TIGHT CLOSURE IN NON–EQUIDIMENSIONAL RINGS

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1. Introduction

Throughout our discussion, all rings are commutative, Noetherian and have an identity element. The notion of the *tight closure* of an ideal was developed by M. Hochster and C. Huneke in [HH1] and has yielded many elegant and powerful results in commutative algebra. The theory leads to the notion of *F–rational* rings, defined by R. Fedder and K.-i. Watanabe as rings in which parameter ideals are tightly closed, see [FW]. Over a field of characteristic zero, rings of F–rational type are now known to be precisely those having rational singularities by the work of K. E. Smith and N. Hara, see [Sm, Ha].

We begin by recalling a theorem of Hochster and Huneke which states that a local ring (R, m) which is a homomorphic image of a Cohen–Macaulay ring is F–rational if and only if it is equidimensional and has a system of parameters which generates a tightly closed ideal, [HH2, Theorem 4.2 (d)]. This leads to the question of whether a local ring in which a single system of parameters generates a tightly closed ideal must be equidimensional (and hence F–rational), #19 of Hochster's "Twenty Questions" in [Ho]. Rephrased, can a non–equidimensional ring have a system of parameters which generates a tightly closed ideal – we show it cannot for some classes of non–equidimensional rings.

A key point is that in equidimensional rings, tight closure has the so–called "colon capturing" property. This property does not hold in non–equidimensional rings. A study of these issues leads to a new closure operation, that we call *NE closure*. This closure does possess the colon capturing property even in non–equidimensional rings, and agrees with tight closure when the ring is equidimensional. We shall show that an excellent local ring R is F–rational if and only if it has a system of parameters which generates an NE–closed ideal.

2. Notation and terminology

Let R be a Noetherian ring of characteristic $p > 0$. We shall always use the letter e to denote a variable nonnegative integer, and q to denote the

eth power of p, i.e., $q = p^e$. For an ideal $I = (x_1, \dots, x_n) \subseteq R$, we let $I^{[q]} = (x_1^q, \dots, x_n^q)$. We shall denote by R^{o} the complement of the union of the minimal primes of R. For an ideal $I \subseteq R$ and an element x of R, we say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^{\mathrm{o}}$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$. If $I = I^*$, we say I is *tightly closed*.

An ideal $I \subseteq R$ is said to be a *parameter ideal* if $I = (x_1, \dots, x_n)$ such that the images of $x_1, \dots, x_n$ form a system of parameters in R_P, for every prime P containing I. The ring R is said to be *F–rational* if every parameter ideal of R is tightly closed.

We next recall some well known results.

Theorem 2.1.
(1) *An F–rational ring R is normal. If in addition R is assumed to be the homomorphic image of a Cohen–Macaulay ring, then R is Cohen–Macaulay.*
(2) *A local ring (R, m) which is the homomorphic image of a Cohen–Macaulay ring is F–rational if and only if it is equidimensional and the ideal generated by one system of parameters is tightly closed.*
(3) *Let $P_1, \dots, P_n$ be the minimal primes of the ring R. Then for an ideal $I \subseteq R$ and $x \in R$, we have $x \in I^*$ if and only if for $1 \le i \le n$, its image $\overline{x}$ is in $(IR/P_i)^*$, the tight closure here being computed in the domain R/P_i.*

Proof. (1) and (2) are part of [HH2, Theorem 4.2] and (3) is observed as [HH1, Proposition 6.25 (a)]. □

3. Main results

Lemma 3.1. *Let $P_1, \dots, P_n$ be the minimal primes of the ring R. Then for a tightly closed ideal I, we have $I = \bigcap_{i=1}^n (I + P_i)$.*

Proof. That $I \subseteq \bigcap_{i=1}^n (I + P_i)$, is trivial. For the other containment note that if $x \in \bigcap_{i=1}^n (I + P_i)$, then $\overline{x} \in (IR/P_i)^*$ for $1 \le i \le n$. Now by Theorem 2.1 (3), we get that $x \in I^* = I$. □

The following theorem, although it has some rather strong hypotheses, does show that no ideal generated by a system of parameters is tightly closed in the non–equidimensional rings

$$R = K[[X_1, \dots, X_n, Y_1, \dots, Y_m]]/(X_1, \dots, X_n) \cap (Y_1, \dots, Y_m)$$

where $m, n \ge 1$ and $m \ne n$.

Theorem 3.2. *Let (R, m) be a non–equidimensional local ring, with the minimal primes partitioned into the sets $\{P_i\}$ and $\{Q_j\}$, such that* $\dim R = \dim R/P_i > \dim R/Q_j$*, for all i and j. Let P and Q be the intersections, $P = \bigcap P_i$ and $Q = \bigcap Q_j$. If $I \subseteq P + Q$ is an ideal of R which is generated* *by a system of parameters, then I cannot be tightly closed. In particular, if $P+Q=m$, then no ideal of R generated by a system of parameters is tightly closed.*

Proof. Suppose not, let $I=(p_1+q_1,\dots,p_n+q_n)$ be a tightly closed ideal of R, where $p_1+q_1,\dots,p_n+q_n$ is a system of parameters with $p_i\in P$ and $q_i\in Q$. Note that we have

$$(I+P)\cap(I+Q)\subseteq(\cap(I+P_i))\cap(\cap(I+Q_j))=I$$

using Lemma 3.1. Consequently $(I+P)\cap(I+Q)\subseteq I$, and so $p_i,q_i\in I$. In particular, $p_i=r_1(p_1+q_1)+\cdots+r_n(p_n+q_n)$. We first note that if $r_i\notin m$ then $q_i\in(p_1+q_1,\dots,p_{i-1}+q_{i-1},p_i,p_{i+1}+q_{i+1},\dots,p_n+q_n)$, but then $p_i\in P$ is a parameter, which is impossible since $\dim R=\dim R/P$. Hence $r_i\in m$, and so $1-r_i$ is a unit. From this we may conclude that $p_i\in(p_1+q_1,\dots,p_{i-1}+q_{i-1},q_i,p_{i+1}+q_{i+1},\dots,p_n+q_n)$, and so the ideal I may be written as $I=(p_1+q_1,\dots,p_{i-1}+q_{i-1},q_i,p_{i+1}+q_{i+1},\dots,p_n+q_n)$.

Proceeding this way, we see that $I=(q_1,\dots,q_n)$, i.e., $I\subseteq Q$. But then each $Q_j=m$, a contradiction. □

Remark 3.3. We would next like to discuss briefly the case where the non–equidimensional local ring (R,m) is of the form $R=S/(P\cap Q)$ where S is a regular local ring with primes P and Q of different height. Then R has minimal primes $\overline{P}$ and $\overline{Q}$ where, without loss of generality, $\dim R=\dim R/\overline{P}>\dim R/\overline{Q}$. If I is an ideal of S whose image $\overline{I}$ in R is a tightly closed ideal, we see that $I+(P\cap Q)=(I+P)\cap(I+Q)$ by Lemma 3.1, and so it would certainly be enough to show that this cannot hold when $\overline{I}$ is generated by a system of parameters for R. One can indeed prove this in the case S/P is Cohen–Macaulay, and S/Q is a discrete valuation ring, which is Theorem 3.6 below. However if we drop the hypothesis that S/P be Cohen–Macaulay, this is no longer true: see Example 3.8.

Lemma 3.4. *If I, P, and Q are ideals of S, satisfying the condition that $I+(P\cap Q)=(I+P)\cap(I+Q)$, then $I\cap(P+Q)=(I\cap P)+(I\cap Q)$.*

Proof. Let $i=p+q\in I\cap(P+Q)$, where $p\in P$ and $q\in Q$. Then $i-p=q\in(I+P)\cap(I+Q)=I+(P\cap Q)$ and so $i-p=q=\tilde{i}+r$ where $\tilde{i}\in I$ and $r\in P\cap Q$. Finally note that $i=(i-\tilde{i})+\tilde{i}\in(I\cap P)+(I\cap Q)$, since $i-\tilde{i}=p+r\in I\cap P$ and $\tilde{i}=q-r\in I\cap Q$. □

Lemma 3.5. *Let M be an S–module and N, a submodule. If $x_1,\dots,x_n$ are elements of S which form a regular sequence on M/N, then*

$$(x_1,\dots,x_n)M\cap N=(x_1,\dots,x_n)N.$$

In particular, if I and J are ideals of S and I is generated by elements which form a regular sequence on S/J, then $I\cap J=IJ$.

Proof. We shall proceed by induction on n, the number of elements. If $n = 1$, the result is simple. For the inductive step, note that if we have $u = x_1m_1 + \cdots + x_km_k \in (x_1, \ldots, x_k)M \cap N$ with $m_i \in M$, then since x_k is not a zero divisor on the module $M/((x_1, \ldots, x_{k-1})M + N)$, we get that $m_k \in (x_1, \ldots, x_{k-1})M + N$. Consequently, $u \in (x_1, \ldots, x_{k-1})M + x_kN$. □

Theorem 3.6. *Let $S = K[[X_1, \ldots, X_n, Y]]$ with ideal $Q = (X_1, \ldots, X_n)S$, and ideal P satisfying the condition that S/P is Cohen–Macaulay. Then if $R = S/(P \cap Q)$ is a non–equidimensional ring, no ideal of R generated by a system of parameters can be tightly closed.*

Proof. Let I be an ideal of S generated by elements which map to a system of parameters in R. If the image of I is a tightly closed ideal in R, we have $(I + P) \cap (I + Q) = I + (P \cap Q)$ as ideals of S, by Lemma 3.1. Any element of the maximal ideal of S, up to multiplication by units, is either in Q, or is of the form $Y^h + q$, where $q \in Q$. Since I cannot be contained in Q, one of its generators has the form $Y^h + q$. Choosing the generator amongst these which has the least such positive value of h, and subtracting suitable multiples of this generator, we may assume that the other generators are in Q. We then have $I = (Y^h + q_1, q_2, \ldots, q_d)S$, where $q_i \in Q$, $h > 0$, and $d = \dim R = \dim S/P$. By a similar argument we may write P as $P = (Y^t + r_1, r_2, \ldots, r_k)S$, where $r_i \in Q$. Since we are assuming that the image of I is a tightly closed ideal in R, Theorem 3.2 shows that I is not contained in $P + Q$, and so we conclude $h < t$.

We then have $Y^t + Y^{t-h}q_1 \in I \cap (P + Q)$, and so by Lemma 3.4

$$Y^t + Y^{t-h}q_1 \in (I \cap P) + (I \cap Q).$$

By Lemma 3.5, $I \cap P = IP$ and consequently

$$Y^t \in IP + Q = (Y^t + r_1)(Y^h + q_1) + Q = Y^{t+h} + Q.$$

However this is impossible since $h > 0$. □

Remark 3.7. Note that in the proof above we used that if $\overline{I}$ is a tightly closed ideal, we must have $I + (P \cap Q) = (I + P) \cap (I + Q)$, and then showed that this cannot hold when $\overline{I}$ is generated by a system of parameters for R in the case S/P is Cohen–Macaulay, and S/Q is a discrete valuation ring. When S/P is not Cohen–Macaulay, this approach no longer works as seen from the following example.

Example 3.8. Let $S = K[[T, X, Y, Z]]$, and consider the two prime ideals $Q = (T, X, Y)$ and $P = (TY - XZ, T^2X - Z^2, TX^2 - YZ, X^3 - Y^2)$. Then S/Q is a discrete valuation ring, although S/P is not Cohen–Macaulay. To see this, observe that

$$S/P \cong K[[U^2, U^3, UT, T]] \subseteq K[[T, U]]$$

where T and U are indeterminates and $x \mapsto U^2$, $y \mapsto U^3$, $z \mapsto UT$ and $t \mapsto T$. (Lower case letters denote the images of the corresponding variables.)

Then $R = S/(P \cap Q)$ is a non–equidimensional ring and the image of $I = (Z, X - T)$ in R is $\overline{I} = (z, x - t)$ which is generated by a system of parameters for R. We shall see that $I + (P \cap Q) = (I + P) \cap (I + Q)$.

Since $I + Q = (T, X, Y, Z)$ is just the maximal ideal of S, we get that $(I + P) \cap (I + Q) = I + P = (Z, X - T, XY, X^3, Y^2)$. It can be verified (using Macaulay, or even otherwise) that

$$P \cap Q = (TY - XZ, TX^2 - YZ, X^3 - Y^2, T^3X - TZ^2)$$

and so

$$I + (P \cap Q) = (Z, X - T, XY, X^3, Y^2) = (I + P) \cap (I + Q).$$

For the ring R, although it does not follow from any of the earlier results, we can show that no system of parameters generates a tightly closed ideal.

We can actually prove the graded analogue of Theorem 3.6 without the requirement that S/P is Cohen–Macaulay.

Theorem 3.9. *Let $S = K[X_1, \ldots, X_n, Y]$ with ideal $Q = (X_1, \ldots, X_n)S$, and P a homogeneous unmixed ideal with $\dim S/P \geq 2$. Then no homogeneous system of parameters of the ring $R = S/(P \cap Q)$ generates a tightly closed ideal.*

Proof. Let I be an ideal of S generated by homogeneous elements which map to a system of parameters in R, and assume that the image of I is a tightly closed ideal of R.

As in the proof of Theorem 3.6, there is no loss of generality in taking as homogeneous generators for I, the elements $Y^h + q_1,\ q_2, \ldots, q_d$ where $q_i \in Q$, and $h > 0$, and for P the elements $Y^t + r_1,\ r_2, \ldots, r_k$, where $r_i \in Q$. One can easily formulate a graded analogue of Theorem 3.2 and then since we are assuming that the image of I is a tightly closed ideal in R, it follows that I is not contained in $P + Q$. Hence we conclude $h < t$.

The assumption implies that

$$Y^t + r_1 \in (I + P) \cap (I + Q) = I + (P \cap Q) = I + (r_2, \ldots, r_k) + (Y^t + r_1)Q$$

and so $Y^t + r_1 \in I + (r_2, \ldots, r_k)$. Hence

$$I + (P \cap Q) = I + P = I + (r_2, \ldots, r_k).$$

If S/P is Cohen–Macaulay, the proof is identical to that of Theorem 3.6, and so we may assume S/P is not Cohen–Macaulay. Consequently $(IS/P)^*$ is strictly bigger that IS/P. Let $F \in S$ be a homogeneous element such that its image is in $(IS/P)^*$ but not in IS/P. Note that if $F \in I + Q$, then $\overline{F} \in (IR)^*$, and so $F \in I + P$, a contradiction. Hence we conclude that $F \notin I + Q = (Y^h, X_1, \ldots, X_n)$ and so $F = Y^i + G$ where $i < h$ and $G \in Q$.

Next note that $Y^{h-i}F = Y^h + GY^{h-i} \in I + (P \cap Q) = I + (r_2, \ldots, r_k)$, and so $GY^{h-i} - q_1 \in (q_2, \ldots, q_d, r_2, \ldots, r_k)$. We then have

$$\overline{F} \in (IS/P)^* = ((Y^h + GY^{h-i}, q_2, \ldots, q_d)S/P)^* = (Y^{h-i}F, q_2, \ldots, q_d)S/P)^*.$$

By a degree argument, we see that $\overline{F} \in ((q_2, \ldots, q_d)S/P)^*$. However this means that $\overline{F}$ is in the radical of the ideal $(q_2, \ldots, q_d)S/P$, which contradicts the fact that $(FY^{h-i}, q_2, \ldots, q_d) = IS/P$ is primary to the homogeneous maximal ideal of S/P. $\square$

4. NE closure

For Noetherian rings of characteristic p we shall define a new closure operation on ideals, the *NE closure*, which will agree with tight closure when the ring is equidimensional. In non–equidimensional local rings, tight closure no longer has the so–called *colon–capturing* property, and the main point of NE closure is to recover this property. This often forces the NE closure of an ideal to be larger than its tight closure and at times even larger than its radical, see Example 5.5. More precisely let (R, m) be an excellent local ring with a system of parameters $x_1, \ldots, x_n$. Then when R is equidimensional we have $(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^*$, but this does not hold in general. The NE closure (denoted by $I^\bigstar$ for an ideal $I \subseteq R$) will have the property that $(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^\bigstar$.

Definition 4.1. We shall say that a minimal prime ideal P of a ring R is *absolutely minimal* if $\dim R/P = \dim R$. When Spec R is connected, $R^\bullet$ shall denote the complement in R of the union of all the absolutely minimal primes. If $R = \prod R_i$, we define $R^\bullet = \prod R_i^\bullet$. The *NE closure* $I^\bigstar$ of an ideal I is given by

$$I^\bigstar = \{x \in R : \text{there exists } c \in R^\bullet \text{ with } cx^{[q]} \in I^{[q]} \text{ for all } q \gg 0\}.$$

The following proposition and its proof are analogous to the statements for tight closure in equidimensional rings, see [HH2, Theorem 4.3].

Proposition 4.2. *Let R be a complete local ring of characteristic p, with a system of parameters $x_1, \ldots, x_n$. Then*

(1) $(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^\bigstar$.

(2) $(x_1, \ldots, x_k)^\bigstar : x_{k+1} = (x_1, \ldots, x_k)^\bigstar$.

(3) *If* $(x_1,\dots,x_{k+1})^\bigstar = (x_1,\dots,x_{k+1})$, *then* $(x_1,\dots,x_k)^\bigstar = (x_1,\dots,x_k)$.
(4) *If* $(x_1,\dots,x_n)^\bigstar = (x_1,\dots,x_n)$ *or* $(x_1,\dots,x_{n-1})^\bigstar = (x_1,\dots,x_{n-1})$, *then R is Cohen–Macaulay.*

Proof. (1) We may represent R as a module-finite extension of a regular subring A of the form $A = K[[x_1,\dots,x_n]]$ where K is a field. Let t be the torsion free rank of R as an A–module, and consider $A^t \subseteq R$. Then R/A^t is a torsion A–module and there exists $c \in A$, nonzero, such that $cR \subseteq A^t \subseteq R$. We note that c cannot be in any absolutely minimal prime P of R, since for any such P, R/P is of dimension n and is module-finite over $A/A \cap P$, and so $A \cap P = 0$. Now if $u \in (x_1,\dots,x_k) : x_{k+1}$ then for some $r_i \in R$, $ux_{k+1} = \sum_{i=1}^k r_i x_i$. Taking qth powers, and multiplying by c we get $cu^q x_{k+1}^q = \sum_{i=1}^k c r_i^q x_i^q$. But now cu^q and each of cr_i^q are in A^t and x_i^q form a regular sequence on A^t. Hence $cu^q \in (x_1^q,\dots,x_k^q)$ and so $u \in (x_1,\dots,x_k)^\bigstar$.

(2) If $ux_{k+1} \in (x_1,\dots,x_k)^\bigstar$ then for some $c_0 \in R^\bullet$, $c_0(ux_{k+1})^q \in (x_1^q,\dots,x_k^q)$ for all sufficiently large q, i.e., $c_0 u^q x_{k+1}^q = \sum_{i=1}^k r_i x_i^q$ for $q \gg 0$. Multiplying this by our earlier choice of c, we again have a relation on x_i^q's with coefficients in A^t, namely $cc_0 u^q x_{k+1}^q = \sum_{i=1}^k c r_i x_i^q$ for $q \gg 0$, and so $cc_0 u^q \in (x_1^q,\dots,x_k^q)$ for $q \gg 0$. Since $cc_0 \in R^\bullet$ we get $u \in (x_1,\dots,x_k)^\bigstar$.

(3) Let $J = (x_1,\dots,x_k)$. Then $J^\bigstar \subseteq (x_1,\dots,x_{k+1})$ and so $J^\bigstar \subseteq J + x_{k+1}R$. If $u \in J^\bigstar$, $u = j + x_{k+1}r$ for $j \in J$ and $r \in R$. This means $r \in J^\bigstar : x_{k+1}$ which equals $J^\bigstar$ by (2). Hence we get $J^\bigstar = J + x_{k+1}J^\bigstar$. Now by Nakayama's lemma we get $J^\bigstar = J$.

(4) This follows from (2) and (3) since, under either of the hypotheses, the system of parameters $x_1,\dots,x_n$ is a regular sequence. □

The above proposition, coupled with results on F–rationality, gives us the following theorem:

Theorem 4.3. *Let R be a complete local ring of characteristic p, with a system of parameters which generates a NE–closed ideal. Then R is F–rational.*

Proof. From the previous proposition the ring is Cohen–Macaulay, and in particular, equidimensional. For equidimensional rings, tight closure agrees with NE closure, and the result follows from Theorem 2.1 (2). □

We shall extend this result to excellent local rings once we develop the theory of test elements for NE closure. The following proposition lists some properties of NE closure.

Proposition 4.4. *Let R be a ring of characteristic p, and I an ideal of R.*
(1) $0^\bigstar$ *is the intersection of the absolutely minimal prime ideals of R.*
(2) *If $I = I^\bigstar$ then for any ideal J, $(I : J)^\bigstar = I : J$.*

(3) *If* $R = \prod R_i$ *and* $I = \prod I_i$*, then* $I^\bigstar = \prod I_i^\bigstar$*.*
(4) *For rings* R *and* S *and a homomorphism* $h : R \to S$ *satisfying the condition* $h(R^\bullet) \subseteq S^\bullet$*, we have* $h(I^\bigstar) \subseteq (IS)^\bigstar$*.*
(5) $x \in I^\bigstar$ *if and only if* $\overline{x} \in (IR/P)^\bigstar$ *for every absolutely minimal prime ideal* P *of* R*.*

Proof. (1), (2), (3) and (4) follow easily from the definitions. For (5) note that if P is absolutely minimal, $h : R \to R/P$ meets the condition of (4), so $x \in I^\bigstar$ implies that its image is in the NE closure of IR/P. For the converse, fix for every absolutely minimal P_i, $d_i \notin P_i$ but in every other minimal prime of R. If $\overline{x} \in (IR/P_i)^\bigstar$ for every absolutely minimal P_i, then there exist elements $\overline{c}_i$ with $\overline{c_i x^q} \in (IR/P_i)^{[q]}$. We can lift each $\overline{c}_i$ to $c_i \in R$ with $c_i \notin P_i$. Then $c_i x^q \in I^{[q]} + P_i$ for all i, for sufficiently large q. Multiplying each of these equations with the corresponding d_i, we get $c_i d_i x^q \in I^{[q]} + \mathfrak{N}$, since $d_i P_i$ is a subset of every minimal prime ideal and so is in the nilradical, $\mathfrak{N}$. If $\mathfrak{N}^{[q']} = 0$, taking q' powers of these equations gives us $(c_i d_i)^{q'} x^q \in I^{[q]}$ for all i, for sufficiently large q. Set $c = \sum (c_i d_i)^{q'}$. By our choice of c_i's and d_i's, $c \in R^\bullet$, and the above equations put together give us $cx^q \in I^{[q]}$ for all sufficiently large q. □

We note that NE closure need not be preserved once we localize, i.e., it is quite possible that $x \in I^\bigstar$, but $x \notin (IR_P)^\bigstar$. Examples of this abound in non–equidimensional rings, but there are some positive results about NE closure being preserved under certain maps which we examine in the next few propositions.

Proposition 4.5. *If* $h : (R, m) \to (S, n)$ *is a faithfully flat homomorphism of local rings then for an ideal* I *of* R*, if* $x \in I^\bigstar$*, then its image* $h(x)$ *is in* $(IS)^\bigstar$*. In particular if* $\hat{R}$ *denotes the completion of* R *at its maximal ideal,* $x \in I^\bigstar$ *implies* $x \in (I\hat{R})^\bigstar$*.*

Proof. By Proposition 4.4 (4), it suffices to check that $h(R^\bullet) \subseteq S^\bullet$. This is equivalent to the assertion that the contraction of every absolutely minimal prime of S is an absolutely minimal prime of R. Now let P be an absolutely minimal prime of S, and p denote its contraction to R. Then since $R \to S$ is faithfully flat, by a change of base, so is $R/p \to S/pS$. This gives dim S/pS = dim R/p + dim S/mS. Also, faithful flatness of h implies that dim S = dim R + dim S/mS. But P was an absolutely minimal prime of S, so dim S = dim S/P = dim S/pS, since $pS \subseteq P$. Putting these equations together, we get dim R/p = dim R, and so p is an absolutely minimal prime of R. □

Proposition 4.6. *Let* R *and* S *be Noetherian rings of characteristic* p*, and* $R \to S$ *a homomorphism such that for every absolutely minimal prime* Q

of S, its contraction to R, Q^c, contains an absolutely minimal prime of R. Assume one of the following holds:
(1) *R is finitely generated over an excellent local ring, or is F–finite, or*
(2) *R is locally excellent and S has a locally stable test element, (or S is local), or*
(3) *S has a completely stable test element (or S is a complete local ring).*
Then if $x \in I^\bigstar$ for I an ideal of R, the image of x in S is in $(IS)^\bigstar$.

Proof. It suffices to check $x \in (IS/Q)^\bigstar$ for every absolutely minimal primes Q of S, by Proposition 4.4 (5). But $(IS/Q)^\bigstar = (IS/Q)^*$ since S/Q is equidimensional. If $P \subseteq Q^c$ is an absolutely minimal prime of R, then $x \in I^\bigstar$ implies $\overline{x} \in (IR/P)^\bigstar = (IR/P)^*$. The result now follows by applying [HH2, Theorem 6.24] to the map $R/P \to S/Q$. □

5. NE–test elements

Definition 5.1. We shall say $c \in R^\bullet$ is a *q'–weak NE–test element for R* if for all ideals I of R and $x \in I^\bigstar$, $cx^q \in I^{[q]}$ for all $q \geq q'$. We may often use the phrase *weak NE–test element* and suppress the q'.

For a local ring (R, m), $c \in R^\bullet$ is a *weak completion stable NE–test element for R* if it is a weak NE–test element for $\hat{R}$, the completion of R at its maximal ideal.

Our definition of a completion stable weak NE–test element is different from the notion of a completely stable weak test element for tight closure, where it is required that the element serve as a weak test element in the completion of every local ring of R. The reason for this, of course, is that localization is no longer freely available to us, since $R^\bullet$ often does not map into $(R_P)^\bullet$.

Note also that since $\hat{R}$ is faithfully flat over R, a weak completion stable NE–test element for R is also a weak NE–test element for R.

Proposition 5.2. *If for every absolutely minimal prime P of R, R/P has a weak test element, then R has a weak NE–test element.*

Proof. Fix for every absolutely minimal prime P_i an element d_i not in P_i but in every other minimal prime of R. Let $\mathfrak{N}$ denote the nilradical of R and fix q' such that $\mathfrak{N}^{[q']} = 0$. If $\overline{c}_i$ is a weak test element for R/P_i, we may pick $c_i \notin P_i$ which maps to it under $R \to R/P$. We claim $c = \sum (c_i d_i)^{q'}$ is a weak NE–test element for R. If $x \in I^\bigstar$, then $\overline{x} \in (IR/P_i)^\bigstar$ for all P_i absolutely minimal. Since $\overline{c}_i$ is a weak test element for R/P_i, we have $\overline{c_i x^q} \in (IR/P_i)^{[q]}$ for all i, for sufficiently large q, i.e., $c_i x^q \in I^{[q]} + P_i$. Multiplying this by d_i, summing over all i and taking the q' power as in the proof of Proposition

4.4 (5), we get that $cx^q \in I^{[q]}$. It is easy to see that $c \in R^\bullet$ and so is a weak NE–test element. $\square$

Proposition 5.3. *Every excellent local ring of characteristic p has a weak completion stable NE–test element.*

Proof. If R is an excellent local domain, it has a completely stable weak test element, see [HH2, Theorem 6.1]. Hence each R/P_i for P_i absolutely minimal, has a completely stable weak test element, say $\overline{c}_i$. Let c_i, d_i, q' and c be as in the proof of the previous proposition. If $x \in (I\hat{R})^\bigstar$, we have $\overline{x} \in (I\hat{R}/P_i\hat{R})^\bigstar$. Since R/P_i is equidimensional and excellent, its completion $\hat{R}/P_i\hat{R}$ is also equidimensional. (We use here the fact that the completion of a universally catenary equidimensional local ring is again equidimensional, [HIO, Page 142]). Hence NE closure agrees with tight closure in $\hat{R}/P_i\hat{R}$, and we get $\overline{x} \in (I\hat{R}/P_i\hat{R})^*$. This gives $\overline{c_i x^q} \in (I\hat{R}/P_i\hat{R})^{[q]}$ for all i, for sufficiently large q. As in the previous proof, we then get that $cx^q \in (I\hat{R})^{[q]}$, and so is a weak completion stable NE–test element. $\square$

We can now extend Theorem 4.3 to the case where R is excellent local, without requiring it to be complete.

Theorem 5.4. *Let (R, m) be an excellent local ring of characteristic p with a system of parameters $x_1, \dots, x_n$. Then*
(1) $(x_1, \dots, x_k) : x_{k+1} \subseteq (x_1, \dots, x_k)^\bigstar$.
(2) $(x_1, \dots, x_k)^\bigstar : x_{k+1} = (x_1, \dots, x_k)^\bigstar$.
(3) *If* $(x_1, \dots, x_{k+1})^\bigstar = (x_1, \dots, x_{k+1})$, *then* $(x_1, \dots, x_k)^\bigstar = (x_1, \dots, x_k)$.
(4) *If* $(x_1, \dots, x_{n-1})^\bigstar = (x_1, \dots, x_{n-1})$ *then R is Cohen–Macaulay.*
(5) *If* $(x_1, \dots, x_n)^\bigstar = (x_1, \dots, x_n)$ *then R is F–rational.*

Proof. Since R has a weak completion stable NE–test element, if there is a counterexample to any of the above claims, we can preserve this while mapping to $\hat{R}$. But all of the above are true for complete local rings as follows from Proposition 4.2 and Theorem 4.3. $\square$

Example 5.5. Let $R = K[[X, Y, Z]]/(X) \cap (Y, Z)$. Then $y, x - z$ is a system of parameters for R and $0 :_R (y) = (x)$. That tight closure fails here to "capture colons" is seen from the fact that $x \notin 0^* = 0$. However $0^\bigstar = (x)$, and we certainly have $0 :_R (y) \subseteq 0^\bigstar$.

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