Divisor class groups of graded hypersurfaces

Anurag K. Singh and Sandra Spiroff

ABSTRACT. We demonstrate how some classical computations of divisor class groups can be obtained using the theory of rational coefficient Weil divisors and related results of Watanabe.

1. Introduction

The purpose of this note is to provide a simple technique to compute divisor class groups of affine normal hypersurfaces of the form

$$k[z, x_1, \ldots, x_d]/(z^n - g) \,,$$

where g is a weighted homogeneous polynomial in $x_1, \ldots, x_d$ of degree relatively prime to n. We use the theory of rational coefficient Weil divisors due to Demazure [**3**] and related results of Watanabe [**14**]. This provides an alternative approach to various classical examples found in Samuel's influential lecture notes [**10**], as well as to computations due to Lang [**6**] and Scheja and Storch [**11**]. While the computations we present here are subsumed by those of [**11**], our techniques are different. A key point in our approach is that the projective variety defined by a hypersurface as above is weighted projective space over k, and this makes for straightforward, elementary calculations.

Watanabe [**14**, page 206] pointed out that $\mathbb{Q}$-divisor techniques can be used to recover the classification of graded factorial domains of dimension two, originally due to Mori [**8**]. Robbiano has applied similar methods to a study of factorial and almost factorial schemes in weighted projective space [**9**].

2. $\mathbb{Q}$-divisors

We review some material from [**3**] and [**14**]. Let k be a field, and let X be a normal irreducible projective variety over k, with rational function field $k(X)$.

A *rational coefficient Weil divisor* or a $\mathbb{Q}$-*divisor* on X is a $\mathbb{Q}$-linear combination of irreducible subvarieties of X of codimension one. Let $D = \sum n_i V_i$ be a $\mathbb{Q}$-divisor, where V_i are distinct. Then $\lfloor D \rfloor$ is defined as

$$\lfloor D \rfloor = \sum \lfloor n_i \rfloor V_i \,,$$

where $\lfloor n \rfloor$ denotes the greatest integer less than or equal to n. We set

$$\mathcal{O}_X(D) = \mathcal{O}_X(\lfloor D \rfloor).$$

If each coefficient n_i occurring in D is nonnegative, we say that $D \geqslant 0$.

A $\mathbb{Q}$-divisor D is *ample* if nD is an ample Cartier divisor for some $n \in \mathbb{N}$. In this case, the *generalized section ring* corresponding to D is the ring

$$R(X, D) = \oplus_{j \geqslant 0} H^0(X, \mathcal{O}_X(jD)).$$

If $R = R(X, D)$, then the n-th Veronese subring of $R = R(X, D)$ is the ring

$$R^{(n)} = \oplus_{j \geqslant 0} H^0(X, \mathcal{O}_X(jnD)) = R(X, nD).$$

The following theorem, due to Demazure, implies that a normal $\mathbb{N}$-graded ring R is determined by a $\mathbb{Q}$-divisor on $\operatorname{Proj} R$.

THEOREM 2.1. [**3**, 3.5]. *Let R be an $\mathbb{N}$-graded normal domain, finitely generated over a field R_0. Let T be a homogeneous element of degree 1 in the fraction field of R. Then there exists a unique ample $\mathbb{Q}$-divisor D on $X = \operatorname{Proj} R$ such that*

$$R = \oplus_{j \geqslant 0} H^0(X, \mathcal{O}_X(jD))T^j.$$

We next recall a result of Watanabe, which expresses the divisor class group of R in terms of the divisor class group of X and a $\mathbb{Q}$-divisor corresponding to R.

THEOREM 2.2. [**14**, Theorem 1.6] *Let X be a normal irreducible projective variety over a field. Assume $\dim X \geqslant 1$ and let $D = \sum_{i=1}^r (p_i/q_i)V_i$ be a $\mathbb{Q}$-divisor on X where V_i are distinct irreducible subvarieties, $p_i, q_i \in \mathbb{Z}$ are relatively prime, and $q_i > 0$. Set*

$$R = \oplus_{j \geqslant 0} H^0(X, \mathcal{O}_X(jD))T^j.$$

Then there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\theta} \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(R) \longrightarrow \operatorname{coker} \alpha \longrightarrow 0,$$

where $\theta(1) = \operatorname{lcm}(q_i) \cdot D$, and $\alpha \colon \mathbb{Z} \longrightarrow \oplus_{i=1}^r \mathbb{Z}/q_i\mathbb{Z}$ is the map $1 \mapsto (p_i \mod q_i)_i$.

In the exact sequence above, $\operatorname{coker} \alpha$ is always a finite group. Moreover, if X is projective space, a Grassmannian variety, or a smooth complete intersection in $\mathbb{P}^n$ of dimension at least three, then $\operatorname{Cl}(X) = \mathbb{Z}$. It follows that, in these cases, the divisor class group of $R(X, D)$ is finite for any ample $\mathbb{Q}$-divisor D on X, and hence that $R(X, D)$ is *almost factorial* in the sense of Storch [**12**].

Lipman proved that the divisor class group of a two-dimensional normal local ring R with rational singularities is finite, [**7**, Theorem 17.4]. While this is a hard result, the analogous statement for graded rings is a straightforward application of Theorem 2.2. Indeed, let R be an $\mathbb{N}$-graded normal ring of dimension two, finitely generated over an algebraically closed field R_0, such that R has rational singularities. Then R has a negative a-invariant by [**14**, Theorem 3.3], so $H^1(X, \mathcal{O}_X) = 0$ where $X = \operatorname{Proj} R$. But then X is a curve of genus 0 so it must be $\mathbb{P}^1$, and it follows that the divisor class group of R is finite.

REMARK 2.3. We note some aspects of Watanabe's proof of Theorem 2.2. Let $\operatorname{Div}(X)$ be the group of Weil divisors on X, and let

$$\operatorname{Div}(X, \mathbb{Q}) = \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$$

be the group of $\mathbb{Q}$-divisors. For D as in Theorem 2.2, set $\mathrm{Div}(X, D)$ to be the subgroup of $\mathrm{Div}(X, \mathbb{Q})$ generated by $\mathrm{Div}(X)$ and the divisors

$$\frac{1}{q_1}V_1, \ \ldots, \ \frac{1}{q_r}V_r \,.$$

Each element $E \in \mathrm{Div}(X, D)$ gives a divisorial ideal

$$\oplus_{j \geqslant 0} H^0(X, \mathcal{O}_X(E + jD))T^j$$

of R, and hence an element of $\mathrm{Cl}(R)$. The map $\mathrm{Div}(X, D) \longrightarrow \mathrm{Cl}(R)$ induces a surjective homomorphism

$$\mathrm{Div}(X, D)/\mathrm{Div}(X) \longrightarrow \mathrm{Cl}(R)/\mathrm{image}(\mathrm{Cl}(X)) \,.$$

3. Computing divisor class groups

The divisor class groups of affine surfaces of characteristic p defined by equations of the form $z^{p^n} = g(x, y)$ have been studied in considerable detail; such surfaces are sometimes called *Zariski surfaces*. In [**6**] Lang computed the divisor class group of hypersurfaces of the form $z^{p^n} = g(x_1, \ldots, x_d)$ where g is a homogeneous polynomial of degree relatively prime to p. The proposition below recovers [**6**, Proposition 3.11].

Let $A = k[x_1, \ldots, x_d]$ be a polynomial ring over a field. We say $g \in A$ is a *weighted homogeneous polynomial* if there exists an $\mathbb{N}$-grading on A, with $A_0 = k$, for which g is a homogeneous element.

PROPOSITION 3.1. *Let $R = k[z, x_1, \ldots, x_d]/(z^n - g)$ be a normal hypersurface over a field k, where $g \in k[x_1, \ldots, x_d]$ is a weighted homogeneous polynomial with degree relatively prime to n. Let $g = h_1 \cdots h_r$, where $h_i \in k[x_1, \ldots, x_d]$ are irreducible polynomials. Then*

$$\mathrm{Cl}(R) = (\mathbb{Z}/n\mathbb{Z})^{r-1} \,,$$

and the images of $(z, h_1), \ldots, (z, h_{r-1})$ form a minimal generating set for $\mathrm{Cl}(R)$.

Note that if $n \geqslant 2$, then the hypothesis that R is normal forces $h_1, \ldots, h_r$ to be pairwise coprime irreducible polynomials.

PROOF OF PROPOSITION 3.1. The polynomial ring $k[x_1, \ldots, x_d]$ has a grading under which $\deg x_i = c_i$ for $c_i \in \mathbb{N}$, and the degree of g is an integer m relatively prime to n. We assume, without any loss of generality, that $\gcd(c_1, \ldots, c_d) = 1$. Consider the $\mathbb{N}$-grading on R where $\deg x_i = nc_i$ and $\deg z = m$. Note that under this grading $\deg g = \sum \deg h_i = mn$. The n-th Veronese subring of R is

$$R^{(n)} = k[z^n, x_1, \ldots, x_d]/(z^n - g) = k[x_1, \ldots, x_d] \,,$$

which is a polynomial ring in $x_1, \ldots, x_d$. Let $X = \mathrm{Proj}\, R^{(n)} = \mathrm{Proj}\, R$.

There exist integers s_i, a, and b such that $\sum_{i=1}^d s_i c_i = 1$ and $am + bn = 1$. Consider the $\mathbb{Q}$-divisor on X given by

$$D = b\,\mathrm{div}(\mathbf{x}) + \frac{a}{n}\,\mathrm{div}(g) = b\sum_{i=1}^d s_i V(x_i) + \frac{a}{n}\sum_{i=1}^r V(h_i) \,,$$

where $\mathbf{x} = x_1^{s_1} \cdots x_d^{s_d}$. We claim that

(3.1.1) $$R = \oplus_{j \geqslant 0} H^0(X, \mathcal{O}_X(jD))T^j \,,$$

where $T = z^a \mathbf{x}^b$ is a homogeneous degree 1 element of the fraction field of R. First note that $\lfloor am/n \rfloor = \lfloor (1 - bn)/n \rfloor = -b$, so

$$\lfloor mD \rfloor = bm \operatorname{div}(\mathbf{x}) + \left\lfloor \frac{am}{n} \right\rfloor \operatorname{div}(g) = bm \operatorname{div}(\mathbf{x}) - b \operatorname{div}(g) \,.$$

Consequently $\deg \lfloor mD \rfloor = 0$, and $H^0(X, \mathcal{O}_X(mD))T^m$ is the k-vector space spanned by the element

$$\mathbf{x}^{-bm} g^b T^m = \mathbf{x}^{-bm} (z^n)^b (z^a \mathbf{x}^b)^m = z^{bn+am} = z \,.$$

Let $c = c_t$ for an integer $1 \leqslant t \leqslant d$. Then $ncD = bnc \operatorname{div}(\mathbf{x}) + ac \operatorname{div}(g)$ has degree nc, and $H^0(X, \mathcal{O}_X(ncD))T^{nc}$ contains the element

$$x_t \mathbf{x}^{-bnc} g^{-ac} T^{nc} = x_t \mathbf{x}^{-bnc} (z^n)^{-ac} (z^a \mathbf{x}^b)^{nc} = x_t \,.$$

To prove the claim (3.1.1), it remains to verify that $z, x_1, \ldots, x_d$ are k-algebra generators for the ring $\oplus_{j \geqslant 0} H^0(X, \mathcal{O}_X(jD))T^j$. An arbitrary positive integer j can be written as $um + vn$ for $0 \leqslant u \leqslant n - 1$. We then have

$$\lfloor jD \rfloor = b(um + vn) \operatorname{div}(\mathbf{x}) + \left\lfloor \frac{a(um + vn)}{n} \right\rfloor \operatorname{div}(g)$$
$$= b(um + vn) \operatorname{div}(\mathbf{x}) + (va - ub) \operatorname{div}(g) \,,$$

which has degree vn. Consequently $H^0(X, \mathcal{O}_X(jD))T^j$ vanishes if v is negative, and for nonnegative v, it is spanned by elements

$$\mu \mathbf{x}^{-b(um+vn)} g^{-va+ub} T^{um+vn} = \mu z^u \,,$$

for monomials μ in x_i of degree v. This completes the proof of (3.1.1).

Since nD has integer coefficients, the exact sequence of Theorem 2.2 for the divisor nD and corresponding ring $R^{(n)}$ reduces to

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;\theta\;} \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(R^{(n)}) \longrightarrow 0 \,,$$

where $\theta(1) = nD$. Since $R^{(n)}$ is a polynomial ring, and hence factorial, it follows that nD generates $\operatorname{Cl}(X)$. Next, consider the exact sequence applied to the divisor D and corresponding ring R, i.e., the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\;\theta\;} \operatorname{Cl}(X) \longrightarrow \operatorname{Cl}(R) \longrightarrow \operatorname{coker} \alpha \longrightarrow 0 \,.$$

The lcm of the denominators occurring in D is n, so we once again have $\theta(1) = nD$. Consequently θ is an isomorphism and $\operatorname{Cl}(R) = \operatorname{coker} \alpha$, where

$$\alpha \colon \mathbb{Z} \longrightarrow \bigoplus_1^r \mathbb{Z}/n\mathbb{Z} \qquad \text{with} \qquad \alpha(1) = (a, \ldots, a) \,.$$

Since a and n are relatively prime, it follows that

$$\operatorname{Cl}(R) = (\mathbb{Z}/n\mathbb{Z})^{r-1} \,.$$

We next determine explicit generators for $\operatorname{Cl}(R)$ by Remark 2.3. The $\mathbb{Q}$-divisors

$$E_t = -\frac{1}{n} V(h_t) \qquad \text{for } 1 \leqslant t \leqslant r$$

give a generating set for $\operatorname{Div}(X, D)/\operatorname{Div}(X)$ which surjects onto $\operatorname{Cl}(R)$. Hence the divisorial ideals

$$\mathfrak{p}_t = \oplus_{j \geqslant 0} H^0(X, \mathcal{O}_X(E_t + jD))T^j \qquad \text{where } 1 \leqslant t \leqslant d \,,$$

generate $\mathrm{Cl}(R)$. The computation of $\mathfrak{p}_t$ is straightforward, and we give a brief sketch. First note that

$$\lfloor E_t + mD \rfloor = bm\,\mathrm{div}(x) + \left\lfloor \frac{am-1}{n} \right\rfloor V(h_t) + \sum_{i \neq t} \left\lfloor \frac{am}{n} \right\rfloor V(h_i)$$

$$= bm\,\mathrm{div}(\mathbf{x}) - b\,\mathrm{div}(g)\,,$$

so $H^0(X, \mathcal{O}_X(E_t + mD))T^m$ is the k-vector space spanned by

$$\mathbf{x}^{-bm}g^bT^m = z.$$

Since the degree of each x_i is a multiple of n, we have $\deg h_t = n\gamma$ for some integer γ. We next compute the component of $\mathfrak{p}_t$ in degree $n\gamma$. Note that

$$\lfloor E_t + n\gamma D \rfloor = -V(h_t) + bn\gamma\,\mathrm{div}(\mathbf{x}) + a\gamma\,\mathrm{div}(g)\,,$$

so $H^0(X, \mathcal{O}_X(E_t + n\gamma D))T^{n\gamma}$ is the k-vector space spanned by

$$h_t\mathbf{x}^{-bn\gamma}g^{-a\gamma}T^{n\gamma} = h_t.$$

It is now a routine verification that z, h_t are generators for the ideal $\mathfrak{p}_t$, which, we note, is a height one prime of R. Consequently $\mathrm{Cl}(R)$ is generated by $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. Using $\sim$ to denote linear equivalence, we have

$$nE_t + n\gamma D \sim 0 \qquad \text{and} \qquad \sum_{i=1}^{r} E_i + mD \sim 0\,,$$

implying that $n[\mathfrak{p}_t] = 0$ and $\sum_i [\mathfrak{p}_i] = 0$ in $\mathrm{Cl}(R)$. These correspond to the calculations with divisorial ideals,

$$\mathfrak{p}_t^{(n)} = h_t R \qquad \text{and} \qquad \bigcap_{i=1}^{r} \mathfrak{p}_i = zR\,,$$

and imply, in particular, that $[\mathfrak{p}_1], \dots, [\mathfrak{p}_{r-1}]$ is a generating set for $\mathrm{Cl}(R)$. $\qquad \square$

EXAMPLE 3.2. We use Proposition 3.1 to compute the divisor class group of diagonal hypersurfaces

$$R = k[z, x_1, \dots, x_d]/(z^n - x_1^{m_1} - \cdots - x_d^{m_d})$$

where n is relatively prime to m_i for $1 \leqslant i \leqslant d$, and k is a field of characteristic zero, or of characteristic not dividing each m_i.

By the Jacobian criterion, R has an isolated singularity at the homogeneous maximal ideal $\mathfrak{m}$. Hence if $d \geqslant 4$, then R, as well as its $\mathfrak{m}$-adic completion $\widehat{R}$, are factorial by Grothendieck's parafactoriality theorem [5]; see [2] for a simple proof of Grothendieck's theorem.

Case $d = 3$. The polynomial $g = x_1^{m_1} + x_2^{m_2} + x_3^{m_3}$ is irreducible since $k[x_1, x_2, x_3]/(g)$ is a normal domain by the Jacobian criterion. We set $\deg x_i$ to be $m_1m_2m_3/m_i$. Then g is a weighted homogeneous polynomial of degree $m_1m_2m_3$, which is relatively prime to n, so Proposition 3.1 implies that R is factorial. Since R satisfies the Serre conditions (R_2) and (S_3), the completion $\widehat{R}$ is factorial as well by [4, Korollar 1.5]. The divisor class groups of rational three-dimensional Brieskorn singularities are computed in [1, Chapter IV]; see also [13].

Case $d = 2$. Let $g = x_1^{m_1} + x_2^{m_2}$. If $c = \gcd(m_1, m_2)$, let $m_1 = ac$ and $m_2 = bc$, and set $\deg x_1 = b$ and $\deg x_2 = a$. Let f be an irreducible factor of g. Then f is homogeneous, and hence has the form $\sum a_{ij}x_1^i x_2^j$ where $a_{ij} \in k$ and $bi + cj = \deg f$

for each term occurring in the summation. Since x_1 and x_2 do not divide g, we see that f must contain nonzero terms of the form $a_{0j}x_2^j$ and $a_{i0}x_1^i$. Hence $\deg f$ is a multiple of ab, and it follows that f is a polynomial in x_1^a and x_2^b. Consequently the number of factors of g in $k[x_1, x_2]$ is the number of factors of $s^c + t^c$ in $k[s, t]$ or, equivalently, the number of factors of $1 + t^c$ in $k[t]$.

In particular, if m_1 and m_2 are relatively prime, then g is irreducible and Proposition 3.1 implies that R is factorial. As is well-known, $\widehat{R}$ need not be factorial; see for example, [**10**, Theorem III.5.2].

If k is algebraically closed, then g is a product of c irreducible factors, and so Proposition 3.1 implies that

$$\mathrm{Cl}(R) = (\mathbb{Z}/n\mathbb{Z})^{c-1}\,.$$

REMARK 3.3. The condition that the degree of g is relatively prime to n is certainly crucial in Proposition 3.1. In the absence of this, $\mathrm{Cl}(R)$ need not be finite, for example $\mathbb{C}[z, x_1, x_2, x_3]/(z^3 - x_1^3 - x_2^3 - x_3^3)$ has divisor class group $\mathbb{Z}^6$. However, one can drop the relatively prime condition when considering hypersurfaces of the form $z^n - x_0 g(x_1, \ldots, x_d)$, see also [**6**, Proposition 3.12]:

COROLLARY 3.4. *Let $R = k[z, x_0, \ldots, x_d]/(z^n - x_0 g)$ be a normal hypersurface over a field k, where g is a weighted homogeneous polynomial in $x_1, \ldots, x_d$. Let $g = h_1 \cdots h_r$, where $h_i \in k[x_1, \ldots, x_d]$ are irreducible. Then*

$$\mathrm{Cl}(R) = (\mathbb{Z}/n\mathbb{Z})^r\,,$$

and the images of $(z, h_1), \ldots, (z, h_r)$ form a minimal generating set for $\mathrm{Cl}(R)$.

PROOF. We may choose the degree of x_0 such that $\deg(x_0 g)$ is relatively prime to n. The result then follows from Proposition 3.1. □

We conclude with the following example.

EXAMPLE 3.5. Let k be a field. Corollary 3.4 implies that the divisor class group of the ring $R = k[xy, x^n, y^n]$ is $\mathbb{Z}/n\mathbb{Z}$, since R is isomorphic to the hypersurface

$$k[z, x_0, x_1]/(z^n - x_0 x_1)\,.$$

In [**10**, Chapter III], the divisor class group of R is computed by Galois descent if n is relatively prime to the characteristic of k, and by using derivations if n equals the characteristic of k.

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Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, UT 84112, USA

E-mail address: `singh@math.utah.edu`

Mathematics Department, Seattle University, 901 12th Ave, Seattle, WA 98122, USA

E-mail address: `spiroffs@seattleu.edu`