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A Computation of Tight Closure in Diagonal Hypersurfaces

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1. INTRODUCTION

The aim of this paper is to settle a question about the tight closure of the ideal (x^2, y^2, z^2) in the ring $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$ where K is a field of prime characteristic $p \neq 3$. (Lower case letters denote the images of the corresponding variables.) M. McDermott has studied the tight closure of various irreducible ideals in R and has established that $xyz \in (x^2, y^2, z^2)^*$ when $p < 200$, see [Mc]. The general case however existed as a classic example of the difficulty involved in tight closure computations, see also [Hu, Example 1.2]. We show that $xyz \in (x^2, y^2, z^2)^*$ in arbitrary prime characteristic p, and furthermore establish that $xyz \in (x^2, y^2, z^2)^F$ whenever R is not F-pure, i.e., when $p \equiv 2 \bmod 3$. We move on to generalize these results to the diagonal hypersurfaces $R = K[X_1, \dots, X_n]/(X_1^n + \cdots + X_n^n)$.

These issues relate to the question whether the tight closure I^* of an ideal I agrees with its plus closure, $I^+ = IR^+ \cap R$, where R is a domain over a field of characteristic p and R^+ is the integral closure of R in an algebraic closure of its fraction field. In this setting, we may think of the Frobenius closure of I as $I^F = IR^\infty \cap R$ where R^∞ is the extension of R obtained by adjoining p^eth roots of all nonzero elements of R for $e \in \mathbb{N}$. It is not difficult to see that $I^+ \subseteq I^*$, and equality in general is a formidable open question. It should be mentioned that in the case when I is an ideal generated by part of a system of parameters, the equality is a result

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of K. Smith, see [Sm]. In the above ring $R = K[X,Y,Z]/(X^3 + Y^3 + Z^3)$ where K is a field of characteristic $p \equiv 2 \bmod 3$, if one could show that $I^* = I^F$ for an ideal I, a consequence of this would be $I^F \subseteq I^+ \subseteq I^* = I^F$, by which $I^+ = I^*$. McDermott does show that $I^* = I^F$ for large families of irreducible ideals and our result $xyz \in (x^2, y^2, z^2)^F$, we believe, fills in an interesting remaining case.

2. DEFINITIONS

Our main reference for the theory of tight closure is [HH]. We next recall some basic definitions.

Let R be a Noetherian ring of characteristic $p > 0$. We shall always use the letter e to denote a variable nonnegative integer, and q to denote the eth power of p, i.e., $q = p^e$. We shall denote by F, the Frobenius endomorphism of R, and by F^e, its eth iteration, i.e., $F^e(r) = r^q$. For an ideal $I = (x_1, \dots, x_n) \subseteq R$, we let $I^{[q]} = (x_1^q, \dots, x_n^q)$. Note that $F^e(I)R = I^{[q]}$, where $q = p^e$, as always.

We shall denote by R° the complement of the union of the minimal primes of R.

DEFINITION 2.1. A ring R is said to be *F-pure* if the Frobenius homomorphism $F: M \to M \otimes_R F(R)$ is injective for all R-modules M.

For an element x of R and an ideal I, we say that $x \in I^F$, the *Frobenius closure* of I, if there exists $q = p^e$ such that $x^q \in I^{[q]}$. A normal domain R is F-pure if and only if for all ideals I of R, we have $I^F = I$.

We say that $x \in I^*$, the *tight closure* of I, if there exists $c \in R^\circ$ such that $cx^q \in I^{[q]}$ for all $q = p^e \gg 0$.

It is easily verified that $I \subseteq I^F \subseteq I^*$. Furthermore, I^* is always contained in the integral closure of I and is frequently much smaller.

3. PRELIMINARY COMPUTATIONS

We record some determinant computations we shall find useful. Note that for integers n and m where $m \geq 1$, we shall use the notation

$$\binom{n}{m} = \frac{(n)(n-1)\cdots(n-m+1)}{(m)(m-1)\cdots(1)}.$$

LEMMA 3.1.

$$\det\begin{vmatrix} \binom{n}{a+k} & \binom{n}{a+k-1} & \cdots & \binom{n}{a+2k} \\ \binom{n}{a+k-1} & \binom{n}{a+k} & \cdots & \binom{n}{a+2k-1} \\ & & \cdots & \\ \binom{n}{a} & \binom{n}{a+1} & \cdots & \binom{n}{a+k} \end{vmatrix}$$

$$= \frac{\binom{n}{a+k}\binom{n+1}{a+k}\cdots\binom{n+k}{a+k}}{\binom{a+k}{a+k}\binom{a+k+1}{a+k}\cdots\binom{a+2k}{a+k}}.$$

Proof. This is evaluated in [Mu, p. 682] as well as [Ro]. ∎

LEMMA 3.2. *Let $F(n,a,k)$ denote the determinant of the matrix*

$$M(n,a,k) = \begin{pmatrix} \binom{n}{a} & \binom{n}{a+1} & \cdots & \binom{n}{a+k} \\ \binom{n+2}{a+1} & \binom{n+2}{a+2} & \cdots & \binom{n+2}{a+k+1} \\ & & \cdots & \\ \binom{n+2k}{a+k} & \binom{n+2k}{a+k+1} & \cdots & \binom{n+2k}{a+2k} \end{pmatrix}.$$

Then for $k \geq 1$ we have

$$\frac{F(n,a,k)}{F(n+2,a+2,k-1)} = \binom{n}{a}\prod_{s=1}^{k}\prod_{r=1}^{k}\frac{s(s+2a-n)}{(a+r)(n-a+r)}.$$

Hence

$$F(n,a,k) = \frac{\binom{n}{a}\binom{n+2}{a+2}\cdots\binom{n+2k}{a+2k}}{\binom{a+k}{k}\binom{a+k+1}{k-1}\cdots\binom{a+2k-1}{1}}$$

$$\cdot\frac{\binom{2a-n+k}{k}\binom{2a-n+k+1}{k-1}\cdots\binom{2a-n+2k-1}{1}}{\binom{n-a+k}{k}\binom{n-a+k-1}{k-1}\cdots\binom{n-a+1}{1}}.$$

Proof. We shall perform row operations on $M(n, a, k)$ in order to get zero entries in the first columns from the second row onwards, starting with the last row and moving up. More precisely, from the $(r+1)$st row, subtract the rth row multiplied by $\binom{n+2r}{a+r}/\binom{n+2r-2}{a+r-1}$ starting with $r = k$, and continuing until $r = 2$. The $(r+1, s+1)$st entry of the new matrix, for $r \geq 1$, is

$$\binom{n+2r}{a+r+s} - \frac{\binom{n+2r}{a+r}}{\binom{n+2r-2}{a+r-1}}\binom{n+2r-2}{a+r-1+s}$$
$$= \frac{s(s+2a-n)}{(a+r)(n-a+r)}\binom{n+2r}{a+r+s}.$$

We have only one nonzero entry in the first column, namely $\binom{n}{a}$ and so we examine the matrix obtained by deleting the first row and column. Factoring out $s(s+2a-n)$ from each column for $s = 1, \ldots, k$ and $1/(a+r)(n - a + r)$ from each row for $r = 1, \ldots, k$, we see that

$$\det M(n, a, k) = \binom{n}{a}\prod_{s=1}^{k}\prod_{r=1}^{k}\frac{s(s+2a-n)}{(a+r)(n-a+r)}$$
$$\times \det M(n+2, a+2, k-1).$$

The required result immediately follows. ∎

LEMMA 3.3. *Consider the polynomial ring $T = K[A_1, \ldots, A_m]$ where $I_{r,i}$ denotes the ideal $I_{r,i} = (A_1^i, \ldots, A_r^i)T$ for $r \leq m$. Then*

$$(A_1 \cdots A_{r-1})^{\alpha}(A_1 + \cdots + A_{r-1})^{\beta} \in I_{r-1,\alpha+\gamma} + (A_1 + \cdots + A_{r-1})^{\alpha+\gamma}T$$

for positive integers α, β, and γ implies

$$(A_1 \cdots A_r)^{\alpha}(A_1 + \cdots + A_r)^{\beta+\gamma-1} \in I_{r,\alpha+\gamma} + (A_1 + \cdots + A_r)^{\alpha+\gamma}T.$$

Proof. Consider the binomial expansion of $(A_1 \cdots A_r)^{\alpha}(A_1 + \cdots + A_r)^{\beta+\gamma-1}$ into terms of the form $(A_1 \cdots A_{r-1})^{\alpha}(A_1 + \cdots + A_{r-1})^{\beta+\gamma-1-j}A_r^{\alpha+j}$. Such an element is clearly in $I_{r,\alpha+\gamma}$ whenever

$j \geq \gamma$, and so assume $\gamma > j$. Now

$$\begin{aligned}(A_1 \cdots A_{r-1})^{\alpha} A_r^{\alpha+j}(A_1 + \cdots + A_{r-1})^{\beta+\gamma-1-j} &\in I_{r,\alpha+\gamma} + A_r^{\alpha+j}(A_1 + \cdots + A_{r-1})^{\alpha+2\gamma-1-j}T \\ &\subseteq I_{r,\alpha+\gamma} + (A_1 + \cdots + A_{r-1}, A_r)^{2\alpha+2\gamma-1}T \\ &\subseteq I_{r,\alpha+\gamma} + (A_1 + \cdots + A_r)^{\alpha+\gamma}T.\end{aligned}$$

▮

4. TIGHT CLOSURE

We now prove the main theorem.

THEOREM 4.1. *Let $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$ where $n \geq 3$ and K is a field of prime characteristic p where $p \nmid n$. Then*

$$(x_1 \cdots x_n)^{n-2} \in \left(x_1^{n-1}, \ldots, x_n^{n-1}\right)^*.$$

Note that there are infinitely many $e \in \mathbb{N}$ such that $p^e = q \equiv 1 \bmod n$. By [HH, Lemma 8.16], it suffices to work with powers of p of this form, and show that for all such q we have

$$(x_1 \cdots x_n)^{(n-2)q+1} \in \left(x_1^{(n-1)q}, \ldots, x_n^{(n-1)q}\right).$$

Letting $q = nk + 1$, it suffices to show

$$(x_1 \cdots x_n)^{(n-2)nk} \in \left(x_1^{(n-1)nk}, \ldots, x_n^{(n-1)nk}\right).$$

Let $A_1 = x_1^n, \ldots, A_n = x_n^n$ and note that $A_1 + \cdots + A_n = 0$. In this notation, we aim to show

$$(A_1 \cdots A_n)^{(n-2)k} \in \left(A_1^{(n-1)k}, \ldots, A_n^{(n-1)k}\right).$$

Our task is then effectively reduced to working in the polynomial ring $K[A_1, \ldots, A_{n-1}] \cong K[A_1, \ldots, A_n]/(A_1 + \cdots + A_n)$ where we need to show $(A_1 \cdots A_{n-1}(A_1 + \cdots + A_{n-1}))^{(n-2)k} \in 2I_{n-1,(n-1)k} + (A_1 + \cdots + A_{n-1})^{(n-1)k}$. By repeated use of Lemma 3.3, it suffices to show

$$(A_1A_2)^{(n-2)k}(A_1 + A_2)^k \in \left(A_1^{(n-1)k}, A_2^{(n-1)k}, (A_1 + A_2)^{(n-1)k}\right).$$

We have now reduced our problem to a statement about a polynomial ring in two variables. The required result follows from the next lemma.

LEMMA 4.2. *Let $K[A,B]$ be a polynomial ring over a field K of characteristic $p>0$ and e be a positive integer such that $q=p^e\equiv 1 \bmod n$. If $q=nk+1$, we have*

$$(A,B)^{(2n-3)k}\subseteq I=\left(A^{(n-1)k},B^{(n-1)k},(A+B)^{(n-1)k}\right).$$

In particular, $(AB)^{(n-2)k}(A+B)^k\in I$.

Proof. Note that I contains the following elements: $(A+B)^{(n-1)k}\times A^kB^{(n-3)k}$, $(A+B)^{(n-1)k}A^{k-1}B^{(n-3)k+1},\dots,(A+B)^{(n-1)k}B^{(n-2)k}$. We take the binomial expansions of these elements and consider them modulo the ideal $(A^{(n-1)k},B^{(n-1)k})$. This shows that the following elements are in I:

$$\binom{(n-1)k}{k}A^{(n-1)k}B^{(n-2)k}+\cdots+\binom{(n-1)k}{2k}A^{(n-2)k}B^{(n-1)k},$$
$$\binom{(n-1)k}{k-1}A^{(n-1)k}B^{(n-2)k}+\cdots+\binom{(n-1)k}{2k-1}A^{(n-2)k}B^{(n-1)k},$$
$$\cdots$$
$$\binom{(n-1)k}{0}A^{(n-1)k}B^{(n-2)k}+\cdots+\binom{(n-1)k}{k}A^{(n-2)k}B^{(n-1)k}.$$

The coefficients of $A^{(n-1)k}B^{(n-2)k}$, $A^{(n-1)k-1}B^{(n-2)k+1},\dots,A^{(n-2)k}B^{(n-1)k}$ form the matrix

$$\begin{pmatrix}\binom{(n-1)k}{k} & \binom{(n-1)k}{k+1} & \cdots & \binom{(n-1)k}{2k}\\ \binom{(n-1)k}{k-1} & \binom{(n-1)k}{k} & \cdots & \binom{(n-1)k}{2k-1}\\ & & \cdots & \\ \binom{(n-1)k}{0} & \binom{(n-1)k}{1} & \cdots & \binom{(n-1)k}{k}\end{pmatrix}.$$

To show that all monomials of degree $(2n-3)k$ in A and B are in I, it suffices to show that this matrix is invertible. Since $q=nk+1$ we have $\binom{(n-1)k+r}{k}=(-1)^k\binom{2k-r}{k}$ for $0\le r\le k$, and so by Lemma 3.1, the determinant of this matrix is

$$\frac{\binom{(n-1)k}{k}\binom{(n-1)k+1}{k}\cdots\binom{nk}{k}}{\binom{k}{k}\binom{k+1}{k}\cdots\binom{2k}{k}}$$

$$= (-1)^{k(k+1)}\frac{\binom{2k}{k}\binom{2k-1}{k}\cdots\binom{k}{k}}{\binom{k}{k}\binom{k+1}{k}\cdots\binom{2k}{k}} = 1.$$

▮

With this we complete the proof that $(x_1 \cdots x_n)^{n-2} \in (x_1^{n-1}, \ldots, x_n^{n-1})^*$.

5. FROBENIUS CLOSURE

Let $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$ as before, where the characteristic of K is $p \nmid n$.

LEMMA 5.1. *Let $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$ where K is a field of characteristic p. Then R is F-pure if and only if $p \equiv 1 \bmod n$.*

Proof. This is Proposition 5.21(c) of [HR]. ▮

The main result of this section is the following theorem.

THEOREM 5.2. *Let $R = K[X_1, \ldots, X_n]/(X_1^n + \cdots + X_n^n)$ where K is a field of characteristic p. Then*

$$(x_1 \cdots x_n)^{n-2} \in \left(x_1^{n-1}, \ldots, x_n^{n-1}\right)^F$$

if and only if $p \not\equiv 1 \bmod n$.

One implication follows from Lemma 5.1, and so we need to consider the case $p \not\equiv 1 \bmod n$.

The case $n = 3$ seems to be the most difficult, and we handle that first. Let $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$ where $p \equiv 2 \bmod 3$. We need to show that $xyz \in (x^2, y^2, z^2)^F$.

Let $A = y^3$, $B = z^3$, and so $A + B = -x^3$. We first show that when $p = 2$, we have $xyz \in (x^2, y^2, z^2)^F$ by establishing that $(xyz)^8 \in (x^2, y^2, z^2)^{[8]}$. Note that is suffices to show that $(xyz)^6 \in (x^{15}, y^{15}, z^{15})$, or in other words that $(AB(A + B))^2 \in (A^5, B^5, (A + B)^5)$, but this is easily seen to be true.

We may now assume $p = 6m + 5$ where $m \geq 0$. We shall show that in this case $(xyz)^p \in (x^2, y^2, z^2)^{[p]}$, i.e., that

$$(xyz)^{6m+5} \in \left(x^{12m+10}, y^{12m+10}, z^{12m+10}\right).$$

Note that to establish this, it suffices to show

$$(xyz)^{6m+3} \in \left(x^{12m+9}, y^{12m+9}, z^{12m+9}\right),$$

i.e., that $(AB(A+B))^{2m+1} \in (A^{4m+3}, B^{4m+3}, (A+B)^{4m+3})$.

LEMMA 5.3. *Let $K[A, B]$ be a polynomial ring over a field K of characteristic $p = 6m + 5$ where $m \geq 0$. Then we have*

$$(AB(A+B))^{2m+1} \in I = \left(A^{4m+3}, B^{4m+3}, (A+B)^{4m+3}\right).$$

Proof. To show that $(AB(A+B))^{2m+1} \in I$, we shall show that the following terms grouped together symmetrically from its binomial expansion,

$$f_1 = (AB)^{3m+1}(A+B), f_3 = (AB)^{3m}(A^3 + B^3), \dots,$$
$$f_{2m+1} = (AB)^{2m+1}(A^{2m+1} + B^{2m+1}),$$

are all in the ideal I. Note that I contains the elements $(AB)^m(A + B)^{4m+3}$, $(AB)^{m-1}(A+B)^{4m+5}, \dots, (AB)(A+B)^{6m+1}$, $(A+B)^{6m+3}$. We consider the binomial expansions of these elements modulo (A^{4m+3}, B^{4m+3}), and get the following elements in I:

$$\binom{4m+3}{2m+2} f_1 + \binom{4m+3}{2m+3} f_3 + \cdots + \binom{4m+3}{3m+2} f_{2m+1},$$
$$\binom{4m+5}{2m+3} f_1 + \binom{4m+5}{2m+4} f_3 + \cdots + \binom{4m+5}{3m+3} f_{2m+1},$$
$$\cdots$$
$$\binom{6m+3}{3m+2} f_1 + \binom{6m+3}{3m+3} f_3 + \cdots + \binom{6m+3}{4m+2} f_{2m+1}.$$

The coefficients of $f_1, f_3, \dots, f_{2m+1}$ arising from these terms form the matrix

$$\begin{pmatrix} \binom{4m+3}{2m+2} & \binom{4m+3}{2m+3} & \cdots & \binom{4m+3}{3m+2} \\ \binom{4m+5}{2m+3} & \binom{4m+5}{2m+4} & \cdots & \binom{4m+5}{3m+3} \\ & & \cdots & \\ \binom{6m+3}{3m+2} & \binom{6m+3}{3m+3} & \cdots & \binom{6m+3}{4m+2} \end{pmatrix}.$$

We need to show that this matrix is invertible, but in the notation of Lemma 3.2, its determinant is $F(4m+3, 2m+2, m)$ and is easily seen to be nonzero. ∎

The above lemma completes the case $n = 3$. We may now assume $n \geq 4$ and $p = nk + \delta$ for $2 \leq \delta \leq n-1$. If $k = 0$, i.e., $2 \leq p \leq n-1$, we have

$$\begin{aligned}(x_1 \cdots x_n)^{(n-2)p} &= -(x_1 \cdots x_{n-1})^{(n-2)p} x_n^{(n-2)p-n}(x_1^n + \cdots + x_{n-1}^n) \\ &\in \left(x_1^{(n-1)p}, \dots, x_{n-1}^{(n-1)p}\right).\end{aligned}$$

In the remaining case, we have $n \geq 4$ and $k \geq 1$. To prove that $(x_1 \cdots x_n)^{n-2} \in (x_1^{n-1}, \dots, x_n^{n-1})^F$, we shall show

$$(x_1 \cdots x_n)^{(n-2)p} \in \left(x_1^{(n-1)p}, \dots, x_n^{(n-1)p}\right).$$

This would follow if we could show

$$(x_1 \cdots x_n)^{(n-2)nk} \in \left(x_1^{(n-1)nk+n}, \dots, x_n^{(n-1)nk+n}\right).$$

As before, let $A_1 = x_1^n, \dots, A_n = x_n^n$. It suffices to show that

$$(A_1 \cdots A_n)^{(n-2)k} \in \left(A_1^{(n-1)k+1}, \dots, A_n^{(n-1)k+1}\right).$$

By Lemma 3.3, this reduces to showing

$$\begin{aligned}&(A_1 A_2)^{(n-2)k}(A_1 + A_2)^k \in I \\ &\qquad = \left(A_1^{(n-1)k+1}, A_2^{(n-1)k+1}, (A_1 + A_2)^{(n-1)k+1}\right).\end{aligned}$$

The only remaining ingredient is the following lemma.

LEMMA 5.4. *Let $K[A, B]$ be a polynomial ring over a field K of characteristic $p > 0$ where $p = nk + \delta$ and where $n \geq 4$, $k \geq 1$, and $2 \leq \delta \leq n-1$.*

Then

$$(A, B)^{(2n-3)k} \subseteq I = \left(A^{(n-1)k+1}, B^{(n-1)k+1}, (A+B)^{(n-1)k+1}\right).$$

In particular, $(AB)^{(n-2)k}(A+B)^k \in I$.

Proof. Note that I contains the elements $(A+B)^{(n-1)k+1}A^kB^{(n-3)k-1}$, $(A+B)^{(n-1)k+1}A^{k-1}B^{(n-3)k}, \dots, (A+B)^{(n-1)k+1}B^{(n-2)k-1}$. We take the binomial expansions of these elements and consider them modulo the ideal $(A^{(n-1)k+1}, B^{(n-1)k+1})$. This shows that the following elements are in I:

$$\binom{(n-1)k+1}{k+1}A^{(n-1)k}B^{(n-2)k} + \cdots + \binom{(n-1)k+1}{2k+1}A^{(n-2)k}B^{(n-1)k},$$

$$\binom{(n-1)k+1}{k}A^{(n-1)k}B^{(n-2)k} + \cdots + \binom{(n-1)k+1}{2k}A^{(n-2)k}B^{(n-1)k},$$

$$\cdots$$

$$\binom{(n-1)k+1}{1}A^{(n-1)k}B^{(n-2)k} + \cdots + \binom{(n-1)k+1}{k+1}A^{(n-2)k}B^{(n-1)k}.$$

The coefficients of $A^{(n-1)k}B^{(n-2)k}$, $A^{(n-1)k-1}B^{(n-2)k+1}, \dots, A^{(n-2)k}B^{(n-1)k}$ form the matrix

$$\begin{pmatrix} \binom{(n-1)k+1}{k+1} & \binom{(n-1)k+1}{k+2} & \cdots & \binom{(n-1)k+1}{2k+1} \\ \binom{(n-1)k+1}{k} & \binom{(n-1)k+1}{k+1} & \cdots & \binom{(n-1)k+1}{2k} \\ & & \cdots & \\ \binom{(n-1)k+1}{1} & \binom{(n-1)k+1}{2} & \cdots & \binom{(n-1)k+1}{k+1} \end{pmatrix}.$$

To show that all monomials of degree $(2n-3)k$ in A and B are in I, it suffices to show that this matrix is invertible. The determinant of this matrix is

$$\frac{\binom{(n-1)k+1}{k+1}\binom{(n-1)k+2}{k+1}\cdots\binom{nk+1}{k+1}}{\binom{k+1}{k+1}\binom{k+2}{k+1}\cdots\binom{2k+1}{k+1}}$$

which is easily seen to be nonzero since the characteristic of the field is $p = nk + \delta$ where $2 \leq \delta \leq n-1$. ∎

Remark 5.5. It is worth noting that $xyz \in (x^2, y^2, z^2)^*$ in the ring $R = K[X,Y,Z]/(X^3 + Y^3 + Z^3)$ is, in a certain sense, unexplained. Under mild hypotheses on a ring, tight closure has a "colon-capturing" property: for $x_1, \dots, x_n$ part of a system of parameters for an excellent local (or graded) equidimensional ring A, we have $(x_1, \dots, x_{n-1}):_A x_n \subseteq (x_1, \dots, x_{n-1})^*$ and various instances of elements being in the tight closure of ideals are easily seen to arise from this colon-capturing property.

To illustrate our point, we recall from [Ho, Example 5.7] how $z^2 \in (x, y)^*$ in the ring R above is seen to arise from colon-capturing. Consider the Segre product $T = R\#S$ where $S = K[U, V]$. Then the elements $xv - yu$, xu and yv form a system of parameters for the ring T. This ring is not Cohen–Macaulay as seen from the relation on the parameters

$$(zu)(zv)(xv - yu) = (zv)^2(xu) - (zu)^2(yv).$$

The colon-capturing property of tight closure shows

$$(zu)(zv) \in (xu, yv):_T(xv - yu) \subseteq (xu, yv)^*.$$

There is a retraction $R \otimes_K S \to R$ under which $U \mapsto 1$ and $V \mapsto 1$. This gives us a retraction from $T \to R$ which, when applied to $(zu)(zv) \in (xu, yv)^*$, shows $z^2 \in (x, y)^*$ in R.

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