles $\triangle(ABC)$, $\triangle(EAD)$ and $\triangle(EBA)$.*



Recall that opposite angles of a right triangle are complementary $\angle CAB + \angle ABC = 90°$. Using the fact that the angles at A add up to $180°$, we have

$$\angle BAE = 180° - \angle CAB - \angle DAE = 180° - \angle CAB - \angle ABC = 90°$$

is a right angle. It follows that the area of of the triangle $\triangle(EBA)$ is $\frac{1}{2}c^2$. The area of a trapezoid is the average of its bases times height

$$\mathcal{A}(\diamond(EBCD)) = \frac{a+b}{2} \cdot (a+b) = \frac{a^2 + 2ab + b^2}{2}.$$

On the other hand, it is also the sum of areas of its triangles

$$\mathcal{A}(\diamond(EBCD)) = \mathcal{A}(\triangle(ABC)) + \mathcal{A}(\triangle(EDE)) + \mathcal{A}(\triangle(BAE)) = \frac{ab}{2} + \frac{ab}{2} + \frac{c^2}{2}$$

The difference is

$$0 = \frac{ab}{2} + \frac{ab}{2} - \frac{c^2}{2}$$

which is the Pythagorean theorem.

3. *Show that any rational solution of the polynomial equation with integer coefficients can only have integral solutions.*

$$x^n + c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1 x + c_0 = 0 \tag{1}$$

Use this fact to show that $\sqrt{n}$ is irrational for any integer n that is not the square of another integer.

Suppose that $x = \frac{p}{q}$ is a rational solution where p and q are assumed to have no common factors. By multiplying (1) by q^n we get

$$p^n = -c_{n-1}qp^{n-1} - c_{n-2}q^2p^{n-2} - \cdots - c_1 q^{n-1}p - c_0 q^n.$$

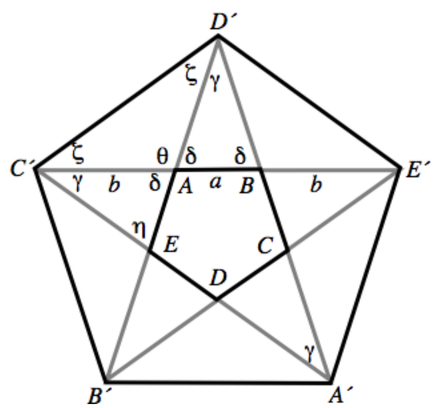
Hence $q|p^n$ (q divides p^n). If $q \neq 1$ it has a prime divisor $q_1|q$ ($q_1 \neq 1$). Hence $q_1|p^n$. The prime must divide one of the p's, which are all the same so $q_1|p$. But then both q and p have the common divisor q_1. Thus it must be the case that $q = 1$ and x is an integer.

Let n be a number that is not the square of an integer. Now, for contradiction, assume that $x = \sqrt{n}$ is rational. It satisfies

$$x^2 - n = 0.$$

We just proved that for such equations, a rational solution x must be an integer. The equation says that $x^2 = n$, or that n is the square of an integer, contrary to what we assumed about n. Thus the hypothesis that x be rational must be false.

4. *Consider the regular pentagram, the symbol of the Pythagoreans. Show that A is a Golden Section of the segment $C'B$ and that B is a Golden Section of the segment $C'E'$.*



First we show that the triangles $\triangle(D'C'E)$ and $\triangle(A'C'D')$ are similar. Let us show that the angles of these triangles coincide. The angles $\zeta + \gamma = \angle D'C'E = \angle D'C'A'$ are equal because they are vertices of both triangles in the figure.

The exterior angle at vertex A of the pentagon $AEDCB$ is δ. We have adding all exterior angles of the pentagon, $5\delta = 360°$ so $\delta = 72°$. In the isosceles triangle, $\triangle(D'AB)$ we have $\gamma = 180° - 2\delta = 180° - 2 \cdot 72° = 36°$. Since the triangle $\triangle(D'AB)$ is similar to $\triangle(A'C'D')$ we have we have $\gamma + \zeta = \delta = 72°$. It follows that $\zeta = \delta - \gamma = 72° - 36° = 36°$. Finally,the third angle in $\triangle(C'AE)$ is $\eta = 180° - \gamma - \delta = 180° - 36° - 36° = 72°$. Hence both triangles $\triangle(D'C'E)$ and $\triangle(A'C'D')$ have corresponding angles $(\zeta, \zeta + \gamma, \eta)$ and $(\gamma, \gamma + \zeta, \gamma + \zeta)$ both equal to $(36°, 72°, 72°)$, thus are similar.

Now by rotation, the triangles $\triangle(C'BD')$ and $\triangle(C'A'E')$ are similar. Observing that the segments $D'A$ and $BB'A$ are angle bisectors of the vertices D' of $\triangle(C'BD')$ and B' of

$\triangle(C'A'E')$, the corresponding lengths of the parts cut by the bisector on the opposite edge have the same ratio, namely

$$\phi = \frac{b}{a} = \frac{b+a}{b} = 1 + \frac{1}{\phi}. \tag{2}$$

But this implies that

$$\phi^2 - \phi - 1 = 0$$

or, taking the positive root of the quadratic equation

$$\phi = \frac{1+\sqrt{5}}{2},$$

which is the Golden Ratio. Equations (2) verify that the lengths in the problem form Golden sections.

5. *For each of the five regular polyhedra, compute the ratio of the edge length to the diameter of the circumscribing sphere. Euclid made these computations in book XIII of* Elements*.*

   The easiest way to do this is to consider the vertices of the polyhedron as vectors in three space symmetrically placed about the origin. The radius of the circumscribing sphere is the length of such vector. The edge length is the distance between two adjacent vectors.

   The tetrahedron can be realized as the convex hull of the four vectors

$$v_1 = (1,1,1),\ v_2 = (-1,-1,1),\ v_3 = (1,-1,-1),\ v_4 = (-1,1,-1).$$

   The circumradius is $C_T = \sqrt{1^2+1^2+1^2} = \sqrt{3}$. The edge length is

$$L_T = |v_1 - v_2| = \sqrt{[1-(-1)]^2 + [1-(-1)]^2 + [1-1]^2} = \sqrt{8} = 2\sqrt{2}.$$

   Thus the desired ratio

$$R_T = \frac{L_T}{D_T} = \frac{L_T}{2C_T} = \frac{2\sqrt{2}}{2\sqrt{3}} = \sqrt{\frac{2}{3}}$$

   The cube can be realized as the convex hull of the six vectors

$$v_1 = (1,1,1),\ v_2 = (1,1,-1),\ v_3 = (1,-1,1),$$
$$v_4 = (1,-1,-1),\ v_5 = (-1,1,1),\ v_6 = (-1,-1,-1).$$

   The circumradius is $C_C = \sqrt{1^2+1^2+1^2} = \sqrt{3}$. The edge length is

$$L_C = |v_1 - v_2| = \sqrt{[1-1]^2 + [1-1]^2 + [1-(-1)]^2} = 2.$$

   Thus the desired ratio

$$R_C = \frac{L_C}{2C_C} = \frac{2}{2\sqrt{3}} = \sqrt{\frac{1}{3}}.$$

   The octohedron can be realized as the convex hull of the six vectors

$$v_1 = (1,0,0),\ v_2 = (-1,0,0),\ v_3 = (0,1,0),$$
$$v_4 = (0,-1,0),\ v_5 = (0,0,1),\ v_6 = (0,0,-1).$$

   The circumradius is $C_O = \sqrt{1^2+0^2+0^2} = 1$. The edge length is

$$L_O = |v_1 - v_3| = \sqrt{[1-0]^2 + [0-(1)]^2 + [0-0]^2} = \sqrt{2}.$$

Thus the desired ratio
$$R_O = \frac{L_O}{2C_O} = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

By Pacioli's construction as you verified in your homework, the icosohedron can be realized as the convex hull of the twelve vectors

$$\begin{gathered} v_1 = (\phi, 1, 0),\ v_2 = (-\phi, 1, 0),\ v_3 = (\phi, -1, 0),\ v_4 = (-\phi, -1, 0), \\ v_5 = (0, \phi, 1),\ v_6 = (0, -\phi, 1),\ v_7 = (0, \phi, -1),\ v_8 = (0, -\phi, -1), \\ v_9 = (1, 0, \phi),\ v_{10} = (-1, 0, \phi),\ v_{11} = (1, 0, -\phi),\ v_{12} = (-1, 0, -\phi). \end{gathered}$$

where $\phi = \frac{1}{2} + \frac{\sqrt{5}}{2}$ is the Golden section. The circumradius is $C_I = \sqrt{\phi^2 + 1^2 + 0^2} = \sqrt{2 + \phi} = \sqrt{\frac{5}{2} + \frac{\sqrt{5}}{2}}$, using $\phi^2 = \phi + 1$. The edge length is

$$L_I = |v_1 - v_3| = \sqrt{[\phi - \phi]^2 + [1 - (-1)]^2 + [0 - 0]^2} = 2.$$

Thus the desired ratio is

$$R_I = \frac{L_I}{2C_I} = \frac{2}{2\sqrt{\frac{5}{2} + \frac{\sqrt{5}}{2}}} = \frac{\sqrt{50 - 10\sqrt{5}}}{10}.$$

Finally, to get the twenty vertices of the icosahedron, we take the centers of the faces of the icosahedron. Since each vertex of the icosahedron has five adjacent triangles, their centers form the vertices of a pentagonal face of the dodecahedron. There are two types of these triangles, those that include the other corner of the rectangle and those that don't. Prototypical of these types adjacent to v_1, multiplying averages by three are

$$\begin{gathered} v_1 + v_3 + v_9 = (\phi, 1, 0) + (\phi, -1, 0) + (1, 0, \phi) = (1 + 2\phi, 0, \phi), \\ v_1 + v_5 + v_9 = (\phi, 1, 0) + (0, \phi, 1) + (1, 0, \phi) = (1 + \phi, 1 + \phi, 1 + \phi). \end{gathered}$$

Thus a list of vertices of a regular dodecahedron is

$$\begin{gathered} w_1 = (1 + 2\phi, 0, \phi),\ w_2 = (-[1 + 2\phi], 0, \phi), \\ w_3 = (1 + 2\phi, 0, -\phi),\ w_4 = (-[1 + 2\phi], 0, -\phi), \\ w_5 = (\phi, 1 + 2\phi, 0),\ w_6 = (-\phi, 1 + 2\phi, 0), \\ w_7 = (\phi, -[1 + 2\phi], 0),\ w_8 = (-\phi, -[1 + 2\phi], 0), \\ w_9 = (0, \phi, 1 + 2\phi),\ w_{10} = (0, -\phi, 1 + 2\phi), \\ w_{11} = (0, \phi, -[1 + 2\phi]),\ w_{12} = (0, -\phi, -[1 + 2\phi]), \\ w_{13} = (1 + \phi, 1 + \phi, 1 + \phi),\ w_{14} = (-[1 + \phi], 1 + \phi, 1 + \phi), \\ w_{15} = (1 + \phi, -[1 + \phi], 1 + \phi),\ w_{16} = (-[1 + \phi], -[1 + \phi], 1 + \phi), \\ w_{17} = (1 + \phi, 1 + \phi, -[1 + \phi]),\ w_{18} = (-[1 + \phi], 1 + \phi, -[1 + \phi]), \\ w_{19} = (1 + \phi, -[1 + \phi], -[1 + \phi]),\ w_{20} = (-[1 + \phi], -[1 + \phi], -[1 + \phi]). \end{gathered}$$

Using $\phi^2 = \phi + 1$, the circumradius is

$$C_D = |w_1| = \sqrt{[1 + 2\phi]^2 + 0^2 + \phi^2} = \sqrt{5\phi^2 + 4\phi + 1} = \sqrt{6 + 9\phi} = \sqrt{\frac{21 + 9\sqrt{5}}{2}}$$

Also, for the other type of center,

$$C_D = |w_{13}| = \sqrt{[1 + \phi]^2 + [1 + \phi]^2 + [1 + \phi]^2} = \sqrt{3\phi^4} = \sqrt{6 + 9\phi} = \sqrt{3}\phi^2.$$

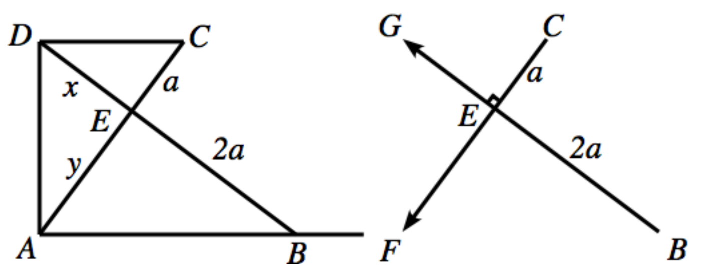
The edge length is

$$L_D = |w_1 - w_{13}| = \sqrt{[\{1+2\phi\} - \{1+\phi\}]^2 + [0 - \{1+\phi\}]^2 + [\phi - \{1+\phi\}]^2}$$
$$= \sqrt{4+4\phi} = \sqrt{4\phi^2} = 2\phi = 2+\sqrt{5}.$$

Thus the desired ratio is

$$R_D = \frac{L_D}{2C_D} = \frac{2\phi}{2\cdot\sqrt{3}\phi^2} = \frac{1}{\sqrt{3}\phi} = \frac{2}{\sqrt{3}(1+\sqrt{5})} = \frac{2(1-\sqrt{5})}{\sqrt{3}(1-5)} = \frac{\sqrt{15}-\sqrt{3}}{6}.$$

6. *This problem describes Plato's solution of the Delian Problem of cube duplication. Given the side of the original cube a, suppose that the one can draw the figure. Prove that x has the property that $2a^3 = x^3$ which solves the cube doubling problem. The figure may be obtained through the use of a mechanical gadget. Let EC have length a and EB have length $2a$ and that the angles at the A, D and E are right angles. Place the gadget in the first quadrant such that B is on the x-axis, the ray $\overrightarrow{EF}$ passes through the origin and the ray $\overrightarrow{EG}$ crosses the y-axis at D. Then rotate the gadget in such a way that D and C have the same y-coordinate. This gadget construction cannot be done only with straightedge and compass and is known as a* verging *solution.*



The right triangles are all similar. $\angle ABE = 90° - \angle EAB = \angle DAE = 90° - \angle EDA = \angle CDE = 90° - \angle ECD$. Thus the right triangles $\triangle AEB \sim \triangle DEA \sim \triangle CED$ are similar by AAA. Thus the ratios of the long to short legs are equal

$$\frac{x}{a} = \frac{y}{x} = \frac{2a}{y},$$

which is the continued mean proportional of Hippocrates of Chios. It follows that

$$x^2 = ay, \qquad y^2 = 2ax$$

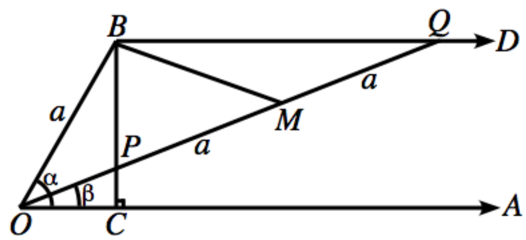
so

$$x^4 = a^2y^2 = 2a^3x$$

which doubles the cube

$$x^3 = 2a^3.$$

7. *This problem describes Nicomedes solution to the trisection of an angle $\alpha = \angle AOB$. Let a be the length of OB. Segment BC is perpendicular to the ray $\overrightarrow{OA}$, and the ray $\overrightarrow{BD}$ is parallel to the ray $\overrightarrow{OA}$. The ray OPQ has been drawn so that the length of PQ is $2a$. Then $\beta = \angle AOQ = \frac{1}{3}\alpha$. [Hint: Consider the midpoint M of PQ.]*



This is a verging solution. Mark on a straightedge a segment PQ of length $2a$ and then slide P on CB and Q on $\overrightarrow{BD}$ until the straightedge passes through O.

First we claim that the length of BM is a. To see this, construct a line through M that is perpendicular to CB. It crosses CB at a point N. Since CB is a mutual perpendicular, the lines NM and BQ are parallel. Because the line OQ crosses three parallel lines BQ, NM and $\overrightarrow{OA}$, the angles $\angle BQP = \angle NMP = \angle AOQ = \beta$. It follows that the triangles $\triangle BQP$ and $\triangle NMP$ are similar. Hence the ratio of lengths of short leg to hypotenuse are equal

$$\frac{\ell(PN)}{\ell(PB)} = \frac{a}{2a} = \frac{1}{2}.$$

Thus the lengths of PN and NB are equal, so the triangles $\triangle PNM$ and $\triangle BNM$ are congruent. Thus the lengths are equal $\ell(BM) = \ell(PM) = a$.

The result follows by computing the angle $\angle BOM$ in two ways. In the first case it is the difference of the two angles at O, namely, $\angle BOM = \alpha - \beta$. Using the similar triangles, $\angle BMO = \angle BMN + \angle PMN = 2\beta$. Finally since the lengths of BO and BM are equal, triangle $\triangle BMO$ is isosceles, therefore $\angle BOM = \angle BMO = 2\beta$. Equating the two angle computations we find $\alpha - \beta = 2\beta$ or $\alpha = 3\beta$, as claimed.

8. *Find two numbers x and y such that $3x + 17y = 1$. Use this result to construct a regular 51-gon by combining a regular triangle with a regular 17-gon inscribed in the same circle.*

3 and 17 are relatively prime ($\gcd(3,17) = 1$). We run the Euclidean algorithm

$$\begin{aligned} 17 &\equiv 2 \mod 3 \qquad & 17 &= 5 \cdot 3 + 2 \\ 3 &\equiv 1 \mod 2 \qquad & 3 &= 1 \cdot 2 + 1. \end{aligned}$$

Working backwards we find

$$1 = 3 - 2 = 3 - (17 - 5 \cdot 3) = 6 \cdot 3 - 1 \cdot 17$$

thus $x = 6$ and $y = -1$. A compass set at the radius can walk about a circle in exactly six steps. Taking every other step yields the vertices of a regular triangle. Gauss constructed the regular 17-gon. To get the angle of a 51-gon we step off x arcs of a side of the triangle and then y arcs of side of the 17-gon. Thus we end up at an angle

$$x\frac{360^\circ}{3} + y\frac{360^\circ}{17} = (17x + 3y)\frac{360^\circ}{51} = \frac{360^\circ}{51}$$

which is the arc of side of the 17-gon if we take $x = 6$ and $y = -1$ so that $3x + 17y = 1$.

9. *Using the Euclidean algorithm, find* $\gcd(1769, 2378)$ *and integers* x *and* y *satisfying*

$$\gcd(1769, 2378) = 1769x + 2378y.$$

Running the Euclidean Algorithm, we find

$$\begin{aligned}
2378 &= 1 \cdot 1769 + 609 \\
1769 &= 2 \cdot 609 + 551 \\
1769 &= 3 \cdot 551 + 116 \\
551 &= 4 \cdot 116 + 87 \\
116 &= 1 \cdot 87 + 29 \\
87 &= 3 \cdot 29 + 0.
\end{aligned}$$

Thus $\gcd(1769, 2378) = 29$. Working backwards,

$$\begin{aligned}
29 &= 116 - 87 \\
&= 116 - (551 - 4 \cdot 116) = 5 \cdot 116 - 551 \\
&= 5 \cdot (1769 - 3 \cdot 551) - 551 = 5 \cdot 1769 - 16 \cdot 551 \\
&= 5 \cdot 1769 - 16 \cdot (1769 - 2 \cdot 609) = 32 \cdot 609 - 11 \cdot 1769 \\
&= 32 \cdot (2378 - 1769) - 11 \cdot 1769 = 32 \cdot 2378 - 43 \cdot 1769
\end{aligned}$$

thus $x = 32$ and $y = -43$.

10. *Show if* $a|c$ *and* $b|c$ *with* $\gcd(a, b) = 1$*, then* $ab|c$*.*

We write $1 = ax + by$ for some integers x and y. Multiplying by c yields

$$c = c \cdot 1 = c(ax + by) = acx + bcy.$$

Now $a|c$ and $b|c$ imply $c = ja$ and $c = kb$ for some integers j and k so

$$c = acx + bcy = a(kb)x + b(ja)y = ab(kx + jy)$$

so $ab|c$.

11. *If* a *and* b *are integers such that* $1 = ax + by$ *for some integers* x *and* y*, then* $\gcd(a, b) = 1$*.*

Suppose x is a common divisor, that is $x|a$ and $x|b$. By the equation $1 = ax + by$ we see that x divides the right side so $x|1$ which implies that $x = 1$. In other words, the every common divisor including the greatest one is 1.

12. *Use Eudoxus's method of exhaustion to prove Archimedes theorem that the area of a circle is the area of a triangle whose height is the radius and whose base is the circumference.*

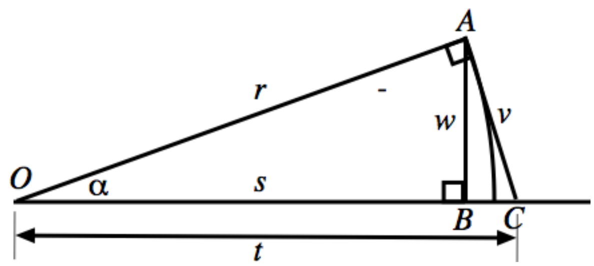
Let r be the radius and C be the circumference. Eudoxus's method is to show that for any two numbers $\beta < \frac{1}{2}rC < \gamma$, the area of the circle A satisfies $\beta < A < \gamma$ and hence $A = \frac{1}{2}rC$ because β and γ may be taken as close to $\frac{1}{2}rC$ as we please. We have to show that the area is greater than any number smaller than $\frac{1}{2}rC$ and less than any number greater than $\frac{1}{2}rC$, hence equal to $\frac{1}{2}rC$. To do this we "exhaust" the circle by regular polygons which inscribe and circumscribe the circle but are are sufficiently close to the circle.

For this purpose, let

$$\alpha_0 = \frac{360°}{6}; \qquad \alpha_n = \frac{360°}{6 \cdot 2^n}$$

denote the central angle of the isosceles triangle of a regular $6 \cdot 2^n$-gon I_n inscribed in the circle. Thus α_{n+1} is half the angle α_n and I_{n+1} has twice the vertices of I_n. Let

O_n be the circumscribing $6 \cdot 2^n$-gon which is rotated $\alpha_n/2$ from I_n so that the midpoint of a side of O_n is the vertex of I_n. Since I_n is inside C is inside O_n we have inequality of areas $a(I_n) \le Aa(O_n)$. Similarly the edge of I_n is a straight line segment inside the corresponding arc of the circle is inside the union of two half edges of O_n. Since a straight line is shorter than any curve connecting its endpoints we have by adding up all pieces, the length $\ell(I_n) \le C$. Similarly since the to half-edges of O_n are outside of C and both are concave toward the center, $C \le \ell(O_n)$. These inequalities are taken as axioms by Archimedes, but may be proved using calculus.



Let $\alpha = \frac{1}{2}\alpha_n$. Then consider a sector of angle α. Let w be half the edge of I_n. (Using trigonometry, $w = r\sin\frac{180\alpha}{\pi}$ using conversion to radians.) Since the arc of the circle is longer, $w \le 60r\alpha$. (From calculus, $\sin\frac{180\alpha}{\pi} \le \frac{180\alpha}{\pi}$ for $\alpha \ge 0$. Also, $\pi \ge 3$.) From the Pythagoren theorem, the long leg $s = \sqrt{r^2 - w^2}$. Half the length of a side of O_n is v. ($v = r\tan\frac{180\alpha}{\pi} \ge \frac{180r\alpha}{\pi}$.) Using similar triangles

$$\frac{v}{r} = \frac{w}{s} = \frac{w}{\sqrt{r^2 - w^2}}.$$

It follows that the ratio of lengths is

$$\frac{\ell(O_n)}{\ell(I_n)} = \frac{v}{w} = \frac{r}{\sqrt{r^2 - w^2}}.$$

Since areas are proportional to the square of the circumference,

$$\frac{a(O_n)}{a(I_n)} = \frac{r^2}{r^2 - w^2}.$$

It follows that the difference in areas is

$$a(O_n) - a(I_n) = a(I_n)\left\{\frac{a(O_n)}{a(I_n)} - 1\right\} = a(I_n)\left\{\frac{r^2}{r^2 - w^2} - 1\right\} = \frac{a(I_n)w^2}{r^2 - w^2} \le \frac{Aw^2}{r^2 - w^2}$$

which tends to zero as n gets large because w tends to zero. To see it, let us assume $2w^2 < r^2$. Then

$$a(O_n) - a(I_n) \le \frac{Aw^2}{r^2 - w^2} \le \frac{Ar^2\alpha^2}{r^2 - \frac{1}{2}r^2} = 2A\alpha^2. \tag{3}$$

Now, suppose we add up all areas of $2n$ halfsectors of I_n to get the area.

$$a(I_n) = 2n \cdot \frac{1}{2}sw \le \frac{1}{2}rC$$

because $s \le r$ and $2nw = \ell(I_n) \le C$. Similarly

$$\frac{1}{2}rC \le 2n \cdot \frac{1}{2}rv = a(O_n)$$

because $2nv = \ell(O_n) \ge C$.

In order to show that A exceeds any number less than $\frac{1}{2}rC$, let us choose any number $\beta < \frac{1}{2}rC$. Now choose n so large that $2w^2 \le 7200r^2\alpha^2 < 1$ and

$$2A\alpha^2 \le \frac{7200A}{n^2} < \frac{1}{2}rC - \beta.$$

This follows from the fact that $\frac{1}{n}$ may be made smaller than any positive number, provided n is large enough. (This fact is called the Archimedean Axiom in real analysis. It is a property of of the real numbers, and may be deduced from completeness. You'll learn about this in Math 3210.)

Now, with this number of sides, using (3),

$$\begin{aligned} A \ge a(I_n) = a(O_n) - [a(O_n) - a(I_n)] &= \frac{1}{2}r\ell(O_n) - [a(O_n) - a(I_n)] \\ \ge \frac{1}{2}rC - [a(O_n) - a(I_n)] > \frac{1}{2}rC - 2A\alpha^2 &> \frac{1}{2}rC - \left[\frac{1}{2}rC - \beta\right] = \beta. \end{aligned}$$

Similarly, to show that A is less than any number greater than $\frac{1}{2}rC$, let us choose any number $\gamma > \frac{1}{2}rC$. Now choose n so large that $7200r^2\alpha^2 < 1$ and

$$2A\alpha^2 \le \frac{7200A}{n^2} < \gamma - \frac{1}{2}rC.$$

Then for polygons with this number of sides, using (3),

$$\begin{aligned} A \le a(O_n) = a(I_n) + [a(O_n) - a(I_n)] &= \frac{1}{2}s\ell(I_n) + [a(O_n) - a(I_n)] \\ \le \frac{1}{2}rC + [a(O_n) - a(I_n)] < \frac{1}{2}rC + 2A\alpha^2 &< \frac{1}{2}rC + \left[\gamma - \frac{1}{2}rC\right] = \gamma. \end{aligned}$$

Thus we have verified Eudoxus conditions and the theorem is proved.

This is a standard way to operate in real analysis. The key idea is that if a number x satisfies $a(I_n) \le x \le a(O_n)$ then the errors $x - a(I_n) \le a(O_n) - a(I_n)$ and $a(O_n) - x \le a(O_n) - a(I_n)$. One proves equalities by showing two numbers are arbitrarily close. As the text points out, the Greeks had a modern handling of real quantities that doesn't rely on formal properties of limits but rather on rigorous $\epsilon - \delta$ type definitions.

13. *Find the greatest common divisor of* 504 *and* 1188 *in two ways.*

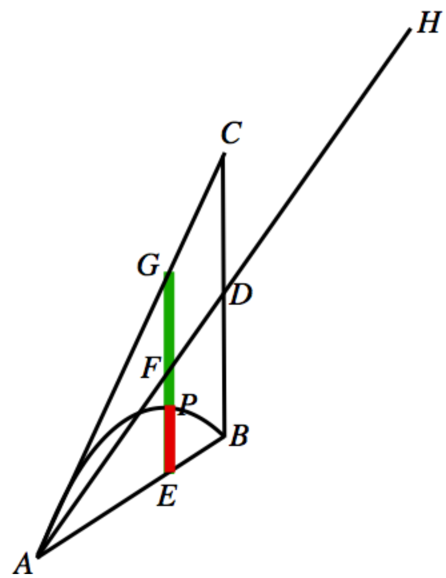
Factoring into primes we find $792 = 2^3 \cdot 3^2 \cdot 7$ and $756 = 2^2 \cdot 3^3 \cdot 11$. The greatest common divisor is $2^2 \cdot 3^2 = 36$.

Using the Euclidean algorithm we find

$$\begin{aligned} 1188 &= 2 \cdot 504 + 180 \\ 504 &= 2 \cdot 180 + 144 \\ 180 &= 1 \cdot 144 + 36 \\ 144 &= 4 \cdot 36 + 0. \end{aligned}$$

Hence $\gcd(504, 1188) = 36$.

14. *Let APB be a segment cut from a vertical parabola. Let AC be the tangent line to the parabola at A and BC a vertical line from B that meets the tangent line at C. Justify the claims in Archimedes proof in* The Method *that the area enclosed by the parabola and the line AB is one third of the area of the triangle $\triangle ABC$. Let D be the midpoint of AC and extend the line AD to H so that the distance AD equals the distance DH. Archimedes idea is to think of the segment AH as a lever with fulcrum D. His idea, anticipating integral calculus, is that the triangle and parabola segment are made up of vertical line segments. He wants to show that the total contribution of the segments from the parabola equals one third of the contributions of the triangle. Consider an arbitrary vertical section EPG. he claims that if a weight equivalent to the length EG is put at the point F of the lever, then it is exactly balanced by a weight equal to the length PE at H. Summing up all vertical lines, the total weight of the parabola at H balances the triangle. Now, observe that the triangle acts as if its total weight were concentrated at the center of gravity, which is on the line AD one third of the distance AD from D. Thus the lever arm of the triangle is on third of the lever arm of the parabola at H, consequently, the total area of the parabola is on third of the area of the triangle. (This was discovered as a palimpsest (overwritten book) in a Constantinople library as late as 1908. The writing underneath the liturgical text turned out be the lost book by Archimedes* The Method of Mechanical Discovery in Geometry.)



Let us assume that the parabola has the equation $y = -x^2$ and the points A and B are located at $x = a$ and $x = b$ respectively. This imples that H is at $x = b + (b - a) = 2b - a$. Let x be the coordinate of EG, $p(x)$ denote the length PE and $t(x)$ be the length of EG. We have to show that the torque from the triangle equals the torque from the parabola, or $t(x)(b - x) = p(x)(b - a)$.

To see this, let us write the equations of the lines. Since AC has slope $-2a$ we have the upper line AC and lower line AB are given by

$$\begin{aligned} u(x) &= -a^2 - 2a(x - a), \\ \ell(x) &= -a^2 + \frac{-b^2 + a^2}{b - a} \cdot (x - a) = -a^2 - (a + b)(x - a). \end{aligned}$$

Hence

$$\begin{aligned}
t(x) &= u(x) - \ell(x) = (b-a)(x-a), \\
p(x) &= -x^2 - \ell(x) = a^2 - x^2 + (a+b)(x-a) = (b-x)(x-a).
\end{aligned}$$

The torques balance because

$$p(x)(b-a) = (b-a)(b-x)(x-a) = t(x)(b-x).$$

15. *Let a and b be positive integers. What condition on a and b is required in order to find integers x and y such that*

$$ax + by = 1?$$

Determine whether the condition to solve this Diophantine equation holds if $a = 5500$ and $b = 2457$, and if it does, solve the equation.

The condition is that $\gcd(a,b) = 1$. If instead $\gcd(a,b) = d > 1$ then if there were a solution we would have $d \mid (ax+by)$ because $d \mid a$ and $d \mid b$, but d does not divide 1.

Let's apply the Euclidean algorithm to find the greatest common divisor.

$$\begin{aligned}
5500 &= 2 \cdot 2457 + 586 \\
2457 &= 4 \cdot 586 + 113 \\
586 &= 5 \cdot 113 + 21 \\
113 &= 5 \cdot 21 + 8 \\
21 &= 2 \cdot 8 + 5 \\
8 &= 1 \cdot 5 + 3 \\
5 &= 1 \cdot 3 + 2 \\
3 &= 1 \cdot 2 + 1 \\
2 &= 2 \cdot 1 + 0
\end{aligned}$$

Thus $\gcd(5500, 2457) = 1$ and the Diophantine equation can be solved. Working backwards,

$$\begin{aligned}
1 &= 3 - 2 \\
&= 3 - (5-3) && = 2 \cdot 3 - 5 \\
&= 2 \cdot (8-5) - 5 && = 2 \cdot 8 - 3 \cdot 5 \\
&= 2 \cdot 8 - 3 \cdot (21 - 2 \cdot 8) && = 8 \cdot 8 - 3 \cdot 21 \\
&= 8 \cdot (113 - 5 \cdot 21) - 3 \cdot 21 && = 8 \cdot 113 - 43 \cdot 21 \\
&= 8 \cdot 113 - 43 \cdot (586 - 5 \cdot 113) && = 223 \cdot 113 - 43 \cdot 586 \\
&= 223 \cdot (2457 - 4 \cdot 586) - 43 \cdot 586 && = 223 \cdot 2457 - 935 \cdot 586 \\
&= 223 \cdot 2457 - 935 \cdot (5500 - 2 \cdot 2457) && = 2093 \cdot 2457 - 935 \cdot 5500
\end{aligned}$$

Since $2093 \cdot 2457 = 5,142,501$ and $935 \cdot 5500 = 5,142,500$, the solution checks. A solution is thus $x = -935$ and $y = 2093$.

16. *Show that* $\sqrt{3}$ *is incommensurable with* 1 *by showing that the geometric version of the Euclidean Algorithm,* anthyphairesis, *fails to converge.*

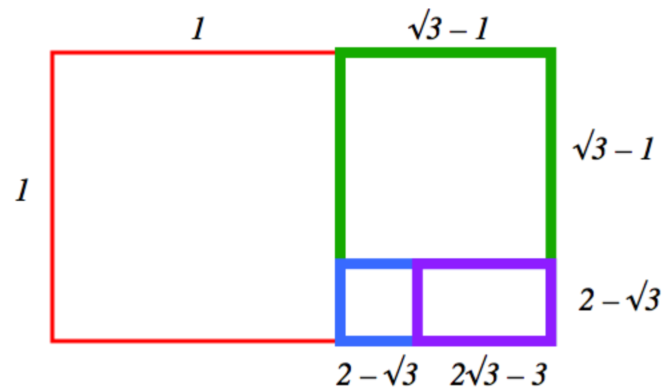
Starting with the rectangle of sides $\sqrt{3}$ and 1, we subtract the shortest side from the longest, and continue in this fashion. The procedure will not stop if after a few steps we obtain a rectangle similar to the original one. Thus $x_{i+1} = \max\{x_i, y_i\} - \min\{x_i, y_i\}$ and $y_{i+1} = \min\{x_i, y_i\}$.

$$\begin{array}{llllll}
x_1 = \sqrt{3} & \approx 1.732 & y_1 = 1 & & \dfrac{y_1}{x_1} \approx 1.732 \\
x_2 = \sqrt{3} - 1 & \approx .732 & y_2 = 1 & & \dfrac{y_2}{x_2} \approx 1.366 \\
x_3 = 1 - (\sqrt{3} - 1) = 2 - \sqrt{3} & \approx .268 & y_3 = \sqrt{3} - 1 & \approx .732 & \dfrac{y_3}{x_3} \approx 2.732 \\
x_4 = (\sqrt{3} - 1) - (2 - \sqrt{3}) = 2\sqrt{3} - 3 & \approx .464 & y_4 = 2 - \sqrt{3} & \approx .268 & \dfrac{x_4}{y_4} \approx 1.732
\end{array}$$

Note that

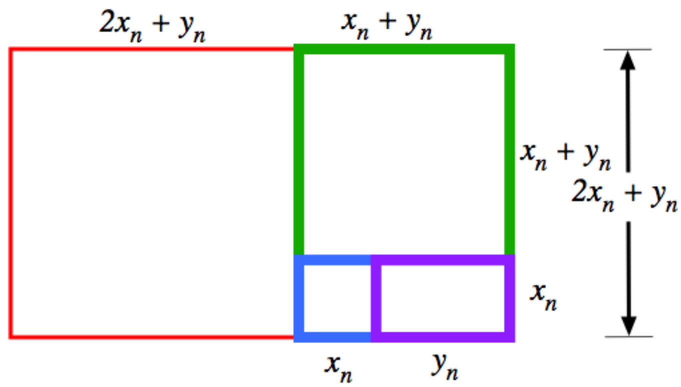
$$\frac{x_4}{y_4} = \frac{2\sqrt{3} - 3}{2 - \sqrt{3}} \cdot \frac{2 + \sqrt{3}}{2 + \sqrt{3}} = \frac{4\sqrt{3} + 6 - 6 - 3\sqrt{3} + 4\sqrt{3}}{4 - 3} = \sqrt{3}.$$

Thus after three steps, the rectangle has the same proportions. The purple rectangle is similar to the red one. The Euclidean algorithm can't stop because every third step we return to a rectangle similar to the original one, so it repeats infinitely often without reaching a square.



17. *Using the geometric Euclidean algorithm for $\sqrt{3}$, find a recursion formula that finds infinitely many integer solutions of Pell's equation given that (x_1, y_1) is an integer solution.*

$$y^2 - 3x^2 = 1$$



In the cubic case, the third rectangle (instead of the second as for $\sqrt{2}$) is proportional to the initial rectangle, as in Problem 16. The anthyphairesis diagram suggests the recursion

$$\begin{aligned} x_{n+1} &= 2x_n + y_n \\ y_{n+1} &= 3x_n + 2y_n \end{aligned}$$

Assuming that

$$y_n^2 - 3x_n^2 = 1$$

we have

$$\begin{aligned} y_{n+1}^2 - 3x_{n+1}^2 &= (3x_n + 2y_n)^2 - 3(2x_n + y_n)^2 \\ &= (9x_n^2 + 12x_n y_n + 4y_n^2) - 3(4x_n^2 + 4x_n y_n + y_n^2) \\ &= -3x_n^2 + y_n^2 = 1 \end{aligned}$$

as desired. y_n/x_n approximates $\sqrt{3} \approx 1.732051$. A few iterates are

$$\begin{array}{lll} y_2 = 7 & x_2 = 4 & \frac{y_2}{x_2} \approx 1.733333 \\ y_3 = 26 & x_3 = 15 & \frac{y_3}{x_3} \approx 1.732143 \\ y_4 = 97 & x_4 = 56 & \frac{y_4}{x_4} \approx 1.732057 \\ y_5 = 326 & x_5 = 209 & \frac{y_5}{x_5} \approx 1.732051 \\ y_6 = 5042 & x_6 = 2911 & \frac{y_6}{x_6} \approx 1.732051 \end{array}$$

Checking, $5042^2 - 3 \cdot 2911^2 = 25421764 - 3 \cdot 8473921 = 25421764 - 25421763 = 1$.

18. *Write $\frac{76}{45}$ as a continued fraction.*

The Euclidean algorithm finds the quotients.

$$\begin{aligned} 76 &= 1 \cdot 45 + 31 \\ 45 &= 1 \cdot 31 + 1431 \qquad\qquad = 2 \cdot 14 + 3 \\ 14 &= 4 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

Thus $\gcd(76,45)=1$. Building up the continued fraction we get

$$\frac{76}{45}=1+\frac{31}{45}=1+\frac{1}{\frac{45}{31}}$$

But

$$\frac{45}{31}=1+\frac{14}{31}=1+\frac{1}{\frac{31}{14}}.$$

But

$$\frac{31}{14}=2+\frac{3}{14}=2+\frac{1}{\frac{14}{3}}.$$

But

$$\frac{14}{3}=4+\frac{2}{3}=4+\frac{1}{\frac{3}{2}}.$$

But

$$\frac{3}{2}=1+\frac{1}{2}.$$

Thus

$$\frac{76}{45}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{1+\frac{1}{2}}}}}$$

19. *Find the continued fraction expansion of* $\sqrt{5}$.

The Euclidean algorithm gives

$$\sqrt{5}=2+(\sqrt{5}-2)$$

so

$$\frac{1}{\sqrt{5}-2}=\sqrt{5}+2=4+(\sqrt{5}-2)$$

From here on in it repeats, thus

$$\sqrt{5}=2+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{\ddots}}}}$$

20. *Using Eudoxus's Method of Exhaustion, prove Euclid's Theorem that circles have area ratio as the squares of their diameters.*

We are to show that if two circles C' and C have areas A' and A and diameters D' and D, resp., then

$$\frac{A'}{A}=\frac{(D')^2}{D^2}.$$

If this were not the case then circle C has area X such that

$$\frac{A'}{X}=\frac{(D')^2}{D^2}$$

where X is not equal to A. Thus either $X<A$ or $X>A$. Euclid shows that neither of these hold, thereby establishing $X=A$. Let's first suppose that $X<A$. The idea is to construct regular polygons P_n and P'_n with $6\cdot 2^n$ sides inscribed in the circles so $P_n\le A$

and $P_n' \le A'$ such that area $X < P_n$. Since polygon areas are proportional to the squares of their circumradii, it follows that

$$A' \ge P_n' = \frac{(D')^2 P_n}{D^2} > \frac{(D')^2 X}{D^2} = A'$$

which is a contradiction. Thus $X < A$ is false.

We may do the analagous argument involving superscribing polyhedra, but Euclid finds a short cut. If $X > A$ we may find a $Y < A'$ such that

$$\frac{Y}{A} = \frac{A'}{X} = \frac{(D')^2}{D^2}$$

Hence we are in the same position as before. Constructing an inscribed polygon P_n' such that $Y < P_n'$ we find

$$A \ge P_n = \frac{D^2 P_n'}{(D')^2} > \frac{D^2 Y}{(D')^2} = A$$

whch is also a contradiction.

Let P_0 be a hexagon inscribed in C with radius r. P_{n+1} is constructed from P_n by doubling the number of sides. Thus it remains to show that the area deficit is a reduced by a fraction $0 < \gamma < 1$ for each step

$$A - P_{n+1} \le \gamma\,(A - P_n)$$

It follows (from an induction argument) that

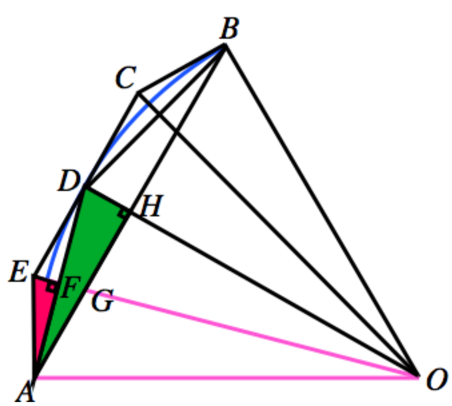
$$A - P_n \le \gamma^n (A - P_0) \tag{4}$$

which may be made smaller than $A - X$ if n is taken large enough. For such a large n, $P_n = A - (A - P_n) > A - (A - X) = X$ as desired.

Finally, it remains to show that (4) holds for these polygons. Starting from P_0, the regular hexagon, P_{n+1} has double the number of vertices as P_n. Consider one of the sectors of P_n where $\alpha = \angle AOB = 360^\circ / v_n$ where $v_n = 6 \cdot 2^n$ is the number of vertices. One of the sides of P_n is AB. The area deficit $A - P_n$ is at least v_n times the area of the triangle $\triangle ADB$ because it is enclosed by the arc ADB and the segment AB. Let Q_n be the circumscribing v_n-gon whose edges are tangent to the circle at the vertices of P_n. Thus one of the edges of Q_{n+1} is the segment EC. The area deficit $A - P_{n+1}$ is less than the v_{n+1} times the area of the triangle $\triangle AED$ because the triangle contains the region bounded by the arc AD and the segment DA. Thus

$$\frac{A - P_{n+1}}{A - P_n} \le \frac{v_{n+1}\mathcal{A}(\triangle AED)}{v_n \mathcal{A}(\triangle ADB)} = \frac{2\mathcal{A}(\triangle AEF)}{\mathcal{A}(\triangle ADH)}$$

since $\triangle AEF$ is half of $\triangle AED$ and $\triangle ADH$ is half of $\triangle ADB$.



The right triangles $\triangle AEF$ and $\triangle ADB$ are similar. To see it, since $\alpha = 360°/v_n$ is the angle $\angle AOB$ of a sector of P_n, then $\angle DOA = \frac{\alpha}{2}$ so its complement is $\angle HAO - 90° - \frac{\alpha}{2}$. $\angle EOA = \frac{\alpha}{4}$. Since FA and FO are perpendicular, its complement is $\angle FAO = 90° - \frac{\alpha}{4}$. Thus angle $\angle FAG = \angle FAO - \angle HAO = \frac{\alpha}{4}$. It follows that $\angle EAF = 90° - \angle FAO = \frac{\alpha}{4}$ also, hence $\triangle AEF$ and $\triangle ADH$ are similar.

The areas of $\triangle AEF$ and $\triangle ADH$ are proportional to the squares of their longer legs. But the length of AF is half of the length of AD which is the hypotenuse of $\triangle ADH$. Thus

$$\frac{2\mathcal{A}(\triangle AEF)}{\mathcal{A}(\triangle ADH)} = \frac{2\mathcal{L}(AF)^2}{\mathcal{L}(AH)^2} = \frac{\mathcal{L}(AH)^2 + \mathcal{L}(DH)^2}{2\mathcal{L}(AH)^2} = \frac{1}{2}\left(1 + \left(\frac{\mathcal{L}(DH)}{\mathcal{L}(AH)}\right)^2\right) \tag{5}$$

Finally we notice that the ratio $\mathcal{L}(DH)/\mathcal{L}(AH)$ (which is $\tan(\frac{\alpha}{4})$) decreases monotonically as $\angle DAH = \frac{\alpha}{4}$ decreases so that the largest that ever is occurs for P_0 where $\alpha = 60°$. The biggest right side of (5) called γ, corresponds to $\alpha = 60°$. $\tan 15° = 2 - \sqrt{3}$ may be computed from triangles, for example by taking the slope of the line through the origin and the midpoint of the points $(1, 0)$ and $(\cos 30°, \sin 30°) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ which are $30°$ apart. Thus for $\alpha = 60°$,

$$\gamma = \frac{1}{2}\left(1 + \left(\frac{\mathcal{L}(DH)}{\mathcal{L}(AH)}\right)^2\right) = \frac{1}{2}\left(1 + \left(2 - \sqrt{3}\right)^2\right) = 4 - 2\sqrt{3} \approx 0.536.$$

Euclid claims that $\gamma = \frac{1}{2}$ works, although I haven't been able to understand his proof of this.

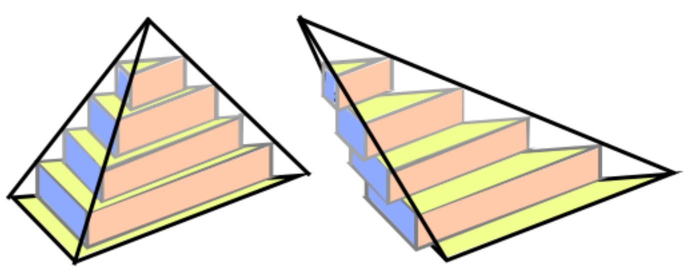
21. *Diophantus gave ingenious solutions to a variety of problems in his book* Arithmetica. *Here is one of them (Book 1, Problem 17). Find four numbers such that when any three of them are added together, their sum is one of four given numbers, say* 20, 22, 24, 27. *[from Burton, p. 220.]*

Let a, b, c and d be the four numbers and $x = a+b+c+d$ be their sum. Then the numbers are $a = x - 20$, $b = x - 22$, $c = x - 24$ and $d = x - 27$. For example, if $b + c + d = 20$ then adding a, $x = a + b + c + d = a + 20$. It follows that

$$x = (x - 20) + (x - 22) + (x - 24) + (x - 27) = 4x - 93$$

Hence $3x = 93$ or $x = 31$. It follows that $a = 31 - 20 = 11$, $b = 9$, $c = 7$ and $d = 4$.

22. *Show that any two tetrahedra with the same base and height can be approximated arbitrarily closely by the same prisms, differently stacked. Deduce that the tetrahedra of the same base and height have equal volume. [Stillwell, problems 4.3.2 and 4.3.3 from your homework.]*



Let Δ denote the tetrahedron. Let us suppose that the on the bottom is a triangle T_0 containing the origin of $\mathbf{R}^3$ whose base has length ℓ and width (height in the $z = 0$ plane) w. Suppose that the apex has coordinates (a, b, h). Suppose that we slice the tetrahedron horizontally into n pieces. The slices are at heights $\frac{kh}{n}$ where $k = 0, 1, 2, \ldots, n$. The intersections of the slice with the tetrahedron are smaller and smaller triangles T_k with length $\frac{n-k}{n}\ell$ and width $\frac{n-k}{n}w$.

The prisms Π_k have bases T_k and heights $\frac{h}{n}$. Thus the volume of the kth prism is

$$V(\Pi_k) = A(\text{base}) \cdot \text{height} = \frac{1}{2} \cdot \frac{(n-k)\ell}{n} \cdot \frac{(n-k)w}{n} \cdot \frac{h}{n} = \frac{(n-k)^2 h\ell w}{2n^3}$$

The stack of prisms P_n consists of $n-1$ prisms whose tops line up with the intersection of the $z = \frac{hk}{n}$ planes. The volume of the stack is

$$V(P_n) = \sum_{k=1}^{n-1} V(\Pi_k) = \sum_{k=1}^{n-1} \frac{(n-k)^2 h\ell w}{2n^3} = \frac{(n-1)n(2n-1)h\ell w}{12n^3}$$

where we have used the formula from Calculus that

$$\sum_{k=1}^{n-1} (n-k)^2 = (n-1)^2 + (n-2)^2 + \cdots 3^2 + 2^2 + 1^2 = \frac{(n-1)n(2n-1)}{6}.$$

If the volume of the stack approaches the volume of the tetrahedron, in modern terminology

$$V(\Delta) = \lim_{n\to\infty} V(P_n) = \frac{h\ell w}{6} = \frac{1}{3}A(T_0)h.$$

The volume of the tetrahedron is one third that of the right prism with the same base T_0.

We must show that $V(P_n)$ approximates the volume of the tetrahedron. Notice, that if the apex is not above the bottom triangle, then the stack may bulge out of the tetrahedron, as in the second diagram. This adds complication but conceptually the same ideas occur in the argument as if the apex was above the base. Let us assume this for now and finish the argument in this case. The general case will be dealt with after that.

When the apex is above the base, then the triangular slices are nested. The first level has base T_1 with height $\frac{h}{n}$. On top of it is the prism with base T_2 and so on. Thus $P_n \subset \Delta$ and

$$\frac{(n-1)n(2n-1)h\ell w}{12n^3} = V(P_n) \le V(\Delta).$$

The outer polyhedron Q_n starts with base T_0, Then base T_1 and so on up to the nth level this time. Indeed Q_n is P_n placed on top of the prism with base T_0. Since $\Delta \subset Q_n$ we have $V(\Delta) \le V(Q_n)$. Also, adding the volume of the bottom prism we find

$$V(Q_n) = V(P_n) + \frac{lwh}{2n} = \frac{(n(n+1)(2n+1)h\ell w}{12n^3}$$

The volume difference is the volume of the new bottom slab

$$V(Q_n) - V(P_n) = \frac{lwh}{2n}$$

which can be made arbitrarily small. This implies that the error made by approximating with the stack is

$$\begin{aligned} V(\Delta) - V(P_n) &\le V(Q_n) - V(P_n) = \frac{lwh}{2n}; \\ V(Q_n) - V(\Delta) &\le V(Q_n) - V(P_n) = \frac{lwh}{2n}. \end{aligned}$$

which tends to zero as n tends to infinity.

Lets show that two tetrahedra Δ and Δ' with the same base, same height and apexes above their bases have the same volume. P_n and P'_n have the same volume because they consist of the same prisms that have been placed at different positions as they are stacked. Thus $V(P_n) = V(P'_n)$. Following Eudoxus to equate ratios, suppose that we select any rational numbers $p < V(\Delta)$ and $V(\Delta) < q$. We have to show that $p < V(\Delta')$ and $V(\Delta') < q$. By the Archmidean principle, we can find n so large that

$$V(Q_n) - V(P_n) = \frac{lwh}{2n} < \min\{V(\Delta) - p, q - V(\Delta)\}.$$

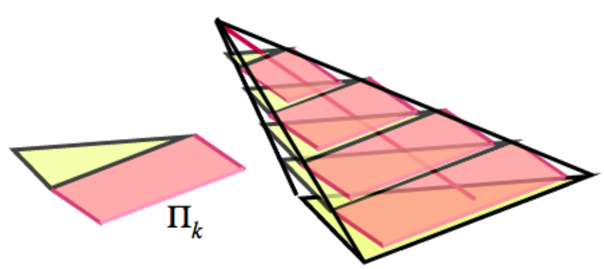
The same estimates apply for Δ' as for Δ because they depend only on n and the dimensions of the bases T'_0 and T_0 which are the same. Thus $V(Q'_n) = V(Q_n)$ too since they are stacks of the same prisms. It follows from $P_n \subset \Delta$ that

$$\begin{aligned} V(\Delta') &= V(P'_n) + [V(\Delta') - V(P'_n)] \\ &\le V(P'_n) + [V(Q'_n) - V(P'_n)] \\ &= V(P_n) + [V(Q_n) - V(P_n)] \\ &< V(\Delta) + [q - V(\Delta)] = q. \end{aligned}$$

Similarly, since $\Delta \subset Q_n$,

$$\begin{aligned} V(\Delta') &= V(Q'_n) - [V(Q'_n) - V(\Delta)] \\ &= V(Q_n) - [V(Q_n) - V(\Delta)] \\ &\ge V(Q_n) - [V(Q_n) - V(P_n)] \\ &> V(\Delta) - [V(\Delta) - p] = p. \end{aligned}$$

Since p and q were arbitrary, it follows that $V(\Delta) = V(\Delta')$, completing the argument for the cases that the apex is above the base.



Now suppose that the apex is not necessarily over the base. We put the origin in the base T_0 and let $V = (a, b, h)$ be the direction from the origin to the apex. Instead of using right

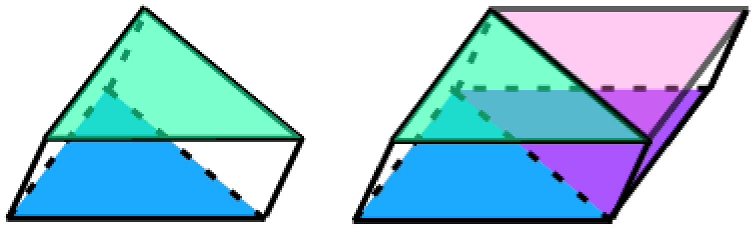


Figure 1: Doubling a skew triangular prism yields a parallelopiped

angled prisms whose sides go vertically, we'll use skew prisms whose sides are parallel to V. Then the argument proceeds exactly as before by making leaning towers of skew prisms instead of straight upward towers. This works if the volume of of a skew prism has the same volume as the right prism, given by area of base times height.

Finally, we argue that the volume of a skew prism is the same as the straight prism. By gluing a copy of the triangle to itself, we double the base. Then we show that a skew parallelopiped, which has double the volume of a skew triangular prism because it is two skew triangular prisms glued together and has volume equal to base times height, proving that the volume of a triangular prism is base times height.

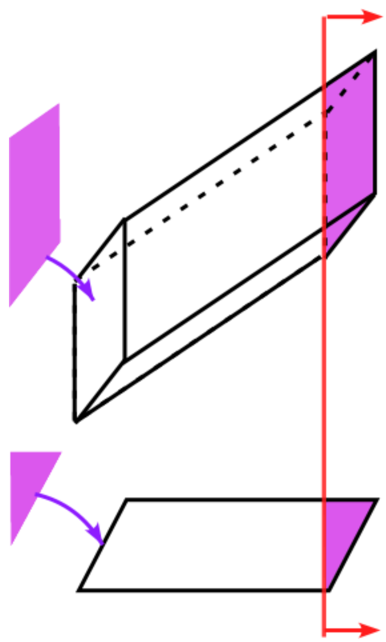


Figure 2: Top and front views of cutting and gluing a parallelopiped

First, we making sure the height is small enough so that the upper face is above the lower one. The top of the skew prism is shifted by a small amount for n large enough. Otherwise, we can cut the skew parallelopiped by horizontal slices with this property, and get the result by adding the slices.

If the upper face is above the lower one, we can cut the parallelopiped in the vertical plane parallel to one of the edges and re-glue the remnant on the other side to make another parallelopiped with the same volume but with two sides vertical. By repeating in the other side direction we get a vertical prism over the same base with the same volume, proving that the skew parallelopiped has volume equal to area of base times height.