ASSOCIATED PRIMES OF LOCAL COHOMOLOGY MODULES

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1. Introduction

Throughout, R will denote a commutative Noetherian ring with a unit element. Let $\mathfrak{a}$ be an ideal of R, and i a non-negative integer. The *local cohomology module* $H^i_{\mathfrak{a}}(R)$ is defined as

$$H^i_{\mathfrak{a}}(R) = \varinjlim_{k \in \mathbb{N}} \operatorname{Ext}^i_R\left(R/\mathfrak{a}^k, R\right),$$

where the maps in the direct limit system are those induced by the natural surjections $R/\mathfrak{a}^{k+1} \longrightarrow R/\mathfrak{a}^k$. If $\mathfrak{a}$ is generated by elements $x_1, \dots, x_n$, then $H^i_{\mathfrak{a}}(R)$ is isomorphic to the ith cohomology module of the extended Čech complex

$$0 \longrightarrow R \longrightarrow \bigoplus_{i=1}^{n} R_{x_i} \longrightarrow \bigoplus_{i<j} R_{x_i x_j} \longrightarrow \cdots \longrightarrow R_{x_1 \cdots x_n} \longrightarrow 0.$$

For an element $f \in R$ and a positive integer m, we use $[f + (x_1^m, \dots, x_n^m)]$ to denote the cohomology class

$$\left[\frac{f}{x_1^m \cdots x_n^m}\right] \in \frac{R_{x_1 \cdots x_n}}{\sum_i R_{x_1 \cdots \hat{x}_i \cdots x_n}} \cong H^n_{\mathfrak{a}}(R).$$

It is easily seen that $[f + (x_1^m, \dots, x_n^m)] = 0$ in $H^n_{\mathfrak{a}}(R)$ if and only if there exists an integer $k \geq 0$, such that

$$f x_1^k \cdots x_n^k \in \left(x_1^{m+k}, \dots, x_n^{m+k}\right) R.$$

Consequently $H^n_{\mathfrak{a}}(R)$ may be also identified with the direct limit

$$\varinjlim_{m \in \mathbb{N}} R/(x_1^m, \dots, x_n^m)R,$$

where the map $R/(x_1^m, \dots, x_n^m) \longrightarrow R/(x_1^{m+1}, \dots, x_n^{m+1})$ is multiplication by the image of the element $x_1 \cdots x_n$.

As these descriptions suggest, $H^i_{\mathfrak{a}}(R)$ is usually not finitely generated as an R-module. However local cohomology modules have useful finiteness properties in certain cases, e.g., for a local ring $(R, \mathfrak{m})$, the modules $H^i_{\mathfrak{m}}(R)$ satisfy the descending chain condition. This implies, in particular, that for all $i \geq 0$,

$$\operatorname{Hom}_R\left(R/\mathfrak{m}, H^i_{\mathfrak{m}}(R)\right) \cong 0 :_{H^i_{\mathfrak{m}}(R)} \mathfrak{m}$$

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is a finitely generated R-module. Grothendieck conjectured that for all ideals $\mathfrak{a} \subset R$, the modules

$$\operatorname{Hom}_R\left(R/\mathfrak{a}, H^i_{\mathfrak{a}}(R)\right) \cong 0 :_{H^i_{\mathfrak{a}}(R)} \mathfrak{a}$$

are finitely generated, [SGA2, Exposé XIII, page 173]. In [Ha, §3] Hartshorne gave a counterexample to this conjecture: Let K be a field and R be the hypersurface

$$K[w, x, y, z]/(wx - yz).$$

Set $\mathfrak{a} = (x, y)$ and consider the local cohomology module $H^2_{\mathfrak{a}}(R)$. It is easily seen that the elements

$$[y^n z^n + (x^{n+1}, y^{n+1})R] \in H^2_{\mathfrak{a}}(R) \quad \text{for} \quad n \geq 0$$

are nonzero, and are killed by the maximal ideal $\mathfrak{m} = (w, x, y, z)$. In fact, they span the module $0 :_{H^2_{\mathfrak{a}}(R)} \mathfrak{m}$ which is a vector space of countably infinite dimension, and so it cannot be finitely generated as an R-module. It follows that $0 :_{H^2_{\mathfrak{a}}(R)} \mathfrak{a}$ is not finitely generated as well.

In [Ha] Hartshorne also began the study of the cofiniteness of local cohomology modules: An R-module M is $\mathfrak{a}$-*cofinite* if $\operatorname{Supp}(M) \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \geq 0$. Some of the work on cofiniteness may be found in the papers [Ch, DM, HK, HM, Kw, Me, Ya], and [Yo]. A related question on the torsion in local cohomology modules was raised by Huneke at the Sundance Conference in 1990, and will be our main focus here.

Question 1.1. [Hu1] Is the number of associated prime ideals of a local cohomology module $H^i_{\mathfrak{a}}(R)$ always finite?

The first results were obtained by Huneke and Sharp.

Theorem 1.2. [HS, Corollary 2.3] *Let R be a regular ring containing a field of positive characteristic, and $\mathfrak{a} \subset R$ an ideal. Then for all $i \geq 0$,*

$$\operatorname{Ass} H^i_{\mathfrak{a}}(R) \subseteq \operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}, R) \qquad (*)$$

In particular, $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ *is a finite set.*

Remark 1.3. The proof of the above theorem relies heavily on the flatness of the Frobenius endomorphism which, by [Ku, Theorem 2.1], characterizes regular rings of positive characteristic. The containment $(*)$ may fail for regular rings of characteristic zero: Let $R = \mathbb{C}[u, v, w, x, y, z]$, and $\mathfrak{a}$ be the ideal generated by the 2×2 minors Δ_i of the matrix

$$M = \begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix}.$$

Then $\operatorname{Ext}^3_R(R/\mathfrak{a}, R) = 0$ since $R/\mathfrak{a}$ has projective dimension two as an R-module. However, as observed by Hochster, the module $H^3_{\mathfrak{a}}(R)$ is nonzero: To see this, consider the linear action of $G = SL_2(\mathbb{C})$ on R, where an element $g \in G$ maps the entries of the matrix M to those of the matrix $g \times M$. The ring of invariants for this action is the polynomial ring $R^G = \mathbb{C}[\Delta_1, \Delta_2, \Delta_3]$. Since $SL_2(\mathbb{C})$ is linearly reductive, the inclusion $R^G \hookrightarrow R$ splits via an R^G-linear retraction, and so

$$H^3_{(\Delta_1,\Delta_2,\Delta_3)}(R^G) \longrightarrow H^3_{\mathfrak{a}}(R)$$

is a split inclusion. Since the module $H^3_{(\Delta_1,\Delta_2,\Delta_3)}(R^G)$ is nonzero, it follows that $H^3_{\mathfrak{a}}(R)$ must be nonzero as well.

While $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ may not be a subset of $\operatorname{Ass} \operatorname{Ext}^i_R(R/\mathfrak{a}, R)$, Question 1.1 does have an affirmative answer for all unramified regular local rings by combining the result of Huneke-Sharp with the following two theorems of Lyubeznik.

Theorem 1.4. [Ly1, Corollary 3.6 (c)] *Let R be a regular ring containing a field of characteristic zero and $\mathfrak{a}$ an ideal of R. Then for every maximal ideal $\mathfrak{m}$ of R, the set of associated primes of a local cohomology module $H^i_{\mathfrak{a}}(R)$, which are contained in the ideal $\mathfrak{m}$, is finite.*

If the regular ring R is finitely generated over a field of characteristic zero, then $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ *is a finite set.*

To illustrate the key point here, consider the case where $R = \mathbb{C}[x_1, \dots, x_n]$, and let D be the ring of $\mathbb{C}$-linear differential operators on R. It turns out that D is left and right Noetherian, that $H^i_{\mathfrak{a}}(R)$ is a finitely generated D-module, and consequently that $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ is finite. Lyubeznik's result below also uses D-modules, though the situation in mixed characteristic is more subtle.

Theorem 1.5. [Ly2, Theorem 1] *If R is an unramified regular local ring of mixed characteristic, and $\mathfrak{a}$ is an ideal of R, then* $\operatorname{Ass} H^i_{\mathfrak{a}}(R)$ *is a finite set.*

So far we have restricted the discussion to local cohomology modules of the form $H^i_{\mathfrak{a}}(R)$. For an R-module M, the local cohomology modules $H^i_{\mathfrak{a}}(M)$ are defined similarly as

$$H^i_{\mathfrak{a}}(M) = \varinjlim_{k\in\mathbb{N}} \operatorname{Ext}^i_R\left(R/\mathfrak{a}^k, M\right), \qquad \text{where} \qquad i \geq 0.$$

If M is a finitely generated R-module, then $H^0_{\mathfrak{a}}(M)$ may be identified with the submodule of M consisting of elements which are killed by a power of the ideal $\mathfrak{a}$, and consequently $H^0_{\mathfrak{a}}(M)$ is a finitely generated R-module. If i is the smallest integer for which $H^i_{\mathfrak{a}}(M)$ is not finitely generated, then the set $\operatorname{Ass} H^i_{\mathfrak{a}}(M)$ is also finite, as proved in [BF] and [KS]. Other positive answers to Question 1.1 include results in small dimensions such as the following theorem due to Marley:

Theorem 1.6. [Ma, Corollary 2.7] *Let R be a local ring and M a finitely generated R-module of dimension at most three. Then* $\operatorname{Ass} H^i_{\mathfrak{a}}(M)$ *is finite for all ideals $\mathfrak{a} \subset R$.*

For some of the other work on this question, we refer the reader to the papers [BKS, BRS, He, Ly3] and [MV].

2. p-torsion

In [Si1] the author constructed a hypersurface for which a local cohomology module has infinitely many associated prime ideals, thereby demonstrating that Question 1.1, in general, has a negative answer. Since the argument is quite elementary, we include it here.

Theorem 2.1. [Si1, §4] *Consider the hypersurface*

$$R = \mathbb{Z}[u, v, w, x, y, z]/(ux + vy + wz)$$

and the ideal $\mathfrak{a} = (x, y, z)R$. Then for every prime integer p, the local cohomology module $H^3_{\mathfrak{a}}(R)$ has a p-torsion element. Consequently $H^3_{\mathfrak{a}}(R)$ has infinitely many associated prime ideals.

Proof. We identify $H^3_{\mathfrak{a}}(R)$ with the direct limit

$$\varinjlim_{k\in\mathbb{N}} R/(x^k, y^k, z^k)R,$$

where the maps are induced by multiplication by the element xyz. For a prime integer p, the fraction

$$\lambda_p = \frac{(ux)^p + (vy)^p + (wz)^p}{p}$$

has integer coefficients, and is therefore an element of R. We claim that the element

$$\eta_p = [\lambda_p + (x^p, y^p, z^p)R] \in H^3_{\mathfrak{a}}(R)$$

is nonzero and p-torsion. Note that $p \cdot \eta_p = [p\lambda_p + (x^p, y^p, z^p)R] = 0$, and what remains to be checked is that η_p is nonzero, i.e., that

$$\lambda_p(xyz)^k \notin \left(x^{p+k}, y^{p+k}, z^{p+k}\right) R \quad \text{for all} \quad k \in \mathbb{N}.$$

We assign weights to the $\mathbb{Z}$-algebra generators of the ring R as follows:

$$\begin{aligned} x &: (1, 0, 0, 0), & u &: (-1, 0, 0, 1),\\ y &: (0, 1, 0, 0), & v &: (0, -1, 0, 1),\\ z &: (0, 0, 1, 0), & w &: (0, 0, -1, 1). \end{aligned}$$

With this grading, λ_p is a homogeneous element of degree $(0, 0, 0, p)$. Now suppose we have a homogeneous equation of the form

$$\lambda(xyz)^k = c_1x^{p+k} + c_2y^{p+k} + c_3z^{p+k},$$

then we must have $\deg(c_1) = (-p, k, k, p)$, i.e., c_1 must be an integer multiple of the monomial $u^py^kz^k$. Similarly c_2 is an integer multiple of $v^pz^kx^k$ and c_3 of $w^px^ky^k$. Consequently

$$\begin{aligned} \lambda(xyz)^k &\in \left(u^py^kz^kx^{p+k},\ v^pz^kx^ky^{p+k},\ w^px^ky^kz^{p+k}\right) R\\ &= (xyz)^k \left(u^px^p,\ v^py^p,\ w^pz^p\right) R, \end{aligned}$$

and so $\lambda \in (u^px^p, v^py^p, w^pz^p)R$. After specializing $u \mapsto 1, v \mapsto 1, w \mapsto 1$, this implies that

$$\frac{x^p + y^p + (-1)^p(x + y)^p}{p} \in (p, x^p, y^p)\, \mathbb{Z}[x, y],$$

which is easily seen to be false. □

This example, however, does not shed light on Question 1.1 in the case of local rings or rings containing a field. Katzman constructed the first examples to demonstrate that Huneke's question has a negative answer in these cases as well, [Ka2]. The equicharacteristic case is discussed here in §3. We next recall a conjecture of Lyubeznik.

Conjecture 2.2. [Ly1, Remark 3.7 (iii)] If R is a regular ring and $\mathfrak{a}$ an ideal, then the local cohomology modules $H^i_{\mathfrak{a}}(R)$ have finitely many associated prime ideals.

This conjecture has been settled for unramified regular local rings by the results of Huneke-Sharp and Lyubeznik mentioned earlier. However it remains open for polynomial rings over the integers, and we discuss some of its implications in this case.

Remark 2.3. Let R be a polynomial ring in finitely many variables over the integers, and let $\mathfrak{a}$ be an ideal of R. Then for every prime integer p, we have a short exact sequence

$$0 \longrightarrow R \xrightarrow{p} R \longrightarrow R/pR \longrightarrow 0,$$

which induces a long exact sequence of local cohomology modules,

$$\cdots \longrightarrow H^{i-1}_{\mathfrak{a}}(R/pR) \xrightarrow{\delta^{i-1}_p} H^i_{\mathfrak{a}}(R) \xrightarrow{p} H^i_{\mathfrak{a}}(R) \longrightarrow H^i_{\mathfrak{a}}(R/pR) \xrightarrow{\delta^i_p} H^{i+1}_{\mathfrak{a}}(R) \xrightarrow{p} \cdots .$$

The image of each connecting homomorphism δ^i_p is annihilated by p, and hence every nonzero element of $\delta^i_p(H^i_{\mathfrak{a}}(R/pR))$ is a p-torsion element. Consequently Lyubeznik's conjectures implies that for all but finitely many prime integers p, we must have $\delta^i_p = 0$ for all $i \geq 0$.

Remark 2.4. Again, let R be a polynomial ring over the integers. Let f_i, g_i be elements of R such that

$$f_1g_1 + f_2g_2 + \cdots + f_ng_n = 0.$$

Consider the ideal $\mathfrak{a} = (g_1, \dots, g_n)R$ and the local cohomology module

$$H^n_{\mathfrak{a}}(R) = \varinjlim_{k \in \mathbb{N}} R/(g_1^k, \dots, g_n^k)R,$$

where the maps in the direct system are induced by multiplication by the element $g_1 \cdots g_n$. For a prime integer p and prime power $q = p^e$, let

$$\lambda_q = \frac{(f_1g_1)^q + \cdots + (f_ng_n)^q}{p}.$$

Then $\lambda_q \in R$, and we set

$$\eta_q = [\lambda_q + (g_1^q, \dots, g_n^q)R] \in H^n_{\mathfrak{a}}(R).$$

It is immediately seen that $p \cdot \eta_q = 0$ and so if η_q is a nonzero element of $H^n_{\mathfrak{a}}(R)$, then it must be a p-torsion element. Hence Lyubeznik's conjecture implies that for all but finitely many prime integers p, the elements η_q must be zero, i.e., for some $k \in \mathbb{N}$, which may depend on $q = p^e$, we have

$$\lambda_q(g_1 \cdots g_n)^k \in (g_1^{q+k}, \dots, g_n^{q+k})R.$$

This motivates the following conjecture:

Conjecture 2.5. Let R be a polynomial ring over the integers, and let f_i, g_i be elements of R such that
$$f_1g_1 + \cdots + f_ng_n = 0.$$
Then for every prime power $q = p^e$, there exists $k \in \mathbb{N}$ such that
$$\frac{(f_1g_1)^q + \cdots + (f_ng_n)^q}{p}(g_1 \cdots g_n)^k \in (g_1^{q+k}, \ldots, g_n^{q+k})R.$$

The above conjecture is easily established if $n = 2$, or if the elements $g_1, \ldots, g_n$ form a regular sequence. The conjecture is also true if $n = 3$, provided the elements f_1, f_2, f_3 form a regular sequence:

Theorem 2.6. [Si2, Theorem 2.1] *Let R be a polynomial ring over the integers and f_i, g_i be elements of R such that f_1, f_2, f_3 form a regular sequence in R and*
$$f_1g_1 + f_2g_2 + f_3g_3 = 0.$$
Let $q = p^e$ be a prime power. Then for $k = q - 1$, we have
$$\frac{(f_1g_1)^q + (f_2g_2)^q + (f_3g_3)^q}{p}(g_1g_2g_3)^k \in (g_1^{q+k}, g_2^{q+k}, g_3^{q+k})R.$$

3. The equicharacteristic case

Recently Katzman constructed the following example in [Ka2]: Let K be an arbitrary field, and consider the hypersurface
$$R = K[s,t,u,v,x,y]/\big(su^2x^2 - (s+t)uxvy + tv^2y^2\big).$$
Katzman showed that the local cohomology module $H^2_{(x,y)}(R)$ has infinitely many associated prime ideals. Since the defining equation of this hypersurface factors as
$$su^2x^2 - (s+t)uxvy + tv^2y^2 = (sux - tvy)(ux - vy),$$
the ring in Katzman's example is not an integral domain. In [SS] Swanson and the author generalize Katzman's construction and obtain families of examples which include examples over normal domains and, in fact, over hypersurfaces with rational singularities:

Theorem 3.1. [SS, Theorem 1.1] *Let K be an arbitrary field, and consider the hypersurface*
$$S = \frac{K[s,t,u,v,w,x,y,z]}{\big(su^2x^2 + sv^2y^2 + tuxvy + tw^2z^2\big)}$$
Then S is a standard $\mathbb{N}$-graded normal domain for which the local cohomology module $H^3_{(x,y,z)}(S)$ has infinitely many associated prime ideals.

If $\mathfrak{m}$ denotes the homogeneous maximal ideal (s,t,u,v,w,x,y,z), then the local cohomology module $H^3_{(x,y,z)}(S_\mathfrak{m})$ has infinitely many associated prime ideals as well.

If K has characteristic zero, then S has rational singularities. If K has positive characteristic, then S is F-regular.

While we refer to [SS] for details as well as more general constructions, we would like to sketch a proof of the above theorem.

Sketch of the proof. Let K be arbitrary field, and consider the hypersurface

$$A = K[s,t,a,b]/(sa^2 + tab + b^2).$$

We use the $\mathbb{N}$-grading on A where $A_0 = K[s,t]$, and a and b have degree 1. For an integer $n \geq 2$, the presentation matrix of the A_0-module $[A/(a^n, b^n)]_n$ is the $(n-1) \times (n-1)$ matrix

$$M_{n-1} = \begin{bmatrix} t & s & & & \\ s & t & s & & \\ & \ddots & \ddots & \ddots & \\ & & s & t & s \\ & & & s & t \end{bmatrix},$$

where the elements s, t, s occur along the three central diagonals, and the other entries are zero. These are special cases of Toeplitz matrices, and the determinants of these matrices will be the source of the infinitely many associated primes.

It is convenient to define $\det M_0 = 1$, and setting $Q_n(s,t) = \det M_n$, it is easily seen that we have

$$Q_0 = 1, \quad Q_1 = t, \quad \text{and} \quad Q_{n+2} = tQ_{n+1} - s^2 Q_n \quad \text{for all} \quad n \geq 0.$$

This recursion enables us to compute a generating function for the polynomials $Q_n(s,t)$, and to check that the polynomials $\{Q_n(s,t)\}_{n \in \mathbb{N}}$ have infinitely many distinct irreducible factors. For example, if $K = \mathbb{C}$ it may be verified that $Q_n(s,t)$ factors as

$$Q_n(s,t) = \prod_{r=1}^{n} \left(t - se^{r\pi i/(n+1)} - se^{-r\pi i/(n+1)} \right) \quad \text{for all} \quad n \in \mathbb{N}.$$

This implies that the set

$$\bigcup_{n \in \mathbb{N}} \operatorname{Ass}_{A_0} [A/(a^n, b^n)]_n$$

is infinite and, in fact, a straightforward computation shows that

$$(a^n, b^n) :_{A_0} sab^{n-1} = (Q_{n-1})A_0 \quad \text{for all} \quad n \in \mathbb{N}.$$

We next consider the ring

$$B = K[s,t,a,b,c]/\big(sa^2 + sb^2 + tab + tc^2\big)$$

with the $\mathbb{N}$-grading where $B_0 = K[s,t] = A_0$, and a, b, and c have degree 1. Note that $B/cB \cong A$, and so

$$(a^n, b^n, c) :_{B_0} sab^{n-1} = (Q_{n-1})B_0 \quad \text{for all} \quad n \in \mathbb{N}.$$

We identify B with the subring of S which is generated, as a K-algebra, by the elements $s, t, a = ux, b = vy$, and $c = wz$. For a fixed integer $n \geq 1$, let

$$\eta_n = \big[s(ux)(vy)^{n-1} z^{n-1} + (x^n, y^n, z^n) \big] \in H^3_{(x,y,z)}(S),$$

where we are using the identification

$$H^3_{(x,y,z)}(S) = \varinjlim_{n\in\mathbb{N}} S/(x^n, y^n, z^n)S.$$

By a multigrading argument, it may be verified that

$$\operatorname{ann}_{S_0}\eta_n = (a^n, b^n, c)B :_{B_0} sab^{n-1} = (Q_{n-1})B_0$$

where $S_0 = B_0 = K[s,t]$. Since the polynomials $\{Q_n(s,t)\}_{n\in\mathbb{N}}$ have infinitely many distinct irreducible factors, it follows that the set

$$\operatorname{Ass}_{S_0} H^3_{(x,y,z)}(S)$$

is infinite. For every prime ideal $\mathfrak{p}$ of S_0 with $\mathfrak{p} \in \operatorname{Ass}_{S_0} H^3_{(x,y,z)}(S)$, there exists a prime ideal $\mathfrak{P} \in \operatorname{Spec} S$ such that $\mathfrak{P} \in \operatorname{Ass}_S H^3_{(x,y,z)}(S)$ and $\mathfrak{P} \cap S_0 = \mathfrak{p}$. Consequently the set $\operatorname{Ass}_S H^3_{(x,y,z)}(S)$ must be infinite as well.

It remains to verify that the hypersurface S has rational singularities (in characteristic zero) or is F-regular (in positive characteristic). In [SS] we show that S is F-regular for an arbitrary field K of positive characteristic. This implies that for all prime integers p, the fiber over $p\mathbb{Z}$ of the map

$$\mathbb{Z} \longrightarrow \frac{\mathbb{Z}[s,t,u,v,w,x,y,z]}{\left(su^2x^2 + sv^2y^2 + tuxvy + tw^2z^2\right)}$$

is an F-rational ring. By [Sm, Theorem 4.3], it then follows that S has rational singularities when K has characteristic zero.

We would like to include here a different proof that S has rational singularities in characteristic zero based on a result from [SW]. We first note that

$$S \cong B[u,v,w,x,y,z]/(ux-a, vy-b, wz-c),$$

and that B is a normal domain. By a result of [BS], if a local (or graded) domain R has rational singularities, then so does $R[u,x]/(ux-a)$, where $a \neq 0$ is a (homogeneous) element of R, see also [HWY, Lemma 3.3]. By repeated use of this, to show that S has rational singularities, it suffices to show that the subring B has rational singularities. In [SW] we obtain a criterion for multigraded rings to have rational singularities. The bigraded case of this criterion is:

Theorem 3.2. *Let R be a normal $\mathbb{N}^2$-graded ring where $R_{\mathbf{0}}$ is a field of characteristic zero, and R is generated over $R_{\mathbf{0}}$ by elements of degrees $(1,0)$ and $(0,1)$. Then R has rational singularities if and only if*

(i) *R is a Cohen-Macaulay ring for which the multigraded $\mathbf{a}$-invariant satisfies $\mathbf{a}(R) < \mathbf{0}$, and*
(ii) *the localizations $R_{\mathfrak{p}}$ have rational singularities for all primes $\mathfrak{p}$ in the set*

$$\operatorname{Spec} R \setminus V(R_{++}), \qquad \textit{where} \qquad R_{++} = \bigoplus_{i>0,j>0} R_{i,j}.$$

To apply the theorem, we consider the $\mathbb{N}^2$-grading on B where s and t have degree $(1,0)$ and a, b, and c have degree $(0,1)$. Then $\mathbf{a}(B) = (-1,-1)$, and a straightforward computation using the Jacobian criterion shows that $B_{\mathfrak{p}}$ is regular for all primes $\mathfrak{p} \in \operatorname{Spec} B \setminus V(B_{++})$. $\square$

4. An application

Let R be a ring of characteristic $p > 0$, and R° denote the complement of the minimal primes of R. For an ideal $\mathfrak{a} = (x_1, \dots, x_n)$ of R and a prime power $q = p^e$, we use the notation $\mathfrak{a}^{[q]} = (x_1^q, \dots, x_n^q)$. The *tight closure* of $\mathfrak{a}$ is the ideal

$$\mathfrak{a}^* = \{z \in R : \text{ there exists } c \in R^\circ \text{ for which } cz^q \in \mathfrak{a}^{[q]} \text{ for all } q \gg 0\},$$

see [HH1]. A ring R is *F-regular* if $\mathfrak{a}^* = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of R and its localizations.

More generally, let F denote the Frobenius functor, and F^e its eth iteration. If an R-module M has presentation matrix (a_{ij}), then $F^e(M)$ has presentation matrix (a_{ij}^q), where $q = p^e$. For modules $N \subseteq M$, we use $N_M^{[q]}$ to denote the image of $F^e(N) \to F^e(M)$. We say that an element $m \in M$ is in the *tight closure of N in M*, denoted N_M^*, if there exists an element $c \in R^\circ$ such that $cF^e(m) \in N_M^{[q]}$ for all $q \gg 0$. While the theory has found several applications, the question whether tight closure commutes with localization remains open even for finitely generated algebras over fields of positive characteristic.

Let W be a multiplicative system in R, and $N \subseteq M$ be finitely generated R-modules. Then

$$W^{-1}(N_M^*) \subseteq (W^{-1}N)^*_{W^{-1}M},$$

where $W^{-1}(N_M^*)$ is identified with its image in $W^{-1}M$. When this inclusion is an equality, we say that *tight closure commutes with localization at W for the pair $N \subseteq M$*. It may be checked that this occurs if and only if tight closure commutes with localization at W for the pair $0 \subseteq M/N$. Following [AHH], we set

$$G^e(M/N) = F^e(M/N)/(0^*_{F^e(M/N)}).$$

An element $c \in R^\circ$ is a *weak test element* if there exists $q_0 = p^{e_0}$ such that for every pair of finitely generated modules $N \subseteq M$, an element $m \in M$ is in N_M^* if and only if $cF^e(m) \in N_M^{[q]}$ for all $q \geq q_0$. By [HH2, Theorem 6.1], if R is of finite type over an excellent local ring, then R has a weak test element.

Proposition 4.1. [AHH, Lemma 3.5] *Let R be a ring of characteristic $p > 0$ and $N \subseteq M$ be finitely generated R-modules. Then the tight closure of $N \subseteq M$ commutes with localization at any multiplicative system W which is disjoint from the set $\bigcup_{e \in \mathbb{N}} \operatorname{Ass} F^e(M)/N_M^{[q]}$.*

If R has a weak test element, then the tight closure of $N \subseteq M$ also commutes with localization at multiplicative systems W disjoint from the set $\bigcup_{e \in \mathbb{N}} \operatorname{Ass} G^e(M/N)$.

Consider a bounded complex $P_\bullet$ of finitely generated projective R-modules,

$$0 \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \xrightarrow{d_1} P_0 \longrightarrow 0.$$

The complex $P_\bullet$ is said to have *phantom homology* at the ith spot if

$$\operatorname{Ker} d_i \subseteq (\operatorname{Im} d_{i+1})^*_{P_i}.$$

The complex $P_\bullet$ is *stably phantom acyclic* if $F^e(P_\bullet)$ has phantom homology at the ith spot for all $i \geq 1$, for all $e \geq 1$. An R-module M has *finite phantom projective dimension* if there exits a bounded stably phantom acyclic complex $P_\bullet$ of projective R-modules, with $H_0(P_\bullet) \cong M$.

Theorem 4.2. [AHH, Theorem 8.1] *Let R be an equidimensional ring of positive characteristic, which is of finite type over an excellent local ring. If $N \subseteq M$ are finitely generated R-modules such that M/N has finite phantom projective dimension, then the tight closure of N in M commutes with localization at W for every multiplicative system W of R.*

The key points of the proof are that for M/N of finite phantom projective dimension, the set $\bigcup_e \operatorname{Ass} G^e(M/N)$ has finitely many maximal elements, and that if $(R, \mathfrak{m})$ is a local ring, then there a positive integer B such that for all $q = p^e$, the ideal $\mathfrak{m}^{Bq}$ kills the local cohomology module

$$H^0_{\mathfrak{m}}\left(G^e(M/N)\right).$$

For more details on this approach to the localization problem, we refer the reader to the papers [AHH, Ho, Ka1, SN], and [Hu2, §12]. Specializing to the case where $M = R$ and $N = \mathfrak{a}$ is an ideal, we note that

$$G^e(R/\mathfrak{a}) \cong R/\big(\mathfrak{a}^{[q]}\big)^*, \quad \text{where} \quad q = p^e.$$

Consider the questions:

Question 4.3. Let R be a Noetherian ring of characteristic $p > 0$, and $\mathfrak{a}$ an ideal of R.

(1) Is the set $\bigcup_{q=p^e} \operatorname{Ass} R/\mathfrak{a}^{[q]}$ finite?
(2) Is the set $\bigcup_{q=p^e} \operatorname{Ass} R/\big(\mathfrak{a}^{[q]}\big)^*$ finite?
(3) For a local domain $(R, \mathfrak{m})$ and an ideal $\mathfrak{a} \subset R$, is there a positive integer B such that

$$\mathfrak{m}^{Bq} H^0_{\mathfrak{m}}\left(R/(\mathfrak{a}^{[q]})^*\right) = 0 \quad \text{for all} \quad q = p^e\,?$$

Katzman proved that affirmative answers to Questions 4.3 (2) and 4.3 (3) imply that tight closure commutes with localization:

Theorem 4.4. [Ka1] *Assume that for every local ring $(R, \mathfrak{m})$ of characteristic $p > 0$ and ideal $\mathfrak{a} \subset R$, the set $\bigcup_q \operatorname{Ass} R/\big(\mathfrak{a}^{[q]}\big)^*$ has finitely many maximal elements. Also, if for every ideal $\mathfrak{a} \subset R$, there exists a positive integer B such that $\mathfrak{m}^{Bq}$ kills*

$$H^0_{\mathfrak{m}}\left(R/(\mathfrak{a}^{[q]})^*\right) \quad \textit{for all} \quad q = p^e,$$

then tight closure commutes with localization for all ideals in Noetherian rings of characteristic $p > 0$.

These issues are the source of our interest in associated primes of Frobenius powers of ideals. It should be mentioned that the situation for *ordinary* powers is well-understood: the set $\bigcup_{n \in \mathbb{N}} \operatorname{Ass} R/\mathfrak{a}^n$ is finite for any Noetherian ring R, see [Br] or [Ra]. In [Ka1] Katzman constructed the first example where $\bigcup_{q=p^e} \operatorname{Ass} R/\mathfrak{a}^{[q]}$ is not finite, thereby settling Question 4.3 (1): For

$$R = K[t, x, y]/\big(xy(x-y)(x-ty)\big),$$

he proved that the set $\bigcup_q \operatorname{Ass} R/(x^q, y^q)$ is infinite. In this example, and some others, $(x^q, y^q)^* = (x, y)^q$ for all $q = p^e$, and so $\bigcup_q \operatorname{Ass} R/(x^q, y^q)^*$ is finite. However we show that Question 4.3 (2) also has a negative answer using the local cohomology examples recorded earlier.

Theorem 4.5 (Singh-Swanson)**.** *Let K be a field of characteristic $p > 0$, and consider the hypersurface*

$$S = \frac{K[s,t,u,v,w,x,y,z]}{\big(su^2x^2 + sv^2y^2 + tuxvy + tw^2z^2\big)}.$$

Then S is F-regular, and the set

$$\bigcup_{q=p^e} \operatorname{Ass} S/\big(x^q, y^q, z^q\big) = \bigcup_{q=p^e} \operatorname{Ass} S/\big(x^q, y^q, z^q\big)^*$$

is infinite.

Proof. The direct system $\{S/(x^q, y^q, z^q)\}_{q=p^e}$ is cofinal with the direct system $\{S/(x^n, y^n, z^n)\}_{n\in\mathbb{N}}$, and so we have

$$H^3_{(x,y,z)}(S) \cong \varinjlim_{q=p^e} S/(x^q, y^q, z^q)S.$$

Using this, it is easily seen that

$$\operatorname{Ass} H^3_{(x,y,z)}(S) \subseteq \bigcup_{q=p^e} \operatorname{Ass} S/(x^q, y^q, z^q)S.$$

By Theorem 3.1 $H^3_{(x,y,z)}(S)$ has infinitely many associated prime ideals, and so $\bigcup_q \operatorname{Ass} S/(x^q, y^q, z^q)S$ must be infinite as well. Since the hypersurface S is F-regular, we have $(x^q, y^q, z^q)^* = (x^q, y^q, z^q)$ for all $q = p^e$. □

Remark 4.6. In [SS] we actually prove a stronger result: There exists an F-regular hypersurface R of characteristic $p > 0$, with an ideal $\mathfrak{a}$, for which the set

$$\bigcup_{q=p^e} \operatorname{Ass} R/\mathfrak{a}^{[q]} = \bigcup_{q=p^e} \operatorname{Ass} R/\big(\mathfrak{a}^{[q]}\big)^*$$

has infinitely many *maximal* elements.

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