F–regularity, F–rationality and F–purity

by

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Dedicated to the memory of my father

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CHAPTER I

INTRODUCTION

Characteristic p methods provide some extremely useful tools for the study of commutative Noetherian rings. The power of the Frobenius morphism for a ring of characteristic $p > 0$ was illustrated by C. Peskine and L. Szpiro in their proof of the local homological conjectures, [PS1, PS2]. They were also able to develop methods of reduction to characteristic p to prove these conjectures for certain rings containing the rational numbers. This was followed by M. Hochster's proof of the existence of big Cohen–Macaulay modules, which used the method of reduction to characteristic p. More recently, M. Hochster and C. Huneke developed the theory of *tight closure* which beautifully brings together and simplifies many characteristic p arguments. To quote from a recent article by W. Bruns, [Br], "'tight closure' can now be regarded as a synonym for 'characteristic p methods in commutative algebra.'"

The theory of tight closure provides stronger formulations of several existing results, and brings together many seemingly unrelated issues. The theory furnishes, for instance, a proof of the Briançon–Skoda theorem on the integral closure of ideals in regular rings. The original work of Briançon and Skoda was motivated by a question of J. Mather about the ring of convergent complex power series in several variables, and made use of deep analytic results. Tight closure theory gives a simpler proof of a much stronger statement. It also yields a theorem generalizing the result [HR1] that the ring of invariants of a linearly reductive group acting on a regular ring is Cohen–Macaulay, and offers improvements on the various local homological theorems. Techniques developed during the study of tight closure have lead to certain strong uniform Artin–Rees theorems, [Hu1], and a proof that the absolute integral closure of a characteristic p domain is a big Cohen–Macaulay algebra, [HH3].

The tight closure of an ideal I is a possibly larger ideal, denoted by I^*, which is always contained in the integral closure of I and is frequently much smaller. Hochster and Huneke observed that for large classes of rings, the geometric notion of rational singularities is analogous to a certain property which could be formulated in terms of tight closure, namely the property that parameter ideals are tightly closed. These rings were eventually named *F–rational* rings by R. Fedder and K.-i. Watanabe. K. E. Smith proved that F–rational rings have rational singularities and the converse, in characteristic zero, is a theorem of N. Hara, see [Sm4, Ha]. Making use of Smith's result, A. Conca and J. Herzog showed that ladder determinantal varieties have rational singularities, [CH].

The theory of tight closure also draws attention to rings in which all ideals are tightly closed, called *weakly F–regular* rings. (The term *F–regular* is reserved for rings all of whose localizations are weakly F–regular.) These properties turn out to be of significant importance, for instance the Hochster–Roberts theorem that direct summands of polynomial rings are Cohen–Macaulay can actually be proved for the much larger class of weakly F–regular rings. A related notion is that of *F–pure* rings, i.e., rings R for which the Frobenius morphism remains injective on tensoring with any R–module. A study of the these properties, namely F–regularity, F–rationality and F–purity constitute the major part of this thesis. Chapter II provides a review of tight closure and related notions and includes some results that we will use later in our study.

Although tight closure is primarily a notion for rings of characteristic p, it has strong connections with the study of the singularities of algebraic varieties over fields of characteristic zero. We have already mentioned the connection between F–rational rings and rational singularities. For $\mathbb{Q}$–Gorenstein rings (i.e., rings for which the canonical module is a torsion element of the divisor class group) essentially of finite type over a field of characteristic zero, we have some more remarkable connections: F–regular type is equivalent to log–terminal singularities and F–pure type implies (and is conjectured to be equivalent to) log–canonical singularities, see [Sm6, Wa5].

Deformation of F–regularity

A natural question that arose with the development of the theory of tight closure was whether the property of F–regularity deforms, i.e., if (R, m, K) is a local ring such that R/tR is F–regular for some nonzerodivisor $t \in m$, must R be F–regular? Hochster and Huneke showed that this is indeed true if the ring R is Gorenstein, and there have been several attempts at extending this result. In Chapter III we show that the property of F–regularity does not deform, and thereby settle this longstanding open question. Specifically, we construct a three dimensional domain R which is not F–regular (or even F–pure), but has a quotient R/tR which is F–regular. Similar examples are also constructed over fields of characteristic zero.

In their proof that F–regularity deforms for Gorenstein rings, Hochster and Huneke show that the properties of F–regularity and F–rationality coincide for Gorenstein rings, and that F–rationality deforms. For $\mathbb{Q}$–Gorenstein rings essentially of finite type over a field of characteristic zero, Smith showed that the property of F–regular type does deform, see [Sm7]. The crucial point is that in this setting F–regular type is equivalent to log–terminal singularities, and log–terminal singularities deform by Kollár's result on "inversion of adjunction", see [Ko]. For $\mathbb{Q}$–Gorenstein rings of characteristic p, a purely algebraic proof that F–regularity deforms was provided by I. Aberbach, M. Katzman, and B. MacCrimmon in [AKM].

A notion closely related to (and frequently the same as) F–regularity is that of *strong F–regularity.* In Chapter III we shall also investigate the deformation of strong F–regularity using the idea of passing to an anti–canonical cover $S = \oplus_{i \geq 0} I^{(i)} X^i$, where I represents the inverse of the canonical module in the divisor class group, $\mathrm{Cl}(R)$. Strong F–regularity is shown to deform in the case that the symbolic powers $I^{(i)}$ satisfy the Serre condition S_3 for all $i \geq 0$, and the ring S is Noetherian.

Failure of F–purity and F–regularity in certain rings of invariants

A subgroup G of the general linear group $GL_n(\mathbb{F}_q)$ has a natural action on the polynomial ring $R = K[X_1, \ldots, X_n]$ (where K is a field containing $\mathbb{F}_q$) by degree preserving ring automorphisms. D. Glassbrenner has shown that when G acts by permuting the variables X_i, then the ring of invariants is F–pure. In Chapter IV we construct examples to demonstrate that the ring of invariants in general need not be F–pure. In these examples G is the symplectic group over a finite field, and the invariant subrings are always complete intersections by the work of D. Carlisle and P. Kropholler [CK]. These examples are of special interest from the point of view of studying the Frobenius closures and tight closures of ideals as contractions from certain extension rings: they provide instances when the socle element modulo an ideal generated by a system of parameters is forced into the expansion of the ideal to a module–finite extension ring which is a *separable* (in fact, Galois) extension. This element is also forced into the expanded ideal in a linearly disjoint *purely inseparable* extension since it is in the Frobenius closure of the ideal. It is noteworthy that the element can be forced into expanded ideals in two such different ways.

For the natural action of the alternating group A_n on the polynomial ring $R = K[X_1, \ldots, X_n]$, where the characteristic p of K is an odd prime, Glassbrenner shows that the invariant subring R^{A_n} is not F–regular when p divides n or $n-1$, [Gl1]. We extend this result by showing that R^{A_n} is F–regular if and only if p does not divide the order of the group A_n.

F–rationality and F–regularity of Veronese subrings

While the property of F–rationality provides an algebraic analogue of the notion of rational singularities, F–regularity is not so well understood geometrically. One approach is to study the variety $X = \operatorname{Proj} R$ for a graded F–regular ring R. The Veronese subrings of R are also homogeneous coordinate rings for X, and so it is interesting to determine when graded rings have F–rational or F–regular Veronese subrings. We investigate this question in Chapter V. (By a graded ring, we mean here a ring $R = \oplus_{n \geq 0} R_n$, which is finitely generated over a field $R_0 = K$. For simplicity, assume that K is algebraically closed.)

The question regarding F–rational Veronese subrings is easily answered: let (R, m, K) be a Cohen–Macaulay graded domain of dimension d, with an isolated singularity at m. Then there exists a positive integer n such that the Veronese subring $R^{(n)}$ is F–rational if and only if $[H^d_m(R)]_0 = 0$. With regard to F–regular Veronese subrings, if R is a normal ring generated by degree one elements over a field, then either R is F–regular, or else no Veronese subring of R is F–regular. This leads to the question: if (R, m, K) is a normal graded ring, generated by degree one elements, with an isolated singularity at m, then under what conditions is R an F–regular ring? It is easily seen that F–regularity forces the *a–invariant*, $a(R)$, to be negative. For rings of dimension two (although not in higher dimensions) this is also a sufficient condition for F–regularity. We construct rings R of dimension $d \geq 3$ with $a(R) = 2 - d$ which are not F–regular, while if $a(R) < 2 - d$, Smith has pointed out that $\operatorname{Proj} R$ is a variety of minimal degree, and the ring R is indeed F–regular, [Sm5, Remark 4.3.1].

The results obtained during the course of this work provide various examples of F–rational rings which are not F–regular, and give a better understanding of F–regularity. Our techniques include Demazure's representation of normal graded rings in terms of Weil divisors with rational coefficients, [De], and related results of Watanabe, [Wa1].

In the last section of Chapter V, we construct a rich family of F–rational rings of characteristic zero, with isolated singularities, which have no F–regular Veronese subrings. We believe these examples will also be of independent interest.

Tight closure in non–equidimensional rings

In Chapter VI we examine issues relating to the tight closure of parameter ideals in non–equidimensional rings. An equidimensional local ring is F–rational if and only if one ideal generated by a full system of parameters is tightly closed. (It then follows that every ideal generated by part of a system of parameters is tightly closed.) The question of whether a non–equidimensional local ring can have a tightly closed ideal generated by a system of parameters was a longstanding open problem and for certain classes of non–equidimensional rings, we can prove that this is not possible. A closely related issue is that tight closure has a "colon capturing" property in equidimensional rings that it does not have in non–equidimensional rings. A study of these issues leads us to define a new closure operation, one that rectifies the absence of the colon capturing property of tight closure in non–equidimensional rings and agrees with tight closure when the ring is equidimensional. We prove that the F–rationality of a local ring is equivalent to a single system of parameters being closed with respect to this new closure operation.

Computations in diagonal hypersurfaces

Consider the ring $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$ where K is a field of prime characteristic $p \neq 3$. M. McDermott computed the tight closure of various irreducible ideals in R, and showed that $xyz \in (x^2, y^2, z^2)^*$ when $p < 200$, [Mc]. (Lower case letters denote the images of the corresponding variables.) The general case however existed as a classic example of the difficulties involved in tight closure computations, see also [Hu2, Example 1.2]. In Chapter VII we show that $xyz \in (x^2, y^2, z^2)^*$ in arbitrary prime characteristic p, a computation which was largely inspired by [Ro]. We furthermore show that $xyz \in (x^2, y^2, z^2)^F$ whenever R is not F–pure, i.e., when $p \equiv 2 \pmod 3$. These results are then generalized to the diagonal hypersurfaces $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$.

These issues relate to the question whether the tight closure I^* of an ideal I agrees with its plus closure, $I^+ = IR^+ \cap R$, where R is a domain over a field of characteristic p and R^+ is the integral closure of R in an algebraic closure of its fraction field. In this setting, we may think of the Frobenius closure of I as $I^F = IR^\infty \cap R$ where R^∞ is the extension of R obtained by adjoining p^e th roots of all nonzero elements of R for $e \in \mathbb{N}$. It is not difficult to see that $I^+ \subseteq I^*$, and equality in general is a formidable open question. It should be mentioned that in the case when I is an ideal generated by part of a system of parameters, the equality is a result of Smith, see [Sm2]. In the above ring $R = K[X,Y,Z]/(X^3+Y^3+Z^3)$ where K is a field of prime characteristic $p \equiv 2 \pmod 3$, if one could show that $I^* = I^F$ for an ideal I, a consequence of this would be $I^F \subseteq I^+ \subseteq I^* = I^F$, by which $I^+ = I^*$. McDermott does show that $I^* = I^F$ for large families of irreducible ideals and our result $xyz \in (x^2, y^2, z^2)^F$, we believe, fills in an interesting remaining case.

CHAPTER II

TIGHT CLOSURE

We provide a brief review of tight closure and related notions, as well as summarize some results we shall find useful in our study. Since one of our main interests is the case of graded rings, we discuss these and, in particular, normal graded rings. Using a result of Demazure, normal graded rings can be interpreted as arising from rational coefficient Weil divisors on projective varieties, and such techniques are turning out to be extremely useful in providing a better understanding of tight closure.

2.1 Notation and conventions

Let R be a Noetherian ring of characteristic $p > 0$. We shall always use the letter e to denote a variable nonnegative integer, and q to denote the eth power of p, i.e., $q = p^e$. We shall denote by F, the Frobenius endomorphism of R, and by F^e, its eth iteration, i.e., $F^e(r) = r^q$. For an ideal $I = (x_1, \ldots, x_n) \subseteq R$, we let $I^{[q]} = (x_1^q, \ldots, x_n^q)$. Note that $F^e(I)R = I^{[q]}$, where $q = p^e$, as always. The *Peskine–Szpiro functor* F^e is crucial to characteristic p methods. Let S denote the ring R viewed as an R–algebra via F^e. Then $S \otimes_R \_$ is a covariant functor from R–modules to S–modules, and so is a covariant functor from R–modules to R–modules! If we consider a map of free modules $R^n \to R^m$ given by the matrix (r_{ij}), applying F^e we get a map $R^n \to R^m$ given by the matrix (r_{ij}^q). For an R–module M, note that the

R–module structure on $F^e(M)$ is $r'(r \otimes m) = r'r \otimes m$, and $r' \otimes rm = r'r^q \otimes m$. For R–modules $N \subseteq M$, we use $N_M^{[q]}$ to denote $\mathrm{Im}(F^e(N) \to F^e(M))$.

For a reduced ring R of characteristic $p > 0$, $R^{1/q}$ shall denote the ring obtained by adjoining all q th roots of elements of R. The ring R is said to be *F–finite* if $R^{1/p}$ is module–finite over R. Note that a finitely generated algebra R over a field K is F–finite if and only if $K^{1/p}$ is a finite field extension of K.

We shall denote by R° the complement of the union of the minimal primes of R. We say $I = (x_1, \ldots, x_n) \subseteq R$ is a *parameter ideal* if the images of $x_1, \ldots, x_n$ form part of a system of parameters in R_P, for every prime ideal P containing I.

2.2 Frobenius closure and tight closure

Definition 2.2.1. Let R be a ring of characteristic p, and I an ideal of R. An element x of R, is said to be in I^F, the *Frobenius closure* of I, if there exists some $q = p^e$ such that $x^q \in I^{[q]}$.

For R–modules $N \subseteq M$ and $u \in M$, we say that $u \in N_M^*$, the *tight closure* of N in M, if there exists $c \in R^\circ$ such that $cu^q \in N_M^{[q]}$ for all $q = p^e \gg 0$.

It is worth recording this when $M = R$, and $N = I$ is an ideal of R. An element x of R is said to be in I^*, the tight closure of I, if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$. If $I = I^*$ we say that the ideal I is *tightly closed.*

It is easily verified that $I \subseteq I^F \subseteq I^*$. Furthermore I^* is always contained in the integral closure of I, [HH2, Theorem 5.2], and is frequently much smaller.

Definition 2.2.2. A ring R is said to be *F–pure* if for all R–modules M, the homomorphism $F : M \to F(M)$ is injective.

A ring R is *weakly F–regular* if every ideal of R is tightly closed, and is *F–regular* if every localization is weakly F–regular. An F–finite ring R is *strongly F–regular* if for every $c \in R^{\circ}$, there exists $q = p^e$ such that the R–linear inclusion $R \to R^{1/q}$ sending 1 to $c^{1/q}$ splits as a map of R–modules. Lastly, R is said to be *F–rational* if every parameter ideal of R is tightly closed.

It follows easily from the definitions that a weakly F–regular ring is F–rational as well as F–pure. We next record some useful results.

Theorem 2.2.3.

(1) *Regular rings are F–regular and F–finite regular rings are strongly F–regular. Strongly F–regular rings are F–regular.*

(2) *Let S be a strongly (weakly) F–regular ring. If R is a subring of S which is a direct summand of S as an R–module, then R is strongly (weakly) F–regular.*

(3) *An F–rational ring R is normal. If, in addition, R is the homomorphic image of a Cohen–Macaulay ring, then it is Cohen–Macaulay.*

(4) *An F–rational Gorenstein ring is F–regular. If it is F–finite, then it is also strongly F–regular.*

(5) *A local ring (R, m) which is the homomorphic image of a Cohen–Macaulay ring is F–rational if and only if it is equidimensional and the ideal generated by one system of parameters is tightly closed.*

(6) *Let R be a reduced excellent local ring of dimension d and characteristic $p > 0$. If $c \in R^{\circ}$ is an element such that R_c is F–rational, then there exists a positive integer N such that $c^N(0^*_{H^d_m(R)}) = 0$.*

(7) *Let $P_1, \ldots, P_n$ be the minimal primes of a ring R. For an ideal $I \subseteq R$ and an element $x \in R$, x is in the tight closure of I if and only if for $1 \leq i \leq n$, its image $\overline{x}$ is in $(IR/P_i)^*$, the tight closure here being computed in the domain R/P_i.*

(8) *The notions of weak F–regularity and F–regularity agree for* $\mathbb{N}$*–graded rings. For F–finite* $\mathbb{N}$*–graded rings, these are also equivalent to strong F–regularity.*

(9) *Let* (R, m) *be an F–finite local ring with a canonical module which, as an element of the divisor class group, is locally of finite order on the punctured spectrum of* R*. Then* R *is weakly F–regular if and only if it is strongly F–regular.*

Proof. For assertions (1)—(5), see [HH1, Theorem 3.1] and [HH4, Theorem 4.2]. Part (6) is a result of Velez [Ve], and (7) is observed in [HH2, Proposition 6.25 (a)]. For (8), see the recent work of Lyubeznik and Smith, [LS, Corollaries 4.3 and 4.4]. Part (9) is the main result of [Ma], see also [Wi]. □

Remark 2.2.4. The equivalence in general of weak F–regularity, F–regularity, and strong F–regularity is a formidable open question. However in the light of results (8) and (9) above, we frequently have no reason to distinguish amongst these notions.

2.3 Graded rings

By an $\mathbb{N}$–*graded* ring (R, m, K), we shall always mean a ring $R = \oplus_{n \geq 0} R_n$ finitely generated over a field $R_0 = K$. We shall denote by $m = R_+$, the homogeneous maximal ideal of R. The *punctured spectrum* of R refers to the set $\operatorname{Spec} R - \{m\}$. By a system of parameters for R, we shall mean a sequence of homogeneous elements of R whose images form a system of parameters for R_m. In specific examples involving homomorphic images of polynomial rings, lower case letters shall denote the images of the corresponding variables, the variables being denoted by upper case letters.

For conventions regarding graded modules and homomorphisms, we follow [GW]. For a graded R–module M, we shall denote by $[M]_i$, the i-th graded piece of M. For graded R–modules M and N, we may define the graded R–module $\underline{\operatorname{Hom}}_R(M, N)$, where $[\underline{\mathrm{Hom}}_R(M,N)]_i$ is the abelian group consisting of all *graded* R–linear homomorphisms from M to $N(i)$ where the convention for the grading shift is $[N(i)]_j = [N]_{i+j}$ for $j \in \mathbb{Z}$. This gives $\underline{\mathrm{Hom}}_R(M,N)$ a natural structure as a graded R–module. The injective hull of K in the category of graded R–modules is $\underline{E}_R(K) = \underline{\mathrm{Hom}}_K(R,K)$. Consequently for graded R–modules M, we have $\underline{\mathrm{Hom}}_R(M, \underline{E}_R(K)) = \underline{\mathrm{Hom}}_K(M,K)$. We shall denote by $\underline{\otimes}$, the graded tensor product.

Definition 2.3.1. Let $R = \oplus_{i \geq 0} R_i$ be an $\mathbb{N}$–graded ring, and n be a positive integer. We shall denote by $R^{(n)}$, the *Veronese subring* of R spanned by all elements of R which have degree a multiple of n, i.e., $R^{(n)} = \oplus_{i \geq 0} R_{in}$.

Note that the ring $R^{(n)}$ is a direct summand of R as an $R^{(n)}$–module and that R is integral over $R^{(n)}$. Hence whenever R is Cohen–Macaulay or normal, so is $R^{(n)}$. We record the following result, see [EGA, Lemme 2.1.6] or [Mum, page 282].

Lemma 2.3.2. *Let R be an $\mathbb{N}$–graded ring. Then there exists a positive integer n such that the Veronese subring $R^{(n)}$ is generated over K by forms of equal degree.*

Recall that the highest local cohomology module $H^d_m(R)$ of R, where $\dim R = d$, may be identified with $\varinjlim R/(x_1^t, \ldots, x_d^t)$ where $x_1, \ldots, x_d$ is a system of parameters for R and the maps are induced by multiplication by $x_1 \cdots x_d$. If R is Cohen–Macaulay, these maps are injective. The R–module $H^d_m(R)$ carries a natural graded structure, namely $\deg[r + (x_1^t, \ldots, x_d^t)] = \deg r - t \sum_{i=1}^d x_i$, where r and x_i are homogeneous elements of R.

Definition 2.3.3. In the above setting, Goto and Watanabe define the *a–invariant* of R as the highest integer $a(R) = a$ such that $[H^d_m(R)]_a$ is nonzero.

When R is a ring of characteristic p, the Frobenius homomorphism of R gives a natural Frobenius action on $H^d_m(R)$ where

$$F : [r + (x_1^t, \ldots, x_d^t)] \mapsto [r^p + (x_1^{pt}, \ldots, x_d^{pt})], \quad \text{see [FW] or [Sm2].}$$

For a graded R–module M, define $M^{(n)} = \oplus_{i\in\mathbb{Z}}[M]_{in}$. With this notation, it follows from [GW, Theorem 3.1.1] that

$$H^d_{m_{R^{(n)}}}(R^{(n)}) \cong (H^d_m(R))^{(n)}.$$

The following theorem, [HH5, Theorem 7.12], indicates the importance of the a–invariant in the study of graded F–rational rings.

Theorem 2.3.4. *An $\mathbb{N}$–graded Cohen–Macaulay normal ring R over a field of prime characteristic p is F–rational if and only if $a(R) < 0$ and the ideal generated by some homogeneous system of parameters for R is Frobenius closed.*

2.4 Rational coefficient Weil divisors

In this section, we review some notation and results from [De], [Wa1] and [Wa3] as well as make a few observations which we shall find useful later in our study.

Definition 2.4.1. By a *rational coefficient Weil divisor* (or a *$\mathbb{Q}$–divisor*) on a normal projective variety X, we mean a $\mathbb{Q}$–linear combination of codimension one irreducible subvarieties of X. For $D = \sum n_i V_i$ with $n_i \in \mathbb{Q}$, we set $[D] = \sum [n_i] V_i$, where $[n]$ denotes the greatest integer less than or equal to n, and define $\mathcal{O}_X(D) = \mathcal{O}_X([D])$.

Let $D = \sum (p_i/q_i) V_i$ where the integers p_i and q_i are relatively prime and $q_i > 0$. We define $D' = \sum ((q_i - 1)/q_i) V_i$ to be the *fractional part* of D. Note that with this definition of D' we have $-[-nD] = [nD + D']$ for any integer n.

Given an ample $\mathbb{Q}$–divisor D (i.e., such that ND is an ample Cartier divisor for some $N \in \mathbb{N}$), we construct the *generalized section ring*:

$$R = R(X, D) = \oplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD)) T^n \subseteq K(X)[T].$$

With this notation, Demazure's result ([De, 3.5]) is:

Theorem 2.4.2. *Let $R = \oplus_{n\geq 0} R_n$ be an $\mathbb{N}$–graded normal ring. Then there exists an ample $\mathbb{Q}$–divisor D on $X = \operatorname{Proj} R$ such that*

$$R \cong \oplus_{n\geq 0} H^0(X, \mathcal{O}_X(nD))T^n \subseteq K(X)[T],$$

where T is a homogeneous element of degree one in the quotient field of R.

Example 2.4.3. Take the $\mathbb{Q}$–divisor $D = (-1/2)V(S) + (1/3)V(T) + (1/5)V(S+T)$ on $\mathbb{P}^1 = \operatorname{Proj} K[S,T]$ where $V(S)$, e.g., denotes the irreducible subvariety defined by the vanishing of S. Fix T as the degree one element. Then

$$R = \oplus_{n\geq 0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(nD))T^n = K[X,Y,Z]/(X^2 + Y^3 + Z^5)$$

where $X = (S^8T^{10})/(S+T)^3, Y = (S^5T^7)/(S+T)^2$, and $Z = (-S^3T^4)/(S+T)$.

Remark 2.4.4. We record a few simple observations. Let $R = R(X,D)$ be as above. Then the Veronese subring $R^{(n)}$ is given by $R^{(n)} \cong R(X, nD)$. For a rational function $f \in K(X)$ we have an isomorphism $R(X,D) \cong R(X, \operatorname{div}(f) + D)$. If R is generated over K by its elements of degree one, we have $R \cong R(X, [D])$. Note that $[D]$ is a Weil divisor, i.e., has integer coefficients.

Let X be a smooth projective variety of dimension d with canonical divisor K_X, and let D be an ample $\mathbb{Q}$–divisor on X. If ω denotes the canonical module of $R = R(X,D)$, we have the following identifications (see [Wa1, Wa3]):

$$[\omega^{(i)}]_n = H^0(X, \mathcal{O}_X(i(K_X + D') + nD)),$$
$$[H^{d+1}_m(\omega^{(i)})]_n = H^d(X, \mathcal{O}_X(i(K_X + D') + nD)).$$

The action of the Frobenius on the nth graded piece of $E_R(K)$, the injective hull of K, can be identified with

$$H^d(X, \mathcal{O}_X(K_X + D' + nD)) \xrightarrow{F} H^d(X, \mathcal{O}_X(p(K_X + D' + nD))).$$

If the ring R is F–pure, this Frobnius action must be injective and so, in particular, $H^d(X, \mathcal{O}_X(p(K_X + D')))$ is nonzero.

2.5 The case of characteristic zero

Hochster and Huneke have also defined notions of tight closure for rings essentially of finite type over a field of characteristic zero, see [HH2, HH6]. However we can also define notions corresponding to F–regularity, F–purity, and F–rationality in characteristic zero, without using a closure operation.

Consider the ring $R = K[X_1, \ldots, X_n]/I$ where K is a field of characteristic zero. Choose a finitely generated $\mathbb{Z}$–algebra A such that $R_A = A[X_1, \ldots, X_n]/I_A$ is a free A–algebra, with $R \cong R_A \otimes_A K$. Note that the fibers of the homomorphism $A \to R_A$ over maximal ideals of A are finitely generated algebras over fields of prime characteristic.

Definition 2.5.1. Let R be a ring finitely generated over a field of characteristic zero. Then R is said to be of *F–regular type* if there exists a finitely generated $\mathbb{Z}$–algebra $A \subseteq K$ and a finitely generated A–algebra R_A such that $R \cong R_A \otimes_A K$, and for all maximal ideals μ in a Zariski dense subset of $\operatorname{Spec} A$, the fiber rings $R_A \otimes_A A/\mu$ are F–regular.

Similarly, R is said to be of *F–pure type* if for all maximal ideals μ in a Zariski dense subset of $\operatorname{Spec} A$, the fiber rings $R_A \otimes_A A/\mu$ are F–pure.

Remark 2.5.2. Some authors use the term F–pure type (F–regular type) to mean that $R_A \otimes_A A/\mu$ is F–pure (F–regular) for all maximal ideals μ in a Zariski dense *open* subset of $\operatorname{Spec} A$.

CHAPTER III

DEFORMATION OF F–REGULARITY

A natural question that arose with the development of the theory of tight closure was whether the property of F–regularity deforms, i.e., if (R, m, K) is a local or $\mathbb{N}$–graded domain such that R/tR is F–regular for some (homogeneous) element $t \in m$, must R be F–regular? (See the Epilogue of [Ho].) Hochster and Huneke showed this to be true if the ring R is Gorenstein, [HH4, Theorem 4.2], and their work has been followed by various attempts at extending this result, see [AKM, Si, Sm7]. We show that F–regularity does not deform by constructing a three dimensional domain R which is not F–regular (or even F–pure), but has a quotient R/tR which is F–regular. We then construct similar examples over fields of characteristic zero.

We also study the deformation of strong F–regularity using the idea of passing to an anti–canonical cover and show that under some rather restrictive assumptions, the property of strong F–regularity does deform.

3.1 F–regularity does not deform

We shall throughout be considering $\mathbb{N}$–graded rings, but local examples can be obtained in all cases by localizing at the homogeneous maximal ideals. The main result of this section is:

Theorem 3.1.1. *There exists an $\mathbb{N}$–graded ring R of dimension three (over a field $R_0 = K$ of characteristic $p > 2$) which is not F–pure, but has an F–regular quotient R/tR where $t \in m$ is a homogeneous nonzerodivisor.*

Specifically, for positive integers m and n with $m - m/n > 2$, consider the ring $R = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix

$$\mathfrak{M}_{m,n} = \begin{pmatrix} A^2 + T^m & B & D \\ C & A^2 & B^n - D \end{pmatrix}.$$

Then the ring R/tR is F–regular, while R fails to be F–regular, and is not even F–pure if p and m are relatively prime.

The ring R is graded by setting the weights of a, b, c, d, and t to be m, $2m$, $2m$, $2mn$, and 2 respectively. This ring is the specialization of a Cohen–Macaulay ring, and so is itself Cohen–Macaulay. The elements t, c and d form a homogeneous system of parameters for R, and so the element $t \in m$ is indeed a nonzerodivisor. We next record the following crucial lemma.

Lemma 3.1.2. *Let m and n be positive integers satisfying $m - m/n > 2$. Consider the ring $R = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix $\mathfrak{M}_{m,n}$. If k is a positive integer such that $k(m - m/n - 2) \geq 1$, we have*

$$(b^n t^{m-1})^{2mk+1} \in (a^{2mk+1},\ d^{2mk+1}).$$

Proof. Let $\tau = A^2 + T^m$, and $\alpha = A^2$. It suffices to working in the polynomial ring $K[\tau, \alpha, B, C, D]$, and show

$$B^{n(2mk+1)}(\tau - \alpha)^{2k(m-1)} \in (\alpha^{mk+1},\ D^{2mk+1}) + \mathfrak{a}$$

where $\mathfrak{a}$ is the ideal generated by the size two minors of the matrix

$$\begin{pmatrix} \tau & B & D \\ C & \alpha & B^n - D \end{pmatrix}.$$

Taking the binomial expansion of $(\tau-\alpha)^{2k(m-1)}$, it suffices to show

$$B^{n(2mk+1)}(\tau,\alpha)^{2k(m-1)} \in (\alpha^{mk+1},\ D^{2mk+1}) + \mathfrak{a}.$$

This would follow if we show that for $0 \le i \le mk+1$, we have

$$B^{n(2mk+1)}\alpha^{mk+1-i}\tau^{mk-2k+i-1} \in (\alpha^{mk+1},\ D^{2mk+1}) + \mathfrak{a},$$

and so it is enough to show that $B^{n(2mk+1)}\tau^{mk-2k+i-1} \in (\alpha^i,\ D^{2mk+1}) + \mathfrak{a}$. Since $\alpha D - B(B^n - D) \in \mathfrak{a}$, it suffices to establish

$$B^{n(2mk+1)}\tau^{mk-2k+i-1} \in (B^i(B^n-D)^i,\ D^{2mk+1},\ B^n\tau - D(C+\tau)).$$

Now work modulo the element $B^i(B^n-D)^i$, and reduce $B^{n(2mk+1)}$, to a polynomial in B and D such that the highest power of B that occurs is less than $i(n+1)$. Consequently it suffices to show

$$B^{n(2mk+1-j)}\tau^{mk-2k+i-1}D^j \in (D^{2mk+1},\ B^n\tau - D(C+\tau)).$$

where $n(2mk+1-j) < i(n+1)$, i.e., $j \ge 2mk + (1-i)(1+1/n)$. With this simplification, we now need to show

$$B^{n(2mk+1-j)}\tau^{mk-2k+i-1} \in (D^{2mk+1-j},\ B^n\tau - D(C+\tau)).$$

We now only need to verify that $mk-2k+i-1 \ge 2mk+1-j$ since, working modulo $B^n\tau - D(C+\tau)$, we can then express $B^{n(2mk+1-j)}\tau^{mk-2k+i-1}$ as a multiple of $D^{2mk+1-j}$. Finally, note that

$$(mk-2k+i-1)-(2mk+1-j) = j - mk - 2k + i - 2 \ge k(m - \frac{m}{n} - 2) - 1 \ge 0.$$

$\square$

Proposition 3.1.3. *Let $S = K[A,B,C,D]/J$ where the characteristic of the field K is a prime $p > 2$, and J is the ideal generated by the size two minors of the matrix*

$$\begin{pmatrix} A^2 & B & D \\ C & A^2 & B^n - D \end{pmatrix}.$$

Then S is an F–regular ring.

Proof. There are various ways to establish this. We may identify S with the ring $\oplus_{i \geq 0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(iD))X^i$ where $\mathbb{P}^1 = \operatorname{Proj} K[X,Y]$, and D is the $\mathbb{Q}$–divisor

$$D = \frac{1}{2}V(X) + \frac{1}{2}V(Y) + \frac{1}{2n}V(X+Y).$$

Under this identification,

$$A \mapsto X, \quad B \mapsto \frac{X^3}{Y}, \quad C \mapsto XY, \quad D \mapsto \frac{X^{3n+1}}{Y^n(X+Y)}.$$

One may now appeal to Watanabe's classification in [Wa3] to conclude that S is F–regular.

For an alternate proof, it is easily verified that S is the Veronese subring $H^{(2n+1)}$ of the hypersurface

$$H = K[A,X,Y]/(A^2 - XY(X^n - Y))$$

where the variables A, X and Y have weights $2n+1$, 2 and $2n$ respectively. Here $B = XY^2$, $C = X(X^n - Y)^2$ and $D = Y^{2n+1}$. Since the characteristic of K is greater than 2, a routine computation shows that the hypersurface H is F–regular, and consequently its direct summand S is also F–regular. (This hypersurface is the cyclic cover $H = S \oplus \omega \oplus \omega^{(2)} \oplus \cdots \oplus \omega^{(2n)}$, where ω the canonical module of S.) $\square$

Proposition 3.1.4. *Let K be a field of characteristic $p > 2$ and consider the ring $R = R_{m,n} = K[A,B,C,D,T]/I$ where I is generated by the size two minors of the* *matrix* $\mathfrak{M}_{m,n}$. *If* $m - m/n > 2$, *then* R *is not F–regular. If in addition* p *and* m *are relatively prime, then* R *is not F–pure.*

Proof. First note that $b^n d^{m-1} \notin (a,d)$. To establish that R is not F–regular we shall show that $b^n t^{m-1} \in (a,d)^*$.

For a suitably large arbitrary positive integer e, let $q = p^e = 2mk + \delta$ where k and δ are integers such that $k(m - m/n - 2) \geq 1$ and $-m + 2 \leq \delta \leq 1$. To see that $b^n t^{m-1} \in (a,d)^*$, it suffices to show that $(b^n t^{m-1})^{q+m-1} \in (a^q, d^q)$ for all $q = p^e$. Since $q + m - 1 = 2mk + \delta + m - 1 \geq 2mk + 1$ and $q \leq 2mk + 1$, it suffices to see that

$$(b^n t^{m-1})^{2mk+1} \in (a^{2mk+1},\ d^{2mk+1}),$$

but this is precisely the assertion of Lemma 3.1.2.

For the second assertion, note that since $p > 2$, the integers p and $2m$ are relatively prime and we may choose a positive integer e such that $q = p^e = 2mk + 1$ for some $k > 0$. Taking a higher power of p, if necessary, we may also assume that $k(m - m/n - 2) \geq 1$. But now $(b^n t^{m-1})^q \in (a^q, d^q)$ by Lemma 3.1.2, and so we have $b^n t^{m-1} \in (a,d)^F$, which shows that R is not F–pure. □

Proof of Theorem 3.1.1. We have already noted that the element $t \in m$ is indeed a nonzerodivisor, and Proposition 3.1.3 establishes that the ring R/tR is F–regular. Since $m - m/n > 2$, Proposition 3.1.4 shows that R fails to be F–regular, and is not F–pure if p and m are relatively prime. □

The examples constructed above also show that the property *F–regular type* does not deform:

Theorem 3.1.5. *For positive integers* m *and* n *satisfying* $m - m/n > 2$, *consider the ring* $R = \mathbb{Q}[A, B, C, D, T]/I$ *where* I *is generated by the size two minors of the* *matrix $\mathfrak{M}_{m,n}$ of Theorem 3.1.1. Then R is not of F–pure type, whereas R/tR is of F–regular type.*

Proof. If p is a prime which does not divide $2m$, the fiber of $\mathbb{Z} \to R_{\mathbb{Z}}$ over p is not F–pure by Proposition 3.1.4. Consequently the ring R is not of F–regular type. Proposition 3.1.3 shows that R/tR is of F–regular type since the fiber of $\mathbb{Z} \to (R/tR)_{\mathbb{Z}}$ over p is F–regular for all primes $p > 2$. □

Remark 3.1.6. R. Fedder first constructed examples to show that F–purity does not deform, see [Fe1]. However Fedder pointed out that his examples were less than satisfactory in two ways: firstly the rings were not integral domains, and secondly his arguments did not work in the characteristic zero setting, i.e., did not comment on the deformation of the property *F–pure type.* In [Si] the author constructed various examples which overcame both these shortcomings, but left at least one issue unresolved — although the rings R were domains (which were not F–pure), the F–pure quotient rings R/tR were not domains. The examples we have constructed here also settle this remaining issue.

3.2 Conditions on fibers

The examples constructed in the previous section are also relevant from the point of view of the behavior of F–regularity under base change. We first recall a theorem of Hochster and Huneke, [HH4, Theorem 7.24].

Theorem 3.2.1. *Let $(A, m, K) \to (R, n, L)$ be a flat local homomorphism of local rings of characteristic p such that A is weakly F–regular, R is excellent, and the generic and closed fibers are regular. Then R is weakly F–regular.*

It is a natural question to ask what properties are inherited by an excellent ring

R if, as above, $(A, m, K) \to (R, n, L)$ is a flat local homomorphism, the ring A is F–regular and the generic and closed fibers are F–regular. Our examples can be used to show that even if (A, m, K) is a discrete valuation ring and the generic and closed fibers of $(A, m, K) \to (R, n, L)$ are F–regular, then R need not be F–regular.

Once again, we construct $\mathbb{N}$–graded examples, and examples with local rings can be obtained by the obvious localizations at the homogeneous maximal ideals. Let $A = K[T]$ be a polynomial ring in one variable, and $R = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix $\mathfrak{M}_{m,n}$. As before, K is a field of characteristic $p > 2$, and m and n are positive integers such that $m - m/n > 2$.

The generic fiber of the inclusion $A \to R$ is (a localization of) R_t, whereas the fiber over the homogeneous maximal ideal of A is R/tR. We have earlier established that R/tR is F–regular, and only need to show that the ring R_t is F–regular. In the following proposition we show that the ring R is, in fact, locally F–regular on the punctured spectrum.

Proposition 3.2.2. *Let K be a field of characteristic $p > 2$. For positive integers m and n consider the ring $R = R_{m,n} = K[A, B, C, D, T]/I$ where I is generated by the size two minors of the matrix $\mathfrak{M}_{m,n}$. Then the ring R_P is F–regular for all primes P in the punctured spectrum* $\operatorname{Spec} R - \{m\}$.

Proof. A routine verification shows that the singular locus of R is $V(J)$ where the defining ideal is $J = (a,\ b,\ c(c+t^m),\ d)$. Consequently we need to show that the two local rings R_P and R_Q are F–regular where $P = (a,\ b,\ c,\ d)$ and $Q = (a,\ b,\ c+t^m,\ d)$.

When we localize at the prime P, we have $d = b^n(a^2+t^m)/(c+a^2+t^m)$ and so R_P is a localization of $K[T, A, B, C]/(A^2(A^2 + T^m) - BC)$ at the prime ideal $(a,\ b,\ c)$. Since $a^2 + t^m$ is a unit, the hypersurface R_P is easily seen to be F–regular.

Localizing at the prime Q, we have $b = a^2(a^2+t^m)/c$ and so R_Q is a localization of

$K[T,A,C,D]/(C^nD(C+A^2+T^m)-A^{2n}(A^2+T^m)^{n+1})$ at the prime ideal $(a,\ c+t^m,\ d)$. Again we have a hypersurface which, it can be easily verified, is F–regular. $\square$

3.3 Anti–canonical covers

Let (R,m,K) be a local or an $\mathbb{N}$–graded normal ring. The *anti–canonical cover* of R is the symbolic Rees ring $S=\oplus_{i\geq 0}I^{(i)}X^i \subseteq R[X]$ where I is an ideal of pure height one which is the inverse of ω in the divisor class group $\mathrm{Cl}(R)$. The ring S need not be Noetherian in general but when it Noetherian, we have the following useful theorem of Watanabe, [Wa4, Theorem 0.1].

Theorem 3.3.1. *Let (R,m,K) be a normal ring for which the anti–canonical cover $S=\oplus_{i\geq 0}I^{(i)}X^i$ is Noetherian. Then R is strongly F–regular (F–pure) if and only if S is strongly F–regular (F–pure).*

Note that if the ring S above is Noetherian and Cohen–Macaulay, then it is also Gorenstein. This can be inferred from a local cohomology calculation in [Wa4], or from [GHNV, Theorem 4.8].

We use these ideas to obtain a positive result regarding the deformation of strong F–regularity. It is known that F–rationality deforms, [HH4, Theorem 4.2 (h)], and our work makes use of this in an essential way.

Theorem 3.3.2. *Let (R,m,K) be a normal local ring for which the anti–canonical cover $S=\oplus_{i\geq 0}I^{(i)}X^i$ is Noetherian, and the symbolic powers $I^{(i)}$ satisfy the Serre condition S_3 for all $i\geq 0$. Then if R/tR is strongly F–regular for some nonzerodivisor $t\in m$, the ring R is also strongly F–regular.*

Proof. We may replace I, if necessary, to ensure that tR is not one of its minimal primes. Since we have assumed that $I^{(i)}$ is S_3, the natural maps give us isomorphisms

$(I/tI)^{(i)} \cong I^{(i)}/tI^{(i)}$ for $i \geq 0$. Hence the ring

$$S/tS \cong R/tR \oplus (I/tI) \oplus (I/tI)^{(2)} \oplus \cdots$$

is an anti–canonical cover for R/tR. Theorem 3.3.1 now shows that S/tS is strongly F–regular, and so is also Cohen–Macaulay. Hence S is Cohen–Macaulay and therefore Gorenstein. As F–rationality deforms and S/tS is strongly F–regular, we get that S is F–rational. However S is Gorenstein, and so is actually strongly F–regular. Finally, R is a direct summand of S and so is strongly F–regular. $\square$

CHAPTER IV

FAILURE OF F–PURITY AND F–REGULARITY IN CERTAIN RINGS OF INVARIANTS

Let $\mathbb{F}_q$ be a finite field of characteristic p, K a field containing it, and take a polynomial ring in n variables, $R = K[X_1, \ldots, X_n]$. The general linear group $GL_n(\mathbb{F}_q)$ has a natural action on R by degree preserving ring automorphisms. L. E. Dickson showed that the subring of elements which are fixed by this group action is a polynomial ring, [Di], though for an arbitrary subgroup G of $GL_n(\mathbb{F}_q)$, the structure of the ring of invariants R^G may be rather mysterious. If the order of the group $|G|$ is relatively prime to the characteristic p of the field, there is an R^G–linear retraction $\rho : R \to R^G$, the *Reynolds operator*. This retraction makes R^G a direct summand of R as an R^G–module, and so R^G is F–regular. However when the characteristic p divides $|G|$, this method no longer applies, and the ring of invariants R^G need not even be Cohen–Macaulay. M.–J. Bertin showed that when R is a polynomial ring in four variables and G is the cyclic group with four elements which acts by permuting the variables in cyclic order, then the ring of invariants R^G is a unique factorization domain which is not Cohen–Macaulay, providing the first example of such a ring, [Ber]. More recently D. Glassbrenner studied the invariant subrings of the action of the alternating group A_n on a polynomial ring in n variables over a field of characteristic p, constructing examples of F–pure rings which are not F–regular, [Gl1, Gl2].

Both these families of examples study rings of invariants of $K[X_1, \ldots, X_n]$ under the action of a subgroup G of the symmetric group on n elements, i.e., an action which permutes the variables, and Glassbrenner shows that for such a group the ring of invariants is F–pure, see [Gl1, Proposition 0.6.7].

We shall construct examples which demonstrate that the ring of invariants for the natural action of a subgroup G of $GL_n(\mathbb{F}_q)$ need not be F–pure. We shall obtain such examples with the group G being the symplectic group over a finite field. These non F–pure invariant subrings are always complete intersections, and are actually hypersurfaces in the case of $G = Sp_4(\mathbb{F}_q) < GL_4(\mathbb{F}_q)$ acting on the polynomial ring $R = K[X_1, X_2, X_3, X_4]$. These examples are particularly interesting if one is attempting to interpret the Frobenius closures and tight closures of ideals as contractions from certain extension rings, since we have an ideal generated by a system of parameters and the socle element modulo this ideal is being forced into the expansion of the ideal to a module–finite extension ring which is a separable (in fact Galois) extension and is also forced into the expanded ideal in a linearly disjoint purely inseparable extension (being in the Frobenius closure of the ideal). It is noteworthy that the element can be forced into expanded ideals in two such different ways.

Our results depend on the work of D. Carlisle and P. Kropholler where they show that the ring of invariants under the natural action of the symplectic group on a polynomial ring is a complete intersection, [CK]. We obtain the precise equations defining these complete intersections in some examples using the program Macaulay, and in some other cases collect enough information to display that the invariant subrings are not F–pure.

The last section of this chapter deals with the alternating group A_n acting on the polynomial ring $R = K[X_1, \ldots, X_n]$ by permuting the variables. We shall assume that the characteristic p of K is an odd prime, and denote by R^{A_n}, the invariant subring of this action. Since R^{A_2} is a polynomial ring we shall always assume $n \geq 3$. If the order of the group $|A_n| = \frac{1}{2}(n!)$ is relatively prime to the characteristic p of the field, the Reynolds operator makes R^{A_n} a direct summand of R as an R^{A_n}–module, and in the language of tight closure, the existence of a retraction is equivalent to the ring R^{A_n} being F–regular, see Lemma 4.3.1. When p divides either n or $n-1$, Glassbrenner has shown that the invariant subring R^{A_n} is no longer F–regular, see [Gl1, Proposition 1.2.5]. We shall extend this result by showing that R^{A_n} is F–regular if and only if p does not divide $|A_n|$.

4.1 Symplectic invariants

We shall summarize in this section the results of Carlisle and Kropholler as presented in [Ben]. Let $\mathbb{F}_q$ be a finite field of characteristic p, and K an infinite field containing it. L. E. Dickson showed that the ring of invariant forms under the natural action of $GL_n(\mathbb{F}_q)$ on the polynomial ring $R = K[X_1, \ldots, X_n]$ is a graded polynomial algebra on the algebraically independent generators $c_{n,i}$, where the $c_{n,i}$ are the coefficients in the equation

$$\prod_{v \in \mathbb{F}_q[X_1,\ldots,X_n]} (T - v) = T^{q^n} - c_{n,n-1}T^{q^{n-1}} + c_{n,n-2}T^{q^{n-2}} - \cdots + (-1)^n c_{n,0}T.$$

When working with a fixed polynomial ring $R = K[X_1, \ldots, X_n]$, we shall drop the first index, and write the generators of $R^{GL_n(\mathbb{F}_q)}$ as $c_0, \ldots, c_{n-1}$, the *Dickson invariants.* It is clear that for any subgroup G of $GL_n(\mathbb{F}_q)$, the ring of invariants R^G is a module–finite extension of the polynomial ring $R^{GL_n(\mathbb{F}_q)} = K[c_0, \ldots, c_{n-1}]$.

Let V be a vector space of dimension $2n$ over the field $\mathbb{F}_q$, with a basis $e_1, \ldots, e_{2n}$, and let B be the non–degenerate alternating bilinear form given by

$$B(\sum a_i e_i, \sum b_j e_j) = a_1 b_2 - a_2 b_1 + \cdots + a_{2n-1}b_{2n} - a_{2n}b_{2n-1}.$$

The symplectic group $G = Sp_{2n}(\mathbb{F}_q)$ is the subgroup of $GL_{2n}(\mathbb{F}_q)$ consisting of the elements which preserve B. We consider the natural action of G on $R = K[X_1, \ldots, X_{2n}]$. In addition to the Dickson invariants, it is easily seen that R^G must contain

$$\xi_i = X_1 X_2^{q^i} - X_2 X_1^{q^i} + \cdots + X_{2n-1} X_{2n}^{q^i} - X_{2n} X_{2n-1}^{q^i}.$$

Carlisle and Kropholler show that the Dickson invariants $c_0, \ldots, c_{2n-1}$ along with the above $\xi_1, \ldots, \xi_{2n}$ form a generating set for R^G, and that there are $2n$ relations, i.e., that R^G is a complete intersection. One may eliminate $c_0, \ldots, c_{n-1}$ and ξ_{2n} using $n+1$ of these relations, after which the remaining $n-1$ relations are

$$\sum_{j=0}^{i-1} (-1)^j \xi_{i-j}^{q^j} c_j = \sum_{j=i+1}^{2n} (-1)^j \xi_{j-i}^{q^i} c_j$$

where $1 \le i \le n-1$ and $c_{2n} = 1$. They furthermore show that $c_0 \in K[\xi_1, \ldots, \xi_{2n-1}]$ which is, in fact, a polynomial ring.

4.2 Rings of invariants which are not F–pure

We shall first show that the ring of invariants of $G = Sp_4(\mathbb{F}_q)$ acting on the polynomial ring $R = K[X_1, X_2, X_3, X_4]$ is not F–pure when $q = 2$ or 3. Note that $Sp_2(\mathbb{F}_q)$ is the same as $SL_2(\mathbb{F}_q)$, and so the ring of invariants in that case is a polynomial ring.

Example 4.2.1. Let $R = K[X_1, X_2, X_3, X_4]$ and $G = Sp_4(\mathbb{F}_q)$ be the symplectic group with its natural action on R. In the notation of the previous section, the ring of invariants is $R^G = K[c_2, c_3, \xi_1, \xi_2, \xi_3]$, where the only relation is

$$\xi_1 c_0 = \xi_1^q c_2 - \xi_2^q c_3 + \xi_3^q.$$

We need to determine c_0 as an element of $K[\xi_1, \xi_2, \xi_3]$. When $q = 2$, it can be verified that $c_0 = \xi_1^5 + \xi_2^3 + \xi_3 \xi_1^2$, and so $\xi_3^2 = \xi_1^6 + \xi_1 \xi_2^3 + \xi_1^3 \xi_3 + \xi_1^2 c_2 + \xi_2^2 c_3$, by which $\xi_3 \in ((\xi_1, \xi_2) R^G)^F$. Since $\xi_3 \notin (\xi_1, \xi_2) R^G$, the ring R^G is not F–pure.

In the case $q = 3$, c_0 can be expressed as an element of $K[\xi_1, \xi_2, \xi_3]$ by the equation $c_0 = \xi_2^8 + \xi_3\xi_1^3\xi_2^4 + \xi_1^6\xi_3^2 + \xi_1^{10}\xi_2^4 - \xi_1^{13}\xi_3 + \xi_1^{20}$. Once again we see that $\xi_3 \in ((\xi_1, \xi_2)R^G)^F$, and so R^G is not F–pure.

Computations with Macaulay helped us determine the precise equations in these examples.

Theorem 4.2.2. *Let $\mathbb{F}_q$ be a finite field of characteristic p, and K an infinite field containing it. Let $G = Sp_{2n}(\mathbb{F}_q)$ be the symplectic group with its natural action on the polynomial ring $R = K[X_1, \ldots, X_{2n}]$. If $n \geq 2$ and $q \geq 4n - 4$, then the ring of invariants R^G is not F–pure.*

Proof. In the notation of the previous section, $R^G = K[c_n, \ldots, c_{2n-1}, \xi_1, \ldots, \xi_{2n-1}]$, where there are exactly $n - 1$ relations, as stated before. Using the relation with $i = 1$, we see that

$$\xi_{2n-1}^q \in (\xi_1^q, \ldots, \xi_{2n-2}^q, \xi_1 c_0)R^G,$$

whereas $\xi_{2n-1} \notin (\xi_1, \ldots, \xi_{2n-2})R^G$. If R^G is indeed F–pure, $\xi_{2n-1}^q \notin (\xi_1^q, \ldots, \xi_{2n-2}^q)R^G$, and so the expression of c_0 as an element of $K[\xi_1, \ldots, \xi_{2n-1}]$ must have a monomial of the form $\xi_1^{a_1}\xi_2^{a_2} \cdots \xi_{2n-1}^{a_{2n-1}}$, with $a_1 \leq q - 2$ and $a_2, \ldots, a_{2n-1} \leq q - 1$. Equating degrees, we have

$$\begin{aligned} \deg c_0 = q^{2n} - 1 &= a_1(q+1) + a_2(q^2+1) + \cdots + a_{2n-1}(q^{2n-1}+1) \\ &= \sum_{i=1}^{2n-1} a_i + \sum_{i=1}^{2n-1} a_i q^i. \end{aligned}$$

Examining this modulo q, we get that $\sum_{i=1}^{2n-1} a_i = \lambda q - 1$, where the bounds on a_i show that $1 \leq \lambda \leq 2n - 2 < q$. Substituting this, we get $q^{2n} = \lambda q + \sum_{i=1}^{2n-1} a_i q^i$. Working modulo q^2, we see that $a_1 = q - \lambda$, and continuing this way we get that

$a_2, \ldots, a_{2n-1} = q - 1$. Hence

$$q^{2n} - 1 = (q - \lambda)(q + 1) + (q - 1)(q^2 + 1) + \cdots + (q - 1)(q^{2n-1} + 1),$$

which simplifies to give $\lambda(q + 1) = 2nq - 2n - q + 3$. Since $\lambda \leq 2n - 2$, this implies that $q \leq 4n - 5$, a contradiction.

Hence R^G is not F–pure. In particular $\xi_{2n-1} \in ((\xi_1, \ldots, \xi_{2n-2})R^G)^F$, the Frobenius closure. □

Corollary 4.2.3. *The ring of invariants R^G of the symplectic group $G = Sp_4(\mathbb{F}_q)$ acting on the polynomial ring $R = K[X_1, X_2, X_3, X_4]$ is not F–pure.*

Proof. We have, in the examples above, treated the case where $q = 2$ or 3. When $q \geq 4$, the result follows from the previous theorem. □

4.3 Rings of invariants of the alternating group

The invariant subring under the natural action of A_n is $R^{A_n} = K[e_1, \ldots, e_n, \Delta]$ where e_i is the elementary symmetric function of degree i in $X_1, \ldots, X_n$, and $\Delta = \prod_{i>j}(X_i - X_j)$. The element Δ is easily seen to be fixed by all even permutations of $X_1, \ldots, X_n$, though not by odd permutations. However its square, Δ^2, is fixed by all permutations, and so is a polynomial in the algebraically independent elements $e_1, \ldots, e_n$. Consequently the invariant subring R^{A_n} is a hypersurface, in particular it is Gorenstein. The elements $e_1, \ldots, e_n$ are an obvious choice as a homogeneous system of parameters for R^{A_n}, and the one–dimensional socle modulo this system of parameters is generated by Δ.

Lemma 4.3.1. *With the above notation, the following are equivalent:*

(1) $R^{A_n} = K[e_1, \ldots, e_n, \Delta]$ *is F–regular.*

(2) *R^{A_n} is a direct summand of $R = K[X_1, \ldots, X_n]$ as an R^{A_n}–module.*

(3) *$\Delta \notin (e_1, \ldots, e_n)R$.*

Proof. (1) $\Rightarrow$ (2) By [HH5, Theorem 5.25], an F–regular ring is a direct summand of any module–finite extension ring.

(2) $\Rightarrow$ (3) Since R^{A_n} is a direct summand of R, we have $(e_1, \ldots, e_n)R \cap R^{A_n} = (e_1, \ldots, e_n)R^{A_n}$.

(3) $\Rightarrow$ (1) The elements $e_1, \ldots, e_n$ form a system of parameters for the Gorenstein ring R^{A_n} and Δ is the socle generator modulo this system of parameters. If Δ is in the tight closure of $(e_1, \ldots, e_n)R^{A_n}$, then $\Delta \in (e_1, \ldots, e_n)R^* = (e_1, \ldots, e_n)R$. Hence Δ cannot be in the tight closure of $(e_1, \ldots, e_n)R^{A_n}$, by which R^{A_n} is F–regular. $\square$

Consequently our aim is to establish that $\Delta \in (e_1, \ldots, e_n)R$, whenever p divides $|A_n|$. We shall henceforth denote this ideal by $I = (e_1, \ldots, e_n)R$.

Lemma 4.3.2. *Let T_j^i denote the sum of all monomials of degree i in $X_j, \ldots, X_n$. Then $T_j^i \in I$ whenever $i \geq j \geq 1$. In particular, $T_i^i \in I$ for all $1 \leq i \leq n$.*

Proof. Observe that $T_j^i = T_{j-1}^i - X_{j-1}T_{j-1}^{i-1}$. Given T_j^i with $i \geq j \geq 1$, we may use this formula to rewrite T_j^i as a sum of terms which are multiples of T_1^i. Since T_1^i is the sum of all the monomials of degree i in $X_1, \ldots, X_n$, it is certainly an element of I, and so $T_j^i \in I$. $\square$

Lemma 4.3.3. *The ideal $I = (e_1, \ldots, e_n)R$ contains the elements: X_n^n, $X_n^{n-1}X_{n-1}^{n-1}$, $X_n^{n-1}X_{n-1}^{n-2}X_{n-2}^{n-2}$, $\ldots$, $X_n^{n-1}X_{n-1}^{n-2}\cdots X_i^{i-1}X_{i-1}^{i-1}$, $\ldots$, $X_n^{n-1}X_{n-1}^{n-2}\cdots X_2X_1$.*

Proof. We shall use the fact that $T_i^i \in I$ for $1 \leq i \leq n$, Lemma 4.3.2. This already says that $X_n^n = T_n^n \in I$, and since I is symmetric in the X_i, we also have $X_{n-1}^n \in I$.

Next, $X_{n-1}^{n-1}T_{n-1}^{n-1} \in I$, but examining this using $X_{n-1}^{n} \in I$ we see that $X_n^{n-1}X_{n-1}^{n-1} \in I$. We proceed by induction.

Since $T_{i-1}^{i-1} \in I$, we know $X_n^{n-1}X_{n-1}^{n-2}\cdots X_i^{i-1}T_{i-1}^{i-1} \in I$, but using the inductive hypothesis, this gives $X_n^{n-1}X_{n-1}^{n-2}\cdots X_i^{i-1}X_{i-1}^{i-1} \in I$. $\square$

Lemma 4.3.4. *In the above notation,* $\Delta \equiv (n!)X_n^{n-1}X_{n-1}^{n-2}\cdots X_2 \pmod{I}$.

Proof. Let $\delta_r = (X_r - X_1)(X_r - X_2)\cdots(X_r - X_{r-1})$. Then $\Delta = \delta_n\delta_{n-1}\cdots\delta_2$. We shall show that $\delta_r \equiv rX_r^{r-1} \pmod{I + (X_{r+1}, \ldots, X_n)R}$ for $2 \leq r \leq n$. Note that for $r = n$, this says $\delta_n \equiv nX_n^{n-1} \pmod{I}$.

Fix r, where $2 \leq r \leq n$. Let f_i be the elementary symmetric function of degree i in the variables $X_1, \ldots, X_{r-1}$. Then $f_i \equiv (-X_r)f_{i-1} \pmod{I + (X_{r+1}, \ldots, X_n))R}$, and using this repeatedly, we see that $f_i \equiv (-X_r)^i \pmod{I + (X_{r+1}, \ldots, X_n))R}$. Consequently

$$\begin{aligned}
\delta_r &= (X_r - X_1)(X_r - X_2)\cdots(X_r - X_{r-1}) \\
&= X_r^{r-1} - X_r^{r-2}(X_1 + \cdots + X_{r-1}) + \cdots + (-1)^{r-1}X_1\cdots X_{r-1} \\
&\equiv X_r^{r-1} - X_r^{r-2}f_1 + \cdots + (-1)^{r-1}f_{r-1} \pmod{I + (X_{r+1}, \ldots, X_n)R} \\
&\equiv X_r^{r-1} - X_r^{r-2}(-X_r) + \cdots + (-1)^{r-1}(-X_r)^{r-1} \pmod{I + (X_{r+1}, \ldots, X_n)R} \\
&\equiv rX_r^{r-1} \pmod{I + (X_{r+1}, \ldots, X_n)R}.
\end{aligned}$$

Since $X_n^n \in I$, when evaluating the term $\delta_n\delta_{n-1} \pmod{I}$, it is enough to consider $\delta_{n-1} \pmod{I + X_nR}$, and so we get $\delta_n\delta_{n-1} \equiv n(n-1)X_n^{n-1}X_{n-1}^{n-2} \pmod{I}$. Proceeding in this manner, one obtains from the above calculations that

$$\Delta = \delta_n\delta_{n-1}\cdots\delta_2 \equiv (n!)X_n^{n-1}X_{n-1}^{n-2}\cdots X_2 \pmod{I}.$$

The point is that since

$$\delta_n\delta_{n-1}\cdots\delta_r \equiv n(n-1)\cdots(r)X_n^{n-1}X_{n-1}^{n-2}\cdots X_r^{r-1} \pmod{I},$$

we have $\delta_n\delta_{n-1}\cdots\delta_r(X_r,\ldots,X_n) \subseteq I$ by Lemma 4.3.3 and so when evaluating $\delta_n\delta_{n-1}\cdots\delta_{r-1} \pmod{I}$, we need only consider $\delta_{r-1} \pmod{I+(X_r,\ldots,X_n)R}$. □

We are now ready to prove the main result of this section.

Theorem 4.3.5. *Let $R = K[X_1,\ldots,X_n]$ be a polynomial ring in n variables over a field k of characteristic p, an odd prime, and let the alternating group A_n act on R by permuting the variables. Then the invariant subring R^{A_n} is F–regular (equivalently, R^{A_n} is a direct summand of R) if and only if the order of the group $|A_n| = \frac{1}{2}(n!)$ is relatively prime to p.*

Proof. As we noted, it suffices to show that $\Delta \in I = (e_1,\ldots,e_n)R$. By Lemma 4.3.4, $\Delta \equiv (n!)X_n^{n-1}X_{n-1}^{n-2}\cdots X_2 \pmod{I}$, and so the result follows. □

Remark 4.3.6. It follows from Glassbrenner's result, [Gl1, Proposition 0.6.7], that R^{A_n} is always F–pure. Consequently when the characteristic p of the field K is an odd prime dividing $|A_n|$, R^{A_n} is an F–pure ring which is not F–regular.

CHAPTER V

F–RATIONALITY AND F–REGULARITY OF VERONESE SUBRINGS

Our objective is to determine when an $\mathbb{N}$–graded ring R has Veronese subrings which are F–rational or F–regular. Our discussion is motivated by the fact that if R is a Cohen–Macaulay ring with an isolated singularity and a negative *a–invariant*, then for all large positive integers n, the Veronese subring $R^{(n)}$ is F–rational, Proposition 5.1.1 below. The existence of F–regular Veronese subrings turns out to be more subtle. Using a result of Watanabe, we show that if R is a normal $\mathbb{N}$–graded ring generated by degree one elements over a field, then either R is F–regular, or else no Veronese subring of R is F–regular. Consequently the problem is reduced to studying when a ring R generated by degree one elements over a field is F–regular; we may assume here that R is F–rational. It is easily seen that the F–rationality of R implies $a(R) < 0$, and for rings of dimension two this also turns out to be a sufficient condition for the F–regularity of R. This is false in higher dimensions, and we construct examples (in prime characteristic, as well as in characteristic zero) of rings generated by degree one elements which are F–rational but not F–regular.

5.1 F–rational Veronese subrings

Proposition 5.1.1. *Let R be an $\mathbb{N}$–graded Cohen–Macaulay domain of dimension d, which is locally F–rational on the punctured spectrum* $\operatorname{Spec} R - m$. *(This is satisfied, in particular, if R has an isolated singularity.) Then $[H^d_m(R)]_0 = 0$ if and only if the Veronese subring $R^{(n)}$ is F–rational for all integers $n \gg 0$. In particular if $a(R) < 0$, then $R^{(n)}$ is F–rational for all integers $n \gg 0$.*

Proof. Note that we have $[H^d_m(R)]_0 \subseteq 0^*_{H^d_m(R)}$, since for $z \in [H^d_m(R)]_0$ we get $cz^q = 0$ for all $q = p^e$, when $c \in m$ is of a sufficiently large degree. Consequently if $R^{(n)}$ is F–rational for some n, we must have $a(R^{(n)}) < 0$, but then $[H^d_m(R)]_0 = 0$.

For the converse first note that since R is F–rational on the punctured spectrum, Theorem 2.2.3 (6) says that $0^*_{H^d_m(R)}$ must be killed by a power of the maximal ideal m, and so is of finite length. As $[H^d_m(R)]_0 = 0$, for large positive integers n we see that $H^d_{m'}(R^{(n)}) \cong (H^d_m(R))^{(n)}$ contains no nonzero element of $0^*_{H^d_m(R)}$ where m' denotes the homogeneous maximal ideal of $R^{(n)}$. If $u \in 0^*_{H^d_{m'}(R^{(n)})}$ then $u \in 0^*_{H^d_m(R)} \cap H^d_{m'}(R^{(n)})$ and so $u = 0$. Hence $R^{(n)}$ is F–rational for $n \gg 0$. $\square$

Example 5.1.2. Let $R = K[X, Y, Z]/(X^2 + Y^3 + Z^5)$ where K is a field of prime characteristic p. We make this a graded ring by setting the weights of x, y and z to be 15, 10 and 6 respectively. We determine the positive integers n for which the Veronese subring $R^{(n)}$ is F–rational. This shall, of course, depend on the characteristic p of R.

First note that $a(R) = -1$ with this grading. If $p \geq 7$, it is easy to verify that the ring R is F–regular, for an interesting proof, see [Fe2, Example 2.9]. Consequently every Veronese subring of R, being a direct summand of R, is also F–regular. For $p = 2$, 3 or 5, $x^p \in (y^p, z^p)$, and so R is not F–rational. It is also easily checked that the action of the Frobenius on $H^2_m(R)$ is injective in degree ≤ -2 with the one exception of $p = 2$ where elements in degree -7 are mapped to zero under the action of the Frobenius, specifically $F(xy^{-1}z^{-2}) = 0$ in $H^2_m(R)$. Recall that $H^2_{m_{R^{(n)}}}(R^{(n)})$ is generated by elements of $H^2_m(R)$ whose degree is a multiple of n. Consequently for $n \geq 2$ the action of the Frobenius on $H^2_{m_{R^{(n)}}}(R^{(n)})$ is injective, with the one exception. Using the arguments in the proof of the above theorem, we see that $R^{(n)}$ is F–rational for all $n \geq 2$, excluding the case when $p = 2$ and $n = 7$.

5.2 Results in dimension two

Theorem 5.2.1. *Let R be an $\mathbb{N}$–graded normal ring of dimension two, which is generated by degree one elements over an algebraically closed field. Then the following statements are equivalent.*

(1) *R is isomorphic to a Veronese subring of a polynomial ring in two variables.*

(2) *R is F–regular.*

(3) *R is F–rational.*

(4) *R has a negative a–invariant.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follow easily. For (4) $\Rightarrow$ (1) note that $X = \operatorname{Proj} R$ is a nonsingular projective curve. Since $[H^2_m(R)]_0 = 0$, we have $H^1(X, \mathcal{O}_X) = 0$ and so X is of genus zero, i.e., $\mathbb{P}^1$. Consequently $R \cong R(\mathbb{P}^1, D)$ where D is a Weil divisor on $\mathbb{P}^1$. Hence D is linearly equivalent to $\mathcal{O}(m)$ for some $m \in \mathbb{N}$ and $R \cong R(\mathbb{P}^1, \mathcal{O}(m)) \cong (K[X_0, X_1])^{(m)}$. $\square$

Theorem 5.2.2. *Let R be an $\mathbb{N}$–graded domain of dimension two, with an isolated singularity, which is finitely generated over an algebraically closed field. If $a(R) < 0$, there exists a positive integer n such that $R^{(n)}$ is isomorphic to a Veronese subring* *of a polynomial ring in two variables over K. In particular, some Veronese subring of R is F–regular.*

Proof. Note that R is excellent and so R', the integral closure R in its fraction field, is module–finite over R. Since R has an isolated singularity, the conductor (i.e., the largest common ideal of R and R') is primary to the maximal ideal of R', by which $R_i = R'_i$ for all $i \gg 0$. We may therefore choose a positive integer k such that $R^{(k)}$ is normal, and then choose an appropriate multiple n of k, by Lemma 2.3.2, such that $R^{(n)}$ is generated by elements of equal degree. We are now in a position to apply the above theorem to conclude that $R^{(n)}$ is isomorphic to a Veronese subring of a polynomial ring in two variables. $\square$

Example 5.2.3. Let $S = K[X,Y,Z]/(X^3 - YZ(Y+Z))$ where K is a field of prime characteristic $p \equiv 1 \pmod 3$. Let $R = K[X, Y^3, Y^2Z, YZ^2, Z^3]/(X^3 - YZ(Y+Z))$ be subring of S. It is proved in [HH5] that R is F–rational but not F–regular, see also [Wa3]. Since $R^{(3)}$ is generated by elements of equal degree, it must be isomorphic to a Veronese subring of a polynomial ring by Theorem 5.2.1. Indeed,

$$R^{(3)} = K[X^3,\ Y^3,\ Y^2Z,\ YZ^2,\ Z^3]/(X^3 - YZ(Y+Z)) \cong K[Y^3,\ Y^2Z,\ YZ^2,\ Z^3].$$

Example 5.2.4. Let $R = K[t, t^4x, t^4x^{-1}, t^4(x+1)^{-1}]$ where K is a field of prime characteristic p. This is one of the examples in [Wa2] of rings which are F–rational but not F–pure; for a different proof see [HH5]. By mapping a polynomial ring onto it, we may write R as

$$R = K[T,\ U,\ V,\ W]/(T^8 - UV,\ T^4(V-W) - VW,\ U(V-W) - T^4W).$$

This is graded by setting the weights of t, u, v and w to be 1, 4, 4 and 4 respectively.

Note that

$$\begin{aligned} R^{(4)} &= K[T^4,\ U,\ V,\ W]/(T^8 - UV,\ T^4(V-W) - VW,\ U(V-W) - T^4W) \\ &= K[S,\ U,\ V,\ W]/(S^2 - UV,\ S(V-W) - VW,\ U(V-W) - SW) \end{aligned}$$

where we relabel T^4 as S. Then $R^{(4)}$ is generated by elements of equal degree, and is isomorphic to $K[X^3, X^2Y, XY^2, Y^3]$ by setting $S = XY(X-Y)$, $U = XY^2$, $V = X(X-Y)^2$, and $W = Y(X-Y)^2$.

By Theorem 5.2.2 we know that a graded normal ring R of dimension two over an algebraically closed field has a Veronese subring $R^{(n)}$ which is F–regular. We next show that if R is a hypersurface, there exists n such that $R^{(n)}$ is actually an F–regular hypersurface.

Theorem 5.2.5. *Let R be an $\mathbb{N}$–graded normal hypersurface of dimension two with $a(R) < 0$. Then there exists a positive integer n such that the Veronese subring $R^{(n)}$ is an F–regular hypersurface.*

Proof. Let $R = K[X, Y, Z]/(f)$ where x, y and z have weights m, n and r respectively. We may assume without any loss of generality that m, n and r have no common factor. If $d = \gcd(m, n)$, then by our assumption d and r are relatively prime. Therefore f must be a polynomial in x, y and z^d. Consequently $R^{(n)}$ is again a hypersurface, and satisfies all the initial hypotheses, and so we may assume that R satisfies the extra hypothesis that m, n and r are pairwise relatively prime. Assume further that $m \geq n \geq r$. We need to consider the two cases a) $n = 1$ and $r = 1$, and b) $m > n > r$. Note that it suffices to show that R is F–rational, since it is indeed a hypersurface.

We first eliminate the case (#) when f is of the form $XH(Y, Z) + G(Y, Z)$. We may take a system of parameters of R of the form x, t where t is the image in R of a polynomial $T \in K[X,Y,Z]$ involving only Y and Z. If R is not F–rational, then since $a(R) < 0$, (x,t) cannot be F–pure. Hence for some $q = p^e$, we have $s^q \in (x^q, t^q)$ while $s \notin (x,t)$. Again, we may assume that s is the image in R of a polynomial $S \in K[X,Y,Z]$ involving only Y and Z. This means that in $K[X,Y,Z]$, we have $S^q \in (X^q, T^q, XH + G)$ but then $S^q \in (T^q, G^q)$ and so $S \in (T,G)$ in $K[X,Y,Z]$, giving us the contradiction $s \in (x,t)$.

a) We have $a(R) = \deg f - (m+n+r) < 0$, and so $\deg f < m + 2$ since $n = r = 1$. This forces f to be of the form (#).

b) Since $a(R) = \deg f - (m+n+r) < 0$, we have $\deg f < m+n+r < 3m$. Hence up to a scalar multiple, f is of the form $XH(Y,Z) + G(Y,Z)$ or $X^2 + G(Y,Z)$. Note that the first case has already been handled.

Now suppose $f = X^2 + G(Y,Z)$. Then $\deg f = 2m < m+n+r$ and so $3 < m < n+r$, consequently G cannot involve a term of the form Y^2Z^l where $l \geq 2$. If G has a term Y^k, then $2m = kn$ and so $n = 1$ or 2. Since $n > r$, we can only have $n = 2$ and $r = 1$, but this too is impossible. Hence f can only be of the form $f = X^2 + aZ^k + bYZ^l + cY^2Z$ where a, b and c are scalars. R is normal, and so c must be non–zero since $l \geq 2$ and $k \geq 2$. It follows that $2m = 2n + r$. If a is non–zero, $2m = rk$ and since r is even, we can only have $r = 2$. But then $m = n+1$, and so r divides either m or n, a contradiction. Hence $a = 0$, and so $f = X^2 + bYZ^l + cY^2Z$. If b were non–zero, then we would have $n + rl = 2n + r$, i.e., $n = r(l-1)$, which forces $r = 1$. However we know r to be even, and so $b = 0$. We are left with $f = X^2 + cY^2Z$ but this is ruled out since R is normal. $\square$

5.3 F–regular Veronese subrings

We begin by recalling a theorem of Watanabe, [Wa3, Theorem 3.4]:

Theorem 5.3.1. *Let D_1 and D_2 be ample $\mathbb{Q}$–divisors on a normal projective variety X. If the fractional parts D_1' and D_2' are equal, then the ring $R(X, D_1)$ is F–regular (F–pure) if and only if the ring $R(X, D_2)$ is F–regular (F–pure).*

Watanabe's theorem gives us the following corollary:

Corollary 5.3.2. *Let R be an $\mathbb{N}$–graded normal ring which is generated by degree one elements over a field. Then either R is F–regular (F–pure), or else no Veronese subring of R is F–regular (F–pure).*

Proof. Since R is generated by its elements of degree one, we have $R = R(X, D)$, where D is a Weil divisor, i.e., has $D' = 0$. Also, $(nD)' = 0$ where n is any positive integer. By the above Theorem, $R = R(X, D)$ is F–regular (F–pure) if and only if $R^{(n)} \cong R(X, nD)$ is F–regular (F–pure). $\square$

As an application of this result, we now construct a family of rings with negative a–invariants, which have no F–pure Veronese subrings. This shows that a result corresponding to Theorem 5.2.2 is no longer true in higher dimensions.

Example 5.3.3. Let $R = K[X_0, \dots, X_d]/(X_0^3 + \cdots + X_d^3)$ with $d \geq 3$, where K is a field of characteristic 2. Then $x_0^2 \in (x_1, \dots, x_d)^*$, since $x_0^4 \in (x_1, \dots, x_d)^{[2]}$. Hence R is not F–pure, and since it is generated by elements of degree one, Corollary 5.3.2 shows that R has no F–regular or F–pure Veronese subrings. Note that $a(R) = 2 - d < 0$.

We can also see that $R^{(n)}$ is not F–pure (for any $n > 0$) by showing that the element $x_0^d(x_1 \cdots x_d)^{n-1}$ is in the Frobenius closure of the ideal

$$I = (x_0^{d-2}x_1^n x_2^{n-1} \cdots x_{d-1}^{n-1},\quad x_0^{d-2}x_2^n x_3^{n-1} \cdots x_d^{n-1},\quad \dots,\quad x_0^{d-2}x_d^n x_1^{n-1} \cdots x_{d-2}^{n-1})R^{(n)},$$

although not in the ideal I itself.

For all $n \geq 2$, the ring $R^{(n)}$ is an example of a graded ring generated by degree one elements (with an isolated singularity and a negative a–invariant) which is F–rational but not F–pure.

Remark 5.3.4. The examples above are not completely satisfactory as they are not valid in the characteristic zero setting. Characteristic zero examples turn out to be much more subtle, and we construct these in the next section.

We again return to the ring $R = K[X,Y,Z]/(X^2 + Y^3 + Z^5)$, and this time determine its F–regular and F–pure Veronese subrings.

Example 5.3.5. Let $R = K[X,Y,Z]/(X^2 + Y^3 + Z^5)$ where K is a field of prime characteristic p, and the grading is as before. We have noted that when $p \geq 7$, the ring R is F–regular, and therefore so is any Veronese subring $R^{(n)}$. We now determine when $R^{(n)}$ is F–regular assuming p is either 2, 3 or 5.

Note that the Veronese subrings $R^{(2)}$, $R^{(3)}$ and $R^{(5)}$ are in fact polynomial rings. Therefore when n is divisible by one of 2, 3 or 5, $R^{(n)}$ is a direct summand of a polynomial ring, and so is F–regular. We show that these are the only instances when $R^{(n)}$ is F–regular, or even F–pure.

Recall from Example 2.4.3 that $R = R(X, D)$ where $X = \operatorname{Proj} K[S,T]$ and $D = (-1/2)V(S) + (1/3)V(T) + (1/5)V(S+T)$. If n is relatively prime to 30, the $\mathbb{Q}$–divisor nD has the same fractional part as D, and so $R^{(n)} \cong R(X, nD)$ is not F–pure or F–regular by Theorem 5.3.1.

We can also construct explicit instances of Frobenius closure to illustrate why $R^{(n)}$ is not F–pure when n is relatively prime to 30. Since n is relatively prime to the weight of y, the ring $R^{(n)}$ has a unique monomial of the form xy^l with $0 < l < n$. Similarly there is a unique integer m with $0 \leq m < n$ such that $y^{l+1}z^m \in R^{(n)}$, and a unique integer r with $0 < r < n$ such that $x^r z \in R^{(n)}$. We claim that

$$x^{r+1}y^{rl+l}z^d \in (x^r y^{rl+l+1}z^m, x^r z^{d+1})^F, \quad \text{and}$$
$$x^{r+1}y^{rl+l}z^d \notin (x^r y^{rl+l+1}z^m, x^r z^{d+1}).$$

The second statement is true in R and so also in $R^{(n)}$, while the first follows from

$$(x^{r+1}y^{rl+l}z^d)^p \in ((x^r y^{rl+l+1}z^m)^p, (x^r z^{d+1})^p) \quad \text{for } p = 2,\ 3 \text{ or } 5.$$

This completes our study of the Veronese subrings of $R = K[X, Y, Z]/(X^2+Y^3+Z^5)$.

Example 5.3.6. We saw that the F–purity and F–regularity of a ring $R = R(X, D)$ depend only on the fractional part D' of the $\mathbb{Q}$–divisor D. This is by no means true of F–rationality and F–injectivity (i.e., the injectivity of the Frobenius action on the highest local cohomology module). As an example of this, consider the $\mathbb{Q}$–divisor $E = (1/2)V(S) + (1/3)V(T) + (1/5)V(S+T)$ on $\operatorname{Proj} K[S, T]$ which has the same fractional part as $D = (-1/2)V(S) + (1/3)V(T) + (1/5)V(S+T)$. Then

$$S = \oplus_{n \geq 0} H^0(X, \mathcal{O}_X(nE))T^n \cong K[A, B, C, T]/I$$

where $I = (AB - T^5,\ BC + CT^3 - BT^5,\ AC + CT^2 - ABT^2)$ and $A = T^3/S$, $B = ST^2$ and $C = ST^5/(S+T)$. If the characteristic of K is 2, 3 or 5, the ring $R = R(X, D) = K[X, Y, Z]/(X^2 + Y^3 + Z^5)$ is not F–rational (or F–injective) as we saw in Example 5.1.2. We claim that the ring S is however F–rational. To see this note that $a(S) < 0$, and so it suffices by Theorem 2.3.4 to verify that the ideal I generated by the homogeneous system of parameters t, $a^{15} + b^{10} + c^6$ is Frobenius closed. However this is easily verified: the ring $S/tS \cong K[A, B, C]/(AB, BC, CA)$ is F–pure since the ideal (AB, BC, CA) is generated by square free monomials, see [HR2, Proposition 5.38].

Remark 5.3.7. Let R be a Cohen–Macaulay ring with an isolated singularity, which is generated by degree one elements over an algebraically closed field. For a two dimensional ring R, a negative a–invariant forces R to be F–regular, although for rings of higher dimensions this is no longer true: in Example 5.3.3 we constructed rings R of dimension $d > 3$, with $a(R) = 2 - d$, which were not F–regular. Smith has pointed out that if R satisfies the stronger condition that $a(R) \leq 1-d$, then Proj R is a variety of minimal degree. These are completely classified (see, for example, [EH]) and it is easily verified that in this case R is F–regular, see [Sm5, Remark 4.3.1].

5.4 Examples in characteristic zero

All our positive results towards the existence of F–rational and F–regular Veronese subrings in prime characteristic do have corresponding statements in the characteristic zero situation. However we have so far not exhibited a normal Cohen–Macaulay ring, generated by degree one elements over a field of characteristic zero, which has an isolated singularity and a negative a–invariant but is not of F–regular type. N. Hara has pointed out to us a geometric argument for the existence of such rings using a blow–up of $\mathbb{P}^2$ at nine points. In this section, we construct a large family of explicit examples of such rings of dimension $d \geq 3$.

Example 5.4.1. Take two relatively prime homogeneous polynomials F and G of degree d in the ring $\mathbb{Z}[X_1, \dots, X_k]$, where $k \geq 3$, such that G is monic in X_k and the monomial X_k^d does not occur in F. Using F and G, construct the hypersurface $S = \mathbb{Q}[S, T, X_1, \dots, X_k]/(SF - TG)$ and let R be the subring of S generated by the elements $sx_1, \dots, sx_k, tx_1, \dots, tx_k$.

For suitably general choices of the polynomials F and G of degree $d = k$ the ring R has only isolated singularities, and we show that it is Cohen–Macaulay with

$a(R) = -1$, and is not of F–regular type. For an explicit example, take $k = 3$, $F = X_1X_2X_3$ and $G = X_1^3 + X_2^3 + X_3^3$.

We shall prove that R is Cohen–Macaulay whenever $d \leq k$. We first show that the Hilbert polynomial multiplicity of R is $d(k-1)+1$, and then construct a system of parameters such that the ring obtained by killing this system of parameters has length $d(k-1)+1$.

We construct a basis for the vector space generated by the monomials of degree $n \gg 0$, $s^i t^{n-i} x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k}$, where the j_r are nonnegative integers which add up to n. The relations permit us to express tx_k^d in terms of other monomials. Let $[u_1, \ldots, u_m]^i$ denote the set $\mathcal{S}$ of monomials of degree i in $u_1, \ldots, u_m$, and for two such sets, let $\mathcal{S} \cdot \mathcal{T}$ denote the product of all possible pairs from $\mathcal{S}$ and $\mathcal{T}$. In this notation, for $n \gg 0$, the following monomials constitute a basis for R_n:

$$[s,t]^n \cdot [x_1, \ldots, x_{k-1}]^n,$$

$$[s,t]^n \cdot [x_1, \ldots, x_{k-1}]^{n-1} \cdot [x_k],$$

$$\ldots$$

$$[s,t]^n \cdot [x_1, \ldots, x_{k-1}]^{n-d+1} \cdot [x_k]^{d-1},$$

$$[s]^n \cdot [x_1, \ldots, x_k]^{n-d} \cdot [x_k]^d.$$

Consequently for large n we have

$$\dim R_n = (n+1)\left\{\binom{n+k-2}{k-2} + \cdots + \binom{n-d+1+k-2}{k-2}\right\} + \binom{n-d+k-1}{k-1}.$$

As a polynomial in n, the leading term of this expression is

$$n\left\{\frac{n^{k-2}}{(k-2)!} + \cdots + \frac{n^{k-2}}{(k-2)!}\right\} + \frac{n^{k-1}}{(k-1)!} = \frac{n^{k-1}(d(k-1)+1)}{(k-1)!},$$

and so the Hilbert polynomial multiplicity of R is $d(k-1)+1$.

The sequence of elements $sx_1,\ sx_2 - tx_1,\ sx_3 - tx_2,\ \ldots,\ sx_k - tx_{k-1}$ is a system of parameters for R. Since we have already verified that the Hilbert polynomial multiplicity of R is $d(k-1)+1$, to prove that R is Cohen–Macaulay when $d \le k$, it suffices to show that the length of the ring T obtained by killing this system of parameters is at most $d(k-1)+1$.

Relabel the generators of T as $a_2 = sx_2,\ a_3 = sx_3,\ \ldots,\ a_k = sx_k,\ a_{k+1} = tx_k$. Note that the relations amongst the a_i include the size two minors of the matrix

$$\begin{pmatrix} 0 & a_2 & \ldots & a_{k-1} & a_k \\ a_2 & a_3 & \ldots & a_k & a_{k+1} \end{pmatrix}.$$

Consequently a generating set for $[T]_{<d}$ is given by

$$\begin{aligned}
\deg 0 \quad &: 1,\\
\deg 1 \quad &: a_2,\ a_3,\ \ldots,\ a_{k+1},\\
\deg 2 \quad &: a_2a_{k+1},\ a_3a_{k+1},\ \ldots,\ a_{k+1}^2,\\
\deg 3 \quad &: a_2a_{k+1}^2,\ a_3a_{k+1}^2,\ \ldots,\ a_{k+1}^3,\\
&\ldots\\
\deg d-1 &: a_2a_{k+1}^{d-2},\ a_3a_{k+1}^{d-2},\ \ldots,\ a_{k+1}^{d-1}.
\end{aligned}$$

In degree d the ring T has d additional independent relations coming from the equations $s^i t^{d-i} f - s^{i-1} t^{d-i+1} g$, for $1 \le i \le d$. Consequently we need $k-d$ generators for the degree d piece of T, and one can check that there are no nonzero elements in degree $d+1$. Hence the length of T is bounded by $d(k-1)+1$, and this completes the proof that R is Cohen–Macaulay.

It only remains to show that R is not of F–regular type when $k \le d$. Consider the fiber A of the map $\mathbb{Z} \to R_{\mathbb{Z}}$ over an arbitrary closed point $p\mathbb{Z}$. Then A is a finitely generated algebra over the finite field $\mathbb{Z}/p\mathbb{Z}$, and it suffices to show that A

is not F–regular. Take the ideal $I = (sx_1, sx_2, \ldots, sx_{k-1}, tx_1, tx_2, \ldots, tx_{k-1})A$. It is easily verified that $(tx_k)^{d-1} \notin I$, and we show that $(tx_k)^{d-1} \in I^*$.

To see $(tx_k)^{d-1} \in I^*$ it suffices to check that $\alpha_q = (tx_k)^{(d-1)(q+1)} \in I^{[q]}$. Using the relation $t^d g - t^{d-1} sf$ where $1 \leq i \leq d$, we may rewrite α_q with lower powers of x_k occurring in the expressions involved. We can proceed in this manner till we are left with terms which involve powers of x_k not greater then $d-1$. Hence α_q is a sum of terms which are multiples of

$$s^i t^{q(d-1)-i} x_1^{j_1} x_2^{j_2} \cdots x_{k-1}^{j_{k-1}}, \quad \text{where } i \leq q(d-1), \quad \text{and } \sum_{r=1}^{k-1} j_r = q(d-1).$$

If $\alpha_q \notin I^{[q]}$, then $j_r < q$ for $1 \leq r \leq k-1$. However on summing these inequalities we get $q(d-1) < q(k-1)$, a contradiction.

Remark 5.4.2. Consider the polynomial ring $K[X_1, \ldots, X_k]$ where $k \geq 3$. It is worth noting that the ring R, as above, is isomorphic to a subring of $K[X_1, \ldots, X_k]$,

$$R = K[X_1F,\ X_2F,\ \ldots,\ X_kF,\ X_1G,\ X_2G,\ \ldots,\ X_kG].$$

We can show that R is Cohen–Macaulay precisely when the degree d of F and G is less than or equal to k. It would certainly be interesting to explore generalizations of this construction.

CHAPTER VI

TIGHT CLOSURE IN NON–EQUIDIMENSIONAL RINGS

We begin by recalling a result of Hochster and Huneke, Theorem 2.2.3 (5), which states that a local ring (R, m) which is a homomorphic image of a Cohen–Macaulay ring is F–rational if and only if it is equidimensional and has a system of parameters which generates a tightly closed ideal. This leads to the question of whether a local ring in which a single system of parameters generates a tightly closed ideal must be equidimensional (and hence F–rational), #19 of Hochster's "Twenty Questions" in [Ho]. Rephrased, can a non–equidimensional ring have a system of parameters which generates a tightly closed ideal – we show it cannot for some classes of non–equidimensional rings.

A key point is that in equidimensional rings, tight closure has the so–called "colon capturing" property. This property does not hold in non–equidimensional rings. A study of these issues leads to a new closure operation, that we call *NE closure*. This closure does possess the colon capturing property even in non–equidimensional rings, and agrees with tight closure when the ring is equidimensional. We shall show that an excellent local ring R is F–rational if and only if it has a system of parameters which generates an NE–closed ideal.

6.1 Main results

Lemma 6.1.1. *Let $P_1, \dots, P_n$ be the minimal primes of the ring R. Then for a tightly closed ideal I, we have $I = \bigcap_{i=1}^{n}(I + P_i)$.*

Proof. One containment is trivial, and for the other note that if $x \in \bigcap_{i=1}^{n}(I + P_i)$, then $\overline{x} \in (IR/P_i)^*$ for $1 \leq i \leq n$. By Theorem 2.2.3 (7), we get that $x \in I^* = I$. $\square$

The following theorem, although it has some rather strong hypotheses, does show that no ideal generated by a system of parameters is tightly closed in the non–equidimensional rings

$$R = K[[X_1, \dots, X_n, Y_1, \dots, Y_m]]/(X_1, \dots, X_n) \cap (Y_1, \dots, Y_m)$$

where $m, n \geq 1$ and $m \neq n$.

Theorem 6.1.2. *Let (R, m) be a non–equidimensional local ring, with the minimal primes partitioned into the sets $\{P_i\}$ and $\{Q_j\}$, such that $\dim R = \dim R/P_i > \dim R/Q_j$, for all i and j. Let P and Q be the intersections, $P = \bigcap P_i$ and $Q = \bigcap Q_j$. If $I \subseteq P + Q$ is an ideal of R which is generated by a system of parameters, then I cannot be tightly closed. In particular, if $P + Q = m$, then no ideal of R generated by a system of parameters is tightly closed.*

Proof. Suppose not, let $I = (p_1 + q_1, \dots, p_n + q_n)$ be a tightly closed ideal of R, where $p_1 + q_1, \dots, p_n + q_n$ is a system of parameters with $p_i \in P$ and $q_i \in Q$. Note that

$$(I + P) \cap (I + Q) \subseteq (\cap(I + P_i)) \cap (\cap(I + Q_j)) = I$$

where the equality follows from Lemma 6.1.1. Consequently $(I + P) \cap (I + Q) \subseteq I$, and so $p_i, q_i \in I$. In particular, $p_i = r_1(p_1 + q_1) + \cdots + r_n(p_n + q_n)$. We first note that if $r_i \notin m$ then $q_i \in (p_1 + q_1, \dots, p_{i-1} + q_{i-1}, p_i, p_{i+1} + q_{i+1}, \dots, p_n + q_n)$, but then $p_i \in P$

is a parameter, which is impossible since $\dim R = \dim R/P$. Hence $r_i \in m$, and so $1-r_i$ is a unit. This shows that $p_i \in (p_1+q_1, \ldots, p_{i-1}+q_{i-1}, q_i, p_{i+1}+q_{i+1}, \ldots, p_n+q_n)$, and so $I = (p_1+q_1, \ldots, p_{i-1}+q_{i-1}, q_i, p_{i+1}+q_{i+1}, \ldots, p_n+q_n)$.

Proceeding this way, we see that $I = (q_1, \ldots, q_n)$, i.e., $I \subseteq Q$. Consequently each $Q_j = m$, a contradiction. $\square$

Remark 6.1.3. We would next like to discuss briefly the case where the non–equidimensional local ring (R, m) is of the form $R = S/(P \cap Q)$ where S is a regular local ring with primes P and Q of different height. Then R has minimal primes $\overline{P}$ and $\overline{Q}$ where, without loss of generality, $\dim R = \dim R/\overline{P} > \dim R/\overline{Q}$. If I is an ideal of S whose image $\overline{I}$ in R is a tightly closed ideal, we see that $I+(P\cap Q) = (I+P)\cap(I+Q)$ by Lemma 6.1.1, and so it would certainly be enough to show that this cannot hold when $\overline{I}$ is generated by a system of parameters for R. One can indeed prove this in the case S/P is Cohen–Macaulay, and S/Q is a discrete valuation ring, Theorem 6.1.6 below. However if we drop the hypothesis that S/P be Cohen–Macaulay, this is no longer true: see Example 6.1.7.

Lemma 6.1.4. *If I, P, and Q are ideals of S, with $I+(P\cap Q) = (I+P)\cap(I+Q)$, then $I \cap (P+Q) = (I\cap P) + (I \cap Q)$.*

Proof. Let $i = p+q \in I \cap (P+Q)$, where $p \in P$ and $q \in Q$. Then $i - p = q \in (I+P)\cap(I+Q) = I + (P \cap Q)$ and so $i - p = q = \tilde{i} + r$ where $\tilde{i} \in I$ and $r \in P \cap Q$. Finally note that $i = (i - \tilde{i}) + \tilde{i} \in (I \cap P) + (I \cap Q)$, since $i - \tilde{i} = p + r \in I \cap P$ and $\tilde{i} = q - r \in I \cap Q$. $\square$

Lemma 6.1.5. *Let M be an S–module and N, be a submodule of M. If $x_1, \ldots, x_n$ are elements of S which form a regular sequence on M/N, then*

$$(x_1, \ldots, x_n)M \cap N = (x_1, \ldots, x_n)N.$$

In particular, if I and J are ideals of S and I is generated by elements which form a regular sequence on S/J, then $I \cap J = IJ$.

Proof. We proceed by induction on n, the number of elements. If $n = 1$, the result is simple. For the inductive step, if $u = x_1 m_1 + \cdots + x_k m_k \in (x_1, \ldots, x_k)M \cap N$ with $m_i \in M$, then since x_k is not a zero divisor on $M/((x_1, \ldots, x_{k-1})M + N)$, we see that $m_k \in (x_1, \ldots, x_{k-1})M + N$. Consequently, $u \in (x_1, \ldots, x_{k-1})M + x_k N$. $\square$

Theorem 6.1.6. *Let $S = K[[X_1, \ldots, X_n, Y]]$ with ideal $Q = (X_1, \ldots, X_n)S$, and ideal P satisfying the condition that S/P is Cohen–Macaulay. Then if $R = S/(P \cap Q)$ is a non–equidimensional ring, no ideal of R generated by a system of parameters can be tightly closed.*

Proof. Let I be an ideal of S generated by elements which map to a system of parameters in R. If the image of I is a tightly closed ideal in R, by Lemma 6.1.1 we have $(I + P) \cap (I + Q) = I + (P \cap Q)$. Any element of the maximal ideal of S, up to multiplication by units, is either in Q, or is of the form $Y^h + q$, where $q \in Q$. Since I cannot be contained in Q, one of its generators has the form $Y^h + q$. Choosing the generator amongst these which has the least such positive value of h, and subtracting suitable multiples of this generator, we may assume that the other generators are in Q. We then have $I = (Y^h + q_1, q_2, \ldots, q_d)S$, where $q_i \in Q$, $h > 0$, and $d = \dim R = \dim S/P$. By a similar argument we may write P as $P = (Y^t + r_1, r_2, \ldots, r_k)S$, where $r_i \in Q$. Since we are assuming that the image of I is a tightly closed ideal in R, Theorem 6.1.2 shows that I is not contained in $P + Q$, and so we conclude $h < t$.

We then have $Y^t + Y^{t-h}q_1 \in I \cap (P + Q)$, and so by Lemma 6.1.4

$$Y^t + Y^{t-h}q_1 \in (I \cap P) + (I \cap Q).$$

By Lemma 6.1.5, $I \cap P = IP$ and consequently

$$Y^t \in IP + Q = (Y^t + r_1)(Y^h + q_1) + Q = Y^{t+h} + Q.$$

However this is impossible since $h > 0$. $\square$

Example 6.1.7. Let $S = K[[T, X, Y, Z]]$, and consider the primes $Q = (T, X, Y)$ and $P = (TY - XZ, T^2X - Z^2, TX^2 - YZ, X^3 - Y^2)$. Then S/Q is a discrete valuation ring, although S/P is not Cohen–Macaulay. To see this, observe that

$$S/P \cong K[[U^2, U^3, UT, T]] \subseteq K[[T, U]]$$

Under this isomorphism, $x \mapsto U^2$, $y \mapsto U^3$, $z \mapsto UT$ and $t \mapsto T$. (Lower case letters denote the images of the corresponding variables.)

$R = S/(P \cap Q)$ is a non–equidimensional ring and the image of $I = (Z, X - T)$ in R is $\overline{I} = (z, x - t)$ which is generated by a system of parameters for R. We shall see that $I + (P \cap Q) = (I + P) \cap (I + Q)$.

Since $I + Q = (T, X, Y, Z)$ is just the maximal ideal of S, we see that

$$(I + P) \cap (I + Q) = I + P = (Z, X - T, XY, X^3, Y^2).$$

It can be verified (using Macaulay, or even otherwise) that

$$P \cap Q = (TY - XZ, TX^2 - YZ, X^3 - Y^2, T^3X - TZ^2),$$

and so

$$I + (P \cap Q) = (Z, X - T, XY, X^3, Y^2) = (I + P) \cap (I + Q).$$

For the ring R, although it does not follow from any of the earlier results, we can show that no system of parameters generates a tightly closed ideal.

We can actually prove the graded analogue of Theorem 6.1.6 without the requirement that S/P is Cohen–Macaulay.

Theorem 6.1.8. *Let $S = K[X_1, \ldots, X_n, Y]$ with ideal $Q = (X_1, \ldots, X_n)S$, and P a homogeneous unmixed ideal with* $\dim S/P \geq 2$. *Then no homogeneous system of parameters of the ring $R = S/(P \cap Q)$ generates a tightly closed ideal.*

Proof. Let I be an ideal of S generated by homogeneous elements which map to a system of parameters in R, and assume that the image of I is a tightly closed ideal of R. As in the proof of Theorem 6.1.6, there is no loss of generality in taking as homogeneous generators for I, the elements $Y^h + q_1,\ q_2, \ldots, q_d$ where $q_i \in Q$, and $h > 0$, and for P the elements $Y^t + r_1,\ r_2, \ldots, r_k$, where $r_i \in Q$. Since we are assuming that the image of I is a tightly closed ideal in R, Theorem 6.1.2 (or rather, its graded analogue) shows that I is not contained in $P + Q$, and so we conclude $h < t$. The assumption implies that

$$Y^t + r_1 \in (I + P) \cap (I + Q) = I + (P \cap Q) = I + (r_2, \ldots, r_k) + (Y^t + r_1)Q$$

and so $Y^t + r_1 \in I + (r_2, \ldots, r_k)$. Hence

$$I + (P \cap Q) = I + P = I + (r_2, \ldots, r_k).$$

If S/P is Cohen–Macaulay, the proof is identical to that of Theorem 6.1.6, and so we may assume S/P is not Cohen–Macaulay. Consequently $(IS/P)^*$ is strictly bigger that IS/P. Let $F \in S$ be a homogeneous element such that its image is in $(IS/P)^*$ but not in IS/P. Note that if $F \in I + Q$, then $\overline{F} \in (IR)^*$, and so $F \in I + P$, a contradiction. Hence $F \notin I + Q = (Y^h, X_1, \ldots, X_n)$ and so $F = Y^i + G$ where $i < h$ and $G \in Q$.

Next note that $Y^{h-i}F = Y^h + GY^{h-i} \in I + (P \cap Q) = I + (r_2, \ldots, r_k)$, and so $GY^{h-i} - q_1 \in (q_2, \ldots, q_d, r_2, \ldots, r_k)$. We then have

$$\overline{F} \in (IS/P)^* = ((Y^h + GY^{h-i}, q_2, \ldots, q_d)S/P)^* = (Y^{h-i}F, q_2, \ldots, q_d)S/P)^*.$$

By a degree argument, we can conclude that $\overline{F} \in ((q_2, \ldots, q_d)S/P)^*$. However this means that $\overline{F}$ is in the radical of the ideal $(q_2, \ldots, q_d)S/P$, which contradicts the fact that $(FY^{h-i}, q_2, \ldots, q_d) = IS/P$ is primary to the homogeneous maximal ideal of the ring S/P.

$\square$

6.2 NE closure

For Noetherian rings of characteristic p we shall define a new closure operation on ideals, the *NE closure*, which will agree with tight closure when the ring is equidimensional. In non–equidimensional local rings, tight closure no longer has the so–called *colon–capturing* property, and the main point of NE closure is to recover this property. This often forces the NE closure of an ideal to be larger than its tight closure and at times even larger than its radical, see Example 6.3.5. More precisely let (R, m) be an excellent local ring with a system of parameters $x_1, \ldots, x_n$. Then $(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^*$ when R is equidimensional, but this does not hold in general. The NE closure (denoted by $I^\bigstar$ for an ideal $I \subseteq R$) will have the property that $(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^\bigstar$.

Definition 6.2.1. We shall say that a minimal prime ideal P of a ring R is *absolutely minimal* if $\dim R/P = \dim R$. When Spec R is connected, $R^\bullet$ shall denote the complement in R of the union of all the absolutely minimal primes. If $R = \prod R_i$, we define $R^\bullet = \prod R_i^\bullet$. The *NE closure* $I^\bigstar$ of an ideal I is given by

$$I^\bigstar = \{x \in R : \text{there exists } c \in R^\bullet \text{ with } cx^{[q]} \in I^{[q]} \text{ for all sufficiently large } q\}.$$

The following proposition and its proof are somewhat similar to the corresponding statements for tight closure in equidimensional rings, see [HH4, Theorem 4.3].

Proposition 6.2.2. *Let R be a complete local ring of characteristic p, with a system of parameters $x_1, \ldots, x_n$. Then*

(1) $(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^{\bigstar}$.

(2) $(x_1, \ldots, x_k)^{\bigstar} : x_{k+1} = (x_1, \ldots, x_k)^{\bigstar}$.

(3) *If* $(x_1, \ldots, x_{k+1})^{\bigstar} = (x_1, \ldots, x_{k+1})$, *then* $(x_1, \ldots, x_k)^{\bigstar} = (x_1, \ldots, x_k)$.

(4) *If* $(x_1, \ldots, x_n)^{\bigstar} = (x_1, \ldots, x_n)$ *or* $(x_1, \ldots, x_{n-1})^{\bigstar} = (x_1, \ldots, x_{n-1})$, *then R is Cohen–Macaulay.*

Proof. (1) We may represent R as a module-finite extension of a regular subring A of the form $A = K[[x_1, \ldots, x_n]]$ where K is a field. Let t be the torsion free rank of R as an A–module, and consider $A^t \subseteq R$. Then R/A^t is a torsion A–module and there exists $c \in A$, nonzero, such that $cR \subseteq A^t \subseteq R$. We note that c cannot be in any absolutely minimal prime P of R, since for any such P, R/P is of dimension n and is module-finite over $A/A \cap P$, and so $A \cap P = 0$. Now if $u \in (x_1, \ldots, x_k) : x_{k+1}$ then for some $r_i \in R$, $ux_{k+1} = \sum_{i=1}^k r_i x_i$. Taking qth powers, and multiplying by c we get $cu^q x_{k+1}^q = \sum_{i=1}^k c r_i^q x_i^q$. But now cu^q and each of cr_i^q are in A^t and x_i^q form a regular sequence on A^t. Hence $cu^q \in (x_1^q, \ldots, x_k^q)$ and so $u \in (x_1, \ldots, x_k)^{\bigstar}$.

(2) If $ux_{k+1} \in (x_1, \ldots, x_k)^{\bigstar}$ then for some $c_0 \in R^{\bullet}$, $c_0(ux_{k+1})^q \in (x_1^q, \ldots, x_k^q)$ for all sufficiently large q, i.e., $c_0 u^q x_{k+1}^q = \sum_{i=1}^k r_i x_i^q$ for $q \gg 0$. Multiplying this by our earlier choice of c, we again have a relation on x_i^q's with coefficients in A^t, namely $cc_0 u^q x_{k+1}^q = \sum_{i=1}^k c r_i x_i^q$ for $q \gg 0$, and so $cc_0 u^q \in (x_1^q, \ldots, x_k^q)$ for $q \gg 0$. Since $cc_0 \in R^{\bullet}$ we get $u \in (x_1, \ldots, x_k)^{\bigstar}$.

(3) Let $J = (x_1, \ldots, x_k)$. Then $J^{\bigstar} \subseteq (x_1, \ldots, x_{k+1})$ and so $J^{\bigstar} \subseteq J + x_{k+1}R$. If $u \in J^{\bigstar}$, $u = j + x_{k+1}r$ for $j \in J$ and $r \in R$. This means $r \in J^{\bigstar} : x_{k+1}$ which equals $J^{\bigstar}$ by (2). Hence we get $J^{\bigstar} = J + x_{k+1}J^{\bigstar}$. Now by Nakayama's lemma we

get $J^\bigstar = J$.

(4) This follows from (2) and (3) since, under either of the hypotheses, the system of parameters $x_1, \ldots, x_n$ is a regular sequence. □

The above proposition, coupled with results on F–rationality, gives us the following theorem:

Theorem 6.2.3. *Let R be a complete local ring of characteristic p, with a system of parameters which generates an NE–closed ideal. Then R is F–rational.*

Proof. From the previous proposition the ring is Cohen–Macaulay and in particular, equidimensional. For equidimensional rings, tight closure agrees with NE closure, and the result follows from Theorem 2.2.3 (5). □

We shall extend this result to excellent local rings once we develop the theory of test elements for NE closure. The following proposition lists some properties of NE closure.

Proposition 6.2.4. *Let R be a ring of characteristic p, and I an ideal of R.*

(1) *$0^\bigstar$ is the intersection of the absolutely minimal prime ideals of R.*

(2) *If $I = I^\bigstar$ then for any ideal J, $(I : J)^\bigstar = I : J$.*

(3) *If $R = \prod R_i$ and $I = \prod I_i$, then $I^\bigstar = \prod I_i^\bigstar$.*

(4) *If $h : R \to S$ is a ring homomorphism with $h(R^\bullet) \subseteq S^\bullet$, then $h(I^\bigstar) \subseteq (IS)^\bigstar$.*

(5) *$x \in I^\bigstar$ if and only if $\overline{x} \in (IR/P)^\bigstar$ for every absolutely minimal prime P of R.*

Proof. (1), (2), (3) and (4) follow easily from the definitions. For (5) note that if P is absolutely minimal, $h : R \to R/P$ meets the condition of (4), so $x \in I^\bigstar$ implies that its image is in the NE closure of IR/P. For the converse, fix for every absolutely minimal P_i, $d_i \notin P_i$ but in every other minimal prime of R. If $\overline{x} \in (IR/P_i)^{\bigstar}$ for every absolutely minimal P_i, then there exist elements $\overline{c}_i$ with $\overline{c_i x^q} \in (IR/P_i)^{[q]}$. We can lift each $\overline{c}_i$ to $c_i \in R$ with $c_i \notin P_i$. Then $c_i x^q \in I^{[q]} + P_i$ for all i, for sufficiently large q. Multiplying each of these equations with the corresponding d_i, we get $c_i d_i x^q \in I^{[q]} + \mathfrak{N}$, since $d_i P_i$ is a subset of every minimal prime ideal and so is in the nilradical, $\mathfrak{N}$. If $\mathfrak{N}^{[q']} = 0$, taking q' powers of these equations gives us $(c_i d_i)^{q'} x^q \in I^{[q]}$ for all i, for sufficiently large q. Set $c = \sum (c_i d_i)^{q'}$. By our choice of c_i's and d_i's, $c \in R^{\bullet}$, and the above equations put together give us $cx^q \in I^{[q]}$ for all sufficiently large q. $\square$

We note that NE closure need not be preserved once we localize, i.e., it is quite possible that $x \in I^{\bigstar}$, but $x \notin (IR_P)^{\bigstar}$. Examples of this abound in non–equidimensional rings, but there are some positive results about NE closure being preserved under certain maps which we examine in the next few propositions.

Proposition 6.2.5. *If $h : (R, m) \to (S, n)$ is a faithfully flat homomorphism of local rings then for an ideal I of R, if $x \in I^{\bigstar}$, then its image $h(x)$ is in $(IS)^{\bigstar}$. In particular if $\hat{R}$ denotes the completion of R at its maximal ideal, $x \in I^{\bigstar}$ implies $x \in (I\hat{R})^{\bigstar}$.*

Proof. By Proposition 6.2.4 (4), it suffices to check that $h(R^{\bullet}) \subseteq S^{\bullet}$. This is equivalent to the assertion that the contraction of every absolutely minimal prime of S is an absolutely minimal prime of R. Now let P be an absolutely minimal prime of S, and p denote its contraction to R. Then since $R \to S$ is faithfully flat, by a change of base, so is $R/p \to S/pS$. This gives dim $S/pS =$ dim $R/p +$ dim S/mS. Also, faithful flatness of h implies that dim $S =$ dim $R +$ dim S/mS. But P was an absolutely minimal prime of S, so dim $S =$ dim $S/P =$ dim S/pS, since $pS \subseteq P$. Putting these equations together, we get dim $R/p =$ dim R, and so p is an absolutely minimal prime of R. $\square$

Proposition 6.2.6. *Let R and S be Noetherian rings of characteristic p, and $R \to S$ a homomorphism such that for every absolutely minimal prime Q of S, its contraction to R, Q^c, contains an absolutely minimal prime of R. Assume one of the following holds:*

(1) *R is finitely generated over an excellent local ring, or is F–finite, or*

(2) *R is locally excellent and S has a locally stable test element, (or S is local), or*

(3) *S has a completely stable test element (or S is a complete local ring).*

Then if $x \in I^\bigstar$ for I an ideal of R, the image of x in S is in $(IS)^\bigstar$.

Proof. It suffices to check $x \in (IS/Q)^\bigstar$ for every absolutely minimal primes Q of S, by Proposition 6.2.4 (5). But $(IS/Q)^\bigstar = (IS/Q)^*$ since S/Q is equidimensional. If $P \subseteq Q^c$ is an absolutely minimal prime of R, then $x \in I^\bigstar$ implies $\overline{x} \in (IR/P)^\bigstar = (IR/P)^*$. The result now follows by applying [HH4, Theorem 6.24] to the map $R/P \to S/Q$. $\square$

6.3 NE–test elements

Definition 6.3.1. We shall say $c \in R^\bullet$ is a *q'–weak NE–test element for R* if for all ideals I of R and $x \in I^\bigstar$, $cx^q \in I^{[q]}$ for all $q \geq q'$. We may often use the phrase *weak NE–test element* and suppress the q'.

For a local ring (R, m), $c \in R^\bullet$ is a *weak completion stable NE–test element for R* if it is a weak NE–test element for $\hat{R}$, the completion of R at its maximal ideal.

Our definition of a completion stable weak NE–test element is different from the notion of a completely stable weak test element for tight closure, where it is required that the element serve as a weak test element in the completion of every local ring of R. The reason for this, of course, is that localization is no longer freely available to us, since $R^\bullet$ often does not map into $(R_P)^\bullet$.

Note also that since $\hat{R}$ is faithfully flat over R, a weak completion stable NE–test element for R is also a weak NE–test element for R.

Proposition 6.3.2. *If for every absolutely minimal prime P of R, R/P has a weak test element, then R has a weak NE–test element.*

Proof. Fix for every absolutely minimal prime P_i an element d_i not in P_i but in every other minimal prime of R. Let $\mathfrak{N}$ denote the nilradical of R and fix q' such that $\mathfrak{N}^{[q']} = 0$. If $\overline{c}_i$ is a weak test element for R/P_i, we may pick $c_i \notin P_i$ which maps to it under $R \to R/P$. We claim $c = \sum (c_i d_i)^{q'}$ is a weak NE–test element for R. If $x \in I^\star$, then $\overline{x} \in (IR/P_i)^\star$ for all P_i absolutely minimal. Since $\overline{c}_i$ is a weak test element for R/P_i, we have $\overline{c_i x^q} \in (IR/P_i)^{[q]}$ for all i, for sufficiently large q, i.e., $c_i x^q \in I^{[q]} + P_i$. Multiplying this by d_i, summing over all i and taking the q' power as in the proof of Proposition 6.2.4 (5), we get that $cx^q \in I^{[q]}$. It is easy to see that $c \in R^\bullet$ and so is a weak NE–test element. $\square$

Proposition 6.3.3. *Every excellent local ring of characteristic p has a weak completion stable NE–test element.*

Proof. If R is an excellent local domain, it has a completely stable weak test element, see [HH4, Theorem 6.1]. Hence each R/P_i for P_i absolutely minimal, has a completely stable weak test element, say $\overline{c}_i$. Let c_i, d_i, q' and c be as in the proof of the previous proposition. If $x \in (I\hat{R})^\star$, we have $\overline{x} \in (I\hat{R}/P_i\hat{R})^\star$. Since R/P_i is equidimensional and excellent, its completion $\hat{R}/P_i\hat{R}$ is also equidimensional. (We use here the fact that the completion of a universally catenary equidimensional local ring is again equidimensional, [HIO, Page 142]). Hence NE closure agrees with tight closure in $\hat{R}/P_i\hat{R}$, and we get $\overline{x} \in (I\hat{R}/P_i\hat{R})^*$. This gives $\overline{c_i x^q} \in (I\hat{R}/P_i\hat{R})^{[q]}$ for all i, for sufficiently large q. As in the previous proof, we then get that $cx^q \in (I\hat{R})^{[q]}$, and so is a weak completion stable NE–test element. $\square$

We can now extend Theorem 6.2.3 to the case where R is excellent local, without requiring it to be complete.

Theorem 6.3.4. *Let (R, m) be an excellent local ring of characteristic p with a system of parameters $x_1, \ldots, x_n$. Then*

(1) $(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^\bigstar$.

(2) $(x_1, \ldots, x_k)^\bigstar : x_{k+1} = (x_1, \ldots, x_k)^\bigstar$.

(3) *If $(x_1, \ldots, x_{k+1})^\bigstar = (x_1, \ldots, x_{k+1})$, then $(x_1, \ldots, x_k)^\bigstar = (x_1, \ldots, x_k)$.*

(4) *If $(x_1, \ldots, x_{n-1})^\bigstar = (x_1, \ldots, x_{n-1})$ then R is Cohen–Macaulay .*

(5) *If $(x_1, \ldots, x_n)^\bigstar = (x_1, \ldots, x_n)$ then R is F–rational.*

Proof. Since R has a weak completion stable NE–test element, if there is a counterexample to any of the above claims, we can preserve this while mapping to $\hat{R}$. But all of the above are true for complete local rings as follows from Proposition 6.2.2 and Theorem 6.2.3. $\square$

Example 6.3.5. Let $R = K[[X, Y, Z]]/(X) \cap (Y, Z)$. Then $y, x - z$ is a system of parameters for R and $0 :_R (y) = (x)$. That tight closure fails here to "capture colons" is seen from the fact that $x \notin 0^* = 0$. However $0^\bigstar = (x)$, and we certainly have $0 :_R (y) \subseteq 0^\bigstar$.

CHAPTER VII

COMPUTATIONS IN DIAGONAL HYPERSURFACES

We settle a question about the tight closure of the ideal (x^2, y^2, z^2) in the ring $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$ where K is a field of prime characteristic $p \neq 3$. M. McDermott has studied the tight closure of various irreducible ideals in R, and has established that $xyz \in (x^2, y^2, z^2)^*$ when $p < 200$, see [Mc]. The general case however existed as a classic example of the difficulty involved in tight closure computations, see also [Hu2, Example 1.2]. We show that $xyz \in (x^2, y^2, z^2)^*$ in arbitrary prime characteristic p, and furthermore establish that $xyz \in (x^2, y^2, z^2)^F$ whenever R is not F–pure, i.e., when $p \equiv 2 \pmod 3$. We move on to generalize these results to the diagonal hypersurfaces $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$.

These issues relate to the question whether the tight closure I^* of an ideal I agrees with its plus closure, $I^+ = IR^+ \cap R$, where R is a domain over a field of characteristic p and R^+ is the integral closure of R in an algebraic closure of its fraction field. In this setting, we may think of the Frobenius closure of I as $I^F = IR^\infty \cap R$ where R^∞ is the extension of R obtained by adjoining p^eth roots of all nonzero elements of R for $e \in \mathbb{N}$. It is not difficult to see that $I^+ \subseteq I^*$, and equality in general is a formidable open question. It should be mentioned that in the case when I is an ideal generated by part of a system of parameters, the equality is a result of K. Smith, see [Sm2]. In the above ring $R = K[X,Y,Z]/(X^3+Y^3+Z^3)$ where K is a field of characteristic $p \equiv 2 \pmod 3$, if one could show that $I^* = I^F$ for an ideal I, a consequence of this would be $I^F \subseteq I^+ \subseteq I^* = I^F$, by which $I^+ = I^*$. McDermott does show that $I^* = I^F$ for large families of irreducible ideals and our result $xyz \in (x^2,y^2,z^2)^F$, we believe, fills in an interesting remaining case.

7.1 Preliminary calculations

We record some determinant computations we shall find useful. Note that for integers n and m where $m \geq 1$, we shall use the notation:

$$\binom{n}{m} = \frac{(n)(n-1)\cdots(n-m+1)}{(m)(m-1)\cdots(1)}.$$

Lemma 7.1.1.

$$\det\begin{vmatrix} \binom{n}{a+k} & \binom{n}{a+k+1} & \cdots & \binom{n}{a+2k} \\ \binom{n}{a+k-1} & \binom{n}{a+k} & \cdots & \binom{n}{a+2k-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \binom{n}{a} & \binom{n}{a+1} & \cdots & \binom{n}{a+k} \end{vmatrix} = \frac{\binom{n}{a+k}\binom{n+1}{a+k}\cdots\binom{n+k}{a+k}}{\binom{a+k}{a+k}\binom{a+k+1}{a+k}\cdots\binom{a+2k}{a+k}}.$$

Proof. This is evaluated in [Mui, page 682] as well as [Ro]. □

Lemma 7.1.2. *Let $F(n,a,k)$ denote the determinant of the matrix*

$$M(n,a,k) = \begin{pmatrix} \binom{n}{a} & \binom{n}{a+1} & \cdots & \binom{n}{a+k} \\ \binom{n+2}{a+1} & \binom{n+2}{a+2} & \cdots & \binom{n+2}{a+k+1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \binom{n+2k}{a+k} & \binom{n+2k}{a+k+1} & \cdots & \binom{n+2k}{a+2k} \end{pmatrix}.$$

Then for $k \geq 1$ we have

$$\frac{F(n,a,k)}{F(n+2,a+2,k-1)} = \binom{n}{a} \prod_{s=1}^{k} \prod_{r=1}^{k} \frac{s(s+2a-n)}{(a+r)(n-a+r)}.$$

Hence

$$F(n,a,k) = \frac{\binom{n}{a}\binom{n+2}{a+2}\cdots\binom{n+2k}{a+2k}}{\binom{a+k}{k}\binom{a+k+1}{k-1}\cdots\binom{a+2k-1}{1}} \cdot \frac{\binom{2a-n+k}{k}\binom{2a-n+k+1}{k-1}\cdots\binom{2a-n+2k-1}{1}}{\binom{n-a+k}{k}\binom{n-a+k-1}{k-1}\cdots\binom{n-a+1}{1}}.$$

Proof. We shall perform row operations on $M(n,a,k)$ in order to get zero entries in the first column from the second row onwards, starting with the last row and moving up. More precisely, from the $(r+1)$ th row, subtract the r th row multiplied by $\binom{n+2r}{a+r}/\binom{n+2r-2}{a+r-1}$ starting with $r = k$, and continuing until $r = 2$. The $(r+1, s+1)$ th entry of the new matrix, for $r \geq 1$, is

$$\binom{n+2r}{a+r+s} - \frac{\binom{n+2r}{a+r}}{\binom{n+2r-2}{a+r-1}} \binom{n+2r-2}{a+r-1+s} = \frac{s(s+2a-n)}{(a+r)(n-a+r)} \binom{n+2r}{a+r+s}.$$

We have only one nonzero entry in the first column, namely $\binom{n}{a}$ and so we examine the matrix obtained by deleting the first row and column. Factoring out $s(s+2a-n)$ from each column for $s = 1, \ldots, k$ and $1/(a+r)(n-a+r)$ from each row for $r = 1, \ldots, k$, we see that

$$\det\, M(n,a,k) = \binom{n}{a} \prod_{s=1}^{k} \prod_{r=1}^{k} \frac{s(s+2a-n)}{(a+r)(n-a+r)} \det\, M(n+2,a+2,k-1).$$

The required result immediately follows. $\square$

Lemma 7.1.3. *Consider the polynomial ring $T = K[A_1, \ldots, A_m]$ where $I_{r,i}$ denotes the ideal $I_{r,i} = (A_1^i, \ldots, A_r^i)T$ for $r \leq m$. Then*

$$(A_1 \cdots A_{r-1})^{\alpha}(A_1 + \cdots + A_{r-1})^{\beta} \;\in\; I_{r-1,\alpha+\gamma} + (A_1 + \cdots + A_{r-1})^{\alpha+\gamma} T$$

for positive integers α, β, and γ implies

$$(A_1 \cdots A_r)^{\alpha}(A_1 + \cdots + A_r)^{\beta+\gamma-1} \ \in \ I_{r,\alpha+\gamma} + (A_1 + \cdots + A_r)^{\alpha+\gamma} T.$$

Proof. Consider the binomial expansion of $(A_1 \cdots A_r)^{\alpha}(A_1+\cdots+A_r)^{\beta+\gamma-1}$ into terms of the form $(A_1 \cdots A_{r-1})^{\alpha}(A_1 + \cdots + A_{r-1})^{\beta+\gamma-1-j} A_r^{\alpha+j}$. Such an element is clearly in $I_{r,\alpha+\gamma}$ whenever $j \geq \gamma$, and so assume $\gamma > j$. Now

$$\begin{aligned}
&(A_1 \cdots A_{r-1})^{\alpha} A_r^{\alpha+j} (A_1 + \cdots + A_{r-1})^{\beta+\gamma-1-j} \\
&\qquad\qquad \in I_{r,\alpha+\gamma} + A_r^{\alpha+j}(A_1 + \cdots + A_{r-1})^{\alpha+2\gamma-1-j} T \\
&\qquad\qquad \subseteq I_{r,\alpha+\gamma} + (A_1 + \cdots + A_{r-1}, A_r)^{2\alpha+2\gamma-1} T \\
&\qquad\qquad \subseteq I_{r,\alpha+\gamma} + (A_1 + \cdots + A_r)^{\alpha+\gamma} T.
\end{aligned}$$

□

7.2 A computation of tight closure

We now prove the main theorem.

Theorem 7.2.1. *Let $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$ where $n \geq 3$ and K is a field of prime characteristic p where $p \nmid n$. Then*

$$(x_1 \cdots x_n)^{n-2} \ \in \ (x_1^{n-1}, \ldots, x_n^{n-1})^*.$$

Note that there are infinitely many $e \in \mathbb{N}$ such that $p^e = q \equiv 1 \pmod{n}$. By [HH2, Lemma 8.16], it suffices to work with powers of p of this form, and show that for all such q we have

$$(x_1 \cdots x_n)^{(n-2)q+1} \ \in \ (x_1^{(n-1)q}, \ldots, x_n^{(n-1)q}).$$

Letting $q = nk+1$, it suffices to show

$$(x_1 \cdots x_n)^{(n-2)nk} \in (x_1^{(n-1)nk}, \ldots, x_n^{(n-1)nk}).$$

Let $A_1 = x_1^n, \ldots, A_n = x_n^n$ and note that $A_1 + \cdots + A_n = 0$. In this notation, we aim to show

$$(A_1 \cdots A_n)^{(n-2)k} \in (A_1^{(n-1)k}, \ldots, A_n^{(n-1)k}).$$

Working in the polynomial ring $K[A_1, \ldots, A_{n-1}] \cong K[A_1, \ldots, A_n]/(A_1 + \cdots + A_n)$, we need to show

$$(A_1 \cdots A_{n-1}(A_1 + \cdots + A_{n-1}))^{(n-2)k} \in I_{n-1,(n-1)k} + (A_1 + \cdots + A_{n-1})^{(n-1)k}.$$

By repeated use of Lemma 7.1.3, it suffices to show

$$(A_1A_2)^{(n-2)k}(A_1 + A_2)^k \in (A_1^{(n-1)k},\ A_2^{(n-1)k},\ (A_1 + A_2)^{(n-1)k}).$$

We have now reduced our problem to a statement about a polynomial ring in two variables. The required result follows from the next lemma.

Lemma 7.2.2. *Let $K[A, B]$ be a polynomial ring over a field K of characteristic $p > 0$ and e be a positive integer such that $q = p^e \equiv 1 \pmod{n}$. If $q = nk + 1$, we have*

$$(A, B)^{(2n-3)k} \subseteq I = (A^{(n-1)k},\ B^{(n-1)k},\ (A + B)^{(n-1)k}).$$

In particular, $(AB)^{(n-2)k}(A + B)^k \in I$.

Proof. Note that I contains the following elements: $(A + B)^{(n-1)k}A^kB^{(n-3)k}$, $(A+B)^{(n-1)k}A^{k-1}B^{(n-3)k+1}, \ldots, (A+B)^{(n-1)k}B^{(n-2)k}$. We take the binomial expansions of these elements and consider them modulo the ideal $(A^{(n-1)k}, B^{(n-1)k})$. This shows that the following elements are in I :

$$\begin{array}{l}
\binom{(n-1)k}{k}A^{(n-1)k}B^{(n-2)k} \quad + \quad \cdots \quad + \quad \binom{(n-1)k}{2k}A^{(n-2)k}B^{(n-1)k}, \\
\binom{(n-1)k}{k-1}A^{(n-1)k}B^{(n-2)k} \quad + \quad \cdots \quad + \quad \binom{(n-1)k}{2k-1}A^{(n-2)k}B^{(n-1)k}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\binom{(n-1)k}{0}A^{(n-1)k}B^{(n-2)k} \quad + \quad \cdots \quad + \quad \binom{(n-1)k}{k}A^{(n-2)k}B^{(n-1)k}.
\end{array}$$

The coefficients of $A^{(n-1)k}B^{(n-2)k}, A^{(n-1)k-1}B^{(n-2)k+1}, \ldots, A^{(n-2)k}B^{(n-1)k}$ form the matrix:

$$\begin{pmatrix}
\binom{(n-1)k}{k} & \binom{(n-1)k}{k+1} & \cdots & \binom{(n-1)k}{2k} \\
\binom{(n-1)k}{k-1} & \binom{(n-1)k}{k} & \cdots & \binom{(n-1)k}{2k-1} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\binom{(n-1)k}{0} & \binom{(n-1)k}{1} & \cdots & \binom{(n-1)k}{k}.
\end{pmatrix}.$$

To show that all monomials of degree $(2n-3)k$ in A and B are in I, it suffices to show that this matrix is invertible. Since $q = nk+1$ we have $\binom{(n-1)k+r}{k} = (-1)^k\binom{2k-r}{k}$ for $0 \le r \le k$, and so by Lemma 7.1.1, the determinant of this matrix is

$$\frac{\binom{(n-1)k}{k}\binom{(n-1)k+1}{k}\cdots\binom{nk}{k}}{\binom{k}{k}\binom{k+1}{k}\cdots\binom{2k}{k}} = (-1)^{k(k+1)}\frac{\binom{2k}{k}\binom{2k-1}{k}\cdots\binom{k}{k}}{\binom{k}{k}\binom{k+1}{k}\cdots\binom{2k}{k}} = 1.$$

$\square$

With this we complete the proof that $(x_1\cdots x_n)^{n-2} \in (x_1^{n-1}, \ldots, x_n^{n-1})^*$.

7.3 A computation of Frobenius closure

Let $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$ as before, where the characteristic of K is $p \nmid n$.

Lemma 7.3.1. *Let $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$ where K is a field of characteristic p. Then R is F–pure if and only if $p \equiv 1 \pmod{n}$.*

Proof. This is Proposition 5.21 (c) of [HR2]. □

The main result of this section is the following theorem.

Theorem 7.3.2. *Let* $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$ *where* K *is a field of characteristic* p. *Then*

$$(x_1 \cdots x_n)^{n-2} \in (x_1^{n-1}, \ldots, x_n^{n-1})^F.$$

if and only if $p \not\equiv 1 \pmod{n}$.

One implication follows from Lemma 7.3.1, and so we need to consider the case $p \not\equiv 1 \pmod{n}$.

The case $n = 3$ seems to be the most difficult, and we handle that first. Let $R = K[X,Y,Z]/(X^3 + Y^3 + Z^3)$ where $p \equiv 2 \pmod{3}$. We need to show that $xyz \in (x^2, y^2, z^2)^F$.

Let $A = y^3, B = z^3$ and so $A + B = -x^3$. We first show that when $p = 2$, we have $xyz \in (x^2, y^2, z^2)^F$ by establishing that $(xyz)^8 \in (x^2, y^2, z^2)^{[8]}$. Note that it suffices to show that $(xyz)^6 \in (x^{15}, y^{15}, z^{15})$, or in other words that $(AB(A+B))^2 \in (A^5, B^5, (A+B)^5)$, but this is easily seen to be true.

We may now assume $p = 6m + 5$ where $m \geq 0$. We shall show that in this case $(xyz)^p \in (x^2, y^2, z^2)^{[p]}$, i.e., that

$$(xyz)^{6m+5} \in (x^{12m+10},\ y^{12m+10},\ z^{12m+10}).$$

Note that to establish this, it suffices to show

$$(xyz)^{6m+3} \in (x^{12m+9},\ y^{12m+9},\ z^{12m+9}),$$

i.e., that $(AB(A+B))^{2m+1} \in (A^{4m+3},\ B^{4m+3},\ (A+B)^{4m+3})$.

Lemma 7.3.3. *Let $K[A,B]$ be a polynomial ring over a field K of characteristic $p = 6m+5$ where $m \geq 0$. Then we have*

$$(AB(A+B))^{2m+1} \in I = (A^{4m+3},\ B^{4m+3},\ (A+B)^{4m+3}).$$

Proof. To show that $(AB(A+B))^{2m+1} \in I$, we shall show that the following terms grouped together symmetrically from its binomial expansion,

$$f_1 = (AB)^{3m+1}(A+B),\ f_3 = (AB)^{3m}(A^3+B^3),\ \ldots,$$
$$f_{2m+1} = (AB)^{2m+1}(A^{2m+1}+B^{2m+1}),$$

are all in the ideal I. Note that I contains the elements $(AB)^m(A+B)^{4m+3}$, $(AB)^{m-1}(A+B)^{4m+5}, \ldots, (AB)(A+B)^{6m+1}, (A+B)^{6m+3}$. We consider the binomial expansions of these elements modulo (A^{4m+3}, B^{4m+3}), and get the following elements in I :

$$\begin{array}{l}
\binom{4m+3}{2m+2} f_1 \quad + \quad \binom{4m+3}{2m+3} f_3 \quad + \quad \cdots \quad + \quad \binom{4m+3}{3m+2} f_{2m+1}, \\
\binom{4m+5}{2m+3} f_1 \quad + \quad \binom{4m+5}{2m+4} f_3 \quad + \quad \cdots \quad + \quad \binom{4m+5}{3m+3} f_{2m+1}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\binom{6m+3}{3m+2} f_1 \quad + \quad \binom{6m+3}{3m+3} f_3 \quad + \quad \cdots \quad + \quad \binom{6m+3}{4m+2} f_{2m+1}.
\end{array}$$

The coefficients of $f_1, f_3, \ldots, f_{2m+1}$ arising from these terms form the matrix:

$$\begin{pmatrix}
\binom{4m+3}{2m+2} & \binom{4m+3}{2m+3} & \cdots & \binom{4m+3}{3m+2} \\
\binom{4m+5}{2m+3} & \binom{4m+5}{2m+4} & \cdots & \binom{4m+5}{3m+3} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\binom{6m+3}{3m+2} & \binom{6m+3}{3m+3} & \cdots & \binom{6m+3}{4m+2}
\end{pmatrix}.$$

We need to show that this matrix is invertible, but in the notation of Lemma 7.1.2, its determinant is $F(4m+3, 2m+2, m)$ and is easily seen to be nonzero. $\square$

The above lemma completes the case $n = 3$. We may now assume $n \geq 4$ and $p = nk + \delta$ for $2 \leq \delta \leq n-1$. If $k = 0$ i.e., $2 \leq p \leq n-1$, we have

$$\begin{aligned}(x_1 \cdots x_n)^{(n-2)p} &= -(x_1 \cdots x_{n-1})^{(n-2)p} x_n^{(n-2)p-n}(x_1^n + \cdots + x_{n-1}^n)\\ &\in (x_1^{(n-1)p}, \ldots, x_{n-1}^{(n-1)p}).\end{aligned}$$

In the remaining case, we have $n \geq 4$ and $k \geq 1$. To prove that $(x_1 \cdots x_n)^{n-2} \in (x_1^{n-1}, \ldots, x_n^{n-1})^F$, we shall show

$$(x_1 \cdots x_n)^{(n-2)p} \in (x_1^{(n-1)p}, \ldots, x_n^{(n-1)p}).$$

This would follow if we could show

$$(x_1 \cdots x_n)^{(n-2)nk} \in (x_1^{(n-1)nk+n}, \ldots, x_n^{(n-1)nk+n}).$$

As before, let $A_1 = x_1^n, \ldots, A_n = x_n^n$. It suffices to show that

$$(A_1 \cdots A_n)^{(n-2)k} \in (A_1^{(n-1)k+1}, \ldots, A_n^{(n-1)k+1}).$$

By Lemma 7.1.3, this reduces to showing

$$(A_1A_2)^{(n-2)k}(A_1 + A_2)^k \in I = (A_1^{(n-1)k+1},\ A_2^{(n-1)k+1},\ (A_1 + A_2)^{(n-1)k+1}).$$

The only remaining ingredient is the following lemma.

Lemma 7.3.4. *Let $K[A, B]$ be a polynomial ring over a field K of characteristic $p > 0$ where $p = nk + \delta$ where $n \geq 4$, $k \geq 1$ and $2 \leq \delta \leq n-1$. Then*

$$(A, B)^{(2n-3)k} \subseteq I = (A^{(n-1)k+1},\ B^{(n-1)k+1},\ (A + B)^{(n-1)k+1}).$$

In particular, $(AB)^{(n-2)k}(A + B)^k \in I$.

Proof. Note that I contains the elements: $(A+B)^{(n-1)k+1}A^kB^{(n-3)k-1}$, $(A+B)^{(n-1)k+1}A^{k-1}B^{(n-3)k}, \ldots,\ (A+B)^{(n-1)k+1}B^{(n-2)k-1}$. We take the binomial expansions of these elements and consider them modulo the ideal $(A^{(n-1)k+1}, B^{(n-1)k+1})$. This shows that the following elements are in I :

$$\begin{array}{l}
\binom{(n-1)k+1}{k+1}A^{(n-1)k}B^{(n-2)k} \quad + \quad \cdots \quad + \quad \binom{(n-1)k+1}{2k+1}A^{(n-2)k}B^{(n-1)k}, \\
\binom{(n-1)k+1}{k}A^{(n-1)k}B^{(n-2)k} \quad + \quad \cdots \quad + \quad \binom{(n-1)k+1}{2k}A^{(n-2)k}B^{(n-1)k}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\binom{(n-1)k+1}{1}A^{(n-1)k}B^{(n-2)k} \quad + \quad \cdots \quad + \quad \binom{(n-1)k+1}{k+1}A^{(n-2)k}B^{(n-1)k}.
\end{array}$$

The coefficients of $A^{(n-1)k}B^{(n-2)k}, A^{(n-1)k-1}B^{(n-2)k+1}, \ldots, A^{(n-2)k}B^{(n-1)k}$ form the matrix:

$$\begin{pmatrix}
\binom{(n-1)k+1}{k+1} & \binom{(n-1)k+1}{k+2} & \cdots & \binom{(n-1)k+1}{2k+1} \\
\binom{(n-1)k+1}{k} & \binom{(n-1)k+1}{k+1} & \cdots & \binom{(n-1)k+1}{2k} \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\binom{(n-1)k+1}{1} & \binom{(n-1)k+1}{2} & \cdots & \binom{(n-1)k+1}{k+1}
\end{pmatrix}.$$

To show that all monomials of degree $(2n-3)k$ in A and B are in I, it suffices to show that this matrix is invertible. The determinant of this matrix is

$$\frac{\binom{(n-1)k+1}{k+1}\binom{(n-1)k+2}{k+1}\cdots\binom{nk+1}{k+1}}{\binom{k+1}{k+1}\binom{k+2}{k+1}\cdots\binom{2k+1}{k+1}}$$

which is easily seen to be nonzero since the characteristic of the field is $p = nk+\delta$ where $2 \leq \delta \leq n-1$. $\square$

Remark 7.3.5. Although we established that $xyz \in (x^2, y^2, z^2)^*$ in the ring $R = K[X,Y,Z]/(X^3+Y^3+Z^3)$, this is, in a certain sense, unexplained. Under mild hypotheses on a ring, tight closure has a "colon–capturing" property: for $x_1, \ldots x_n$

part of a system of parameters for an excellent local (or graded) equidimensional ring A, we have $(x_1, \dots x_{n-1}) :_A x_n \subseteq (x_1, \dots x_{n-1})^*$ and various instances of elements being in the tight closure of ideals are easily seen to arise from this colon–capturing property.

To illustrate our point, we recall from [Ho, Example 5.7] how $z^2 \in (x,y)^*$ in the ring R above is seen to arise from colon–capturing. Consider the Segre product $T = R\#S$ where $S = K[U,V]$. Then the elements $xv - yu$, xu and yv form a system of parameters for the ring T. This ring is not Cohen–Macaulay as seen from the relation on the parameters:

$$(zu)(zv)(xv - yu) = (zv)^2(xu) - (zu)^2(yv).$$

The colon–capturing property of tight closure shows

$$(zu)(zv) \in \quad (xu, yv) :_T (xv - yu) \quad \subseteq (xu, yv)^*.$$

There is a retraction $R \otimes_K S \to R$ under which $U \mapsto 1$ and $V \mapsto 1$. This gives us a retraction from $T \to R$ which, when applied to $(zu)(zv) \in (xu, yv)^*$, shows $z^2 \in (x,y)^*$ in R.

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ABSTRACT

F–regularity, F–rationality and F–purity

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Examples are constructed to show that the property of F–regularity does not deform. Specifically, we exhibit a three dimensional domain which is not F–regular or even F–pure, but has a quotient by a principal ideal which is F–regular.

We show that the invariant subring for the action of the symplectic group on a polynomial ring is, in general, not F–pure. This shows that the socle element modulo an ideal can be forced into the expansion of the ideal in a separable extension, as well as in a linearly disjoint purely inseparable extension.

Conditions are examined under which graded rings have Veronese subrings which are F–rational or F–regular. The results obtained give us various techniques of constructing F–rational rings which are not F–regular.

For certain classes of local non–equidimensional rings, we prove the conjecture that no ideal generated by a system of parameters can be tightly closed. A new closure operation is constructed, which agrees with tight closure for equidimensional

rings, and rectifies the absence of the colon–capturing property of tight closure in non–equidimensional rings.

We compute the Frobenius closure and tight closure of certain ideals in diagonal hypersurfaces. This enables us to establish the equality of tight closure and plus closure for these ideals.