Foundations of Analysis II
Week 5

Domingo Toledo

University of Utah

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Homework

Compute some Fourier Series

$f : \mathbb{R} \to \mathbb{R}$

$(\mathbb{R} \to \mathbb{C})$

periodic $\quad f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}$

$\sin x$, $\cos x$, tg

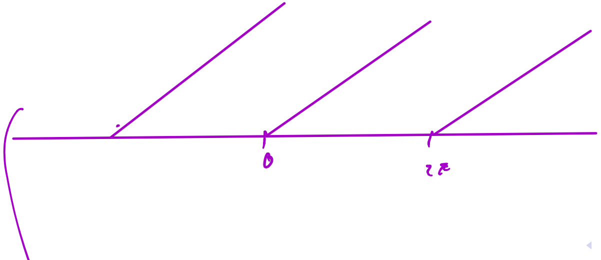
cont, diff, analytic

$\sin nx$

$\cos nx$

$\cos x + \cos 3x + \dots$

extend by $f(x+2\pi)$
$= f(x)$



$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$$

$$\sum_{-\infty}^{\infty} = \begin{matrix} 2 \\ 0 \end{matrix}$$

$$\frac{(-1)^a + 1}{\sum} \qquad a \qquad \boxed{\sum_{-\infty}^{\infty} \frac{1}{n^a}}_{\neq 0}$$

a even $\quad 2\sum_{1}^{\infty} 1/n^a$

a odd $\quad 0$

$$\sum 1/n^3$$

$\boxed{\sum 1/n^2 = }$

1970's

$\sum 1/n^3$ irrational

Fejer's Theorem

- Cesaro sums: given $\{s_n\}$, define

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N + 1}$$

- $\{s_n\}$ is Cesaro summable if $\{\sigma_n\}$ converges
- $\{s_n\}$ convergent $\Rightarrow$ Cesaro summable
- Not conversely.

$a_0, a_1, a_2, a_3, \ldots$

$\boxed{S_n = a_0 + \cdots + a_n}$ $\quad \sum_0^\infty a_n \overset{def}{=} \lim_{n\to\infty} S_n$

$\sigma_N = \dfrac{S_0 + S_1 + \cdots + S_N}{N+1}$

Theorem

f continuous $\Rightarrow$ $\sigma_N(f;x) \to f$ *uniformly.*

S_n converges $\Rightarrow$ $\{\sigma_n\}$ converges

$S_n \to S$

$$\frac{S_0 + S_1 + \cdots + S_n + \underbrace{S_{n+1} + \cdots\cdots + S_N}_{\sim S(N+1-n)}}{N+1}$$

$a_n = (-1)^n$

a_n: $1, -1, 1, -1, \ldots$

s_i: $1, 0, 1, 0, 1, 0$

σ: $\cancel{1, \tfrac{1}{2}, \ldots}$

1

$1, 0, 1, 0, 1, 0$

1

$\frac{1}{2}$

$2/3,$

$2/4$

$1/2$

$\sigma_n \to 1/2$

f periodic fn $\mathbb{R} \to \mathbb{R}$

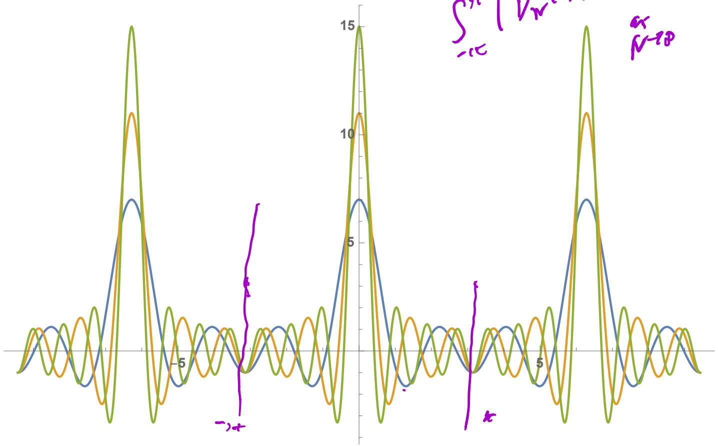
$$S_N(f, x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t)\, \underline{D_N(t)}\, dt$$

Dirichlet's Kernel

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(t)\,dt = 1$$

$$\int_{-\pi}^{\pi} |D_N(t)|\,dt \to \infty \quad \text{as } N\to\infty$$

Out[ ]=



Fejér : Cesaro converge

fact

$\sigma_N(f,x) \longrightarrow f$ uniformly

$$\sigma_N(f,x) = \frac{S_0(f,x) + S_1(f,x) + \cdots + S_N(f,x)}{N+1}$$

$$\frac{1}{N+1}\sum_{n=0}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\, D_n(t)\, dt\right)$$

$$= \frac{1}{N+1}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t) \underbrace{\left(\sum_{n=0}^{N} D_n(t)\right)}_{\overset{\text{def}}{=} K_N(t)} dt$$

Fejer’s kernel

$$D_n(t) = \frac{\sin(n+1/2)t}{\sin(t/2)}$$

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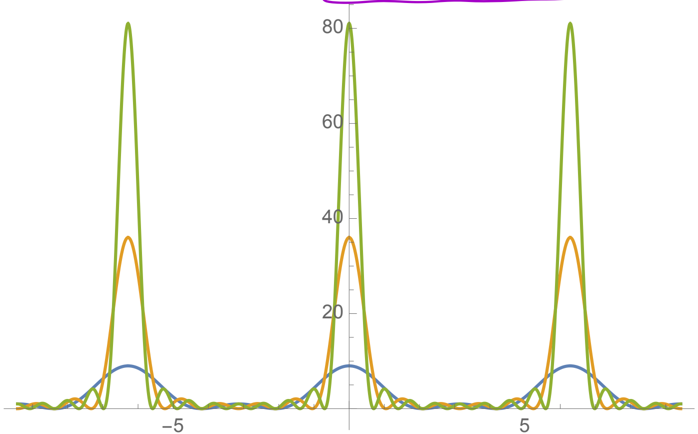

D_0, D_1, D_2

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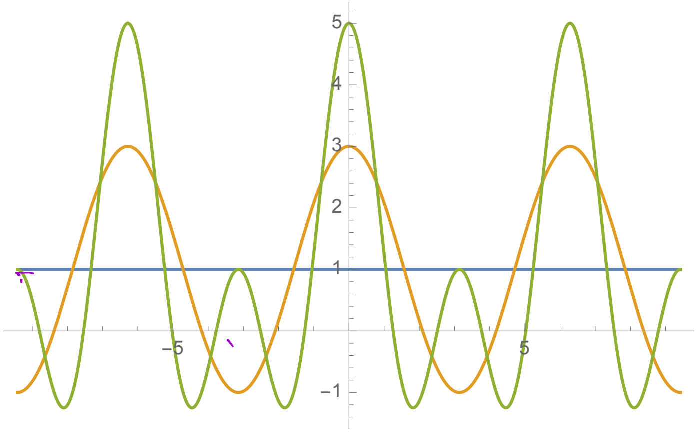

$D_0, (D_0 + D_1)/2, (D_0 + D_1 + D_2)/3$

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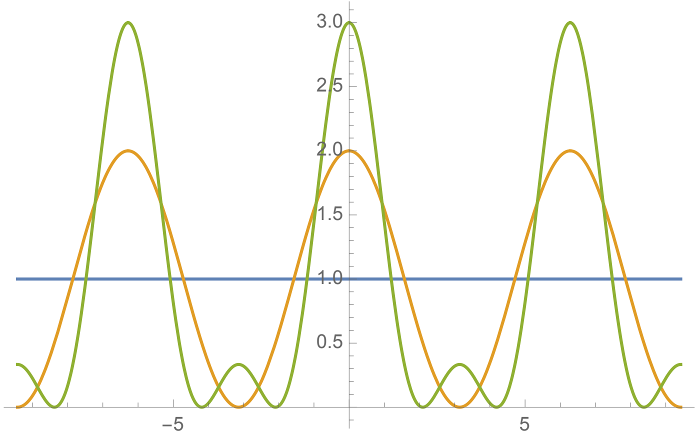

$$\frac{1}{N+1}\sum_{n=0}^{N}\frac{\sin(n+\frac12)t\ \sin(t/2)}{\sin(t/2)^2}$$

$$\underline{\sin(n+\tfrac12 t)\sin(t/2)}$$

$$\cos(nt) = \cos[(n+\tfrac12)t - t/2]$$

$$\cos(n+1)t = \cos[(n+\tfrac12)t + t/2]$$

$$\cos nt = \cos(n+\tfrac12)t\cos(t/2) + \sin(n+\tfrac12)t\sin(t/2)$$

$$\cos(n+1)t = \text{———} - \sin(n+\tfrac12)t\sin t/2$$

$$\sin(n+\tfrac12)t\sin(t/2) = \frac{\cos nt - \cos(n+1)t}{2}$$

$$\frac{1}{N+1}\sum_{0}^{N}\frac{\sin(n+\frac12)t\sin(t/2)}{\sin^2 t/2} = \frac{1}{N+1}\frac{1-\cos(N+1)t}{2\sin^2 t/2}$$

$1-\cos\theta$

$\cos(0t) - \cos(1t)$
$\cos(1t) - \cos(2t)$
$\cos 2t - \cos 3t$
$\vdots$
$\cos Nt - \cos(N+1)t$

"telescoping sum"

$$\frac{1}{N+1}\frac{1-\cos(N+1)t}{2\sin^2 t/2} = \frac{2\sin^2(\frac{N+1}{2}t)}{2\sin^2 t/2}$$

HW

like the Kanto..., ...change by Rolz

Rudin Chap 8 Exer 15

Redo (a)/(b)/(c)

a) $K_N(t)\geq 0$

2) $\frac{1}{2\pi}\int K_N(t)\,dt = 1$

3) $K_N(t) \leq \frac{1}{N+1}\frac{2}{1-\cos\delta}$, $\delta\leq|t|\leq\pi$

$f(x^+), f(x^-)$ exist

$$\Rightarrow \sigma_N(f,x)\to\frac12(f(x^+)+f(x^-))$$

$\forall \delta > 0$ $\to 0$ unif on $|x| \geq \delta$

Uniform Approximation by Polynomials

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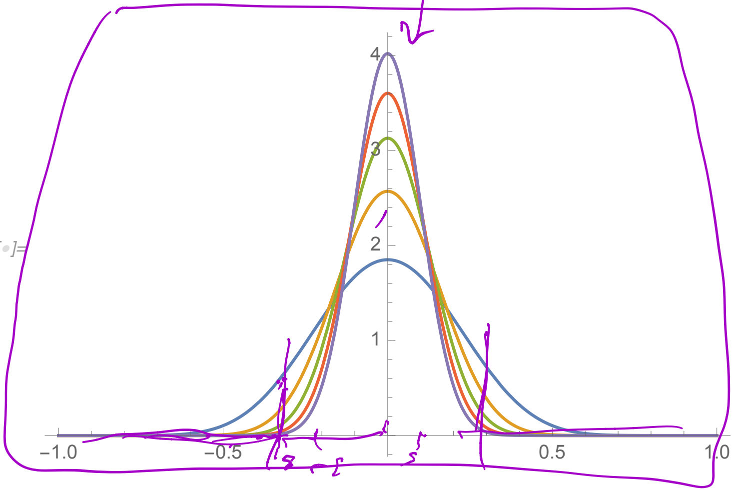
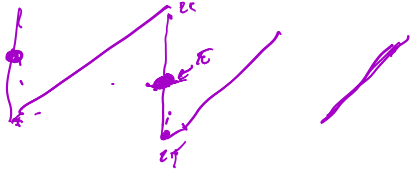


if S_N conv

$\Rightarrow \sigma_N$ conv $\to \dfrac{f(x^+)+f(x^-)}{2}$



L^2-Convergence and Parseval's Theorem

$$\sum 1/n^2 = \pi^2/6$$

f Riemann integrable, ~~[illegible]~~

$$\int_{-\pi}^{\pi} [f(x)]^2 \, dx \quad \text{exists}$$

$$\sum_{-\infty}^{\infty} |c_n|^2 \leq \int_{-\pi}^{\pi} [f(x)]^2 \, dx$$

Thm 8.16 Rudin

actually have =.

$$\sum c_n e^{inx} \to f \quad \text{in } L^2$$

$$\Big\| \sum_{-N}^{N} c_n e^{inx} - f \Big\|_2 \to 0 \quad \text{as } N \to \infty$$

Idea of pf.

$\forall \varepsilon > 0$

1) f R-int $\to \exists$ cont h

s.t. $\|f - h\|_2 < \varepsilon$

2) Stone-Weierstrass

$\exists P$ = trigonom poly

s.t. $\|h - P\|_\infty < \varepsilon$

$\Rightarrow \|h - P\|_2 < \varepsilon$?

$\|h - P\|_2 = \left(\int_{-\pi}^{\pi} |h(x) - P(x)|^2 \, dx \right)^{1/2}$

$\leq \left(\varepsilon^2 \cdot (2\pi)^2 \right)^{1/2}$

$\cancel{2\pi}\,\varepsilon$

$\|f\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx \right)^{1/2}$

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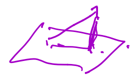
approx f by trig polys.

$\|f - h\|$ $\|h - P\|$ $\|f - S_N\|$

f, h, P, S_N



f k

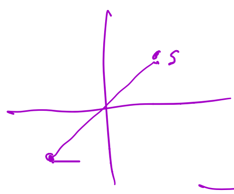
$\Rightarrow \| f - \sum t_n e^{\cdots} \| < \varepsilon$

$\Rightarrow \| f - S_N(G_r) \| < \varepsilon$

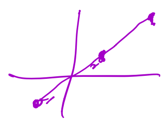
$S_N \to f$ in L^2

$\left(\Gamma(\,),\ \zeta \right)$ $\mathbb{C}$

$\xi(s)$ $\xi(s) = \xi(1-s)$



$s \to 1-s$



Γ, ζ

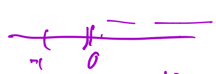
$\mathbb{C}$ $\sum \frac{1}{n^s}$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

$x > 0$

$x < 0$ diverges at 0

$$\Gamma(x+1) = x\,\Gamma(x)$$



$-1 < x < 0$

$\Gamma(x+1) = x\,\Gamma(x)$

$\Gamma(x) = \frac{\Gamma(x+1)}{x}$

$\Gamma(x) \in \mathbb{R} - \{0, -1, -2, \ldots\}$

$$\int_0^\infty t^{x-1} e^{-t}\, dt$$

$$\int_0^1 t^{x-1}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} \ldots\right) dt$$

$\int_0^1 \quad + \int_1^\infty$ OK

$t^{x-1}(1 - t + \frac{t^2}{2} - \dots)$

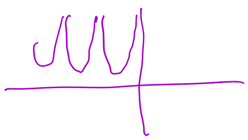
$t^{x-1} - t^x + \dots$ OK

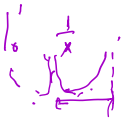
t^{x-1}

$-1 < x < 0$

$\int_0^1 t^{x-1} - (\int_0^1 t^{x} - \dots)$ OK $\quad \int_1^\infty \dots$

$\frac{t^x}{x}\Big|_0^1 \quad \frac{1}{x}$

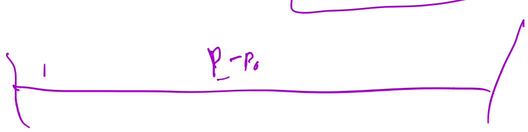




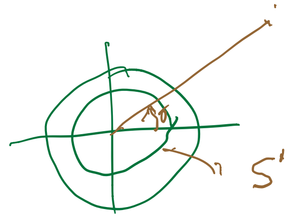
Calculus in several variables.

$\min\{d(p_0, p) : p \in L\}$

$\boxed{p - p_0 \perp L}$



$\|x\| = 1$



$\mathbb{R}^n \quad (\underset{>0}{r}, \Theta \in S^{n-1})$

Θ = vector of length 1

$r, \Theta \to r\Theta$

$f(r,\theta)\ dr\,d\theta$

$r\Delta r \Delta\theta$

$\iint f(r)\,(\ )\ = \int_0^{2\pi}\int_0^{\infty} f(r)\ r\,dr\,d\theta$

darea

$\mathbb{R}^n$

$\int_{S^{n-1}}\int_0^{\infty} f(r)\,\boxed{r^{n-1}}\,dr\,d\Theta \qquad r^{n-1}\Delta\Theta\Delta r$

$\boxed{= V(S^{n-1})\int_0^{\infty} f(r)\,r^{n-1}\,dr}$

$V(S^{n-1})\int_0^{\infty} e^{-r^2} r^{n-1}\,dr$

$= \iiint_{\mathbb{R}^n} e^{-(x_1^2+\cdots+x_n^2)}$

$(\sqrt{\pi})^n = \pi^{n/2}$

$\mathrm{vol}\{S^{n-1}(r)\} = r^{n-1}\,\mathrm{vol}(S^{n-1}(1))$

$\boxed{r = \text{dist from } 0 = \sqrt{x_1^2+\cdots+x_n^2}}$

$\Gamma(\tfrac12)$ from $\iint e^{-r^2}\,r\,dr\,d\theta$

$\Gamma(\tfrac12) = \int_{-\infty}^{\infty} e^{-x^2}dx \qquad \Gamma(\tfrac12)^2 = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy$

Knowing length $S^1 \longrightarrow$ compute $\Gamma(\tfrac12)$

compute vol $(S^{n-1}) \longleftarrow$ knowing Γ

$\mathrm{vol}(S^{n-1}_1) \qquad \mathrm{vol}(B^n)$

$\{\|x\| = 1\} \qquad \{\|x\| \le 1\}$

$$\mathrm{vol}\, B^n(1) = \int_0^1 \mathrm{vol}(S^{n-1}(r))\, dr$$

$$V(B^2) = \int_0^1 2\pi r\, dr = \pi r^2 \Big|_0^1 = \pi$$

$$\frac{d}{dr} V(B^n(r)) = V(S^{n-1}(r))$$

$\boxed{V_n(B^n) \quad V_{n-1}(S^{n-1})}$

Start from

$$\boxed{\pi^{n/2} = V(S^{n-1}) \int_0^\infty e^{-r^2} r^{n-1}\, dr}$$

$\Gamma(\ )$

$\rightarrow$ formula for $V(S^{n-1})$

$V(B^n)$

$\cancel{2\pi \ \pi}$

Factorials power of π.

$2\pi r,\ 4\pi r^2, \ldots$

$2\pi,\ 4\pi, \ldots$

$\pi,\ \frac{4}{3}\pi, \ldots$

α / π irrational, $f: \mathbb{R} \to \mathbb{R}$ periodic

$f(x + 2\pi) = f(x)$

$\forall x \in \mathbb{R}$

$$\boxed{\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + n\alpha)} = \boxed{\frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt}$$

$\{n\alpha\}$

α

N. average of f on $\{n\alpha\}$ is

$\approx$ average over S^1

$\{n\alpha\}$ are uniformly distributed in S^1

Differentiable Functions of Several Variables

$f: \mathbb{R}^m \to \mathbb{R}^n$

"differentiable"

- Simplest Example:
  Linear transformations $A : \mathbb{R}^m \to \mathbb{R}^n$
- $\mathbb{R}^n$ is a *Vector Space*
- So is $\mathcal{C}[0, 1]$, $L^2[0, 1]$, etc.
- What's the same? What's different?

Vector space X

Set X operations, $X \times X \to X$

Sum $x, y \mapsto x+y$

scalar mult $\mathbb{R} \times X \to X$

$r, x \mapsto rx$

($\mathbb{R}$-vector space)

Subject to all [illegible]

$\forall x,y,z \in X \quad (x+y)+z = x+(y+z)$

$x+y = y+x$

$r(x+y) = rx+ry$

...

Vector Spaces

Ex $\mathbb{R}^n$ Finite dim

$C[0,1]$

L^1, L^2

infinite dimension

d

$\mathbb{R}$-vector space.

$\mathbb{C}$-vector space.

$\mathbb{R}$ - vector space

$\mathbb{F}$ any vector space

$\mathbb{F}$ = a field

Vector Space Vocabulary

- Linear combinations

$x_1, \ldots, x_k \in X$

$\alpha_1, \ldots, \alpha_n \in \mathbb{R}$

Ex. $\alpha_1 x_1 + \cdots + \alpha_k x_k \in X$ is called a linear comb of $x_1, \ldots, x_k$

- Subspaces

subset $Y \subset X$ is called a (vector) subspace if closed under linear combina if $x, y \in Y$, $\alpha, \beta \in \mathbb{R}$ $\Longrightarrow \alpha x + \beta y \in Y$

Y is a vector space

$x \to -x \quad x + (-x) = 0$

0

$-x$

$-x = (-1)x$

$\langle S \rangle$ a a subspace of X.

$\{\alpha_1 x_1 + \cdots + \alpha_n x_n : x_1, \ldots, x_n \in S, \alpha_1, \ldots, \alpha_n \in \mathbb{R}\}$

$S \subset X$ subset

- Span

span of S

$\langle S \rangle$

$=$ {all linear combs of elements of S}

- Linear Independence

$\{x_1, \ldots, x_k\}$ is linearly indep

$\Longleftrightarrow$ when $\alpha_1 x_1 + \cdots + \alpha_k x_k = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$

- $\boxed{\text{Basis}}$ $\boxed{S' \to \text{Span},\ \to \text{linearly indep}}$

- Dimension $\boxed{\#(\text{Basis})}$

Ex $\dim \mathbb{R}^n = n$

#e of terms

$e_1, \ldots, e_n$

$$(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$$

$$(0 \cdots \underset{i}{1} \cdots 0)$$

Every spanning set contains a basis

Every fin indep set is contained in a basis

Any two bases have same #.

$\dim X < \infty \iff \exists$ finite spanning set.

$\dim C[0,1] = \infty$

$\dim \mathbb{R}^n = n$

$\dim \{\text{polynomials of degree} \leq n\}$

$= \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in \mathbb{R}\}$

$\infty \to C(\mathbb{R}) \to \infty$ dim.

$\cup\cup$

$\infty \to P(\mathbb{R})$ $\{a_0 + a_1 x + \cdots + a_n x^n$ some $n\}$

≤ 2

$\{a_0 + a_1 x + a_2 x^2\}$

$\cup$

$n+1 \leftarrow P^n$

deg $\leq n$

$1, x, x^2$ basis

$a_0 + a_1 x + a_2 x^2$ as fun on $\mathbb{R} \to$ get a_0, a_1, a_2

Linear transformations

X, Y vec spaces

$A : X \to Y$ is called linear fn

iff $A(x+y) = Ax + Ay \quad \forall x, y \in X$

$A(\alpha x) = \alpha A x \quad \forall \alpha \in \mathbb{R}, x \in X.$

$\Updownarrow$

$\forall x, y \in X, \alpha, \beta \in \mathbb{R}$

$$A(\alpha x+\beta y)=\alpha Ax+\beta Ay$$

Linear Transformations of Finite Dimensional Spaces

- Matrix of a Linear transformation $A : \mathbb{R}^m \to \mathbb{R}^n$

matrix

$m \times n$

$$\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \to \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \right. \mapsto \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

i-th column

$$A \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

$|\cdot|$ and $\{$linear funcs $\mathbb{R}^n \to \mathbb{R}^m\}$

$\{m \times n$ matrices$\}$

- Matrix of linear $A : X \to Y$ with respect to bases:
- Choose bases $\{e_1, \ldots, e_m\}$ for X and $\{f_1, \ldots, f_n\}$ for Y.

given $\Rightarrow \exists! \{a_{ij}\}$ $i = \{1, \ldots, n\}$, $j = \{1, \ldots, m\}$

$$A e_j = \sum a_{ij} f_i$$

Ex: $P^n \to P^{n-1}$

$$\boxed{p \to p'}$$

$1, x, \ldots, x^n$ $\qquad x^d$

$\frac{d}{c}$ $\frac{d}{1}$ $\frac{d}{c^i x^{c-1}}$

$1, x, \ldots, x^n$ $\quad P^n$

$1, x, \ldots, x^{n-1}$ $\quad P^{n-1}$

$$\begin{pmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & \ddots & \ddots \\ 0 & & & \end{pmatrix}$$

$$\begin{array}{cc} & n+1 \\ n & \begin{pmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & \ddots & \\ & & & & n \end{pmatrix} \end{array}$$

Invertible Linear Transformations

- X finite dimensional, $A : X \to X$ linear
- Then A is one-to-one $\Leftrightarrow$ A is onto.

$A : X \to X$ Proof.

$(Ax = 0 \Rightarrow x = 0) \Leftrightarrow$ 1-1

$Ax = Ay \Leftrightarrow x = y$

$A(x-y) \Leftrightarrow x - y = 0$

If A 1:1 $\Rightarrow$

$e_1, \dots, e_n$ basis for X

$\Rightarrow$ $Ae_1, \dots, Ae_n$ indep in X

$\Rightarrow$ Complete

$\Rightarrow$ onto

$A : X \to X$ onto

$e_1, \dots, e_n$ basis for X

$\Rightarrow$ $Ae_1, \dots, Ae_n$ span X

$\Rightarrow$ Basis for X.

$$x = \alpha_1 Ae_1 + \dots + \alpha_n Ae_n$$

$$= A(\alpha_1 e_1 + \dots + \alpha_n e_n)$$

$$\boxed{\begin{gathered} Ax = y \\ A^{-1} y = x \end{gathered}}$$

The Space $\boxed{L(X, Y)}$

$= \{ A : X \to Y : \text{linear maps} \}$

is a ~~ve~~ vector space.

$(A+B)(x) = Ax + Bx$

$(\alpha A)(x) = \alpha A x.$

Norm of $A \in L(\mathbb{R}^m, \mathbb{R}^n)$

$\mathbb{R}^m$, $\mathbb{R}^n$ with 2-norm
Euclidean

$|x| = \sqrt{x_1^2 + \dots + x_m^2}$ $\in \mathbb{R}^m$

Def $\|A\| = \sup\{|Ax| : |x| = 1\}$

(Rmk: $\sup\{|Ax| : |x| \leq 1\}$

$\|A(\alpha x)\| = |\alpha| \|Ax\|$

dle

1) $|Ax| \le \|A\| \, |x|$

2) $\exists$ min C s.t. $\boxed{|Ax| \le C|x|}$

$\sup_{x \neq 0} \left\{ \frac{|Ax|}{|x|} \right\}$

$$\frac{|Ax|}{|x|} = \left| A\left(\frac{x}{|x|}\right) \right|$$

$x \neq 0 \iff \frac{x}{|x|}$ lies in S^{n-1}

$$|Ax| \le C|x| \iff \frac{|Ax|}{|x|} \le C \iff \left|A\left(\frac{x}{|x|}\right)\right| \le C$$

$|Ax|$ cont fn of x, max on $\{|x|=1\}$
↓ compact

A lin
$\Rightarrow$ A cont
$\Rightarrow$ $\{|Ax| : |x| \le 1\}$ bdd
↑ compact
$\Rightarrow$ $\|A\|$ defined

- $A \in L(\mathbb{R}^m, \mathbb{R}^n)$
  $\Rightarrow$ A is Lipschitz
  $\Rightarrow$ A is uniformly continuous.

$$|Ax| \le \|A\|\,|x|$$

uniformly cont.

ε, $\delta = \varepsilon / \|A\|$ → unif cont

- $A, B \in L(\mathbb{R}^m, \mathbb{R}^n) \;\Rightarrow\; \|A + B\| \le \|A\| + \|B\|$.

- $A \in L(\mathbb{R}^M, \mathbb{R}^n)$, $B \in L(\mathbb{R}^n, \mathbb{R}^k) \;\Rightarrow\; \|BA\| \le \|B\|\,\|A\|$

- $L(\mathbb{R}^m, \mathbb{R}^n)$ is a *normed* vectorspace.

- $L(\mathbb{R}^n, \mathbb{R}^n)$ is a *normed algebra*