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Associated Primes of Local Cohomology Modules and of Frobenius Powers

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Dedicated to Professor Melvin Hochster on the occasion of his sixtieth birthday

1 Introduction

Let R be a commutative Noetherian ring and $\mathfrak{a} \subset R$ an ideal. In [15], Huneke asked whether the number of associated prime ideals of a local cohomology module $H^n_{\mathfrak{a}}(R)$ is always finite. In [29], the first author constructed an example of a hypersurface

$$R = \frac{\mathbb{Z}[u, v, w, x, y, z]}{(ux + vy + wz)} \tag{1.1}$$

for which the local cohomology module $H^3_{(x,y,z)}(R)$ has a p-torsion element for every prime integer p, and consequently has infinitely many associated prime ideals. However, this example does not address Huneke's question for rings containing a field, nor does it yield an example over a local ring. More recently, Katzman constructed the following example in [19]: let K be an arbitrary field and consider the hypersurface

$$S = \frac{K[s, t, u, v, x, y]}{\left(su^2x^2 - (s + t)uxvy + tv^2y^2\right)}. \tag{1.2}$$

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Katzman showed that the local cohomology module $H^2_{(x,y)}(S)$ has infinitely many associated prime ideals. Since the defining equation of this hypersurface factors, the ring in Katzman's example is not an integral domain. In this paper, we generalize Katzman's construction and obtain families of examples which include examples over normal domains and even over hypersurfaces with rational singularities.

Theorem 1.1. Let K be an arbitrary field. Then there exists a standard graded hypersurface R with $[R]_0 = K$, which is a unique factorization domain and contains an ideal $\mathfrak{a}$ such that a local cohomology module $H^n_{\mathfrak{a}}(R)$ has infinitely many associated prime ideals.

If K has characteristic zero, there exist such examples where, furthermore, R has rational singularities. If K has positive characteristic, R may be chosen to be F-regular. If $\mathfrak{m}$ denotes the homogeneous maximal ideal of R, then $H^n_{\mathfrak{a}}(R_{\mathfrak{m}})$ has infinitely many associated prime ideals as well. □

There are affirmative answers to Huneke's question if the ring R is regular, but, as our theorem indicates, the hypothesis of regularity cannot be weakened substantially. The first results were obtained by Huneke and Sharp who proved that if R is a regular ring containing a field of prime characteristic, then the set of associated prime ideals of $H^n_{\mathfrak{a}}(R)$ is finite, [17, Corollary 2.3]. Lyubeznik established that $H^n_{\mathfrak{a}}(R)$ has finitely many associated prime ideals if R is a regular local ring containing a field of characteristic zero, or an unramified regular local ring of mixed characteristic, see [21, Corollary 3.6(c)] and [23, Theorem 1], respectively. Marley proved that if R is a local ring, then for any finitely generated R-module M of dimension at most three, any local cohomology module $H^n_{\mathfrak{a}}(M)$ has finitely many associated primes, [24, Corollary 2.7]. If i is the smallest integer for which $H^i_{\mathfrak{a}}(M)$ is not a finitely generated R-module, then the set $\operatorname{Ass} H^i_{\mathfrak{a}}(M)$ is finite, as proved in [3, 20]. For some of the other work on this question, we refer the reader to [4, 5, 8, 22, 25, 32].

In Section 2, we establish a relationship between the associated primes of Frobenius powers of an ideal and the associated primes of a local cohomology module over an auxiliary ring. Recall that for an ideal $\mathfrak{a}$ in a ring R of prime characteristic $p > 0$, the *Frobenius powers* of $\mathfrak{a}$ are the ideals $\mathfrak{a}^{[p^e]} = (x^{p^e} \mid x \in \mathfrak{a})$, where $e \in \mathbb{N}$. The finiteness of the associated primes of the ideals $\mathfrak{a}^{[p^e]}$ is related to the localization problem in tight closure theory discussed in Section 6. In [18], Katzman constructed the first example where the set $\bigcup_e \operatorname{Ass} R/\mathfrak{a}^{[p^e]}$ is infinite. The question however remained whether the set $\bigcup_e \operatorname{Ass} R/(\mathfrak{a}^{[p^e]})^*$ is finite or if it has finitely many maximal elements—this has strong implications for the localization problem, see [1, 9, 18, 28] or [16, Section 12]. As an application of our results on local cohomology, we settle this question in Section 6 with the following theorem.

Theorem 1.2. There exists an F-regular unique factorization domain R of characteristic $p > 0$, with an ideal $\mathfrak{a}$, for which the set

$$\bigcup_{e\in\mathbb{N}} \operatorname{Ass} \frac{R}{\mathfrak{a}^{[p^e]}} = \bigcup_{e\in\mathbb{N}} \operatorname{Ass} \frac{R}{\left(\mathfrak{a}^{[p^e]}\right)^*} \tag{1.3}$$

has infinitely many maximal elements. □

2 General constructions

Let $\mathfrak{a} = (x_1, \dots, x_n)$ be an ideal of a ring R. For an integer $r \geq 0$, the local cohomology module $H^r_{\mathfrak{a}}(R)$ may be computed as the rth cohomology module of the extended Čech complex

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^{n} R_{x_i} \longrightarrow \bigoplus_{i<j} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1\cdots x_n} \longrightarrow 0. \tag{2.1}$$

For positive integers m_i and an element $f \in R$, we will use $[f + (x_1^{m_1}, \dots, x_n^{m_n})]$ to denote the cohomology class

$$\left[\frac{f}{x_1^{m_1} \cdots x_n^{m_n}}\right] \in H^n_{\mathfrak{a}}(R) = \frac{R_{x_1\cdots x_n}}{\sum R_{x_1\cdots\widehat{x_i}\cdots x_n}}. \tag{2.2}$$

It is easily seen that $[f + (x_1^{m_1}, \dots, x_n^{m_n})] \in H^n_{\mathfrak{a}}(R)$ is zero if and only if there exist integers $k_i \geq 0$ such that

$$f x_1^{k_1} \cdots x_n^{k_n} \in \left(x_1^{m_1+k_1}, \dots, x_n^{m_n+k_n}\right)R. \tag{2.3}$$

Consequently, $H^n_{\mathfrak{a}}(R)$ may also be computed as the direct limit

$$H^n_{\mathfrak{a}}(R) \cong \varinjlim_{m\in\mathbb{N}} \frac{R}{(x_1^m, \dots, x_n^m)R}, \tag{2.4}$$

where the maps in the direct system are induced by multiplication by the element $x_1 \cdots x_n$. We may regard an element $[f + (x_1^m, \dots, x_n^m)] \in H^n_{\mathfrak{a}}(R)$ as the class of $f + (x_1^m, \dots, x_n^m)R$ in this direct limit.

We next record two results which illustrate the relationship between associated primes of local cohomology modules and associated primes of generalized Frobenius powers of ideals.

Proposition 2.1. Let R be a Noetherian ring and $\{M_i\}_{i\in I}$ a direct system of R-modules. Then

$$\operatorname{Ass}\left(\varinjlim M_i\right) \subseteq \bigcup_{i\in I} \operatorname{Ass} M_i. \tag{2.5}$$

In particular, if $\mathfrak{a} = (x_1, \dots, x_n)$ is an ideal of R, then for any infinite set $\mathbb{S}$ of positive integers,

$$\operatorname{Ass} H^n_{\mathfrak{a}}(R) \subseteq \bigcup_{m\in\mathbb{S}} \operatorname{Ass} \frac{R}{(x_1^m, \dots, x_n^m)}. \tag{2.6}$$

□

Proof. Let $\mathfrak{p} = \operatorname{ann} m$ for some element $m \in \varinjlim M_i$. If $z \in \mathfrak{p}$, then $zm = 0$, and so there exists $i \in I$ such that m is the image of $m_i \in M_i$ and $zm_i = 0$. Since $\mathfrak{p}$ is finitely generated, there exists $j \geq i$ such that $m_i \mapsto m_j \in M_j$ and $\mathfrak{p}m_j = 0$. Consequently,

$$\mathfrak{p} \subseteq 0 :_R m_j \subseteq 0 :_R m = \mathfrak{p}, \tag{2.7}$$

that is, $\mathfrak{p} = \operatorname{ann} m_j \in \operatorname{Ass} M_j$. ■

It immediately follows that whenever $H^n_{\mathfrak{a}}(R)$ has infinitely many associated prime ideals, the set $\bigcup_m \operatorname{Ass} R/(x_1^m, \dots, x_n^m)$ is infinite as well. The converse is false, as we will see in Remark 4.7.

Proposition 2.2. Let A be an $\mathbb{N}$-graded ring which is generated, as an A_0-algebra, by elements $t_1, \dots, t_n$ of degree 1 which are nonzerodivisors in A. Let R be the extension ring

$$R = \frac{A[u_1, \dots, u_n, x_1, \dots, x_n]}{(u_1x_1 - t_1, \dots, u_nx_n - t_n)}. \tag{2.8}$$

Let $m_1, \dots, m_n$ be positive integers and $f \in A$ a homogeneous element. Then, for arbitrary integers $k_i \geq 0$,

$$(t_1^{m_1}, \dots, t_n^{m_n})A :_{A_0} f = (x_1^{m_1+k_1}, \dots, x_n^{m_n+k_n})R :_{A_0} fx_1^{k_1} \cdots x_n^{k_n}. \tag{2.9}$$

Consequently, if we consider the element $\eta = [f + (x_1^{m_1}, \dots, x_n^{m_n})]$ of the local cohomology module $H^n_{(x_1,\dots,x_n)}(R)$, then

$$(t_1^{m_1}, \dots, t_n^{m_n})A :_{A_0} f = \operatorname{ann}_{A_0} \eta. \tag{2.10}$$

□

Proof. The inclusion $\subseteq$ is easily verified. For the other inclusion, let $e_i \in \mathbb{Z}^{n+1}$ be the unit vector with 1 as its ith entry, and consider the $\mathbb{Z}^{n+1}$-grading on R, where $\deg x_i = e_i$ and $\deg u_i = e_{n+1} - e_i$ for all $1 \leq i \leq n$. If $f \in A_r$, then, as an element of R, the degree of f is re_{n+1}. The subring A is a direct summand of R since

$$A_j = R_{(0,\dots,0,j)} \quad \text{for } j \geq 0, \tag{2.11}$$

and $A = \oplus_{j\geq 0} R_{(0,\dots,0,j)}$. Now if $h \in A_0$ is an element such that

$$hfx_1^{k_1} \cdots x_n^{k_n} \in \left(x_1^{m_1+k_1}, \dots, x_n^{m_n+k_n}\right)R, \tag{2.12}$$

then there exist homogeneous elements $c_1, \dots, c_n \in R$ such that

$$hfx_1^{k_1} \cdots x_n^{k_n} = c_1 x_1^{m_1+k_1} + \cdots + c_n x_n^{m_n+k_n}. \tag{2.13}$$

Comparing degrees, we must have $\deg c_1 = (-m_1, k_2, \dots, k_n, r)$, and so c_1 is an A_0-linear combination of monomials μ of the form

$$\mu = u_1^{l_1+m_1} u_2^{l_2} \cdots u_n^{l_n} x_1^{l_1} x_2^{l_2+k_2} \cdots x_n^{l_n+k_n}, \tag{2.14}$$

where $l_i \geq 0$ and $m_1 + l_1 + \cdots + l_n = r$. Consequently,

$$\begin{aligned} \mu x_1^{m_1+k_1} &= \left(u_1x_1\right)^{l_1+m_1} \left(u_2x_2\right)^{l_2} \cdots \left(u_nx_n\right)^{l_n} x_1^{k_1} \cdots x_n^{k_n} \\ &= t_1^{l_1+m_1} t_2^{l_2} \cdots t_n^{l_n} x_1^{k_1} \cdots x_n^{k_n}, \end{aligned} \tag{2.15}$$

and so $c_1 x_1^{m_1+k} \in (x_1^{k_1} \cdots x_n^{k_n} t_1^{m_1})R$. Similar computations for $c_2, \dots, c_n$ show that

$$hfx_1^{k_1} \cdots x_n^{k_n} \in x_1^{k_1} \cdots x_n^{k_n} \left(t_1^{m_1}, \dots, t_n^{m_n}\right)R. \tag{2.16}$$

Multiplying by $u_1^{k_1} \cdots u_n^{k_n}$ and using that A is a direct summand of R, we get

$$\begin{aligned} hft_1^{k_1} \cdots t_n^{k_n} &\in t_1^{k_1} \cdots t_n^{k_n} \left(t_1^{m_1}, \dots, t_n^{m_n}\right)R \cap A \\ &= t_1^{k_1} \cdots t_n^{k_n} \left(t_1^{m_1}, \dots, t_n^{m_n}\right)A. \end{aligned} \tag{2.17}$$

Since the elements $t_i \in A$ are nonzerodivisors, the required result follows. ■

We next record two results which will be used in the proof of Theorem 2.6.

Lemma 2.3. Let M be a square matrix with entries in a ring R. Then the minimal primes of the ideal $(\det M)R$ are precisely the minimal primes of the cokernel of the matrix M. □

Proof. Let C denote the cokernel of M, that is, we have an exact sequence

$$R^n \xrightarrow{M} R^n \longrightarrow C \longrightarrow 0. \tag{2.18}$$

For a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$, note that $C_\mathfrak{p} = 0$ if and only if $R^n_\mathfrak{p} \xrightarrow{M} R^n_\mathfrak{p}$ is surjective or, equivalently, is an isomorphism. This occurs if and only if $\det M$ is a unit in $R_\mathfrak{p}$, and so we have

$$V(\det M) = \operatorname{Supp} C. \tag{2.19}$$

■

Lemma 2.4. Let R be an $\mathbb{N}$-graded ring and M a $\mathbb{Z}$-graded R-module. For every integer r and prime ideal $\mathfrak{p} \in \operatorname{Ass}_{R_0} M_r$, there exists a homogeneous prime ideal $\mathfrak{P} \in \operatorname{Ass}_R M$ such that $\mathfrak{P} \cap R_0 = \mathfrak{p}$. Consequently, if the set $\operatorname{Ass}_{R_0} M$ is infinite, then so is the set $\operatorname{Ass}_R M$. □

Proof. Let $\mathfrak{p} = \operatorname{ann}_{R_0} m$ for some element $m \in M_r$. There is no loss of generality in replacing M by the cyclic module $R/\mathfrak{a} \cong mR$, in which case $\mathfrak{p} = \mathfrak{a} \cap R_0$. The isomorphism

$$\frac{R}{(\mathfrak{a} + R_+)} \cong \frac{R_0}{\mathfrak{p}} \tag{2.20}$$

shows that $\mathfrak{a} + R_+$ is a prime ideal of R. Let $\mathfrak{P}$ be a minimal prime of $\mathfrak{a}$ which is contained in $\mathfrak{a} + R_+$. Then $\mathfrak{P} \in \operatorname{Min}_R R/\mathfrak{a} \subseteq \operatorname{Ass}_R R/\mathfrak{a}$, and $\mathfrak{P} \cap R_0 = \mathfrak{p}$ since $(\mathfrak{a} + R_+) \cap R_0 = \mathfrak{p}$. ■

Definition 2.5. Let d be a positive even integer and $r_0, \dots, r_d$ elements of a ring A_0. The nth *multidiagonal matrix with respect to* $r_0, \dots, r_d$ will refer to the $n \times n$ matrix

$$M_n = \begin{bmatrix} r_{d/2} & \cdots & r_0 & & & & \\ \vdots & \ddots & & \ddots & & & \\ r_d & & \ddots & & \ddots & & \\ & \ddots & & \ddots & & r_0 & \\ & & \ddots & & \ddots & \vdots & \\ & & & r_d & \cdots & r_{d/2} \end{bmatrix}, \tag{2.21}$$

where the elements $r_0, \dots, r_d$ occur along the $d + 1$ central diagonals, and all the other entries are zero. (These multidiagonal matrices are special cases of Töplitz matrices.)

Theorem 2.6. Let d be an even positive integer, $r_0, \ldots, r_d$ elements of a domain A_0, $a \geq 0$ an integer, and M_n the nth multidiagonal matrix with respect to $r_0, \ldots, r_d$. Let u, v, x, y be variables over A_0, and $\mathbb{S} \subseteq \mathbb{N}$ a subset such that

$$\bigcup_{n\in\mathbb{S}} \operatorname{Min}\left(\det M_{n-a-d/2}\right) \tag{2.22}$$

is an infinite set. If

$$A = \frac{A_0[x, y]}{\left((xy)^a\left(r_0x^d + r_1x^{d-1}y + \cdots + r_dy^d\right)\right)}, \tag{2.23}$$

then $\bigcup_{n\in\mathbb{S}} \operatorname{Ass} A/(x^n, y^n)$ is an infinite set.

Furthermore, if r_0 and r_d are nonzero elements of A_0, then for

$$R = \frac{A_0[u, v, x, y]}{\left(r_0(ux)^d + r_1(ux)^{d-1}(vy) + \cdots + r_d(vy)^d\right)}, \tag{2.24}$$

the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated primes.

If $(A_0, \mathfrak{m})$ is a local domain or if $(A_0, \mathfrak{m})$ is a graded domain and $\det M_n$ is a homogeneous element for all $n \geq 0$, then these issues are preserved under localizations of A and R at the respective maximal ideals $(\mathfrak{m} + (x, y))A$ and $(\mathfrak{m} + (u, v, x, y))R$. $\square$

Proof. Consider the A_0-module $[A/(x^n, y^n)]_{n-1+a+d/2}$ for $n > a + d$. A generating set for this module is given by the $n - a - d/2$ monomials

$$x^{a+d/2}y^{n-1}, x^{a+d/2+1}y^{n-2}, \ldots, x^{n-1}y^{a+d/2}. \tag{2.25}$$

There are $n - a - d/2$ relations amongst these monomials, arising from the equations

$$(xy)^a\left(r_0x^d + r_1x^{d-1}y + \cdots + r_dy^d\right)x^iy^{n-1-a-d/2-i} = 0, \tag{2.26}$$

where $0 \leq i \leq n - 1 - a - d/2$. Using this, it is easily checked that the presentation matrix for $[A/(x^n, y^n)]_{n-1+a+d/2}$ is precisely the multidiagonal matrix $M_{n-a-d/2}$. By Lemma 2.3, whenever $\det M_{n-a-d/2}$ is nonzero, its minimal primes are the minimal primes of $[A/(x^n, y^n)]_{n-1+a+d/2}$, and so

$$\bigcup_{n\in\mathbb{S}} \operatorname{Ass}_{A_0}\left[\frac{A}{(x^n, y^n)}\right]_{n-1+a+d/2} \tag{2.27}$$

is an infinite set. Using Lemma 2.4, the set $\bigcup_{n\in\mathbb{S}} \operatorname{Ass} A/(x^n, y^n)$ is infinite as well.

Note that xy is a nonzerodivisor in $A_0[x,y]/(r_0x^d + r_1x^{d-1}y + \cdots + r_dy^d)$ whenever r_0 and r_d are nonzero elements of A_0. The set $\operatorname{Ass}_{A_0} H^2_{(x,y)}(R)$ is infinite by Proposition 2.2. Since $A_0 = R_0$, Lemma 2.4 implies that the set $\operatorname{Ass}_R H^2_{(x,y)}(R)$ is infinite. ∎

Remark 2.7. We demonstrate how Katzman's examples from [18, 19] follow from Theorem 2.6. Let K be an arbitrary field and consider the polynomial ring $A_0 = K[t]$. Let M_n be the nth multidiagonal matrix with respect to the elements $r_0 = 1$, $r_1 = -(1+t)$, and $r_2 = t$. An inductive argument shows that

$$\det M_n = (-1)^n\left(1 + t + t^2 + \cdots + t^n\right) = (-1)^n \frac{t^{n+1}-1}{t-1} \quad \forall n \geq 1. \tag{2.28}$$

It is easily verified that $\bigcup_{n\in\mathbb{N}} \operatorname{Min}(\det M_n)$ is an infinite set and, if K has characteristic $p > 0$, that the set $\bigcup_{e\in\mathbb{N}} \operatorname{Min}(\det M_{p^e-2})$ is also infinite. Theorem 2.6 now gives us the main results of [19]: the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated primes, where

$$R = \frac{K[t,u,v,x,y]}{\left(u^2x^2 - (1+t)uxvy + tv^2y^2\right)}. \tag{2.29}$$

Similarly, graded or local examples may be obtained using (1.2), in which case $H^2_{(x,y)}(S)$ and $H^2_{(x,y)}(S_m)$ have infinitely many associated primes.

If K has characteristic $p > 0$, consider the hypersurface

$$A = \frac{K[t,x,y]}{\left(xy\left(x^2 - (1+t)xy + ty^2\right)\right)}, \tag{2.30}$$

where $\mathfrak{a} = 1$ in the notation of Theorem 2.6. The theorem now implies that the Frobenius powers of the ideal $(x,y)A$ have infinitely many associated primes, as first proved by Katzman in [18].

3 Tridiagonal matrices

The results of Section 2 demonstrate how multidiagonal matrices give rise to associated primes of local cohomology modules and of Frobenius powers of ideals. One of the goals of this paper is to construct an integral domain A of characteristic $p > 0$, with an ideal $\mathfrak{a}$, such that the set $\bigcup_e \operatorname{Ass} A/\mathfrak{a}^{[p^e]}$ is infinite. To obtain such examples directly from Theorem 2.6, we need the set $\bigcup_e \operatorname{Min}(\det M_{p^e-d/2})$ to be infinite, since the domain hypothesis forces $\mathfrak{a} = 0$ in the notation of the theorem. In Section 7, we show that $\bigcup_e \operatorname{Min}(\det M_{p^e-d/2})$ can indeed be infinite when $d = 4$.

In Proposition 3.1, we prove that $\bigcup_e \operatorname{Min}(\det M_{p^e-d/2})$ is finite whenever $d = 2$, see also [18, Lemma 10]. Nevertheless, the main results of our paper rely heavily on an analysis of multidiagonal matrices with $d = 2$, which we undertake next.

In the notation of Definition 2.5, multidiagonal matrices with $d = 2$ have the form

$$M_n = \begin{bmatrix} r_1 & r_0 & & & \\ r_2 & r_1 & r_0 & & \\ & \ddots & \ddots & \ddots & \\ & & r_2 & r_1 & r_0 \\ & & & r_2 & r_1 \end{bmatrix}. \tag{3.1}$$

It is convenient to define $\det M_0 = 1$, and it is easily seen that

$$\det M_{n+2} = r_1 \det M_{n+1} - r_0 r_2 \det M_n \quad \forall\, n \geq 0. \tag{3.2}$$

While we will not be using it here, we mention that

$$\det M_n = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} r_1^{n-2i} (r_0 r_2)^i. \tag{3.3}$$

Consider the generating function for $\det M_n$:

$$G(x) = \sum_{n \geq 0} \left(\det M_n\right) x^n. \tag{3.4}$$

By the recursion formula,

$$\sum_{n \geq 0} \left(\det M_{n+2}\right) x^{n+2} = r_1 \sum_{n \geq 0} \left(\det M_{n+1}\right) x^{n+2} - r_0 r_2 \sum_{n \geq 0} \left(\det M_n\right) x^{n+2}, \tag{3.5}$$

and substituting $G(x)$ and solving, we get

$$G(x) = \sum_{n \geq 0} \left(\det M_n\right) x^n = \frac{1}{1 - r_1 x + r_0 r_2 x^2}. \tag{3.6}$$

Proposition 3.1. Let r_0, r_1, and r_2 be elements of a ring R of prime characteristic $p > 0$. For each $n \in \mathbb{N}$, let M_n be the nth multidiagonal matrix with respect to r_0, r_1, and r_2. Then, for any integer $e \geq 1$,

$$\det M_{p^e-1} = \left(\det M_{p-1}\right)^{1+p+\cdots+p^{e-1}}. \tag{3.7}$$

Consequently, the set $\bigcup_e \operatorname{Min}(\det M_{p^e-1})$ is finite. □

Proof. Let $1 - r_1 x + r_0 r_2 x^2 = (1 - \alpha x)(1 - \beta x)$ for some elements α and β in a suitable extension of R. The generating function $G(x)$ can be written as

$$G(x) = \sum_{n \geq 0} \left(\det M_n\right) x^n = \frac{1}{(1 - \alpha x)(1 - \beta x)} = \sum_{i,j \geq 0} \alpha^i \beta^j x^{i+j}, \tag{3.8}$$

and consequently,

$$\det M_{p-1} = \sum_{i=0}^{p-1} \alpha^i \beta^{p-1-i}, \qquad \det M_{p^e-1} = \sum_{i=0}^{p^e-1} \alpha^i \beta^{p^e-1-i}. \tag{3.9}$$

Using this,

$$\begin{aligned} \left(\det M_{p-1}\right)^{1+p+\cdots+p^{e-1}} &= \prod_{j=0}^{e-1} \left(\sum_{i=0}^{p-1} \alpha^i \beta^{p-1-i} \right)^{p^j} \\ &= \prod_{j=0}^{e-1} \left(\sum_{i=0}^{p-1} \alpha^{ip^j} \beta^{(p-1-i)p^j} \right) \\ &= \sum_{k=0}^{p^e-1} \alpha^k \beta^{p^e-1-k} = \det M_{p^e-1}. \end{aligned} \tag{3.10}$$

■

We next consider a special family of tridiagonal matrices: let $K[s,t]$ be a polynomial ring over a field K, and consider the $n \times n$ multidiagonal matrices

$$M_n = \begin{bmatrix} t & s & & & \\ s & t & s & & \\ & \ddots & \ddots & \ddots & \\ & & s & t & s \\ & & & s & t \end{bmatrix}. \tag{3.11}$$

In the notation of Definition 2.5, we have $d = 2$, $r_1 = t$, and $r_0 = r_2 = s$. Setting $Q_n(s,t) = \det M_n$, we have

$$Q_0 = 1, \qquad Q_1 = t, \qquad Q_{n+2} = tQ_{n+1} - s^2 Q_n \quad \forall n \geq 0. \tag{3.12}$$

Note that the polynomials $Q_n(s,t)$ are relatively prime to s. Using the specialization $P_n(t) = Q_n(1,t)$, we get polynomials $P_n(t) \in K[t]$ satisfying the recursion

$$P_0(t) = 1, \qquad P_1(t) = t, \qquad P_{n+2}(t) = tP_{n+1}(t) - P_n(t) \quad \forall n \geq 0. \tag{3.13}$$

Each $P_n(t)$ is a monic polynomial of degree n, and in Lemma 3.3, we establish that the number of distinct irreducible factors of the polynomials $\{P_n(t)\}_{n\in\mathbb{N}}$ is infinite. As $Q_n(s,t) = s^n P_n(t/s)$ for all $n \geq 0$, this also establishes that the number of distinct irreducible factors of the polynomials $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ is infinite.

Lemma 3.2. Let K be an algebraically closed field and consider the polynomials $P_n(t) = \det M_n \in K[t]$ for $n \geq 1$ as above.

(1) If ξ is a nonzero element of K with $\xi \neq \pm 1$, then $P_n(\xi + \xi^{-1}) = 0$ if and only if $\xi^{2n+2} = 1$.

(2) The number of distinct roots of P_n which are different from 0 and ± 1 is half the number of distinct $(2n+2)$th roots of unity different from ± 1.

(3) If $2n+2$ is invertible in K, then $P_n(t)$ has n distinct roots of the form $\xi + \xi^{-1}$, where $\xi^{2n+2} = 1$ and $\xi \neq \pm 1$.

(4) The elements 2 or -2 are roots of $P_n(t)$ if and only if the characteristic of K is a positive prime p which divides $n+1$.

(5) If the characteristic of K is an odd prime p, then $P_{q-2}(t)$ has $q-2$ distinct roots for all $q = p^e$. If $p = 2$, then $P_{q-2}(t)$ has $q/2 - 1$ distinct roots. □

Proof. (1) Consider the generating function of the polynomials $P_n(t)$:

$$G(t,x) = \sum_{n\geq 0} P_n(t)x^n = \frac{1}{1 - xt + x^2} \in K[t][[x]]. \tag{3.14}$$

If $\xi \neq 0$ and $\xi \neq \pm 1$, then

$$\begin{aligned} \sum_{n\geq 0} P_n\left(\xi + \xi^{-1}\right)x^n &= \frac{1}{1 - x\left(\xi + \xi^{-1}\right) + x^2} \\ &= \frac{1}{\left(\xi^{-1} - x\right)\left(\xi - x\right)} \\ &= \frac{1}{\left(\xi - \xi^{-1}\right)\left(\xi^{-1} - x\right)} - \frac{1}{\left(\xi - \xi^{-1}\right)\left(\xi - x\right)} \\ &= \frac{\xi}{\xi - \xi^{-1}} \sum_{n\geq 0} (\xi x)^n - \frac{\xi^{-1}}{\xi - \xi^{-1}} \sum_{n\geq 0} \left(\xi^{-1}x\right)^n \in K[[x]]. \end{aligned} \tag{3.15}$$

Equating the coefficients of x^n, we have

$$P_n\left(\xi + \xi^{-1}\right) = \frac{\xi^{n+1} - \xi^{-(n+1)}}{\xi - \xi^{-1}} = \frac{\xi^{2n+2} - 1}{\xi^n\left(\xi^2 - 1\right)}, \tag{3.16}$$

and the assertion follows.

(2) We observe that

$$\xi+\frac{1}{\xi}-\left(\eta+\frac{1}{\eta}\right)=\xi-\eta-\frac{\xi-\eta}{\xi\eta}=(\xi-\eta)\left(1-\frac{1}{\xi\eta}\right), \tag{3.17}$$

and so $\xi+\xi^{-1}=\eta+\eta^{-1}$ if and only if ξ equals η or η^{-1}.

(3) Since $2n+2$ is invertible in K, the polynomial $X^{2n+2}-1=0$ has $2n$ distinct roots ξ with $\xi\neq\pm1$. These give the n distinct roots $\xi+\xi^{-1}$ of the degree-n polynomial $P_n(t)$.

(4) Using the generating function above,

$$G(2,x)=\frac{1}{1-2x+x^2}=(1-x)^{-2}=1+2x+3x^2+\cdots, \tag{3.18}$$

and so $P_n(\pm2)=0$ if and only if $n+1=0$ in K.

(5) The case when p is odd follows immediately from (2). If $p=2$, the equation $X^{2q-2}-1=(X^{q-1}-1)^2=0$ has $q-2$ distinct roots ξ with $\xi\neq1$, which ensures that $P_{q-2}(t)$ has at least $q/2-1$ distinct roots. It follows from (4) that 0 is not a root of $P_{q-2}(t)$, so these must be all the roots. ■

Lemma 3.3. Let K be an arbitrary field. Then the number of distinct irreducible factors of the polynomials $\{P_n(t)\}_{n\in\mathbb{N}}$ is infinite. If K has characteristic $p>0$ and $q=p^e$ varies over the powers of p, then the polynomials $\{P_{q-2}(t)\}_{q=p^e}$ have infinitely many distinct irreducible factors.

Consequently the number of distinct irreducible factors of the homogeneous polynomials $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ as well as $\{Q_{q-2}(s,t)\}_{q=p^e}$ is also infinite. □

Proof. It follows from Lemma 3.2 that $\{P_n(t)\}_n$ as well as $\{P_{q-2}(t)\}_{q=p^e}$ have infinitely many distinct irreducible factors in $K[t]$. ■

4 Examples over integral domains

We can now construct a domain which has a local cohomology module with infinitely many associated primes.

Theorem 4.1. Let K be an arbitrary field and consider the integral domain

$$R=\frac{K[s,t,u,v,x,y]}{\left(su^2x^2+tuxvy+sv^2y^2\right)}. \tag{4.1}$$

Then the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated prime ideals. Also, $H^2_{(x,y)}(R_m)$ has infinitely many associated primes for the local domain R_m, where $m=(s,t,u,v,x,y)R$.

If $\mathbb{S}$ is any infinite set of positive integers, then the set $\bigcup_{m\in\mathbb{S}} \operatorname{Ass} R/(x^m, y^m)$ is infinite; in particular, if K has characteristic $p > 0$, then $\bigcup_{e\in\mathbb{N}} \operatorname{Ass} R/(x^{p^e}, y^{p^e})$ is infinite. The same conclusions hold if the hypersurface R is replaced by its specialization $R/(s-1)$ or by the localization $R_{\mathfrak{m}}$. □

Proof. The assertions regarding local cohomology follow from Theorem 2.6 and Lemma 3.3. These, along with Proposition 2.1, imply the results for generalized Frobenius powers of ideals; see also the remark below. ■

Remark 4.2. Specializing $s = 1$ and working with the hypersurface

$$S = \frac{R}{(s-1)} = \frac{K[t, u, v, x, y]}{(u^2x^2 + tuxvy + v^2y^2)}, \tag{4.2}$$

similar arguments show that $H^2_{(x,y)}(S)$ has infinitely many associated primes. This gives an example of a four-dimensional integral domain S for which $H^2_{(x,y)}(S)$ has infinitely many associated prime ideals. However, it remains an open question whether a local cohomology module $H^i_{\mathfrak{a}}(T)$ has infinitely many associated primes, where T is a *local* ring of dimension four. This is of interest in view of Marley's results that the local cohomology of a Noetherian local ring of dimension less than four has finitely many associated primes [24].

For the assertion regarding the associated primes of generalized Frobenius powers of an ideal, the hypersurface R of Theorem 4.1 can be modified to obtain a three-dimensional local domain or a two-dimensional nonlocal domain.

Theorem 4.3. Let K be an arbitrary field and consider the integral domain

$$A = \frac{K[s, t, x, y]}{(sx^2 + txy + sy^2)}. \tag{4.3}$$

Then the set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n, y^n)$ is infinite. The same conclusion holds if A is replaced by the specialization $A/(s-1)$ or by the localization $A_{(s,t,x,y)}$. □

The proof of the theorem is again an immediate consequence of Theorem 2.6 and Lemma 3.3, but we feel it is of interest to explicitly determine the infinite set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n, y^n)$ at least in this one example, and we record the result as Theorem 4.6. If K has characteristic $p > 0$, this theorem also shows that the set $\bigcup_{e\in\mathbb{N}} \operatorname{Ass} A/(x^{p^e}, y^{p^e})$ is finite. We next record some preliminary computations which will be needed in determining the associated primes of the ideals $(x^n, y^n)A$, and will also be used later in Section 5.

Lemma 4.4. Consider the polynomial ring $K[s,t,x,y]$ and integers $m, n \geq 1$. Then

(1) $xy^{n-1}Q_{n-1} \in (x^n, y^n, sx^2 + txy + sy^2)$,
(2) $(x^n, y^n, sx^2 + txy + sy^2) : (s^m xy^{n-1}) = (x, y, Q_{n-1})$,
(3) $(x^n, y^n, x^2 + txy + y^2) : (xy^{n-1}) = (x, y, P_{n-1})$. □

Proof. (1) The case $n = 1$ holds trivially. Using the equation $tQ_i = Q_{i+1} + s^2 Q_{i-1}$ for $1 \leq i \leq n-2$, we get

$$\begin{aligned}
&\left(sx^2 + txy + sy^2\right)(sx)^{n-2-i}y^i Q_i \\
&\quad = s^{n-1-i}x^{n-i}y^i Q_i + s^{n-1-i}x^{n-2-i}y^{i+2}Q_i \\
&\qquad + s^{n-2-i}x^{n-1-i}y^{i+1}Q_{i+1} + s^{n-i}x^{n-1-i}y^{i+1}Q_{i-1},
\end{aligned} \tag{4.4}$$

and taking an alternating sum gives us

$$\begin{aligned}
&\sum_{i=0}^{n-2}(-1)^i\left(sx^2 + txy + sy^2\right)(sx)^{n-2-i}y^i Q_i \\
&\quad = s^{n-1}x^n Q_0 + (-1)^{n-2}xy^{n-1}Q_{n-1} + (-1)^{n-2}sy^n Q_{n-2}.
\end{aligned} \tag{4.5}$$

This shows that $xy^{n-1}Q_{n-1} \in (x^n, y^n, sx^2 + txy + sy^2)$.

(2) If $n = 1$, we have the unit ideal on each side of the asserted equality, so we may assume that $n \geq 2$ for the rest of this proof. It is easy to verify that

$$sxy^{n-1}(x,y) \subseteq \left(x^n, y^n, sx^2 + txy + sy^2\right). \tag{4.6}$$

Let $h \in K[s,t]$ be an element such that

$$hs^m xy^{n-1} \in \left(x^n, y^n, sx^2 + txy + sy^2\right). \tag{4.7}$$

Using the grading where $\deg s = \deg t = 0$ and $\deg x = \deg y = 1$, there exist elements α, β, and $d_0, \ldots, d_{n-2}$ in $K[s,t]$ with

$$\begin{aligned}
&hs^m xy^{n-1} \\
&\quad = \alpha x^n + \beta y^n + \left(d_0 x^{n-2} - d_1 x^{n-3}y + \cdots + (-1)^{n-2}d_{n-2}y^{n-2}\right)\left(sx^2 + txy + sy^2\right).
\end{aligned} \tag{4.8}$$

Comparing coefficients of $x^{n-1}y, x^{n-2}y^2, \ldots, xy^{n-1}$, we get

$$\begin{aligned}
&sd_1 - td_0 = 0, \\
&sd_{i+2} - td_{i+1} + sd_i = 0 \quad \forall\, 0 \leq i \leq n-4, \\
&(-1)^{n-2}\left(td_{n-2} - sd_{n-3}\right) = hs^m.
\end{aligned} \tag{4.9}$$

In particular,

$$d_1 = \left(\frac{t}{s}\right) d_0, \qquad d_{i+2} = \left(\frac{t}{s}\right) d_{i+1} - d_i \quad \forall\, 0 \le i \le n-4, \tag{4.10}$$

and consequently, $d_i = d_0 P_i(t/s)$ for $0 \le i \le n-2$, where the P_i are the polynomials defined recursively in Section 3. This gives us

$$\begin{aligned} hs^m &= (-1)^{n-2}\left(td_0P_{n-2}\left(\frac{t}{s}\right) - sd_0P_{n-3}\left(\frac{t}{s}\right)\right) \\ &= (-1)^{n-2} sd_0 P_{n-1}\left(\frac{t}{s}\right), \end{aligned} \tag{4.11}$$

and so $hs^{m+n-2} = (-1)^{n-2} d_0 Q_{n-1}$. Since s and Q_{n-1} are relatively prime in $K[s,t]$, we see that h is a multiple of Q_{n-1}.

(3) This case is the inhomogeneous case of (2) and is left to the reader. ■

Lemma 4.5. Let $A = K[s,t,x,y]/(sx^2 + txy + sy^2)$ and let $n \ge 1$ be an arbitrary integer.

(1) $s^{i-1}x^iy^{n-i} \in (x^n, y^n, xy^{n-1})$ for all $1 \le i \le n$. In particular,

$$s^{n-1}(x,y)^n \subseteq \left(x^n, y^n, xy^{n-1}\right), \qquad s^n(x,y)^n \subseteq \left(x^n, y^n, sxy^{n-1}\right). \tag{4.12}$$

(2) Also, $t^n(x,y)^n \subseteq (x^n, y^n, sxy^{n-1})$.

(3) If $n \ge 2$, the ideal (x^n, y^n, sxy^{n-1}) has a primary decomposition

$$\left(x^n, y^n, sxy^{n-1}\right) = (x,y)^n \cap \left(x^n, y^n, sxy^{n-1}, s^n, t^n\right). \tag{4.13}$$

□

Proof. For (1), we use induction on i to show that $s^{i-1}x^iy^{n-i} \in (x^n, y^n, xy^{n-1})$. This is certainly true if $i = 1$ and, assuming the result for integers less than i, observe that

$$\begin{aligned} s^{i-1}x^iy^{n-i} &= -s^{i-2}x^{i-2}y^{n-i}\left(txy + sy^2\right) \\ &= -s^{i-2}tx^{i-1}y^{n-i+1} - s^{i-1}x^{i-2}y^{n-i+2} \in \left(x^n, y^n, xy^{n-1}\right). \end{aligned} \tag{4.14}$$

Next, the equation $txy = -(sx^2 + sy^2)$ gives us

$$t(x,y)^n \subseteq \left(x^n, y^n\right) + s(x,y)^n, \tag{4.15}$$

and using this inductively, we get

$$t^n(x,y)^n \subseteq \left(x^n, y^n\right) + s^n(x,y)^n \subseteq \left(x^n, y^n, sxy^{n-1}\right), \tag{4.16}$$

which proves (2).

We next use the grading on the hypersurface A, where $\deg s = \deg t = 0$ and $\deg x = \deg y = 1$. If α and β are nonzero homogeneous elements of A with $\alpha s^n + \beta t^n \in (x,y)^n$, then α and β must have degree at least n, and therefore belong to the ideal $(x,y)^n$. This shows that

$$\left(s^n, t^n\right) \cap (x,y)^n = \left(s^n, t^n\right)(x,y)^n, \tag{4.17}$$

and using (1) and (2), we get

$$\left(s^n, t^n\right) \cap (x,y)^n \subseteq \left(x^n, y^n, sxy^{n-1}\right). \tag{4.18}$$

The intersection asserted in (3) follows immediately from this, and it remains to verify that the ideals $\mathfrak{q}_1 = (x,y)^n$ and $\mathfrak{q}_2 = (x^n, y^n, sxy^{n-1}, s^n, t^n)$ are indeed primary ideals. The radical of $\mathfrak{q}_2$ is the maximal ideal (s,t,x,y), so $\mathfrak{q}_2$ is a primary ideal. Using the earlier grading, any homogeneous zerodivisor in the ring $A/\mathfrak{q}_1$ must have positive degree, and hence must be nilpotent. Consequently, $\mathfrak{q}_1$ is a primary ideal as well. ■

Theorem 4.6. Let $A = K[s,t,x,y]/(sx^2+txy+sy^2)$, where K is a field. Then $\operatorname{Ass} A/(x^2,y^2) = \{(x,y),(t,x,y)\}$ and

$$\operatorname{Ass} \frac{A}{\left(x^n, y^n\right)} = \left\{(x,y),(s,t,x,y)\right\} \cup \operatorname{Ass} \frac{A}{\left(x,y,Q_{n-1}\right)} \quad \text{for } n \geq 3. \tag{4.19}$$

In particular, $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n,y^n)$ is an infinite set. If K is an algebraically closed field, let

$$\mathcal{S} = \left\{\left(x, y, t - s\xi - s\xi^{-1}\right)A \mid \xi \in K,\ \xi^n = 1 \text{ for some } n \geq 1,\ \xi \neq \pm 1\right\}. \tag{4.20}$$

In the case that K has characteristic zero,

$$\bigcup_{n\geq 1} \operatorname{Ass} \frac{A}{\left(x^n, y^n\right)} = \left\{(x,y),(t,x,y),(s,t,x,y)\right\} \cup \mathcal{S}, \tag{4.21}$$

and if K has positive characteristic, then

$$\bigcup_{n\geq 1} \operatorname{Ass} \frac{A}{\left(x^n, y^n\right)} = \left\{(x,y),(t,x,y),(s,t,x,y),(t-2s,x,y),(t+2s,x,y)\right\} \cup \mathcal{S}. \tag{4.22}$$

□

Proof. It is easily checked that $(x,y)^2 \cap (x^2,y^2,t)$ is a primary decomposition of (x^2,y^2), so we need to compute $\operatorname{Ass} A/(x^n,y^n)$ for $n \geq 3$. By Lemma 4.4(2), we have an exact sequence

$$0 \longrightarrow \frac{A}{(x,y,Q_{n-1})} \xrightarrow{\cdot sxy^{n-1}} \frac{A}{(x^n,y^n)} \longrightarrow \frac{A}{(x^n,y^n,sxy^{n-1})} \longrightarrow 0, \tag{4.23}$$

and consequently,

$$\operatorname{Ass}\frac{A}{(x,y,Q_{n-1})} \subseteq \operatorname{Ass}\frac{A}{(x^n,y^n)} \subseteq \operatorname{Ass}\frac{A}{(x,y,Q_{n-1})} \cup \operatorname{Ass}\frac{A}{(x^n,y^n,sxy^{n-1})}. \tag{4.24}$$

By Lemma 4.5, $\operatorname{Ass} A/(x^n,y^n,sxy^{n-1}) = \{(x,y),(s,t,x,y)\}$, and so it suffices to verify that the prime ideals $\mathfrak{p}_1 = (x,y)$ and $\mathfrak{p}_2 = (s,t,x,y)$ are indeed associated primes of $A/(x^n,y^n)$. This follows since $\mathfrak{p}_1$ is a minimal prime of (x^n,y^n) and

$$\mathfrak{p}_2 = \left(x^n,y^n\right) : (xy)^{n-1}. \tag{4.25}$$

If K is an algebraically closed field, the polynomials $Q_i(s,t)$ split into linear factors determined by the roots of $P_i(t)$, which are computed in Lemma 3.2. ■

Remark 4.7. If K is a field of characteristic $p > 0$ and A is the hypersurface (4.3), we proved that the set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n,y^n)$ is infinite. However, the set $\bigcup_{e\in\mathbb{N}} \operatorname{Ass} A/(x^{p^e},y^{p^e})$ is finite since, by Theorem 4.6,

$$\operatorname{Ass}\frac{A}{\left(x^{p^e},y^{p^e}\right)} = \left\{(x,y),(s,t,x,y)\right\} \cup \operatorname{Ass}\frac{A}{(x,y,Q_{p^e-1})} \quad \text{for } p^e \geq 3, \tag{4.26}$$

and Q_{p^e-1} is a power of Q_{p-1} by Proposition 3.1. The set $\operatorname{Ass} H^2_{(x,y)}(R)$ is finite as well by using Proposition 2.1. Consequently, we have a strict inclusion

$$\operatorname{Ass} H^2_{(x,y)}(R) \subsetneq \bigcup_{n\in\mathbb{N}} \operatorname{Ass}\frac{R}{(x^n,y^n)R}. \tag{4.27}$$

The set $\bigcup_{n\in\mathbb{N}} \operatorname{Ass} A/(x^n,y^n)$ has been explicitly computed in Theorem 4.6, and we next observe that the only associated prime of $H^2_{(x,y)}(R)$ is the maximal ideal $\mathfrak{m} = (s,t,x,y)$. The module $H^2_{(x,y)}(R)$ is generated over R by the elements $\eta_q = [1 + (x^q,y^q)]$ for $q = p^e$, and it suffices to show that η_q is killed by a power of $\mathfrak{m}$. It is immediately seen that x^q and y^q kill η_q, and for the remaining cases, note that

$$s^q\eta_q = \left[s^q x^{2q} + \left(x^{3q},y^q\right)\right] = 0, \qquad t^q\eta_q = \left[t^q x^q y^q + \left(x^{2q},y^{2q}\right)\right] = 0. \tag{4.28}$$

5 F-regular and unique factorization domain examples

In Theorem 4.1, we proved that for the hypersurface (4.1), the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated prime ideals. This ring R, while being a domain, is not normal. In Theorem 5.1, we construct examples over normal hypersurfaces, in fact over hypersurfaces of characteristic zero with rational singularities, as well as over F-regular hypersurfaces of positive characteristic. F-regularity is a notion arising from the theory of tight closure developed by Hochster and Huneke in [10]. A brief discussion may be found in Section 6, though for details of the theory and its applications, we refer the reader to [10, 11, 12, 16].

Theorem 5.1. Let K be an arbitrary field, and consider the hypersurface

$$S = \frac{K[s,t,u,v,w,x,y,z]}{(su^2x^2 + sv^2y^2 + tuxvy + tw^2z^2)}. \tag{5.1}$$

Then S is a normal domain for which the local cohomology module $H^3_{(x,y,z)}(S)$ has infinitely many associated prime ideals. This is preserved if we replace S by $S/(s-1)$ or by the localization $S_{(s,t,u,v,w,x,y,z)}$. If K has characteristic zero, then S has rational singularities, and if K has characteristic $p > 0$, then S is F-regular. □

Proof. We defer the proof that S has rational singularities or is F-regular, see Lemma 5.3. Normality follows from this or may be proved directly using the Jacobian criterion. Let B be the subring of S generated, as a K-algebra, by the elements $s, t, a = ux$, $b = vy$, and $c = wz$, that is,

$$B = \frac{K[s,t,a,b,c]}{(sa^2 + sb^2 + tab + tc^2)}. \tag{5.2}$$

For integers $n \geq 1$, let

$$\eta_n = \left[s(ux)(vy)^{n-1} + (x^n, y^n, z)\right] \in H^3_{(x,y,z)}(S). \tag{5.3}$$

Using $S_0 = K[s,t]$ as the subring of S of elements of degree zero, Proposition 2.2 implies that

$$\operatorname{ann}_{S_0}\eta_n = (a^n, b^n, c)B :_{S_0} sab^{n-1}, \tag{5.4}$$

and then Lemma 4.4(2) give us

$$(a^n, b^n, c)B :_{S_0} sab^{n-1} = (Q_{n-1})S_0, \tag{5.5}$$

where the Q_i are the polynomials defined recursively in Section 3. Using Lemmas 2.4 and 3.3, it follows that $H^3_{(x,y,z)}(S)$ has infinitely many associated prime ideals. ∎

It remains to prove that the hypersurface S in Theorem 5.1 has rational singularities or is F-regular, depending on the characteristic. The results of [30] provide a direct proof that the hypersurface S has rational singularities in characteristic zero. However, instead of relying on this, we prove here that if K has positive characteristic, then S is F-regular. Using [31, Theorem 4.3], it then follows that S has rational singularities when K has characteristic zero. We first record an elementary lemma.

Lemma 5.2. Let $(S, \mathfrak{m})$ be an $\mathbb{N}$-graded Gorenstein domain of dimension d, finitely generated over a field $[S]_0 = K$ of characteristic $p > 0$, and let $\eta \in H^d_{\mathfrak{m}}(S)$ denote a socle generator. Let $c \in R$ be a nonzero element such that S_c is regular. Then S is F-regular if and only if there exists an integer $e \geq 1$ such that η belongs to the S-span of $cF^e(\eta)$. □

Proof. If S is F-regular, then the zero submodule of $H^d_{\mathfrak{m}}(S)$ is tightly closed, that is, $0^*_{H^d_{\mathfrak{m}}(S)} = 0$, and so there exists a positive integer e such that $cF^e(\eta) \neq 0$. Since η generates the socle of $H^d_{\mathfrak{m}}(S)$, which is one-dimensional, η must belong to the S-span of $cF^e(\eta)$.

Conversely, assume that η belongs to the S-span of $cF^e(\eta)$ for some $e \geq 1$. Then $cF^e(\eta) \neq 0$, and so the Frobenius morphism $F : H^d_{\mathfrak{m}}(S) \longrightarrow H^d_{\mathfrak{m}}(S)$ is injective. It follows from [14, Proposition 6.11] that the ring S is F-pure. By [11, Theorem 6.2], the element c has a power which is a test element but then, since S is F-pure, c itself must be a test element. The condition $cF^e(\eta) \neq 0$ implies that $\eta \notin 0^*_{H^d_{\mathfrak{m}}(S)}$. Consequently, $0^*_{H^d_{\mathfrak{m}}(S)} = 0$, and it follows that S is F-regular. ∎

Lemma 5.3. Let K be a field and consider the hypersurface (5.1). If K has characteristic $p > 0$, then S is F-regular. If K has characteristic zero, then S has rational singularities. □

Proof. We first consider the case where K has characteristic $p > 0$. It is easily checked that S_{twz} is a regular ring. We may compute $H^7_{\mathfrak{m}}(S)$ using the Čech complex with respect to the system of parameters $s, u, x, v, y, w - t, z - t$. The socle of $H^7_{\mathfrak{m}}(S)$ is spanned by the element

$$\eta = \left[t^4 + (s, u, x, v, y, w - t, z - t)\right] \in H^7_{\mathfrak{m}}(S). \tag{5.6}$$

Since S_{twz} is regular, it suffices, by Lemma 5.2, to show that η belongs to the S-span of $twzF^e(\eta)$ for some $e \geq 1$, that is, that

$$t^4\big(suxvy(w-t)(z-t)\big)^{q-1} \in \big(twzt^{4q}, s^q, u^q, x^q, v^q, y^q, (w-t)^q, (z-t)^q\big)S \tag{5.7}$$

for some $q = p^e$. We will consider here the case $p \geq 5$, and the interested reader may verify that (5.7) holds with $q = 2^3$ and $q = 3^2$ in the remaining cases $p = 2$ and $p = 3$, respectively. It suffices to show that

$$t^4(suxvy)^{p-1} \in \big(t^{4p+3}, s^p, u^p, x^p, v^p, y^p, w-t, z-t\big)S. \tag{5.8}$$

Working in the polynomial ring $A = K[s,t,u,v,x,y]$, it is enough to check that $t^4(suxvy)^{p-1} \in \mathfrak{a} + (t^{5p-1})A$, where

$$\mathfrak{a} = \big(x^p, y^p, su^2x^2 + sv^2y^2 + tuxvy + t^5\big)A. \tag{5.9}$$

We observe that

$$\begin{aligned} t^{5p-1} &\equiv t^4\big(su^2x^2 + sv^2y^2 + tuxvy\big)^{p-1} \bmod \mathfrak{a} \\ &= t^4 \sum_{i,j} \binom{p-1}{i}\binom{p-1-i}{j} \big(su^2x^2\big)^i \big(sv^2y^2\big)^j (tuxvy)^{p-1-i-j} \bmod \mathfrak{a} \\ &= t^4 \sum_{i,j} \binom{p-1}{i}\binom{p-1-i}{j} s^{i+j} t^{p-1-i-j} (ux)^{p-1+i-j} (vy)^{p-1-i+j} \bmod \mathfrak{a}. \end{aligned} \tag{5.10}$$

The only terms which contribute $\bmod(x^p, y^p)$ are those for which $i = j$, and so

$$t^{5p-1} \equiv t^4 \sum_{i=0}^{(p-1)/2} \binom{p-1}{i}\binom{p-1-i}{i} s^{2i} t^{p-1-2i} (uxvy)^{p-1} \bmod \mathfrak{a}. \tag{5.11}$$

When $2i < p-1$, the corresponding summand in the above expression is a multiple of $t^5(uxvy)^{p-1}$, which is an element of $\mathfrak{a}$. Thus

$$t^{5p-1} \equiv t^4 \binom{p-1}{(p-1)/2} s^{p-1} (uxvy)^{p-1} \bmod \mathfrak{a}. \tag{5.12}$$

Since the binomial coefficient occurring above is a unit, $t^4(suxvy)^{p-1} \in \mathfrak{a} + (t^{5p-1})A$, which completes the proof that S is F-regular.

It remains to show that S has rational singularities in the case where K has characteristic zero. By [31, Theorem 4.3], it suffices to show that S has F-rational type, that is, that for all but finitely many prime integers p, the fiber over $p\mathbb{Z}$ of the map

$$\mathbb{Z} \longrightarrow \frac{\mathbb{Z}[s,t,u,v,w,x,y,z]}{\left(su^2x^2 + sv^2y^2 + tuxvy + tw^2z^2\right)} \tag{5.13}$$

is an F-rational ring. While this is indeed true for *all* prime integers p, our earlier computation for $p \geq 5$ certainly suffices. ■

We next construct unique factorization domains with similar behavior.

Theorem 5.4. Let K be an arbitrary field, and consider the hypersurface

$$T = \frac{K[r,s,t,u,v,w,x,y,z]}{\left(su^2x^2 + sv^2y^2 + tuxvy + rw^2z^2\right)}. \tag{5.14}$$

Then T is a unique factorization domain for which the local cohomology module $H^3_{(x,y,z)}(T)$ has infinitely many associated prime ideals. This is preserved if T is replaced by the localization at its homogeneous maximal ideal. The hypersurface T has rational singularities if K has characteristic zero, and is F-regular in the case of positive characteristic. □

Proof. It is easily verified that T is a normal domain; in particular, the element $t - r \in T$ is a nonzerodivisor. Note that

$$\frac{T}{(t-r)} \cong \frac{K[s,t,u,v,w,x,y,z]}{\left(su^2x^2 + sv^2y^2 + tuxvy + tw^2z^2\right)} \tag{5.15}$$

is F-regular or F-rational by Lemma 5.3. The rational singularity property deforms by [6], and F-regularity deforms for Gorenstein rings by [11, Corollary 4.7]. It follows that T has rational singularities if K has characteristic zero, and is F-regular in the case of positive characteristic.

We next prove that T is a unique factorization domain. Consider the multiplicative system $W \subset T$ generated by the elements w and z. Since W is generated by prime elements, by Nagata's theorem, it suffices to verify that $W^{-1}T$ is a unique factorization domain, see [27, Theorem 6.3] or [7, Corollary 7.3]. But

$$W^{-1}T = K\left[s,t,u,v,x,y,w,w^{-1},z,z^{-1}\right] \tag{5.16}$$

is a localization of a polynomial ring, and hence is a unique factorization domain.

For integers $n \geq 1$, consider

$$\eta_n = \left[s(ux)(vy)^{n-1} + \left(x^n, y^n, z\right)\right] \in H^3_{(x,y,z)}(T). \tag{5.17}$$

As in the proof of Theorem 5.1, we use Proposition 2.2 and Lemma 4.4(2) to compute $\operatorname{ann}_{T_0}\eta_n$, where $T_0 = K[r,s,t]$. Setting $a = ux$, $b = vy$, and $c = wz$, we see that

$$\operatorname{ann}_{T_0}\eta_n = (Q_{n-1})T_0. \tag{5.18}$$

By Lemmas 2.4 and 3.3, it follows that $H^3_{(x,y,z)}(T)$ has infinitely many associated prime ideals. ∎

6 An application to tight closure theory

Let R be a ring of characteristic $p > 0$ and let R° denote the complement of the minimal primes of R. For an ideal $\mathfrak{a} = (x_1, \ldots, x_n)$ of R and a prime power $q = p^e$, we use the notation $\mathfrak{a}^{[q]} = (x_1^q, \ldots, x_n^q)$. The *tight closure* of $\mathfrak{a}$ is the ideal $\mathfrak{a}^* = \{z \in R \mid$ there exists $c \in R^\circ$ for which $cz^q \in \mathfrak{a}^{[q]}$ for all $q \gg 0\}$, see [10]. A ring R is *F-regular* if $\mathfrak{a}^* = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of R and its localizations.

More generally, let F denote the Frobenius functor and F^e its eth iteration. If an R-module M has presentation matrix (a_{ij}), then $F^e(M)$ has presentation matrix (a_{ij}^q), where $q = p^e$. For modules $N \subseteq M$, we use $N_M^{[q]}$ to denote the image of $F^e(N) \to F^e(M)$. We say that an element $m \in M$ is in the tight closure of N in M, denoted N_M^*, if there exists an element $c \in R^\circ$ such that $cF^e(m) \in N_M^{[q]}$ for all $q \gg 0$. While the theory has found several applications, the question whether tight closure commutes with localization remains open even for finitely generated algebras over fields of positive characteristic.

Let W be a multiplicative system in R and $N \subseteq M$ finitely generated R-modules. Then

$$W^{-1}(N_M^*) \subseteq (W^{-1}N)^*_{W^{-1}M}, \tag{6.1}$$

where $W^{-1}(N_M^*)$ is identified with its image in $W^{-1}M$. When this inclusion is an equality, we say that *tight closure commutes with localization at W for the pair* $N \subseteq M$. It may be checked that this occurs if and only if tight closure commutes with localization at W for the pair $0 \subseteq M/N$. Following [1], we set

$$G^e(M/N) = \frac{F^e(M/N)}{0^*_{F^e(M/N)}}. \tag{6.2}$$

An element $c \in R^{\circ}$ is a *weak test element* if there exists $q_0 = p^{e_0}$ such that for every pair of finitely generated modules $N \subseteq M$, an element $m \in M$ is in N^*_M if and only if $cF^e(m) \in N^{[q]}_M$ for all $q \geq q_0$. By [11, Theorem 6.1], if R is of finite type over an excellent local ring, then R has a weak test element.

Proposition 6.1 [1, Lemma 3.5]**.** Let R be a ring of characteristic $p > 0$ and $N \subseteq M$ finitely generated R-modules. Then the tight closure of $N \subseteq M$ commutes with localization at any multiplicative system W which is disjoint from the set $\bigcup_{e\in\mathbb{N}} \operatorname{Ass} F^e(M)/N^{[q]}_M$.

If R has a weak test element, then the tight closure of $N \subseteq M$ also commutes with localization at multiplicative systems W disjoint from the set $\bigcup_{e\in\mathbb{N}} \operatorname{Ass} G^e(M/N)$. $\square$

Consider a bounded complex $P_\bullet$ of finitely generated projective R-modules:

$$0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_1} P_0 \longrightarrow 0. \tag{6.3}$$

The complex $P_\bullet$ is said to have *phantom homology* at the ith spot if

$$\operatorname{Ker} d_i \subseteq \left(\operatorname{Im} d_{i+1}\right)^*_{P_i}. \tag{6.4}$$

The complex $P_\bullet$ is *stably phantom acyclic* if $F^e(P_\bullet)$ has phantom homology at the ith spot for all $i \geq 1$, for all $e \geq 1$. An R-module M has *finite phantom projective dimension* if there exists a bounded stably phantom acyclic complex $P_\bullet$ of projective R-modules with $H_0(P_\bullet) \cong M$.

Theorem 6.2 [1, Theorem 8.1]**.** Let R be an equidimensional ring of positive characteristic, which is of finite type over an excellent local ring. If $N \subseteq M$ are finitely generated R-modules such that M/N has finite phantom projective dimension, then the tight closure of N in M commutes with localization at W for every multiplicative system W of R. $\square$

The key points of the proof are that for M/N of finite phantom projective dimension, the set $\bigcup_e \operatorname{Ass} G^e(M/N)$ has finitely many maximal elements, and that if $(R, \mathfrak{m})$ is a local ring, then there is a positive integer B such that for all $q = p^e$, the ideal $\mathfrak{m}^{Bq}$ kills the local cohomology module

$$H^0_{\mathfrak{m}}\left(G^e\left(\frac{M}{N}\right)\right). \tag{6.5}$$

For more details on this approach to the localization problem, we refer the reader to the papers [1, 9, 18, 28] and [16, Section 12]. Specializing to the case where $M = R$ and $N = \mathfrak{a}$

is an ideal, we note that

$$G^e\left(\frac{R}{\mathfrak{a}}\right) \cong \frac{R}{\left(\mathfrak{a}^{[q]}\right)^*}, \quad q = p^e. \tag{6.6}$$

This raises the following questions.

Question 6.3 [9, page 90]. Let R be a Noetherian ring of characteristic $p > 0$ and $\mathfrak{a}$ an ideal of R.

(1) Does the set $\bigcup_q \operatorname{Ass} R/\mathfrak{a}^{[q]}$ have finitely many maximal elements?
(2) Does $\bigcup_q \operatorname{Ass} R/(\mathfrak{a}^{[q]})^*$ have finitely many maximal elements?
(3) For a complete local domain $(R, \mathfrak{m})$ and an ideal $\mathfrak{a} \subset R$, is there a positive integer B such that

$$\mathfrak{m}^{Bq} H^0_{\mathfrak{m}}\left(\frac{R}{\left(\mathfrak{a}^{[q]}\right)^*}\right) = 0 \quad \forall q = p^e? \tag{6.7}$$

Katzman proved that affirmative answers to Question 6.3(2) and (3) imply that tight closure commutes with localization.

Theorem 6.4 [18]**.** Assume that for every local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ and ideal $\mathfrak{a} \subset R$, the set $\bigcup_q \operatorname{Ass} R/(\mathfrak{a}^{[q]})^*$ has finitely many maximal elements. Also, if for every ideal $\mathfrak{a} \subset R$ there exists a positive integer B such that $\mathfrak{m}^{Bq}$ kills

$$H^0_{\mathfrak{m}}\left(\frac{R}{\left(\mathfrak{a}^{[q]}\right)^*}\right) \quad \forall q = p^e, \tag{6.8}$$

then tight closure commutes with localization for all ideals in Noetherian rings of characteristic $p > 0$. □

These issues are, of course, the source of our interest in associated primes of Frobenius powers of ideals. It should be mentioned that the situation for *ordinary* powers is well understood: $\bigcup_n \operatorname{Ass} R/\mathfrak{a}^n$ is finite for any Noetherian ring R, see [2, 26]. However, for Frobenius powers, Katzman showed that the maximal elements of $\bigcup_q \operatorname{Ass} R/\mathfrak{a}^{[q]}$ need not form a finite set, thereby settling Question 6.3(1). We recall the example from [18], discussed earlier in Remark 2.7: if

$$A = \frac{K[t, x, y]}{(xy(x - y)(x - ty))}, \tag{6.9}$$

then the set $\bigcup_q \operatorname{Ass} R/(x^q, y^q)$ is infinite. In this example $(x^q, y^q)^* = (x, y)^q$ for all $q = p^e$, and so, in contrast, $\bigcup_q \operatorname{Ass} A/(x^q, y^q)^*$ is finite.

Remark 6.5. In Theorem 5.1, we constructed an F-regular ring S for which the set $\operatorname{Ass} H^3_{(x,y,z)}(S)$ is infinite. By Proposition 2.1, we have

$$\operatorname{Ass} H^3_{(x,y,z)}(S) \subseteq \bigcup_{q=p^e} \operatorname{Ass} \frac{S}{(x^q, y^q, z^q)}, \tag{6.10}$$

and it follows that $\bigcup_q \operatorname{Ass} S/(x^q, y^q, z^q)$ must be infinite. Since S is F-regular, we have $(x^q, y^q, z^q)^* = (x^q, y^q, z^q)$ for all $q = p^e$, and so $\bigcup_q \operatorname{Ass} S/(x^q, y^q, z^q)^*$ is infinite. The question remains whether $\bigcup_q \operatorname{Ass} R/(\mathfrak{a}^{[q]})^*$ has finitely many *maximal* elements for arbitrary rings R of characteristic $p > 0$, and we next show that this has a negative answer as well, thereby settling Question 6.3(2).

Theorem 6.6. Let K be a field of characteristic $p > 0$, and consider

$$R = \frac{K[t, u, v, w, x, y, z]}{(u^2x^2 + v^2y^2 + tuxvy + tw^2z^2)}. \tag{6.11}$$

Then R is an F-regular ring, and the set

$$\bigcup_{e \in \mathbb{N}} \operatorname{Ass} \frac{R}{(x^{p^e}, y^{p^e}, z^{p^e})} = \bigcup_{e \in \mathbb{N}} \operatorname{Ass} \frac{R}{(x^{p^e}, y^{p^e}, z^{p^e})^*} \tag{6.12}$$

has infinitely many maximal elements. □

Proof. By Lemma 5.3, the hypersurface (5.1) is F-regular, and therefore so is its localization

$$S_s = \frac{K\left[\dfrac{t}{s}, u, v, w, x, y, z, s, \dfrac{1}{s}\right]}{\left(u^2x^2 + v^2y^2 + \dfrac{t}{s}uxvy + \dfrac{t}{s}w^2z^2\right)}. \tag{6.13}$$

The ring S_s has a $\mathbb{Z}$-grading, where $\deg s = 1$, $\deg 1/s = -1$, and the remaining generators t/s, u, v, w, x, y, and z have degree 0. By [10, Proposition 4.12] a direct summand of an F-regular ring is F-regular, and so $R \cong [S_s]_0$ is F-regular.

For $q = p^e$, consider the ideals of R:

$$\mathfrak{a}_q = (x^q, y^q, z^q)R :_R t^q u v^{q-2} x^2 y^{q-1} z^{q-1}. \tag{6.14}$$

Let $R_0 = K[t]$. As in the proof of Theorem 5.1, we may use Proposition 2.2 and Lemma 4.4(3) to verify that

$$\begin{aligned} \mathfrak{a}_q \cap R_0 &= \left(x^q, y^q, z^q\right)R :_{R_0} t^q(ux)(vy)^{q-2}xyz^{q-1} \\ &= \left(x^{q-1}, y^{q-1}, z\right)R :_{R_0} t^q(ux)(vy)^{q-2} \\ &= P_{q-2} :_{R_0} t^q, \end{aligned} \tag{6.15}$$

where the P_i are the polynomials defined recursively in Section 3. In particular, this shows that $\mathfrak{a}_q \neq R$ for $q \gg 0$. It is immediately seen that $x, y, z \in \sqrt{\mathfrak{a}_q}$, and we claim that $u, v, w \in \sqrt{\mathfrak{a}_q}$. To see that $u \in \mathfrak{a}_q$, note that

$$u\left(t^q uv^{q-2}x^2y^{q-1}z^{q-1}\right) = t^q\left(u^2x^2\right)v^{q-2}y^{q-1}z^{q-1} \in \left(y^q, z^q\right). \tag{6.16}$$

Next, observe that

$$(vy)^2 \in ux(ux, vy)R + zR, \tag{6.17}$$

and so

$$(vy)^{q-1} \in (ux)^{q-2}(ux, vy)R + zR. \tag{6.18}$$

Using this,

$$v\left(t^q uv^{q-2}x^2y^{q-1}z^{q-1}\right) = t^q(vy)^{q-1}ux^2z^{q-1} \in \left(x^q, z^q\right), \tag{6.19}$$

and so $v \in \mathfrak{a}_q$. Finally, it is easily verified that $w^{q-1} \in \mathfrak{a}_q$, that is,

$$w^{q-1}\left(st^q uv^{q-2}x^2y^{q-1}z^{q-1}\right) \in \left(x^q, y^q, z^q\right) \tag{6.20}$$

since $t^q(wz)^{q-1} \in (x^{q-2}, y)$. We have now established

$$\operatorname{Min}\left(\mathfrak{a}_q\right) = \operatorname{Min}\left((u, v, w, x, y, z)R + \left(P_{q-2} :_{R_0} t^q\right)R\right), \tag{6.21}$$

and so the minimal primes of $\mathfrak{a}_q$ are maximal ideals of R. By Lemma 3.3, the union $\bigcup_q \operatorname{Min}(\mathfrak{a}_q)$ is an infinite set, and so we conclude that $\bigcup_q \operatorname{Ass} R/(x^q, y^q, z^q)$ has infinitely many maximal elements. ∎

Remark 6.7. We would like to point out that the ring R in Theorem 6.6 is a unique factorization domain if $K = \mathbb{Z}/p\mathbb{Z}$, where p is a prime with $p \equiv 3 \bmod 4$ or, more generally, if K does not contain a square root of -1. In this case, the polynomial $u^2x^2 + v^2y^2$ is irreducible, so $f = uxvy + w^2z^2 \in R$ is a prime element. The ring R_f is a localization of $K[u, v, w, x, y, z]$, hence is a unique factorization domain. By Nagata's theorem, it follows that R is a unique factorization domain.

For examples which do not depend on the field K, the interested reader may verify that

$$S = \frac{K(r)[t, u, v, w, x, y, z]}{\left(u^2x^2 + v^2y^2 + tuxvy + rw^2z^2\right)} \tag{6.22}$$

is an F-regular unique factorization domain for which the set

$$\bigcup_{e\in\mathbb{N}} \operatorname{Ass} \frac{S}{\left(x^{p^e}, y^{p^e}, z^{p^e}\right)} = \bigcup_{e\in\mathbb{N}} \operatorname{Ass} \frac{S}{\left(x^{p^e}, y^{p^e}, z^{p^e}\right)^*} \tag{6.23}$$

has infinitely many maximal elements.

7 Examples of small dimension

We analyze multidiagonal matrices with $d = 4$ and use these computations to obtain low-dimensional examples of integral domains of characteristic $p > 0$, where the set of associated primes of Frobenius powers of an ideal is infinite. The example in Theorem 4.1, after specializing $s = 1$, is an integral domain of dimension four. We construct here a hypersurface A of dimension two, which is an integral domain and has an ideal $(x, y)A$ for which $\bigcup_e \operatorname{Ass} A/(x^{p^e}, y^{p^e})$ is infinite. In view of Proposition 3.1, to construct such an example using Theorem 2.6, we need to consider multidiagonal matrices with $d \geq 4$.

We start with the polynomial ring $A_0 = K[t]$ over a field K. Let $d = 4$ and consider the matrices M_n of multidiagonal form with respect to $r_0 = r_4 = 1$, $r_2 = t$, and $r_1 = r_3 = 0$, that is,

$$M_n = \begin{bmatrix} t & 0 & 1 & & & & \\ 0 & t & 0 & 1 & & & \\ 1 & 0 & t & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 0 & t & 0 & 1 \\ & & & 1 & 0 & t & 0 \\ & & & & 1 & 0 & t \end{bmatrix}. \tag{7.1}$$

We again use the convention $\det M_0 = 1$, and have $\det M_1 = t$, $\det M_2 = t^2$, $\det M_3 = t^3 - t$, and the recursion

$$\det M_{n+4} = t \det M_{n+3} - t \det M_{n+1} + \det M_n \quad \forall n \geq 0. \tag{7.2}$$

Using this, the generating function for $\det M_n$ is easily computed to be

$$\begin{aligned} G(x) &= \sum_{n \geq 0} \left(\det M_n \right) x^n \\ &= \frac{1}{1 - tx + tx^3 - x^4} \\ &= \frac{1}{(1-x)(1+x)\left(1 - tx + x^2\right)}. \end{aligned} \tag{7.3}$$

Set $F_n(t) = \det M_n$, which is a monic polynomial of degree n. We need to analyze the distinct irreducible factors of the polynomials $\{F_n(t)\}$.

Lemma 7.1. Let K be an algebraically closed field, and consider the polynomials $F_n(t) = \det M_n \in K[t]$ as above.

(1) Let ξ be a nonzero element of K with $\xi \neq \pm 1$. If n is an odd integer, then

$$F_n\left(\xi + \xi^{-1}\right) = \frac{\left(\xi^{n+3} - 1\right)\left(\xi^{n+1} - 1\right)}{\xi^n \left(\xi^2 - 1\right)^2}, \tag{7.4}$$

and so $F_n(\xi + \xi^{-1}) = 0$ if and only if $\xi^{n+3} = 1$ or $\xi^{n+1} = 1$.

(2) If n is an odd integer and $(n+3)(n+1)$ is invertible in K, then the polynomial $F_n(t)$ has n distinct roots of the form $\xi + \xi^{-1}$, where $\xi \neq \pm 1$, and either $\xi^{n+3} = 1$ or $\xi^{n+1} = 1$.

(3) If the characteristic of K is an odd prime p, then $F_{q-2}(t)$ has $q - 2$ distinct roots for all $q = p^e$. □

Proof. (1) Consider the generating function for the polynomials $F_n(t)$:

$$G(x) = \sum_{n \geq 0} F_n(t) x^n = \frac{1}{(1-x)(1+x)\left(1 - tx + x^2\right)} \in K[t][[x]]. \tag{7.5}$$

If $\xi \in K$ with $\xi \neq 0$ and $\xi \neq \pm 1$, then

$$\begin{aligned}\sum_{n \geq 0} F_n\left(\xi + \xi^{-1}\right)x^n &= \frac{1}{(1-x)(1+x)(1-\xi x)\left(1-\xi^{-1}x\right)} \\ &= \frac{\sum x^n}{2\left(2-\xi-\xi^{-1}\right)} + \frac{\sum (-x)^n}{2\left(2+\xi+\xi^{-1}\right)} \\ &\quad + \frac{\xi^3 \sum (\xi x)^n}{\left(\xi^2-1\right)\left(\xi-\xi^{-1}\right)} + \frac{\xi^{-3} \sum \left(\xi^{-1}x\right)^n}{\left(\xi^{-2}-1\right)\left(\xi^{-1}-\xi\right)}.\end{aligned} \tag{7.6}$$

Comparing the coefficients of x^n and simplifying, we obtain the asserted formula for $F_n(\xi + \xi^{-1})$.

(2) As we observed earlier in the proof of Lemma 3.2(2), $\xi + \xi^{-1} = \eta + \eta^{-1}$ if and only if ξ equals η or η^{-1}. The only common roots of the polynomials $X^{n+3} - 1 = 0$ and $X^{n+1} - 1 = 0$ are ± 1. Since $n + 3$ is invertible in the field K, the polynomial $X^{n+3} - 1 = 0$ has $n + 1$ distinct roots ξ with $\xi \neq \pm 1$. These give the $(n + 1)/2$ distinct roots $\xi + \xi^{-1}$ of $F_n(t)$. Similarly, the roots of $X^{n+1} - 1 = 0$ contribute $(n-1)/2$ other distinct roots of $F_n(t)$. But then we have $(n+1)/2 + (n-1)/2 = n$ distinct roots of the degree-n polynomial $F_n(t)$, which, then, must be all its roots.

(3) Since $n = q - 2$ is odd and $(n + 3)(n + 1) = (q + 1)(q - 1)$ is invertible in K, it follows from (2) that $F_{q-2}(t)$ has $q - 2$ distinct roots. ■

As a consequence of Lemma 7.1, we immediately have the following lemma.

Lemma 7.2. Let K be an arbitrary field of characteristic $p > 2$. Then the polynomials $\{F_{q-2}(t)\}_{q=p^e}$ have infinitely many distinct irreducible factors. □

Theorem 7.3. Let K be an arbitrary field of characteristic $p > 2$, and consider the integral domain

$$A = \frac{K[t, x, y]}{\left(x^4 + tx^2y^2 + y^4\right)}. \tag{7.7}$$

Then the set $\bigcup_{e \in \mathbb{N}} \operatorname{Ass} A/(x^{p^e}, y^{p^e})$ is infinite. □

Proof. The hypersurface A arises from Theorem 2.6 using the matrices M_n of multidiagonal form with respect to $r_0 = r_4 = 1$, $r_2 = t$, and $r_1 = r_3 = 0$. By Lemma 7.2, the set $\bigcup_e \operatorname{Min}(\det M_{p^e-2})$ is infinite, and so the result follows. ■

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References

[1] I. M. Aberbach, M. Hochster, and C. Huneke, *Localization of tight closure and modules of finite phantom projective dimension*, J. reine angew. Math. **434** (1993), 67–114.

[2] M. Brodmann, *Asymptotic stability of* $\operatorname{Ass}(M/I^n M)$, Proc. Amer. Math. Soc. **74** (1979), no. 1, 16–18.

[3] M. P. Brodmann and A. L. Faghani, *A finiteness result for associated primes of local cohomology modules*, Proc. Amer. Math. Soc. **128** (2000), no. 10, 2851–2853.

[4] M. P. Brodmann, M. Katzman, and R. Y. Sharp, *Associated primes of graded components of local cohomology modules*, Trans. Amer. Math. Soc. **354** (2002), no. 11, 4261–4283.

[5] M. P. Brodmann, Ch. Rotthaus, and R. Y. Sharp, *On annihilators and associated primes of local cohomology modules*, J. Pure Appl. Algebra **153** (2000), no. 3, 197–227.

[6] R. Elkik, *Singularités rationnelles et déformations*, Invent. Math. **47** (1978), no. 2, 139–147 (French).

[7] R. M. Fossum, *The Divisor Class Group of a Krull Domain*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 74, Springer-Verlag, New York, 1973.

[8] M. Hellus, *On the set of associated primes of a local cohomology module*, J. Algebra **237** (2001), no. 1, 406–419.

[9] M. Hochster, *The localization question for tight closure*, Commutative Algebra (International Conference, Vechta, 1994), Vechtaer Universitätsschriften, vol. 13, Verlag Druckerei Rucke GmbH, Cloppenburg, 1994, pp. 89–93.

[10] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), no. 1, 31–116.

[11] ———, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), no. 1, 1–62.

[12] ———, *Tight closure of parameter ideals and splitting in module-finite extensions*, J. Algebraic Geom. **3** (1994), no. 4, 599–670.

[13] ———, *Tight closure in equal characteristic zero*, in preparation.

[14] M. Hochster and J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Adv. Math. **13** (1974), 115–175.

[15] C. Huneke, *Problems on local cohomology*, Free Resolutions in Commutative Algebra and Algebraic Geometry (Sundance, Utah, 1990), Res. Notes Math., vol. 2, Jones and Bartlett, Massachusetts, 1992, pp. 93–108.

[16] ———, *Tight Closure and Its Applications*, CBMS Regional Conference Series in Mathematics, vol. 88, American Mathematical Society, Rhode Island, 1996.

[17] C. L. Huneke and R. Y. Sharp, *Bass numbers of local cohomology modules*, Trans. Amer. Math. Soc. **339** (1993), no. 2, 765–779.

[18] M. Katzman, *Finiteness of* $\bigcup_e \operatorname{Ass} F^e(M)$ *and its connections to tight closure*, Illinois J. Math. **40** (1996), no. 2, 330–337.

[19] ———, *An example of an infinite set of associated primes of a local cohomology module*, J. Algebra **252** (2002), no. 1, 161–166.

[20] K. Khashyarmanesh and Sh. Salarian, *On the associated primes of local cohomology modules*, Comm. Algebra **27** (1999), no. 12, 6191–6198.

[21] G. Lyubeznik, *Finiteness properties of local cohomology modules (an application of* D*-modules to commutative algebra)*, Invent. Math. **113** (1993), no. 1, 41–55.

[22] ———, *Finiteness properties of local cohomology modules: a characteristic-free approach*, J. Pure Appl. Algebra **151** (2000), no. 1, 43–50.

[23] ———, *Finiteness properties of local cohomology modules for regular local rings of mixed characteristic: the unramified case*, Comm. Algebra **28** (2000), no. 12, 5867–5882.

[24] T. Marley, *The associated primes of local cohomology modules over rings of small dimension*, Manuscripta Math. **104** (2001), no. 4, 519–525.

[25] T. Marley and J. C. Vassilev, *Cofiniteness and associated primes of local cohomology modules*, J. Algebra **256** (2002), no. 1, 180–193.

[26] L. J. Ratliff Jr., *On prime divisors of* I^n, n *large*, Michigan Math. J. **23** (1976), no. 4, 337–352.

[27] P. Samuel, *Lectures on Unique Factorization Domains*, Tata Institute of Fundamental Research Lectures on Mathematics, no. 30, Tata Institute of Fundamental Research, Bombay, 1964.

[28] R. Y. Sharp and N. Nossem, *Ideals in a perfect closure*, *linear growth of primary decompositions*, *and tight closure*, to appear in Trans. Amer. Math. Soc.

[29] A. K. Singh, p*-torsion elements in local cohomology modules*, Math. Res. Lett. **7** (2000), no. 2-3, 165–176.

[30] A. K. Singh and K. Watanabe, *Multigraded rings*, *rational singularities*, *and diagonal subalgebras*, in preparation.

[31] K. E. Smith, F*-rational rings have rational singularities*, Amer. J. Math. **119** (1997), no. 1, 159–180.

[32] R. Tajarod and H. Zakeri, *On the local-global principle and the finiteness of associated primes of local cohomology modules*, Math. J. Toyama Univ. **23** (2000), 29–40.

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