Foundations of Analysis II
Week 10

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Integration

Chp 10 Rudin

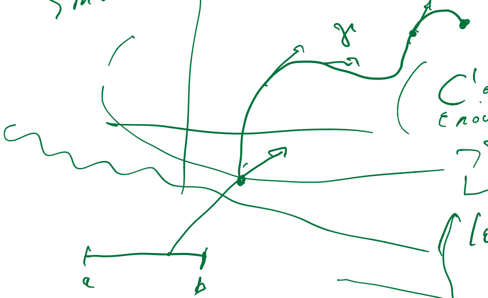
k - dim integrals in $\mathbb{R}^n$ $k \le n$

- Need to define
  - Integrands
  - Domains of integration
  - Integrals

- Model: Line integrals in $\mathbb{R}^2$. Given
  - $U \subset \mathbb{R}^2$ opem and $\mathcal{C}^1$-functions $p, q : U \to \mathbb{R}$.
  - $\gamma : [a, b] \to U$ parametrized $\mathcal{C}^1$-curve, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$
  - Define

$$\int_\gamma pdx + qdy = \int_a^b p(\gamma(t))\gamma_1'(t) + q(\gamma(t))\gamma_2'(t)dt$$

smooth curve $C^{?}$



Smooth $= \boxed{C^{\infty}}$ | lazy

(C^1 or C^2 enough)

$\gamma : [a,b] \to \mathbb{R}^2$

(later may want: $\gamma'(t) \neq 0 \ \forall t$)

usual def

$$\int_\gamma p\,dx + q\,dy$$

$$= \int_a^b \left(p(x(t),y(t))\,\frac{dx}{dt} + q(x(t),y(t))\,\frac{dy}{dt} \right) dt$$

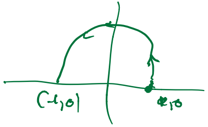
$$\gamma(t) = (x(t), y(t)) = (\gamma_1(t), \gamma_2(t))$$

Examples

What is $p\,dx + q\,dy$?

$\gamma(t) = (\cos t, \sin t)$

$p\,dx + q\,dy = y\,dx + x\,dy$



$\int_\gamma y\,dx + x\,dy$ $\quad \gamma(t) = (\cos t, \sin t)$, $0 \le t \le \pi$

$= \int_0^\pi [(\sin t)(-\sin t) + (\cos t)\cos t]\,dt$

$= \int_0^\pi (-\sin^2 t + \cos^2 t)\,dt$

$= \int_0^\pi \cos 2t\,dt$

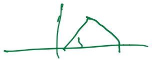
$= \frac{\sin 2t}{2}\Big|_0^\pi = 0$

ϕ

$[0, \pi/4]$

$\frac{\sin 2t}{2}\Big|_0^{\pi/4} = \frac{\sin \pi/2}{2} = 1/2$

$(\cos t, \sin t)$ $\quad 0 \le t \le 2\pi$



$$\boxed{x\,dy + y\,dx = d(xy)}$$

Language to make sense

$$\boxed{x\,dy - y\,dx}$$

$\gamma(t)$
$= \cos t, \sin t$
$0 \le t \le 2\pi$

$$\int_\gamma x\,dy + y\,dx = \int_\gamma d(xy) = xy\Big|_{\gamma(a)}^{\gamma(b)} = 0$$

$p\,dx + q\,dy = df$

in part $\int_\gamma df = 0$ if γ is closed

$$\int_0^{2\pi} \left(\cos t\,(\cos t) - \sin t\,(-\sin t)\right) dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = 2\pi$$

What is $p\,dx + q\,dy$?

$\boxed{\text{Differential one-forms}}$

- Smooth $= C^\infty$ I won't ~~worry~~ count derivatives.
- Smooth one-form on open $U \subset \mathbb{R}^n$ means a function

$$\boxed{\omega} : \underset{x}{U} \times \underset{v}{\mathbb{R}^n} \to \mathbb{R}$$

written

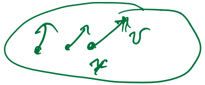
$$\omega_x(v) \quad \text{rather than} \quad \omega(x, v)$$

where $x \in U$, $v \in \mathbb{R}^n$ and

- $\omega_x(v)$ is smooth in x and $\boxed{\text{linear in } v.}$

$$\omega : U \times \mathbb{R}^n$$

$$(x, v) \longrightarrow \omega(x, v) = \omega_x(v)$$



x, v

$x \in U$

$v \in \mathbb{R}^n$

(x, v) v tangent to U at x.

- Usually think of
  - x a point in U
  - v a tangent vector to $\mathbb{R}^n$ based at the point $x \in U$.

Def

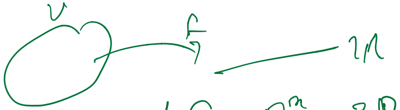
a 1-form on U ~~A~~ function $\omega : U \times \mathbb{R}^n \to \mathbb{R}$

$(x, v) \longrightarrow \omega(x, v)$

smooth in x, linear in v

(mental picture: v is a tangent vector at x)

Example



$d_x f : \mathbb{R}^n \to \mathbb{R}$ linear

$d_x f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$

- If $f : U \to \mathbb{R}$ is smooth, then df is a smooth one-form on U.
- Note $d_x f(v)$ is smooth in x, linear in v.
- Usually write

$d(xy) = y\,dx + x\,dy$

$\boxed{p\,dx + q\,dy}$

$v = (v_1, \dots, v_n)$

$$\boxed{df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i}$$

$$(d_x f)(v) = \frac{\partial f}{\partial x_1}(x)\, v_1 + \frac{\partial f}{\partial x_2}(x)\, v_2 + \dots + \frac{\partial f}{\partial x_n}(x)\, v_n$$

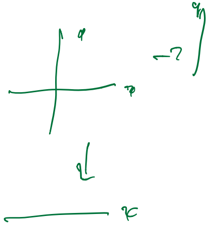
What's dx_i?

in $\mathbb{R}^2$ x, y

$x : \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \to x$

$y : \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \to y$

- Let $x_i : \mathbb{R}^n \to \mathbb{R}$ be the i^{th} coordinate function:
- If $x = (x_1, \dots, x_n)$, then $x_i(x) = x_i$
- dx_i is literally the differential of x_i
- Check that



$$x_i(x+h) - x_i(x) = d_x x_i(h) + o(|h|),$$

where $o(|h|) = 0$ in this case.

h_i

$$(x+h)_i = x_i + h_i$$
$$- \quad (x)_i \quad = x_i$$
$$h_i \quad = (dx_i)(h)$$

Look at $\mathbb{R}^2$

$x : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \to x$

x is diff $\quad x((x+h, y+k)) - x(x,y)$

$= x+h - x = \underline{h}$

$=$ linear $+ 0$

$=$ linear $+ o\left(\sqrt{h^2+k^2}\right)$

x is diff

$\boxed{dx(h,k) = h}$

~~$p\,dx + q\,dy$~~ $\quad p(x,y)\,dx + q(x,y)\,dy$

$U \subset \mathbb{R}^2 \qquad x : (x,y) \to x$

$x : U \to \mathbb{R}$

$\subset \mathbb{R}^2$

$\underline{(x,y)} \to x \in \mathbb{R}$

$\in \mathbb{R}^2 \to$

better notation

$\pi_x : (x,y) \to x$

$\pi_y : (x,y) \to y$

$d\pi_x \;,\; d\pi_y$

Correct

$\longrightarrow \; p(x,y)\,d\pi_x + q(x,y)\,d\pi_y$

but

tradition $\quad p(x,y)\,dx + q(x,y)\,dy$

$d\pi_i(h) = h_i$

$(0, \dots, 1_{j}, \dots 0)$

$d\pi_i(e_j)$

$=$

$\pi_i(x_1, \dots, x_n) = x_i$

$(d\pi_i)(e_j)$

$= \pi_i(x + e_j)$

$- \pi_i(x) \; =$

$d\pi_i$

$\pi_i(x_1, \dots, x_n) = x_i$

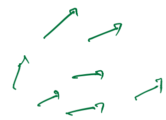
$h = (h_1, \dots, h_n)$

$\pi_i(x+h) = x_i + h_i$

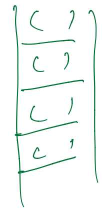
$- \pi_i(x) \quad - x_i$

h_i

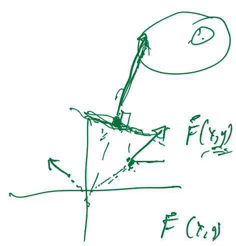
Vector fields



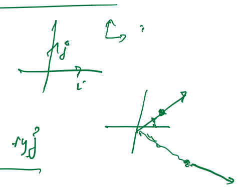
$p\vec{i} + q\vec{j}$



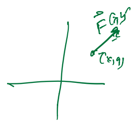
$\mathbb{R}^n \leftrightarrow L(\mathbb{R}^n, \mathbb{R})$



$\vec{F}(x,y) = x\vec{i} + y\vec{j}$



$\vec{F}(x,y) = \text{vector}$



$\vec{F}(x,y)$

$= p\vec{i} + q\vec{j}$

$(u,v) = u\vec{i} + v\vec{j}$

$F(x,y) \cdot (u\vec{i} + v\vec{j})$

$\underline{F(x,y)\cdot(u\vec{i}+v\vec{j})} = \underset{C^\infty}{\omega_{(x,y)}}\underset{\text{linear}}{(u\vec{i}+v\vec{j})}$

func of x, y, u, v
smooth in x, y
linear in u, v

$\omega : U \times \mathbb{R}^n \to \mathbb{R}$

$(x, v) \to \omega_x(v)$

ex: $\vec{F}(x)$ vector field $(x,v) \to \underline{F(x)\cdot v}$

$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

$df(v) = \underline{(\nabla f)\cdot v}$

$x\vec{i} + y\vec{j}$

$\nabla\left(\frac{x^2+y^2}{2}\right)$

$\left(d\left(\frac{x^2+y^2}{2}\right)\right)^{(u,v)} = \boxed{\left(x\,dx + y\,dy\right)(u,v)}$

$x u + y v$

$= (x\vec{i} + y\vec{j})\cdot(u\vec{i} + v\vec{j})$

Dual Basis

$$(dx_i)(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Another interpretation:
- $dx_1, \ldots, dx_n \in L(\mathbb{R}^n, \mathbb{R})$ is the basis for $L(\mathbb{R}^n, \mathbb{R})$ dual to the standard basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$.
- This means

$dx_i \in L(\mathbb{R}^n, \mathbb{R})$

$$dx_i(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

$$(dx_i)(e_j) = \delta_{ij}$$

$$= \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

- Strictly speaking should write $d_x x_i$, but it is independent of $x \in U$.

Explicit Expressions Using Components

- ω smooth one-form on U
  $\Rightarrow$
  there exists a unique collection of smooth functions $p_1 \ldots, p_n : U \to \mathbb{R}$ such that

$$\boxed{\omega = \sum_{i=1}^{n} p_i \, dx_i}$$

v

- In fact

$$p_i(x) = \omega_x(e_i)$$

$v = v_1 e_1 + \ldots + v_n e_n$

$= (v_1, \ldots, v_n)$

$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\omega_x(v) = \sum \omega_x(e_i) v_i$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$L\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (a_1 \cdots a_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$(1\ 0 \cdots 0)$

$(0, 1, 0 \cdots 0)$

$\vdots$

$$\begin{matrix} 1 & 0 & \\ 0 & 1 & \\ \vdots & 0 & \cdots \\ 0 & \vdots & \\ e_1 & e_2 & \cdots\ e_n \end{matrix}$$

Colum $\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$ $\mathbb{R}^n$

rows $(a_1 \cdots a_n)$

$(x_1 \cdots x_n)$

in $L(\mathbb{R}^n, \mathbb{R})$

- If $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, then

$$\omega_x(v) = \sum_{i+1}^{n} p_i(x) v_i$$

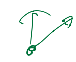
- In this notation, if $\gamma = (\gamma_1, \ldots, \gamma_n) : [a, b] \to U$ is a smooth curve, then

$$\int_\gamma \omega = \int_a^b \left(\sum_{i=1}^{n} p_i(\gamma(t))\, \gamma_i'(t) \right) dt$$

- Need to make the notation more concise.

1-form on $U \subset \mathbb{R}^2$

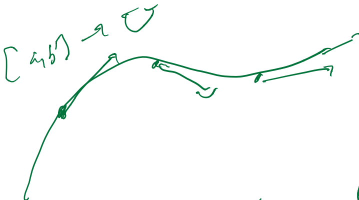
$p(x,y)\,dx + q(x,y)\,dy \qquad p, q : U \to \mathbb{R}$ smooth



(x,y)

$(u,v) \to p(x,y)\,u + q(x,y)\,v$

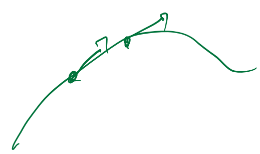
$\gamma : [a,b] \to U$



$t \to \gamma(t), \; \gamma'(t)$
$\in U \qquad \in \mathbb{R}^2$

Tangent to γ at $\gamma(t)$

~~$\omega \; \mathbb{R}^2$~~



$(\gamma(t), \gamma'(t))$
$\;\; = x \qquad = v$

given $\omega_x(v)$

$\gamma(t)$

$[a,b] \to \omega_{\gamma(t)}(\gamma'(t))$

linear

integrand

$$\boxed{\int_a^b \omega_{\gamma(t)}(\gamma'(t))\,dt}$$

integrals

$$\boxed{\omega_\gamma(\gamma') \longrightarrow \int_\gamma \omega}$$

domain of interest

parameterizations

Take Rudin's def of k-form on U (defi 10.11 p 254)

Specialize $k=1$

$$\omega = \sum a_i(x)\, dx_i$$

a func assign

$$\Phi : I \longrightarrow U$$

$$I = [a,b]$$

$$\sum_i \int_a^b a_i(\Phi(t)) \frac{\partial x_i}{\partial t}\, dt$$

Rudin

$$\frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(u_1, \ldots, u_k)} = \begin{vmatrix} \frac{\partial x_{i_1}}{\partial u_1} & \frac{\partial x_{i_1}}{\partial u_2} & \cdots \\ \vdots & \vdots & \\ \frac{\partial x_{i_k}}{\partial u_1} & \frac{\partial x_{i_k}}{\partial u_2} & \end{vmatrix}$$

= the $i_1 \ldots i_k$ minor

of ∂x_1

Φ

$$(x_1, \ldots, x_n) = \Phi(u_1, \ldots, u_k)$$

$k \leq n$

$$x_1 = \Phi_1(u_1, \ldots, u_k)$$

$\vdots$

$$x_n = \Phi_n(u_1, \ldots, u_k)$$

$$\Phi : \underbrace{I \times \cdots \times I}_{k} \to U$$

$$\sum a_i(x)\, dx_i$$

$$\begin{cases} k=1 \quad \underline{I} \xrightarrow{\underline{\phi}} U \end{cases}$$

$$\sum_{i=1}^{n} \int_{\underline{I}} a_i(\phi(u)) \frac{dx_i}{\partial u} du$$

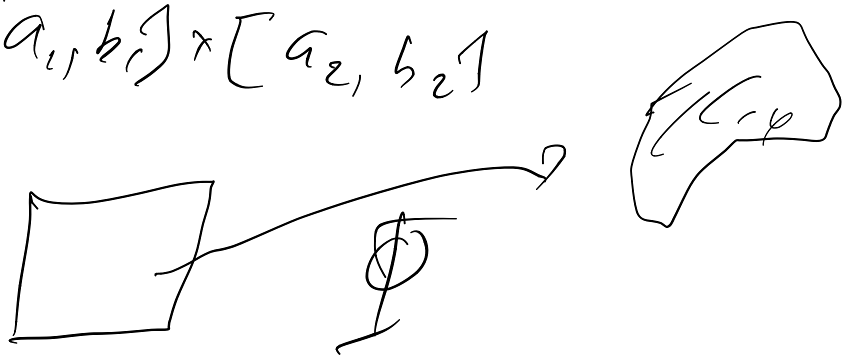
$\underline{I} = [a,b]$

$$\sum_{i=1}^{n} \int_{a}^{b} a_i \left[\underline{\phi}(u) \right] \frac{dx_i}{du} du$$

$\underline{\Phi}$ parametrizatu

$k=2 \qquad \sum_{i<j} a_{ij}(x)\, dx_i \wedge dx_j$

$[a_1, b_1] \times [a_2, b_2]$



$$\sum_{i<j} \int_{a_1}^{b_1}\int_{a_2}^{b_2} a_{ij}\left[\Phi(u_1,u_2)\right] \begin{vmatrix} \dfrac{\partial x_i}{\partial u_1} & \dfrac{\partial x_i}{\partial u_2} \\[2ex] \dfrac{\partial x_j}{\partial u_1} & \dfrac{\partial x_j}{\partial u_2} \end{vmatrix} du_1\, du_2$$

$a_{ij}(\Phi)$

Start again

$U \subset \mathbb{R}^n$ open $\qquad x = (x_1, \dots, x_n)$

1-form $\omega : U \times \mathbb{R}^n \to \mathbb{R}$ $\quad v = (v_1, \dots, v_n)$

$(x, v) \to \omega_x(v)$

smooth in x, linear in v.

Ex ① $f : U \to \mathbb{R}$ smooth func

$df : U \times \mathbb{R}^n \to \mathbb{R}$

$(x, v) \to d_x f(v) \quad \Big(f(x+v) - f(x) = \underbrace{d_x f(v)}_{\text{linear in } v} + o(|v|) \Big)$

② special case: $f(x, v) = x_i$

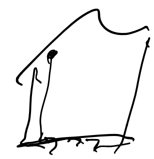
$d_x f(v) = (x+v)_i - (x)_i = v_i \Rightarrow \underset{\text{indep of } x}{d_x f(v) = v_i}$

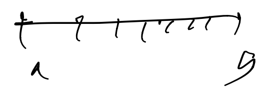
call it dx_i $\quad \boxed{d_x x_i(v) = v_i}$

Example

$\int_a^b f(t)\,dt$ =

$\int_a^b f(t)$





$\lim \{ \underbrace{\sum f(\xi_i)}_{\text{height}} \underbrace{(t_i - t_{i-1})}_{\text{base}} \}$

t = length (+c)

t = length + c

$\Delta t = \Delta$ length

dt = length $(\vec{e_1})$

$\boxed{t_i - t_{i-1} = \text{length } [t_{i-1}, t_i]}$

$dt(e) = 1$

$f(t)\,dt$ 1-form

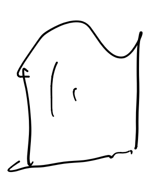
Change $t = g(s)$ $s \in [c,d]$ $[c,d] \xrightarrow{g} [a,b]$

1-1, onto

$$\int_c^d f(g(s))\,dg = \int_c^d f(g(s))\,g'(s)\,ds$$

t = length

g = length new



$\frac{g'(s)ds}{g(s)}$ = length + c

Integrate 1-forms over $[a,b]$

(not function)

$$\sum f(\xi_i)(t_i - t_{i-1})$$

signs of length

Change of variables

$t = g(s)$

$$f(t)\,dt \longrightarrow f(g(s))\,dg$$

$$= f(g(s))\,g'(s)\,ds$$

$f(t)\,dt$ func of x, v $\qquad x, v \to f(x)\,dt(v) =$

~~$f(x)\,dt(v$~~ $\qquad x, v \to f(x)\,v$

$g: (x, v) \longrightarrow (g(x), g'(s)v)$

$= g(x)$

Pull-back of form

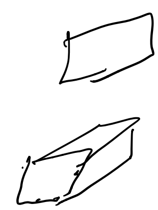
$$\boxed{g^*(f\,dt) = f(g(s))\,g'(s)\,ds}$$

over $\mathbb{R}$ don't integrate functions

Integrate 1-forms

Same in $\mathbb{R}^k$

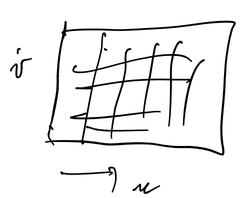
"box" $D = I^k$



$$\int_D f(x,y)\,dx\,dy$$

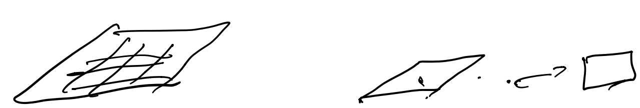
$$\int_D f(x_1,\dots,x_n)\,dx_1\dots dx_n$$

Actually integral of a k-form.



$$\sum \underbrace{f(\xi_{ij}, \eta_{ij})}\,\underbrace{(u_i - u_{i-1})(v_j - v_{j-1})}_{\text{area}}$$

2-form



over $[a,b]$ integrate 1-form $f(t)\,dt$

on $[a_1,b_1]\times[a_2,b_2]$ integr. 2-form $f(t_1,t_2)\,dt_1\,dt_2$

$\vdots$

$[a_1,b_1]\times\cdots[a_k,b_k]$ — k-form

$dt_1\wedge\cdots\wedge dt_k$

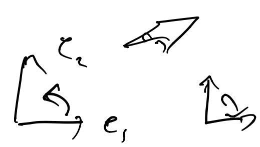
$dt_1\wedge dt_2 = -dt_2\wedge dt_1$ orientation.

$$\iint f(t_1,t_2)\,dt_1\,dt_2$$

2-form $f(t_1,t_2)\,\underbrace{dt_1\wedge dt_2}$

area + orientation

$dt_1\wedge dt_2 = -dt_2\wedge dt_1$



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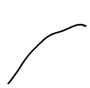
k-form on $\mathbb{R}^k$



Pull-back of differential forms

- Write $A^1(U)$ for the collection of smooth one-forms on U.
- If $V \subset \mathbb{R}^n$ is open and $f : V \to U$ is smooth, define the *pull-back*
$$f^* : A^1(U) \to A^1(V)$$
by
$$(f^*\omega)_x(v) = \omega_{f(x)}(d_x f(v))$$

Change of variables for double in

Single $\underbrace{[c_1, d_1]\times[c_2, d_2]}_{I_1} \xrightarrow{\Phi} \underbrace{[a_1, b_1]\times[a_2, b_2]}_{I_2}$

$$\iint_{I_2} f(t_1,t_2)\, dt_1\, dt_2 = \iint_{I_1} f(\Phi(u_1,u_2))$$

$$\iint f(\Phi(u_1,u_2)) \underbrace{\begin{vmatrix} \frac{\partial t_1}{\partial u_1} & \frac{\partial t_1}{\partial u_2} \\ \frac{\partial t_2}{\partial u_1} & \frac{\partial t_2}{\partial u_2} \end{vmatrix}}_{\text{no absolute value..}} du_1\, du_2$$

$t_1 = t_1(u_1,u_2)$

$t_2 = t_2(u_1,u_2)$

$$dt_1 = \frac{\partial t_1}{\partial u_1} du_1 + \frac{\partial t_1}{\partial u_2} du_2$$

$$dt_2 = \frac{\partial t_2}{\partial u_1} du_1 + \frac{\partial t_2}{\partial u_2} du_2$$

$$dt_1 \wedge dt_2 = (\quad) \wedge (\quad)$$

$du_1 \wedge du_2$

$= -du_2 \wedge du_1$

$du_1 \wedge du_1 = -du_1 \wedge du_1 = 0$

$$\cancel{\frac{\partial t_1}{\partial u_1}\frac{\partial t_2}{\partial u_1} du_1 \wedge du_1} + \frac{\partial t_1}{\partial u_1}\frac{\partial t_2}{\partial u_2} du_1 \wedge du_2$$

$$\frac{\partial t_1}{\partial u_2}\frac{\partial t_2}{\partial u_1} du_2 \wedge du_1 + (\quad)\, \cancel{du_2 \wedge du_2}$$

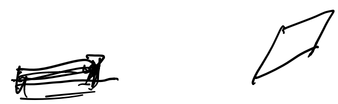
$$\left(\frac{d f_1}{\partial x_1} \frac{d f_2}{\partial x_2} - \frac{d f_1}{\partial x_2} \frac{\partial f_2}{\partial x_1} \right) dx_1 \wedge dx_2$$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

Without thinking:

in $\mathbb{R}^n$ $\quad dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ $\quad \boxed{dx_i \wedge dx_j = -dx_j \wedge dx_i}$

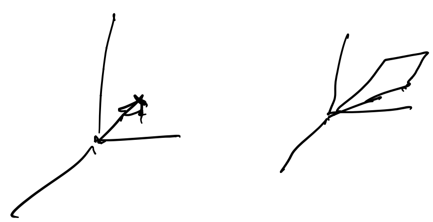
Geometric picture



based at x_0

~~lens in $\mathbb{R}^3$~~ $\quad$ intervals in $\mathbb{R}^3$

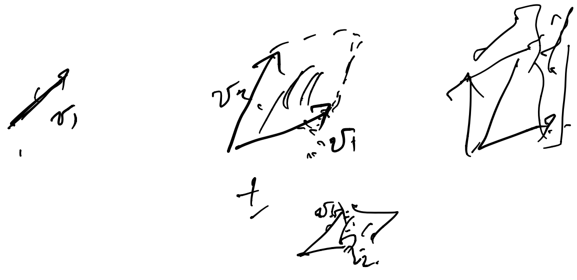
p $\quad$ parallelog. in $\mathbb{R}^3$



in $\mathbb{R}^n$ $\quad v_1, \dots, v_k$ linearly indep. vectors.

"parallelepiped" $P(v_1, \dots, v_k)$

$= \{t_1 v_1 + t_2 v_2 + \dots + t_k v_k, 0 \le t_i \le 1\}$

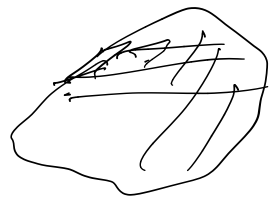


"oriented"

parametr. $v_1, \dots, v_k$

else by sign of permutn

area



Grassmann Alg

expedient to define it in terms of an ON basis $e_1, \ldots, e_n$ for $\mathbb{R}^n = \Lambda^1\mathbb{R}^n$

from these are symbols

$$e_{i_1} \wedge \cdots \wedge e_{i_k} \quad 1 \le i_1 < i_2 < \cdots < i_k \le n$$

for $\Lambda^k \mathbb{R}^n$

they suggest how to define multiplication

- In terms of components, choose coordinates
  - $(t_1, \ldots, t_m)$ for $\mathbb{R}^m$, basis $\bar{e}_1, \ldots, \bar{e}_m$ dual to $dt_1, \ldots dt_m$
  - $(x_1, \ldots, x_n)$ for $\mathbb{R}^n$, basis $e_1, \ldots, e_n$ dual to $dx_1, \ldots dx_n$
  - $f : V \to U$ given explicitly by $x = f(t)$, that is

$$x_i = f_i(t_1, \ldots, t_m) \quad i = 1, \ldots, n$$

- Then

$$f^*(dx_i) = df_i \ \text{ for } \ i = 1, \ldots, n$$

- more precisely, for all $t \in V$ have

$$(f^*(dx_i))_t = d_t f_i$$

- Check definition

$$(f^* dx_i)_t(\bar{e}_j) = (dx_i)(d_t f(\bar{e}_j)) = \frac{\partial f_i}{\partial t_j}(t)$$

- This means

$$(f^* dx_i)_t = \sum_{j=1}^{m} \frac{\partial f_i}{\partial t_j}(t) dt_j$$

- In other words,

$$\boxed{f^* dx_i = df_i} \tag{1}$$

Back to Line Integrals

- Let
  - U be open in $\mathbb{R}^n$
  - ω be a smooth one-form on U.
  - $\gamma : [a, b] \to U$ be a smooth curve.
- Then
$$\gamma^*\omega(t) = \omega_{\gamma(t)}(\gamma'(t))dt$$
- **Define**
$$\int_\gamma \omega = \int_a^b \gamma^*(\omega) \tag{2}$$

- We recover a concise form of

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))dt$$

- Which in turn was a concise form of

$$\int_\gamma \omega = \int_a^b \left(\sum_{i=1}^{n} p_i(\gamma(t))\, \gamma_i'(t) \right) dt$$

Independence of Parametrization

- $\gamma : [a, b] \to U$ smooth curve.
- $\phi : [c, d] :\to [a, b]$ smooth, strictly increasing and surjective.
- $\tilde{\gamma} = \gamma \circ \phi : [c, d] \to U$
- Then for all $\omega \in A^1(U)$

$$\int_{\tilde{\gamma}} \omega = \int_{\gamma} \omega$$

Special Case: $\omega = df$

- In this case

$$\int_\gamma df = \int_a^b (d_{\gamma(t)} f)(\gamma'(t))\, dt$$

which by the chain rule and fundamental theorem of calculus is

$$\int_a^b \frac{d}{dt}(f(\gamma(t))dt = f(b) - f(a)$$

- In other words, integral depends only on the endpoints of γ
- Loosely: “path independent”.

Higher Dimensions

- For 1-dimensional integration in $\mathbb{R}^n$ we made precise:
  - Integrands: smooth one-forms $\omega \in A^1(U)$.
  - Domains of integration: smooth maps $\gamma : [a, b] \to U$.
  - Integral: $\int_a^b \gamma^*\omega = \int_a^b \omega_{\gamma(t)}(\gamma'(t))dt = \int_a^b func(t)dt$.
- To define integral need *pull-back of one-forms*
- To prove integral independent of parametrization need change of variable formula for integrals.

- In higher dimensions we need the k-dimensional analogues.
- Start with domains of integration:
  - Let $\mathbf{I}^k$ denote the cartesian product of k intervals:

$$\mathbf{I}^k = \prod_{i=1}^{k} [a_i, b_i], \quad a_i, b_i \in \mathbb{R}, \quad a_i < b_i. \tag{3}$$

  - Let $\sigma : \mathbf{I}^k \to \mathbb{R}^n$ be a smooth map

- $C \subset \mathbb{R}^k$ is compact,
- $f : C \to \mathbb{R}^n$ a map.
- Say f is smooth
  $\iff$
  there exists an open set $U \subset \mathbb{R}^k$, $C \subset U$, such that f extends to a smooth map $g : U \to \mathbb{R}^n$.

- Next define the integrands: smooth k-forms.
- Should be linear functions on a space that contains the tangent spaces to the images of the maps σ
- Given vectors $v_1, \ldots, v_k \in \mathbb{R}^n$ linearly independent, want a way to manipulate the subspaces

$$\{t_t v_1 + \cdots + t_k v_k \ : \ 0 \leq t \leq 1\} \subset \mathbb{R}^n \tag{4}$$

(the parallelipiped spanned by $v_1, \ldots, v_k$.)

The Grassmann Algebra

- For each k, $1 \leq k \leq n$ want a symbol

$$v_1 \wedge v_2 \wedge, \cdots \wedge v_k$$

  that represents the parallelipiped (4).
- Operations on these symbols that reflect the geometry.
- Example:

$$v_2 \wedge v_1 \wedge v_3 \cdots \wedge v_k = -v_1 \wedge v_2 \cdots \wedge v_k$$

  reflecting change of orientation.

Define The Grassmann Algebra

- Start with $\mathbb{R}^n$ and an ON basis $e_1, \ldots, e_n$
- For each increasing sequence I of k integers

$1 \le k \le n$

$$I = 1 \le i_1 < i_2 < \cdots < i_k \le n$$

define a symbol

$$\boxed{e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}}$$

- There are $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ such symbols.

"k-th exterior power of $\mathbb{R}^n$"

- For $k = 0, \dots, n$ define spaces $\Lambda^k = \boxed{\Lambda^k(\mathbb{R}^n)}$ by
- $\Lambda^0 = \mathbb{R}$

- For $1 \leq k \leq n$,

  wedge $\bigwedge$ or lambda Λ

  $\Lambda^k =$ the $\mathbb{R}$- vector space with basis $\{e_I : \text{ card } (I) = k\}$

- $\dim(\Lambda^k) = \begin{pmatrix} n \\ k \end{pmatrix}$

- $\Lambda^1 = \mathbb{R}^n$ with basis $e_1, \dots, e_n$

$k=0 \quad \mathbb{R} \qquad \underline{e_\emptyset = 1}$

$k=1 \qquad e_1, \dots, e_n \quad \mathbb{R}^n$

$k=2 \qquad \{e_i \wedge e_j : i<j\} \qquad e_1\wedge e_2 \quad e_1\wedge e_3 \dots e_1\wedge e_n$

$e_2 \wedge e_3 \dots$

$\binom{n}{k} \qquad \vdots \qquad \binom{n}{2} = \frac{n(n-1)}{2}$

$k=n \qquad e_1 \wedge \dots \wedge e_n$

Mat: $e_i \wedge e_j = - e_j \wedge e_i$

$\Rightarrow e_i \wedge e_i = 0 \quad \Rightarrow$

$e_1 \wedge e_2$ know e

define $e_2 \wedge e_1 = - e_1 \wedge e_2$

$e_3 \wedge e_1 \wedge e_2 \wedge e_3 \wedge e_1 \wedge e_4$

$= e_2 \wedge e_1$

$e_1 \wedge e_1 \wedge e_3 = 0$

$e_2 \wedge e_3 \wedge e_1 = - e_2 \wedge e_1 \wedge e_3 = + e_1 \wedge e_2 \wedge e_3$

$e_{i_1} \wedge \cdots \wedge e_{i_k}$ basis k-vectors $i_1 < \cdots < i_k$ $\quad I = i_1 \dots i_k$, $i_1 < \cdots < i_k$

$$\Big(\sum_{I \text{ inc}} a_I e_I\Big) \wedge \Big(\sum_{J \text{ inc}} b_J e_J\Big)$$

$J = j_1 \dots j_l$, $j_1 < \cdots < j_l$

$$= \sum_{I, J} a_I b_J (e_I \wedge e_J)$$

$= 0$ if $I \cap J \neq \emptyset$

$\pm e_K$ $\quad K = I \cup J$ increasing order

$\varepsilon(I, J)$

$$\varepsilon(I,J) = \begin{cases} 0 & \text{if } I \cap J \neq \emptyset \\ \text{sign of perm} & \end{cases}$$

taking $i_1, \dots, i_k, j_1, \dots, j_l$

but in increasing order

$\{1\ 3\}\ \{2\ 5\}$

$(e_1 \wedge e_3) \wedge (e_2 \wedge e_5)$

$= - e_1 \wedge e_2 \wedge e_3 \wedge e_5$

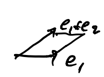
$\varepsilon(\{1,3\}, \{2,5\}) = -1$

Parallelepiped $P(v_1, \dots, v_k)$.

$v_1 \wedge \cdots \wedge v_k$ represents

in $\mathbb{R}^3$ $\quad e_1 \wedge e_2$

$e_1 \wedge (e_1 \wedge e_2)$

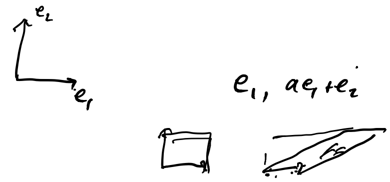


k vecs $v_1, \dots, v_k$ represent a $P(v_1, \dots, v_k)$

$v_1 \wedge \dots \wedge v_k = w_1 \wedge \dots \wedge w_k$ different vecs

lie in the same k-dim subspace of $\mathbb{R}^2$, have same volume.

$(e_{i_1} \wedge \dots \wedge e_{i_k})$



$e_1, \; a e_1 + e_2$

$$\left\| \sum a_j e_j \right\| = \sqrt{\sum a_j^2}$$

$$\| v_1 \wedge \dots \wedge v_k \| = \text{volume of } P(v_1, \dots, v_k)$$

$k=1$ $\quad \|v_1\|$ = length v_1

$k=2$ $\quad \|v_1 \wedge v_2\|$ $\quad \cancel{v_1 = a_{11} e_1 + a_{21} e_2}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \begin{aligned} v_1 &= a_{11} e_1 + a_{21} e_2 \\ v_2 &= a_{12} e_1 + a_{22} e_2 \end{aligned}$$

$$v_1 \wedge v_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} e_1 \wedge e_2$$

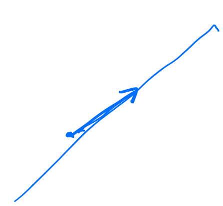
$$\begin{pmatrix} a_{11} & a_{12} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{pmatrix} \quad \begin{aligned} v_1 &= a_{11} e_1 + a_{21} e_2 + \dots + a_{n1} e_n \\ v_2 &= a_{12} e_1 + a_{22} e_2 - a_{n2} e_n \end{aligned}$$

$$v_1 \wedge v_2 = \sum_{i<j} (\;) e_i \wedge e_j$$

$e_1 \wedge e_2$

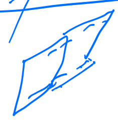
$$\sum_{i<j} \begin{vmatrix} a_{i1} & a_{i2} \\ a_{j1} & a_{j2} \end{vmatrix} e_i \wedge e_j$$

Λ^1 = vectors = $\mathbb{R}^2$



Λ^2 Similar picture

Λ^3

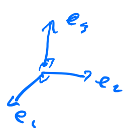


⋮

$e_1, \dots, e_n$ basis

↓ define

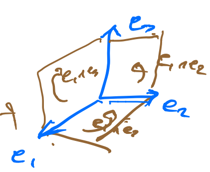
$\Lambda^0 \mid \Lambda^1, \Lambda^2, \Lambda^3, \dots \Lambda^n$



$\{e_i \wedge e_j : i<j\}$

"mental picture"

$e_i \wedge e_j$



k-vectors

$v_1, \dots, v_k$

$v_1 \wedge \dots \wedge v_k$

Imagine k-d $\in \Lambda^k$

Want $v_1 \wedge v_2 = -v_2 \wedge v_1$

Try Λ^2 $\{e_i \wedge e_j : (i,j),\ i<j\}$

$\Lambda^2 = \{\underset{\in\mathbb{R}}{a_{ij}}\, e_i \wedge e_j : i<j\}$

$\underset{\in \Lambda^k}{\alpha} \wedge \underset{\in \Lambda^\ell}{\beta}$

$\Lambda^k \quad \Lambda^\ell \to \Lambda^{k+\ell}$

Λ^1 = original $\mathbb{R}^n$

$\Lambda^2 = \sum a_{ij}\, e_i \wedge e_j$

$\Lambda^3 = \sum a_{ijk}\, e_i \wedge e_j \wedge e_k : i<j<k\}$

⋮

$e_1 \wedge e_2$

$e_2 \wedge e_1 = -e_1 \wedge e_2$

$e_i \wedge e_j \wedge e_k \quad i < j < k$ basis

$e_2 \wedge e_1 \wedge e_3 = -e_1 \wedge e_2 \wedge e_3$

~~$e_6 \wedge e_5 \wedge e_4 \wedge e_1$~~

$=0$

$e_6 \wedge e_5 \wedge e_2 \wedge e_1$

$e_2 \wedge e_6 \wedge e_5 \wedge e_1$

$- e_2 \wedge e_5 \wedge e_6 \wedge e_1$

$1 < 5 < 4 < 7$

$\{ e_i \wedge e_j \wedge e_k \,;\, i < j < k \}$ basis

$$\left(\sum_{i<j} a_{ij} e_i \wedge e_j \right) \wedge \left(\sum b_\ell e_\ell \right)$$

$$= \sum_{(i<j),\ell} a_{ij} b_\ell \, e_i \wedge e_j \wedge e_\ell \qquad e_i \wedge e_j = -e_j \wedge e_i \qquad e_i \wedge e_i = 0$$

$\mathbb{R}^4$

$$\big(a_{12} e_1 \wedge e_2 + a_{13} e_1 \wedge e_3 + a_{14} e_1 \wedge e_4 + a_{23} e_1 \wedge e_3 + a_{24} e_1 \wedge e + a_{34} e_1 \wedge e_3 \big) \wedge \big(b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 \big)$$

$$= \cancel{a_{12} b_1 e_1 \wedge e_2 \wedge e_1} + a_{12} \cancel{e_1 \wedge e_2 \wedge e_2} + \boxed{a_{12} e_1 \wedge e_2 \wedge e_3} + \boxed{a_{12} e_1 \wedge e_2 \wedge e_4}$$
$$+ a_{13} b_1 \cancel{e_1 \wedge e_3 \wedge e_1} + a_{13} b_2 e_1 \wedge e_3 \wedge e_2 + a_{13} b_3 \cancel{e_1 \wedge e_3 \wedge e_3} + \boxed{a_{13} b_4 e_1 \wedge e_3 \wedge e_4}$$

$- a_{13} b_2 e_1 \wedge e_2 \wedge e_3$

$$\sum_{i<j<k} c_{ijk} e_i \wedge e_j \wedge e_k$$

$I = i_1, \dots, i_k$

$i_1 < \dots < i_k$

$J = j_1 < \dots j_\ell$

$e_I = e_{i_1} \wedge \dots \wedge e_{i_k}$

e_J

$e_I \wedge e_J = 0$ if $I \cap J \neq \emptyset$

$\varepsilon(I,J) \quad I, J \to K$

concatenation of I, J

$k+\ell \qquad i_1, i_2, \dots i_k, j_1, j_2 \dots$

$\ell_1 \; \ell_2 \quad \ell_k \; \ell_{k+1} \; \ell_n$

$\boxed{e_I \wedge e_J}$ defined $\forall I, J$

extend by linearity $\left(\sum_{|I|=k} a_I e_I \right) \wedge \left(\sum_{|J|=\ell} b_J e_J \right)$

$$\sum_{I,J} a_I b_J \underbrace{e_I \wedge e_J}_{\varepsilon(I,J)\, e_K}$$

$K = K(I,J)$

$= 0 \quad I \cap J \neq \emptyset$

Conc (I,J)

[illegible]

$(e_1 \wedge e_3 \wedge e_7) \wedge (e_2 \wedge e_4)$

$= e_1 \wedge e_3 \wedge e_7 \wedge e_2 \wedge e_4$

$= e_1 \wedge e_2 \wedge e_3 \wedge e_4 \wedge e_7$

$= e_1 \wedge e_2 \wedge e_4 \wedge e_5 \wedge e_7$

$(e_1 \wedge e_3 \wedge e_7) \wedge (e_2 \wedge e_6)$

$= e_1 \wedge e_3 \wedge e_7 \wedge e_2 \wedge e_6$

$= e_1 \wedge e_2 \wedge e_3 \wedge e_7 \wedge e_6$

$= -e_1 \wedge e_2 \wedge e_3 \wedge e_6 \wedge e_7$

<u>Want</u> vector $v_1, \dots, v_k$ in $\mathbb{R}^n = \Lambda^1 \mathbb{R}^n$

$\underline{v_1 \wedge \dots \wedge v_k} \in \Lambda^k \mathbb{R}^n$ is now def

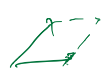
$\underline{v_1 \wedge \dots \wedge v_k} = 0 \iff \{v_1, \dots, v_k\}$ lin dep

for $v_1, \dots, v_k$ $\quad \neq 0 \iff \{ \quad \}$ indep

$t_1 v_1 + t_2 v_2 + \dots + t_k v_k \qquad 0 \le t_i \le 1$

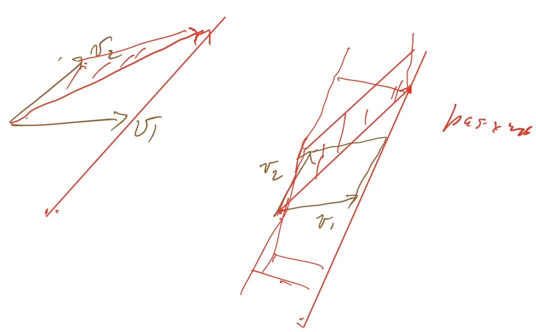
$v_1, v_2 \qquad t_1 v_1 + t_2 v_2$

$v_1 \wedge v_2$



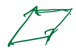
$(v_1 + a v_2) \wedge v_2 = v_1 \wedge v_2 + a \cancel{v_2 \wedge v_2}$

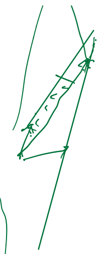
$= v_1 \wedge v_2$



Λ

$v_1 \wedge v_2$
$\shortparallel$
$w_1 \wedge w_2$





$v_1 \wedge v_2 = w_1 \wedge w_2$

$\Leftrightarrow$ $\boxed{\text{parallelogram in same plane, same area, same orient.}}$

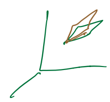
$e_I \wedge e_J$

$v_1 = a_1 e_1 + \dots + a_n e_n$
$v_2 = b_1 e_1 + \dots + b_n e_n$

$v_1 \wedge v_2$

$\mathbb{R}^n \to \Lambda^2 \mathbb{R}^n$
$\Lambda^3 \mathbb{R}^n$

$(v_1 \wedge \dots \wedge v_k) \wedge (v_1 \wedge \dots \wedge v_\ell) = 0$
$k \qquad \ell \qquad \neq 0$
$k+\ell$



$(v_1 \wedge v_2) \wedge (v_3)$

$(a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + \dots)$

$v_3 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} e_1 \wedge e_2 + \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} e_1 \wedge e_3 + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} e_2 \wedge e_3$

$(c_1 e_1 + c_2 e_2 + c_3 e_3)$

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} c_3 - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} c_1$

$= \pm \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

---

e_1, e_2, e_3, e_4

in $\mathbb{R}^3$ $\quad x \wedge e_1 \quad x \wedge e_2$

$v_1 \wedge v_2 + w_1 \wedge w_2$ $\quad x \wedge (e_1 + e_2)$

$\Lambda^2 \mathbb{R}^3 \quad = x \wedge y$

$\Lambda^2 \mathbb{R}^4 \ni \boxed{e_1 \wedge e_2 + e_3 \wedge e_4}$

$\neq x \wedge y$

for any x, y in $\mathbb{R}^4$

~~blades~~ k-dim slabs $\subset \Lambda^k \mathbb{R}^n$

$(x \wedge y) \wedge (x \wedge y) = x \wedge y \wedge x \wedge y = - x \wedge x \wedge y \wedge y = 0$

$(e_1 \wedge e_2 + e_3 \wedge e_4) \wedge (e_1 \wedge e_2 + e_3 \wedge e_4)$

$= e_1 \wedge e_2 \wedge e_3 \wedge e_4 + e_3 \wedge e_4 \wedge e_1 \wedge e_2$

$e_1 \wedge e_2 \wedge e_3 \wedge e_4 \quad e_3 \wedge e_4 \wedge e_1 \wedge e_2 = e_1 \wedge e_2 \wedge e_3 \wedge e_4$

Product

- If $I = \{i_1 < \cdots < i_k\}$ as above, write $|I| = k$
- If $|I| = k$ and $|J| = l$, define $e_I \wedge e_J$ by

$$e_I \wedge e_J = \begin{cases} 0 \text{ if } I \cap J \neq \emptyset, \\ \varepsilon(I, J) e_K \text{ if } I \cap J = \emptyset. \end{cases} \tag{5}$$

- K and $\varepsilon(I, J)$ defined as follows:
  - Let $I \cup J$ denote the sequence $\{i_1, \ldots, i_k, j_1, \ldots j_l\}$
  - K is the sequence $I \cup J$ arranged in increasing order.
  - $\varepsilon(I, J)$ is the sign of the permutation that takes $I \cup J$ to K.

- This determines a product

$$\Lambda^k \times \Lambda^l \to \Lambda^{k+l}$$

- If $a = \sum_I a_I e_I$ and $b = \sum_J b_J e_J$, then

$$a \wedge b = \sum_{I,J} a_I b_J e_I \wedge e_J$$

- This sum can be rewritten, using the definition of $e_I \wedge e_J$ above, as

$$\sum_K c_K e_K$$

  This is $a \wedge b$.

- Multiplication is associative

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

- Distributive law holds

$$(a + b) \wedge c = a \wedge c + b \wedge c$$

- If $a \in \Lambda^k$ and $b \in \Lambda^l$, then

$$b \wedge a = (-1)^{kl} a \wedge b$$

- $\Lambda^k(\mathbb{R}^n)$ has an inner product, with $\{e_I \ : \ |I| = k\}$ as ON basis.
- The corresponding *norm* is

$$|a| = |\sum_I a_i e_I| = \sqrt{\sum_I a_I^2}$$

- If $v_1, \ldots, v_k \in \mathbb{R}^n$ are linearly independent, then

$$v_1 \wedge \cdots \wedge v_k \in \Lambda^k(\mathbb{R}^n)$$

represents the oriented parallelipiped (4)

$$\{t_1 v_1 + \cdots + t_k v_k \ : \ 0 \leq t_i \leq 1\}$$

- The norm

$$|v_1 \wedge \cdots \wedge v_k|$$

is the *k-dimensional volume* of the parallelipiped.

Reality check

- $v_1, \dots, v_k$ are linearly independent and $w =$ linear combination of $v_2, \dots, v_k$, then

$$v_1 \wedge v_2 \wedge \cdots \wedge v_k = (v_1 + w) \wedge v_2 \wedge \cdots \wedge v_k$$

- Same is true for volume

$\| v_1 \wedge \cdots v_k \| =$ volume

- Example: for $k = 2$

$$v_1 \wedge v_2 = (v_1 + \alpha v_2) \wedge v_2 \text{ for all } \alpha \in \mathbb{R}$$

- Picture for area:

- For $k = n$, if $v_i = Ae_i$ for $i = 1, \ldots, n$ then

$$v_1 \wedge \cdots \wedge v_n = \det(A) e_1 \wedge \cdots \wedge e_n$$

- Known $|\det(A)|$ = volume of parallelipiped.

- If $k = 2$, let

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ \dots & \dots \\ a_{n,1} & a_{n,2} \end{pmatrix}$$

- Let $v_1 = \sum_i a_{i,1} e_i$ and $v_2 = \sum_i a_{i,2} e_i$
- Check

$$v_1 \wedge v_2 = \sum_{i<j} \begin{vmatrix} a_{i,1} & a_{i,2} \\ a_{j,1} & a_{j,2} \end{vmatrix} e_i \wedge e_j$$

- For $k = n = 2$ get

  $v_1 \wedge v_2 = \pm$ area of parallelogram $\{t_1 v_1 + t_2 v_2 : 0 \le t_i \le 1\}$

- equivalently

$$v_1 \wedge v_2 = \det(A) e_1 \wedge e_2$$

- equivalently

$$|\det(A)| = \text{ area of parallelogram}$$

- For $k = 2$ and $n = 3$ get $v_1 \wedge v_2$ is the sum

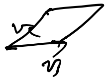
$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} e_1 \wedge e_2 + \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} e_1 \wedge e_3 + \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} e_2 \wedge e_3$$

- This looks like the cross product $v_1 \times v_2$

$$\left(\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}, - \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}, \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} \right)$$

- In any case, the two vectors have the same magnitude:

$$|v_1 \wedge v_2| = |v_1 \times v_2|$$



- So the new formula $|v_1 \wedge v_2|$ and the old formula $|v_1 \times v_2|$ for the area of the parallelogram agree.

- Similarly one can check the case $k = n = 3$

$$|v_1 \wedge v_2 \wedge v_3| = |\det(A)| = |(v_1 \times v_2) \cdot v_3| \quad \text{etc}$$

for the volume of the parallelipiped.

General Formula

- The cases already discussed:
  - $k = 1$, n arbitrary
  - $k = 2$, n arbitrary, particularly $n = 3$,
  - $k = n$, particularly both $= 3$.

  are the most common

- General formula:
  If for $j = 1, \dots, k$, $v_j = \sum_{i=1}^{n} a_{i,j} e_i \in \Lambda^1(\mathbb{R}^n)$,
  then $v_1 \wedge \cdots \wedge v_k$ is given by

$$\sum_{i_1 < \cdots < i_k} \begin{vmatrix} a_{i_1,1} & \dots & a_{i_1,k} \\ \dots & \dots & \dots \\ a_{i_k,1} & \dots & a_{i_k,k} \end{vmatrix} e_{i_1} \wedge \cdots \wedge e_{i_k} \tag{6}$$

Differential k-forms in $\mathbb{R}^n$ ($k \leq n$)

- k-dimensional integrands in $\mathbb{R}^n$ are the differential k-forms.
- $U \subset \mathbb{R}^n$ open.
- A (smooth) differential k- form on U is smooth function
$$\omega : U \times \Lambda^k(\mathbb{R}^n) \to \mathbb{R}$$
written $\omega_x(w)$ for $x \in U$ and $w \in \Lambda^k$, which is smooth in x and linear in w.
- Notation: $A^k(U) = \{\omega \ : \omega \text{ smooth } k - \text{form on } U\}$

- If $e_1, \ldots, e_n$ is an ON basis for $\mathbb{R}^n$, ω is determined by the $\left(\begin{array}{c} n \\ k \end{array} \right)$ functions

$$a_I(x) = \omega_x(e_I)$$

for all $I = \{i_1 < \cdots < i_k\}$.

- Write $e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$ for the basis elements of Λ^k
- Write $dx^I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for the dual basis of $L(\Lambda^k, \mathbb{R})$.

- Then
$$\omega = \sum_{I} a_I(x) dx^I$$
- Explicitly
$$\omega = \sum_{i_1 < \cdots < i_k} a_{i_1, \ldots, i_k}(x) dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{7}$$

- Let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation ($A \in L(\mathbb{R}^m, \mathbb{R}^n)$)
- Define the associated linear transformation

$$\Lambda^k A : \Lambda^k(\mathbb{R}^m) \to \Lambda^k(\mathbb{R}^n)$$

  by

$$\Lambda^k A(e_I) = Ae_{i_1} \wedge Ae_{i_2} \wedge \cdots \wedge Ae_{i_k}$$

- Also called the *induced* linear transformation.

- Often it's easier to say that $\Lambda^k A$ is defined by

$$\Lambda^k A(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k$$

for all $v_1, \ldots, v_k \in \mathbb{R}^m$.

- Since

$$\{v_1 \wedge \cdots \wedge v_k \ : \ v_1, \ldots, v_k \in \mathbb{R}^n\}$$

spans $\Lambda^k(\mathbb{R}^m)$, $\Lambda^k A$ is determined by these values.

- To know that the definition makes sense, that is, $Av_1 \wedge \cdots \wedge Av_k$ depends just on $v_1 \wedge \cdots \wedge v_k$, need

$$v_1 \wedge \cdots \wedge v_k = 0 \Rightarrow Av_1 \wedge \cdots \wedge Av_k = 0$$

- This is equivalent to

  $v_1, \ldots, v_k$ linearly dependent
  $\Rightarrow$
  $Av_1, \ldots, Av_k$ linearly dependent

- Clear

Pull-back

- $V \subset \mathbb{R}^m$, $U \subset \mathbb{R}^n$ open sets
- $f : V \to U$ smooth map
- Pull-back $A^k(U) \to A^k(V)$ is defined by

  $$(f^*\omega)_t(v_1 \wedge \cdots \wedge v_k) = \omega_{f(t)}(d_t f(v_1) \wedge \cdots \wedge d_t f(v_k))$$

  for all $t \in V$ and for all $v_1, \ldots, v_k \in \mathbb{R}^m$

- More concisely

  $$(f^*\omega)_t = \omega_{f(t)} \circ \Lambda^k d_t f$$

  for all $t \in V$.

- In terms of coordinates $t = (t_1, \ldots, t_m)$ and $x = (x_1, \ldots, x_n)$
- $x = f(t) = (f_1(t_1, \ldots, t_m), \ldots, f_n(t_1, \ldots, t_m))$
- $\omega = \sum_I a_I(x) dx^I = \sum_{i_1 < \cdots < i_k} a_{i_1,\ldots,i_k}(x) dx_{i_1} \wedge \ldots dx_{i_k}$
- Then

$$f^*\omega = \sum_{i_1 < \cdots < i_k} a_{i_1,\ldots,i_k}(f(t)) f^*(dx_{i_1}) \wedge \cdots \wedge f^*(dx_{i_k}) \quad (8)$$

- Using (1), this can be rewritten as

$$f^*\omega = \sum_{i_1<\cdots<i_k} a_{i_1,\ldots,i_k}(f(t))(d_t f_{i_1}) \wedge \cdots \wedge (d_t f_{i_k}) \qquad (9)$$

- Writing $df_i = \sum_{j=1}^{m} \frac{\partial f_i}{\partial t_j} dt_j$ and expanding df^I in the same manner as (6) we get an explicit expression for $f^*\omega$ as a sum
$$\sum_J c_J(t) dt^J$$
- Perhaps more useful than an explicit but complicated formula is to observe the multiplicative properties of f^*.

- If $a : U \to \mathbb{R}$ is a smooth function, that is, $a \in A^0(U)$, let
$$f^* : A^0(U) \to A^0(V)$$
be defined by
$$(f^* a)(t) = a(f(t))$$
- Then (8) says

$$f^*\Big(\sum_{I=i_1<\cdots<i_k} a_I \wedge \cdots \wedge dx_{i_k}\Big) = \sum (f^* a_I)(f^* dx_{i_1}) \wedge \cdots \wedge (f^* dx_{i_k})$$

- Suggests the following
- There is a product

$$L(\Lambda^k, \mathbb{R}) \times L(\Lambda^l, \mathbb{R}) \to L(\Lambda^{k+l}, \mathbb{R})$$

  defined just as in (5) using the dual basis dx^I rather than e_I

- Induces a product $A^k(U) \times A^l(U) \to A^{k+l}(U)$.
- If $\omega \in A^k(U)$ and $\eta \in A^l(U)$, then $\omega \wedge \eta \in A^{k+l}(U)$.
- If $f : V \to U$ is smooth, then

$$f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta) \tag{10}$$

Some properties of pull-back

- $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$ as above
- $f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$
- $f : V \to U$ and $g : W \to V$ smoooth maps of open sets. then

$$(f \circ g)^* = g^* \circ f^* : A^k(U) \to A^k(W)$$

Integration over k-cells

- Let $D = \mathbf{I}^k$ be a k-*cell* as in (3)
- Let $\alpha \in A^k(D)$ be a smooth k-form.
- Then
$$\alpha = \phi(t)dt_1 \wedge \cdots \wedge dt_k$$
for some smooth $\phi : D \to \mathbb{R}$, $t = (t_1, \ldots, t_k)$
- Define
$$\int_D \alpha = \int_D \phi(t)dt_1 \ldots dt_k$$
the Riemann integral of ϕ over $D = \mathbf{I}^k$.

- If $\sigma : D \to U$ is smooth and $\omega \in A^k(U)$, define

$$\int_\sigma \omega = \int_D \sigma^*(\omega)$$

- Would like $\int_\sigma \omega$ to be *independent of parametrization*.

- This means that if E is another k-cell and

$$\Phi : E \to D$$

is smooth, bijective, $\det(d\Phi) > 0$ everywhere on E, then

$$\int_{\sigma\circ\Phi} \omega = \int_\sigma \omega$$

- This follows from the *change of variables formula*
- If $\Phi : E \to D$ and $\alpha \in A^k(D)$ as before, then

$$\int_E \Phi^* \alpha = \int_D \alpha$$

- More usual formulation:
- If $\alpha = a(t)dt_1 \wedge, \cdots \wedge dt_k$ then

$$\int_E a(\Phi(t)) \left|\det(d_t\Phi)\right| dt_1 \ldots, dt_k = \int_D a(t)dt_1 \ldots dt_n$$

- Note how the absolute vaue $|\det(d\Phi)|$ appears, rathen than $\det(d\Phi)$. Results from orientation.