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Deformation of F-purity and F-regularity

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Abstract

For a Noetherian local domain (R, m, K), it is an open question whether strong F-regularity deforms. We provide an affirmative answer to this question when the canonical module satisfies certain additional assumptions. The techniques used here involve passing to a Gorenstein ring, using an anti-canonical cover. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Throughout our discussion, all rings are commutative, Noetherian, and have an identity element. Unless specified otherwise, we shall be working with rings containing a field of characteristic $p > 0$. The theory of the *tight closure* of an ideal or a submodule of a module was developed by Melvin Hochster and Craig Huneke in [9] and draws attention to rings which have the property that all their ideals are tightly closed, called *weakly F-regular* rings. This property turn out to be of significant importance, for instance, the Hochster–Roberts theorem of invariant theory that direct summands of polynomial rings are Cohen–Macaulay [12], can actually be proved for the much larger class of weakly F-regular rings. A closely related notion is that of a *strongly F-regular* ring. The recent work of Lyubeznik and Smith (see [14]) shows that the properties of weak F-regularity and strong F-regularity agree for $\mathbb{N}$-graded F-finite rings.

Whether the property of strong F-regularity deforms has been a persistent open question, specifically, if R/xR is strongly F-regular for some nonzerodivisor $x \in m$, is

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R strongly F-regular? A similar question may be posed when (R, m, K) is a ring graded by the nonnegative integers, and $x \in m$ is a homogeneous nonzerodivisor. We examine these questions using the idea of passing to an anti-canonical cover $S = \bigoplus_{i \geq 0} I^{(i)} t^i$ (where I represents the inverse of the canonical module in $\mathrm{Cl}(R)$) and get positive answers when the symbolic powers $I^{(i)}$ satisfy the Serre condition S_3 for all $i \geq 0$ and S is Noetherian. It was earlier known that strong F-regularity deforms for Gorenstein rings [10, Theorem 4.2], and our results here may be considered an extension of those ideas. Another case when it is known that strong F-regularity deforms is in the case of $\mathbb{Q}$-Gorenstein rings, see [1] and for the characteristic zero case, [17]. It is interesting to note that all currently known results on the deformation of strong F-regularity require somewhat restrictive hypotheses which are weaker forms of the Gorenstein hypothesis – while $\mathbb{Q}$-Gorenstein is a natural weakening of the Gorenstein condition, the hypotheses required in our results are most obviously satisfied when R is Gorenstein!

Another notion ubiquitous in characteristic p methods is that of F-purity i.e., the property that for a ring R of characteristic p, the Frobenius map remains injective upon tensoring with any R-module, see [13]. The corresponding question about the deformation of F-purity has a negative answer, the first examples being provided by Fedder in [3]. However, Fedder points out that his examples are less than satisfactory in two ways: firstly the rings are not integral domains, and secondly his arguments provide relevant examples only for finitely many choices of the characteristic p of K. We point out various examples that have already been studied in different contexts which overcome both these shortcomings, but now another shortcoming stands out – although the rings R are domains (which are not F-pure), the F-pure quotient rings R/xR are not domains. The obvious question then arises: if R/xR is an F-pure domain, is R F-pure? An even more interesting question would be whether the property of being an F-pure normal domain deforms.

In Section 6 we shall use cyclic covers to study graded rings of dimension two, and develop a criterion for the strong F-regularity of such rings when the characteristic of the field is zero or a "sufficiently large" prime. By a graded ring we shall mean, unless stated otherwise, a ring graded by the nonnegative integers, with $R_0 = K$, a field. We shall denote by $m = R_+$, the homogeneous maximal ideal of R, and by a system of parameters for R, we shall mean a sequence of homogeneous elements of R whose images form a system of parameters for R_m. In specific examples involving homomorphic images of polynomial rings, lower case letters shall denote the images of the corresponding variables, the variables being denoted by the upper case letters. We shall also implicitly assume our rings are homomorphic images of Gorenstein rings, and so possess canonical modules.

2. Frobenius closure and tight closure

Let R be a Noetherian ring of characteristic $p > 0$. The letter e denotes a variable nonnegative integer, and q its eth power, i.e., $q = p^e$. We shall denote by F, the

Frobenius endomorphism of R, and by F^e, its eth iteration, i.e., $F^e(r) = r^q$. For an ideal $I = (x_1,\dots,x_n) \subseteq R$, let $I^{[q]} = (x_1^q,\dots,x_n^q)$. Note that $F^e(I)R = I^{[q]}$, where $q = p^e$, as always. Let S denote the ring R viewed as an R-algebra via F^e. Then $S \otimes_R -$ is a covariant functor from R-modules to S-modules. If we consider a map of free modules $R^n \to R^m$ given by the matrix (r_{ij}), applying F^e we get a map $R^n \to R^m$ given by the matrix (r_{ij}^q). For an R-module M, note that the R-module structure on $F^e(M)$ is $r'(r \otimes m) = r'r \otimes m$, and $r' \otimes rm = r'r^q \otimes m$. For R-modules $N \subseteq M$, we use $N_M^{[q]}$ to denote $\mathrm{Im}(F^e(N) \to F^e(M))$.

For a reduced ring R of characteristic $p > 0$, $R^{1/q}$ shall denote the ring obtained by adjoining all qth roots of elements of R. The ring R is said to be *F-finite* if $R^{1/p}$ is module-finite over R. Note that a finitely generated algebra R over a field K is F-finite if and only if $K^{1/p}$ is finite over K.

We shall denote by R° the complement of the union of the minimal primes of R. We say $I = (x_1,\dots,x_n) \subseteq R$ is a *parameter ideal* if the images of $x_1,\dots,x_n$ form part of a system of parameters in R_P, for every prime ideal P containing I.

Definition 2.1. Let R be a ring of characteristic p, and I an ideal of R. For an element x of R, we say that $x \in I^F$, the *Frobenius closure* of I, if there exists $q = p^e$ such that $x^q \in I^{[q]}$.

For R-modules $N \subseteq M$ and $u \in M$, we say that $u \in N_M^*$, the *tight closure* of N in M, if there exists $c \in R^\circ$ such that $cu^q \in N_M^{[q]}$ for all $q = p^e \gg 0$. It is worth recording this when $M = R$, and $N = I$ is an ideal of R. For an element x of R, we say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$. If $I^* = I$, we say that the ideal I is *tightly closed*.

It is easily verified that $I \subseteq I^F \subseteq I^*$. Furthermore, I^* is always contained in the integral closure of I, and is frequently much smaller.

Definition 2.2. A ring R is said to be *F-pure* if the Frobenius homomorphism $F: M \to F(M)$ is injective for all R-modules M.

A ring R is said to be *weakly F-regular* if every ideal of R is tightly closed, and is *F-regular* if every localization is weakly F-regular. A weakly F-regular ring is F-pure.

An F-finite ring R is *strongly F-regular* if for every element $c \in R^\circ$, there exists an integer $q = p^e$ such that the R-linear inclusion $R \to R^{1/q}$ sending 1 to $c^{1/q}$ splits as a map of R-modules.

R is said to be *F-rational* if every parameter ideal of R is tightly closed.

The equivalence of the notions of strong F-regularity, F-regularity, and weak F-regularity, in general, is a persistent open problem. It is known that these notions agree when the canonical module is a torsion element of the divisor class group, see [15] and [21], and so when this hypothesis is satisfied, we switch freely between these notions. Furthermore, Lyubeznik and Smith have shown that graded F-finite rings are weakly F-regular if and only if they are strongly F-regular [14, Corollary 4.3].

Theorem 2.3. (1) *An F-finite regular domain of characteristic p is strongly F-regular.*

(2) *A strongly F-regular ring is F-regular, and an F-regular ring is weakly F-regular.*

(3) *Let S be a strongly (weakly) F-regular ring. If R is a subring of S which is a direct summand of S as an R-module, then R is strongly (weakly) F-regular.*

(4) *An F-rational ring R is normal. If in addition, R is assumed to be he homomorphic image of a Cohen–Macaulay ring, then R is Cohen–Macaulay.*

(5) *An F-rational Gorenstein ring is F-regular. If it is F-finite, then it is also strongly F-regular.*

Proof. (1) follows from the fact that the Frobenius endomorphism over a regular local ring is flat. For all the assertions about strong F-regularity, see [8, Theorem 3.1]. The results on F-rationality (4) and (5) are part of [14, Theorem 4.2]. □

3. F-purity does not deform

We recall a useful result [13, Proposition 5.38].

Proposition 3.1. *Let $R = K[X_1,\dots,X_n]/I$ where K is a field of characteristic p and I is an ideal generated by square-free monomials in the variables $X_1,\dots,X_n$. Then R is F-pure.*

In [3] Fedder constructs a family of examples for which F-purity does not deform. He points out that the examples have two shortcomings, firstly that they do not work for large primes, and secondly that the rings are not integral domains. We next review a special case of Fedder's examples and point out that it does serve as a counter-example to the deformation of F-purity for all choices of the prime characteristic p of the field K.

Example 3.2. Let $R = K[U,V,Y,Z]/(UV, UZ, Z(V - Y^2))$. Fedder's work shows that when the characteristic of the field K is 2, then R is not F-pure, although R/yR is F-pure. We shall show here that this does not depend on the prime characteristic p of K, by showing that $y^3z^4 \notin I$ where $I = y^2(u^2 - z^4)R$ but $(y^3z^4)^p \in I^{[p]}$. As for the assertion that R/yR is F-pure, note that $R/yR \cong K[U,V,Z]/(UV,UZ,ZV)$ and so F-purity follows from the above proposition.

In $R/I^{[p]}$ we have $(y^3z^4)^p = vy^{3p-2}z^{4p} = vy^{3p-2}u^{2p} = 0$ and it only remains to show that $y^3z^4 \notin I$. Consider the grading on R where u, v, y and z have weights 2, 2, 1 and 1, respectively. If $y^3z^4 \in I$, we have an equation in $K[U,V,Y,Z]$ of the form

$$Y^3Z^4 = A(Y^2(U^2 - Z^4)) + B(UV) + C(UZ) + D(Z(V - Y^2)).$$

The polynomial A is of degree 1 and so must be a linear combination of Y and Z. Examining this modulo (U,V), we see that $A = -Y$, and so $Y^3U^2 = B(UV)+C(UZ)+$

$D(Z(V-Y^2))$. Dividing through by U, we have $Y^3U = B(V)+C(Z)+D'(Z(V-Y^2))$ where $D = UD'$, but then $Y^3U \in (V,Z)$, a contradiction.

We next use the idea of rational coefficient Weil divisors to construct a large family of two-dimensional normal domains for which F-purity does not deform, independent of the characteristic of the field K. These examples, as presented, are for the graded case but local examples are obtained by the obvious localization at the homogeneous maximal ideal. We recall some notation and results from [2, 18, 19].

Definition 3.3. By a *rational coefficient Weil divisor* on a normal projective variety X, we mean a linear combination of codimension one irreducible subvarieties of X, with coefficients in $\mathbb{Q}$.

Let $K(X)$ denote the field of rational functions on X. Each $f \in K(X)$ gives us a Weil divisor $\operatorname{div}(f)$ by considering its zeros and poles with appropriate multiplicity. For a divisor D, we shall say $D \geq 0$ if each coefficient of D is nonnegative. Recall that for a Weil divisor D on X, we have

$$H^0(X,\mathcal{O}_X(D)) = \{f \in K(X)\colon \operatorname{div}(f)+D \geq 0\}.$$

If $D = \sum n_iV_i$ with $n_i \in \mathbb{Q}$ is a rational coefficient Weil divisor on X, we shall set $[D] = \sum [n_i]V_i$, where $[n]$ denotes the greatest integer less than or equal to n, and define $\mathcal{O}_X(D) = \mathcal{O}_X([D])$. With this notation, Demazure's result is

Theorem 3.4. *Let $R = \bigoplus_{n\geq 0} R_n$ be a graded normal ring where $R_0 = K$ is a field. Then R can be described by a rational coefficient Weil divisor D on $X = \operatorname{Proj}(R)$ satisfying the condition that ND is an ample Cartier divisor for some integer $N > 0$, in the form*

$$R = R(X,D) = \bigoplus_{n\geq 0} H^0(X,\mathcal{O}_X(nD))T^n \subseteq K(X)[T],$$

where T is a homogeneous element of degree one in the quotient field of R. This identification preserves the grading, i.e., $R_n = H^0(X,\mathcal{O}_X(nD))T^n$.

For a rational coefficient Weil divisor $D = \sum (p_i/q_i)V_i$ for p_i and q_i relatively prime integers with $q_i > 0$, we define $D' = \sum ((q_i-1)/q_i)V_i$ to be the *fractional part* of D. Note that with this definition of D' we have, for a positive integer n, $-[-nD] = [nD+D']$.

Let D be a rational coefficient Weil divisor on X such that for some integer $N>0$, ND is an ample Cartier divisor, and let $R = R(X,D)$ as above. One can get the following descriptions of the symbolic powers of the canonical module and their highest local cohomology modules, see [18, 19]. If X is a smooth projective variety of dimension d with canonical divisor K_X and ω is the canonical module of R, we have

$$[\omega^{(i)}]_n = H^0(X,\mathcal{O}_X(i(K_X+D')+nD))T^n,$$

$$[H_m^{d+1}(\omega^{(i)})]_n = H^d(X,\mathcal{O}_X(i(K_X+D')+nD))T^n.$$

It is easily seen that we can identify the action of the Frobenius on the nth graded piece of the injective hull $E_R(K) = [H_m^{d+1}(\omega)]_n$ of K (as a graded R-module) with the map

$$H^d(X, \mathcal{O}_X(K_X + D' + nD))T^n \to H^d(X, \mathcal{O}_X(p(K_X + D' + nD)))T^{pn}.$$

The action of the Frobenius on the socle of $E_R(K)$ corresponds to the piece of this map in degree zero, i.e.,

$$H^d(X, \mathcal{O}_X(K_X + D')) \to H^d(X, \mathcal{O}_X(p(K_X + D'))).$$

Remark 3.5. In the above notation, if the ring R is F-pure, then the action of the Frobenius on $E_R(K)$ is injective and so $H^d(X, \mathcal{O}_X(p(K_X + D')))$ must be nonzero.

Example 3.6. We first fix positive integers n and k such that $1/n + 2/k < 1$. Take $\mathbb{P}^1 = \operatorname{Proj} K[X, Y]$, choose k distinct nonzero elements $\alpha_1, \dots, \alpha_k$ in the field K, and construct the rational coefficient Weil divisor on $\mathbb{P}^1$

$$D = (1/n)V(X - \alpha_1 Y) + \ldots + (1/n)V(X - \alpha_k Y).$$

Next, construct the graded ring $R = R(\mathbb{P}^1, D)$ where

$$R = \bigoplus_{i \geq 0} R_i = \bigoplus_{i \geq 0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(iD))Y^i,$$

see [2]. Let $\Delta_i = (X - \alpha_1 Y) \cdots (X - \alpha_{i-1} Y)(X - \alpha_{i+1} Y) \cdots (X - \alpha_k Y)$ for $1 \leq i \leq k$ and $\Delta = (X - \alpha_1 Y) \cdots (X - \alpha_k Y)$. Then R is generated by the elements $Y, A_1, \dots, A_k$ where $A_i = \Delta_i X Y^n / \Delta$ for $1 \leq i \leq k$. Now, for $i \neq j$, $A_i A_j \in YR$, and it is easily verified that

$$R/YR \cong K[A_1, \dots, A_k]/(A_i A_j : i \neq j).$$

Since $(A_i A_j : i \neq j)$ is an ideal generated by square-free monomials, the ring R/YR is F-pure. To see that R is not F-pure, by Remark 3.5 it suffices to see that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p(K_{\mathbb{P}^1} + D'))) = 0$. Note that by Serre duality, the vector space dual of this is $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((1-p)(K_{\mathbb{P}^1} + D')))$, and it certainly suffices to show that this is zero. Since we picked n and k satisfying $1/n + 2/k < 1$, we get that $\deg(1-p)(K_{\mathbb{P}^1} + D') = k(p-1)(1/n + 2/k - 1)$ is negative, and so $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}((1-p)(K_{\mathbb{P}^1} + D'))) = 0$.

It is easy to see that the ring R is actually F-rational. The elements Y, $A_1 + \ldots + A_k$ form a homogeneous system of parameters for R, and since R/YR is F-pure, the ideal $(Y, A_1 + \ldots + A_k)R$ is its own Frobenius closure. The a-invariant of R is $a(R) = -1$, and so R is F-rational by [11, Theorem 7.12].

4. Cyclic covers and anti-canonical covers

If (R, m, K) is a local or graded normal ring with an ideal I of pure height one satisfying $I^{(n)} \cong R$, by the *cyclic cover* of R with respect to I, we mean the ring

$$S = R \oplus I \oplus I^{(2)} \oplus \ldots \oplus I^{(n-1)}.$$

We use the identification $I^{(n)} = uR$ to give S a natural ring structure, i.e., S is the ring

$$S = R[It, I^{(2)}t^2, \ldots, I^{(i)}t^i, \ldots]/(ut^n - 1) = R[It, \ldots, I^{(n-1)}t^{n-1}]/(ut^n - 1).$$

If the characteristic of K is relatively prime to n, the inclusion of R in S is étale in codimensional one. This holds because for a height one prime P of R, $IR_P = vR_P$ and so $S_P = R_P[vt]/(rv^n t^n - 1)$ where $r \in R_P$ is a unit. This is implicit in [19]. Note that this also shows that S is normal.

This construction is most interesting when $I = \omega$ is the canonical ideal of R, and ω is an element of finite order in the divisor class group $\mathrm{Cl}(R)$. In this case the cyclic cover of R with respect to ω is usually called the *canonical cover*. (As the reader has already discovered, we will sometimes speak of an ideal of pure height one as an element of $\mathrm{Cl}(R)$, although we really mean the class it represents.)

If R is of dimension d, we have the isomorphism

$$\mathrm{Hom_R}(H^d_m(\omega^{(i)}), E_R) \cong \omega^{(1-i)},$$

where E_R is the injective hull of the residue field K, and consequently,

$$\mathrm{Hom_S}(H^d_{m_S}(S), E_S) \cong \mathrm{Hom_R}(H^d_m(S), E_R) \cong S.$$

S is therefore Gorenstein whenever it is Cohen–Macaulay. The following is a very useful result of Watanabe [19, Theorem 2.7].

Theorem 4.1. *Let $(R, m) \to (S, n)$ be a finite local homomorphism of normal rings which is étale in codimension one. Then if R is strongly F-regular (F-pure), so is S.*

An immediate consequence of this is the following theorem:

Theorem 4.2. *Let (R, m, K) be a Cohen–Macaulay normal ring of dimension two with a canonical ideal ω satisfying $\omega^{(n)} \cong R$ for n relatively prime to the characteristic of K. Then R is (strongly) F-regular if and only the cyclic cover $S = R \oplus \omega \oplus \cdots \oplus \omega^{(n-1)}$ is F-rational.*

Proof. Since ω is a torsion element of the divisor class group of R, the results of [15] or [21] show that F-regularity is equivalent to strong F-regularity. If R is F-regular, Theorem 4.1 then shows that S is F-regular and so is certainly F-rational. For the converse, since S is of dimension two our earlier discussion proves that S is Gorenstein, and consequently is F-regular. The ring R, being a direct summand of S, is then F-regular. $\square$

We next consider the case when the canonical ideal ω of the Cohen–Macaulay normal ring (R,m,K) is not necessarily of finite order in the divisor class group $\mathrm{Cl}(R)$. In this case, we construct the *anti-canonical cover* S by taking an ideal I of pure height one, which is the inverse of ω in $\mathrm{Cl}(R)$ and forming the symbolic Rees ring

$$S = \bigoplus_{i\geq 0} I^{(i)}t^i.$$

Note that symbolic Rees rings, in general, need not be Noetherian. However, there is a theorem of Watanabe which is very useful when the ring is indeed Noetherian [20, Theorem 0.1].

Theorem 4.3. *Let (R,m,K) be a normal ring and I an ideal of pure height one, which is the inverse of ω in $\mathrm{Cl}(R)$. Let $S=\bigoplus_{i\geq 0} I^{(i)}t^i$ be the anti-canonical cover as above. Then if S is Noetherian, R is strongly F-regular (F-pure) if and only if S is strongly F-regular (F-pure).*

Note that if S as above is Noetherian and Cohen–Macaulay, then it is Gorenstein. This can be inferred from a local cohomology calculation in [20], and is also [6, Theorem 4.8].

5. Deformation

We now consider the question whether strong F-regularity deforms for local rings, i.e., if R/xR is strongly F-regular for some nonzerodivisor $x \in m$, then is R strongly F-regular? It is known that F-rationality deforms [10, Theorem 4.2 (h)], and our work makes use of this in an essential way.

Theorem 5.1. *Let (R,m,K) be a normal local ring such that if I is an ideal of pure height one representing the inverse of ω in $\mathrm{Cl}(R)$, then the symbolic powers $I^{(i)}$ satisfy the Serre condition S_3 for all $i \geq 0$ and the anti-canonical cover $S=\bigoplus_{i\geq 0} I^{(i)}t^i$ is Noetherian.*

If R/xR is strongly F-regular for some nonzerodivisor $x \in m$, then R is also strongly F-regular.

Proof. We may, if necessary, change I to ensure that xR is not one of its minimal primes. Since we have assumed that $I^{(i)}$ is S_3, the natural maps give us isomorphisms $(I/xI)^{(i)} \cong I^{(i)}/xI^{(i)}$. Hence,

$$S/xS \cong R/xR \oplus (I/xI) \oplus (I/xI)^{(2)} \oplus \cdots$$

serves as an anti-canonical cover for R/xR. Theorem 4.3 now shows that S/xS is strongly F-regular, and so is also Cohen–Macaulay. Hence, S is Cohen–Macaulay and therefore Gorenstein. As F-rationality deforms and S/xS is strongly F-regular, we get that S is

F-rational. However, S is Gorenstein, and so is actually strongly F-regular. Finally, R is a direct summand of S and so is strongly F-regular. □

We next specialize to the case when R has a canonical ideal which is a torsion element of $\mathrm{Cl}(R)$. Note that the work of [1] already shows that F-regularity deforms in this setting, but we believe it is still interesting to examine deformation from the point of view of cyclic covers. Recall that we do not need to distinguish between F-regularity and strong F-regularity in this case. Our previous result needed the rather strong hypothesis that the symbolic powers $I^{(i)}$ are S_3, but in this case it turns out that this hypothesis is equivalent to the F-regularity of R, at least when the characteristic p is "sufficiently large".

Theorem 5.2. *Let (R, m, K) be a normal local ring, where K is a field of characteristic p, such that the canonical ideal ω of R is of finite order in* $\mathrm{Cl}(R)$, *and R/xR is F-regular for some nonzerodivisor $x \in m$.*

If the symbolic powers $\omega^{(i)}$ satisfy the Serre condition S_3, then R is F-regular. The converse is also true provided p does not divide the order of ω in $\mathrm{Cl}(R)$.

Proof. We choose an ideal I of pure height one which gives an inverse for ω in $\mathrm{Cl}(R)$ such that xR is not one of the minimal primes of I. Since ω is of finite order in $\mathrm{Cl}(R)$, so is its inverse I, and the anti–canonical cover $S = \bigoplus_{i \geq 0} I^{(i)} t^i$ is certainly Noetherian. The symbolic powers $I^{(i)}$ are isomorphic to some symbolic powers $\omega^{(j)}$, and consequently satisfy the condition S_3, by which we can conclude that R is F-regular from the previous theorem. For the converse, since R is F-regular, by Theorem 4.2 so is the cyclic cover $S = R \oplus I \oplus I^{(2)} \oplus \cdots \oplus I^{(n-1)}$ where n is the order of I in $\mathrm{Cl}(R)$. Hence the summands $I^{(i)}$ are, in fact, Cohen–Macaulay. □

6. Two-dimensional graded rings

Let (R, m, K) be a graded ring where m is the homogeneous maximal ideal. We shall follow the notation of Goto and Watanabe in [7]. We recall that the highest local cohomology module $H^d_m(R)$ of R, where $\dim R = d$ may be identified with $\varinjlim R/(x_1^t, \ldots, x_d^t)$ where $x_1, \ldots, x_d$ is a system of parameters for R and the maps are induced by multiplication by $x_1 \ldots, x_d$. If R is Cohen–Macaulay, these maps are injective. $H^d_m(R)$ has the natural structure of a graded R–module, where $\deg[r + (x_1^t, \ldots, x_d^t)] = \deg r - \sum_{i=1}^d x_i$, when r and the x_i are taken to be forms. With this grading on $H^d_m(R)$, we define the *a-invariant* of R to be the highest integer a such that $[H^d_m(R)]_a$ is nonzero, see [7].

In general, for graded R-modules M and N, we may define the graded R-module $\underline{\mathrm{Hom}}(M, N)$, where $[\underline{\mathrm{Hom}}(M, N)]_i$ is the abelian group consisting of all *graded* R-linear homomorphisms from M to $N(i)$ where the convention for the grading shift is $[N(i)]_j = [N]_{i+j}$ for all $j \in \mathbb{Z}$. This gives $\underline{\mathrm{Hom}}(M, N)$ a natural structure as a graded R-module. The injective hull of K in the category of graded R-modules is $\underline{E}_R(K) = \underline{\mathrm{Hom}}(R, K)$.

Consequently for all graded R-modules M, we have $\underline{\mathrm{Hom}}(M, \underline{E}_R(K)) = \underline{\mathrm{Hom}}(M, K)$. With this notation, $\omega = \underline{\mathrm{Hom}}(H^d_m(R), K)$ is a *graded canonical module* for R. Note that since $\omega = \underline{\mathrm{Hom}}(H^d_m(R), K)$, we have $\omega = \bigoplus_{i \geq -a} [\omega]_i$.

If R is Gorenstein, it is easy to see that $\omega \cong R(a)$, the isomorphism now being a graded isomorphism.

If R has a graded canonical module ω satisfying $\omega^{(n)} = uR$, set $k = (\deg u)/n$. We may construct a graded cyclic cover S as

$$S = R[\omega t, \ldots, \omega^{(n-1)} t^{n-1}]/(ut^n - 1),$$

where we set $\deg t = -k$. The ring S has a $(1/n)\mathbb{Z}$ grading, and it can be verified that this is in fact a $(1/n)\mathbb{Z}_{\geq 0}$ grading. Consequently, S may be graded by the nonnegative integers, and results holding for such gradings do apply here. We next show that $a(S) = -k$.

Note that we have a graded isomorphism

$$S \cong R \oplus \omega(k) \oplus \omega^{(2)}(2k) \oplus \cdots \oplus \omega^{(n-1)}(nk - k).$$

Using the fact that $\underline{\mathrm{Hom}}(H^d_m(\omega^{(i)}), E_R) \cong \omega^{(1-i)}$, we get the graded isomorphisms

$$\begin{aligned}
\omega_S &\cong \underline{\mathrm{Hom}}(H^d_m(S), K) \\
&\cong \omega \oplus R(-k) \oplus \omega^{(-1)}(-2k) \oplus \cdots \oplus \omega^{(2-n)})(k - nk) \\
&\cong [\omega(k) \oplus R \oplus \omega^{(-1)}(-k) \oplus \cdots \oplus \omega^{(2-n)}(2k - nk)](-k) \\
&\cong [\omega(k) \oplus R \oplus \omega^{(n-1)}(nk - k) \oplus \cdots \oplus \omega^{(2)}(2k)](-k) \\
&\cong S(-k).
\end{aligned}$$

We can conclude from this that $a(S) = -k$.

We next recall a result of Fedder [4, Theorem 2.10].

Let (R, m, K) be a graded ring over a perfect field K. Then $\mathfrak{S} \subseteq \mathrm{der}_K R$ is said to be *D-complete* if for every element $a \in R$ with $D(a) = 0$ for all $D \in \mathfrak{S}$, we have $D(a) = 0$ for all $D \in \mathrm{der}_K R$. If $\mathfrak{S}$ is a D-complete set of homogeneous derivations, we set

$$d_{\mathfrak{S}}(R) = \sup\{\deg(D) : D \in \mathfrak{S}\}.$$

Fedder's Theorem then is

Theorem 6.1. *Let (R, m, K) be a graded two-dimensional normal ring over a perfect field K. Assume that the characteristic p of K satisfies $d_{\mathfrak{S}}(R) < p$ for some D-complete set $\mathfrak{S}$ of homogeneous derivations. Then R is F-rational if and only if it has a negative a-invariant.*

There is a corresponding result in characteristic zero [4, Theorem 3.6].

Theorem 6.2. *Let (R, m, K) be a graded two-dimensional normal ring where K is a field of characteristic zero. Then R is F-rational if and only if it has a negative a-invariant.*

We are now in a position to combine this idea with the theory of cyclic covers to get the following result:

Theorem 6.3. *Let (R,m,K) be a graded two-dimensional normal ring over a perfect field K of characteristic p. Assume that $d_{\mathfrak{S}}(R) < p$ for some D-complete set $\mathfrak{S}$ of homogeneous derivations. If n is the order of the graded canonical module ω in the graded divisor class group of R, also assume that n and p are relatively prime, and let $\omega^{(n)} = uR$. Then R is F-regular if and only if* $\deg u > 0$.

Proof. Consider the graded cyclic cover $S = R[\omega t,\dots,\omega^{(n-1)}t^{n-1}]/(ut^n - 1)$. We observed in Theorem 4.2 that R is F-regular if and only if S is F-rational. By the above result of Fedder, this holds if and only if $a(S) < 0$. But $a(S) = -(\deg u)/n$ and so R is F-regular if and only if $\deg u > 0$. □

The characteristic 0 version follows similarly.

Theorem 6.4. *Let (R,m,K) be a graded two-dimensional normal ring over a field K of characteristic* 0. *Let n be the order of the graded canonical module ω in the graded divisor class group of R and $\omega^{(n)} = uR$. Then R is F-regular if and only if* $\deg u > 0$.

Example 6.5. Let $R = K[t, t^4x, t^4x^{-1}, t^4(x+1)^{-1}]$ where K is a field of characteristic p which we assume to be a suitably large prime. This is a ring which is F-rational but not F-pure, see [11, 19]. By mapping a polynomial ring onto it, we may write R as

$$R = K[T,U,V,W]/(T^8 - UV, T^4(V-W) - VW, U(V-W) - T^4W).$$

This is graded by setting the weights of t,u,v and w to be 1, 4, 4 and 4, respectively. We shall see that R is not F-regular as an application of the above results. Note that with this grading we have $a(R) = -1$. A graded canonical module for R is $\omega = 1/t^3(v,w)R$. It is easy to compute that $\omega^{(2)} = 1/t^6(v^2, vw, w^2)R$ and $\omega^{(3)} = 1/t^9(v^2 - 2vw + w^2)R$. Since $\deg[1/t^9(v^2 - 2vw + w^2)] = -1$, Theorem 6.3 shows that R is not F-regular.

The cyclic cover S of R is isomorphic to $K[T,Y,Z]/(T^4 + YZ^2 - Y^2Z)$ where the inclusion $R \to S$ is obtained by extending the map

$$u \mapsto yz^2, \quad v \mapsto y^3 + yz^2 - 2y^2z, \quad w \mapsto z^3 + y^2z - 2yz^2, \quad t \mapsto t.$$

The ring S has a grading where the weights of t, y and z are 1, $\frac{4}{3}$ and $\frac{4}{3}$, respectively. The a-invariant is $a(S) = \frac{1}{3}$ and corresponds to the fact that S is not F-rational.

Example 6.6. Let $T = K[X,Y,Z]/(X^3 - YZ(Y+Z))$ where K is a field of characteristic p, assumed to be a sufficiently large prime. Let T have the obvious grading where each variable has weight 1. We let R be the subring

$$R = K[X, Y^3, Y^2Z, YZ^2, Z^3]/(X^3 - YZ(Y+Z)).$$

This again is an example from [11], see also [19]. With the inherited grading, $a(R) = -1$ and so R is F-rational. A graded canonical module for R is $\omega = (y, z)R$. It is easily verified that $\omega^{(2)} = (y^2, yz, z^2)R$ and $\omega^{(3)} = R$. Consequently, R is not F-regular.

T is in fact the cyclic cover of R, under the natural inclusion. Of course, $a(T) = 0$ and T is not F-rational.

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