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The F-pure threshold of a determinantal ideal

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— Dedicated to Professor Steven Kleiman and Professor Aron Simis
on the occasion of their 70th birthdays.

Abstract. The F-pure threshold is a numerical invariant of prime characteristic singularities, that constitutes an analogue of the log canonical thresholds in characteristic zero. We compute the F-pure thresholds of determinantal ideals, i.e., of ideals generated by the minors of a generic matrix.

Keywords: F-pure threshold, log canonical threshold, determinantal ideals.

Mathematical subject classification: Primary: 13A35; Secondary: 13C40, 13A50.

1 Introduction

Consider the ring of polynomials in a matrix of indeterminates X, with coefficients in a field of prime characteristic. We compute the F-pure thresholds of determinantal ideals, i.e., of ideals generated by the minors of X of a fixed size.

The notion of *F-pure thresholds* is due to Takagi and Watanabe [18], see also Mustaţă, Takagi, and Watanabe [17]. These are positive characteristic invariants of singularities, analogous to log canonical thresholds in characteristic zero. While the definition exists in greater generality – see the above papers – the following is adequate for our purpose:

Definition 1.1. *Let R be a polynomial ring over a field of characteristic $p > 0$, with the homogeneous maximal ideal denoted by $\mathfrak{m}$. For a homogeneous proper ideal I, and integer $q = p^e$, set*

$$\nu_I(q) = \max \left\{ r \in \mathbb{N} \mid I^r \not\subseteq \mathfrak{m}^{[q]} \right\},$$

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where $\mathfrak{m}^{[q]} = (a^q \mid a \in \mathfrak{m})$. *If* I *is generated by* N *elements, it is readily seen that* $0 \leqslant \nu_I(q) \leqslant N(q-1)$. *Moreover, if* $f \in I^r \setminus \mathfrak{m}^{[q]}$, *then* $f^p \in I^{pr} \setminus \mathfrak{m}^{[pq]}$. *Thus,*

$$\nu_I(pq) \geqslant p\nu_I(q)\,.$$

It follows that $\left\{\nu_I(p^e)/p^e\right\}_{e \geqslant 1}$ *is a bounded monotone sequence; its limit is the* F-pure threshold of I, *denoted* $\operatorname{fpt}(I)$.

The F-pure threshold is known to be rational in a number of cases, see, for example, [2, 3, 4, 9, 16]. The theory of F-pure thresholds is motivated by connections to log canonical thresholds; for simplicity, and to conform to the above context, let I be a homogeneous ideal in a polynomial ring over the field of rational numbers. Using "I modulo p" to denote the corresponding characteristic p model, one has the inequality

$$\operatorname{fpt}(I \text{ modulo } p) \leqslant \operatorname{lct}(I) \qquad \text{for all } \ p \gg 0\,,$$

where $\operatorname{lct}(I)$ denotes the log canonical threshold of I. Moreover,

$$\lim_{p \longrightarrow \infty} \operatorname{fpt}(I \text{ modulo } p) = \operatorname{lct}(I)\,. \tag{1.1.1}$$

These follow from work of Hara and Yoshida [10]; see [17, Theorems 3.3, 3.4].

The F-pure thresholds of defining ideals of Calabi-Yau hypersurfaces are computed in [1]. Hernández has computed F-pure thresholds for binomial hypersurfaces [11] and for diagonal hypersurfaces [12]. In the present paper, we perform the computation for determinantal ideals:

Theorem 1.2. *Fix positive integers* $t \leqslant m \leqslant n$, *and let* X *be an* $m \times n$ *matrix of indeterminates over a field* $\mathbb{F}$ *of prime characteristic. Let* R *be the polynomial ring* $\mathbb{F}[X]$, *and* I_t *the ideal generated by the size* t *minors of* X.

The F*-pure threshold of* I_t *is*

$$\min\left\{\frac{(m-k)(n-k)}{t-k} \mid k = 0, \ldots, t-1\right\}.$$

It follows that the F-pure threshold of a determinantal ideal is independent of the characteristic: for each prime characteristic, it agrees with the log canonical threshold of the corresponding characteristic zero determinantal ideal, as computed by Johnson [15, Theorem 6.1] or Docampo [8, Theorem 5.6] using log resolutions as in Vainsencher [19]. In view of (1.1.1), Theorem 1.2 recovers the calculation of the characteristic zero log canonical threshold.

2 The computations

The primary decomposition of powers of determinantal ideals, i.e., of the ideals I_t^s, was computed by DeConcini, Eisenbud, and Procesi [7] in the case of characteristic zero, and extended to the case of *non-exceptional* prime characteristic by Bruns and Vetter [6, Chapter 10]. By Bruns [5, Theorem 1.3], the intersection of the primary ideals arising in a primary decomposition of I_t^s in non-exceptional characteristics, yields, in all characteristics, the integral closure $\overline{I_t^s}$. We record this below in the form that is used later in the paper:

Theorem 2.1 (Bruns). *Let s be a positive integer, and let $\delta_1, \ldots, \delta_h$ be minors of the matrix X. If*

$$h \leqslant s \ \ \text{and} \ \ \sum_i \deg \delta_i = ts,$$

then

$$\delta_1 \cdots \delta_h \ \in \ \overline{I_t^s}\,.$$

Proof. By [5, Theorem 1.3], the ideal $\overline{I_t^s}$ has a primary decomposition

$$\bigcap_{j=1}^{t} I_j^{((t-j+1)s)}\,.$$

Thus, it suffices to verify that

$$\delta_1 \cdots \delta_h \ \in \ I_j^{((t-j+1)s)}$$

for each j with $1 \leqslant j \leqslant t$. This follows from [6, Theorem 10.4]. □

We will also need:

Lemma 2.2. *Let k be the least integer in the interval $[0, t-1]$ such that*

$$\frac{(m-k)(n-k)}{t-k} \ \leqslant \ \frac{(m-k-1)(n-k-1)}{t-k-1};$$

interpreting a positive integer divided by zero as infinity, such a k indeed exists. Set

$$u \ = \ t(m+n-2k) - mn + k^2\,.$$

Then $t-k-u \geqslant 0$.

Moreover, if k is nonzero, then $t-k+u > 0$; if $k = 0$, then $t(m+n-1) \leqslant mn$.

Proof. Rearranging the inequality above, we have

$$t(m+n-2k-1) \leqslant mn-k^2-k\,,$$

which gives $t-k-u \geqslant 0$. If k is nonzero, then the minimality of k implies that

$$t(m+n-2k+1) > mn-k^2+k\,,$$

equivalently, that $t-k+u > 0$. If $k=0$, the assertion is readily verified. □

Notation 2.3. Let X be an $m \times n$ matrix of indeterminates. Following the notation in [6], for indices

$$1 \leqslant a_1 < \cdots < a_t \leqslant m \qquad \text{and} \qquad 1 \leqslant b_1 < \cdots < b_t \leqslant n\,,$$

we set $[a_1, \ldots, a_t \mid b_1, \ldots, b_t]$ to be the minor

$$\det\begin{pmatrix} x_{a_1b_1} & \ldots & x_{a_1b_t} \\ \vdots & & \vdots \\ x_{a_tb_1} & \ldots & x_{a_tb_t} \end{pmatrix}.$$

We use the lexicographical term order on $R = \mathbb{F}[X]$ with

$$x_{11} > x_{12} > \cdots > x_{1n} > x_{21} > \cdots > x_{m1} > \cdots > x_{mn}\,;$$

under this term order, the initial form of the minor displayed above is the product of the entries on the leading diagonal, i.e.,

$$\operatorname{in}\big([a_1, \ldots, a_t \mid b_1, \ldots, b_t]\big) = x_{a_1b_1}x_{a_2b_2}\cdots x_{a_tb_t}\,.$$

For an integer k with $0 \leqslant k \leqslant m$, we set Δ_k to be the product of minors:

$$\prod_{i=1}^{n-m+1} [1, \ldots, m \mid i, \ldots, i+m-1]$$

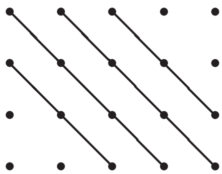
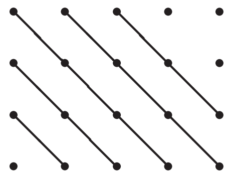
$$\times \prod_{j=2}^{m-k} [j, \ldots, m \mid 1, \ldots, m-j+1] \cdot [1, \ldots, m-j+1 \mid n-m+j, \ldots, n]\,.$$

If $k \geqslant 1$, we set Δ'_k to be

$$\Delta_k \cdot [m-k+1, \ldots, m \mid 1, \ldots, k]\,.$$

Notice that $\deg \Delta_k = mn-k^2-k$ and that $\deg \Delta'_k = mn-k^2$. The element Δ_k is a product of $m+n-2k-1$ minors and Δ'_k of $m+n-2k$ minors.

Example 2.4. We include an example to assist with the notation. In the case $m = 4$ and $n = 5$, the elements Δ_2 and Δ_2' are, respectively, the products of the minors determined by the leading diagonals displayed below:

Δ_2  Δ_2' 

The initial form of Δ_2' is the square-free monomial

$$x_{11}\, x_{12}\, x_{13}\, x_{21}\, x_{22}\, x_{23}\, x_{24}\, x_{31}\, x_{32}\, x_{33}\, x_{34}\, x_{35}\, x_{42}\, x_{43}\, x_{44}\, x_{45}\,.$$

For arbitrary m, n, the initial form of Δ_0 is the product of the mn indeterminates.

Proof of Theorem 1.2. We first show that for each k with $0 \leqslant k \leqslant t-1$, one has

$$\operatorname{fpt}(I_t) \ \leqslant\ \frac{(m-k)(n-k)}{t-k}\,.$$

Let δ_k and δ_t be minors of size k and t respectively. Theorem 2.1 implies that

$$\delta_k^{t-k-1}\delta_t \ \in\ \overline{I_{k+1}^{t-k}}\,,$$

and hence that $\delta_k^{t-k-1} I_t \ \subseteq\ \overline{I_{k+1}^{t-k}}$. By the Briançon-Skoda theorem, see, for example, [13, Theorem 5.4], there exists an integer N such that

$$\left(\delta_k^{t-k-1} I_t\right)^{N+l} \ \in\ I_{k+1}^{(t-k)l}$$

for each integer $l \geqslant 1$. Localizing at the prime ideal I_{k+1} of R, one has

$$I_t^{N+l} \ \subseteq\ I_{k+1}^{(t-k)l} R_{I_{k+1}} \qquad \text{for each } \ l \geqslant 1\,,$$

as the element δ_k is a unit in $R_{I_{k+1}}$. Since $R_{I_{k+1}}$ is a regular local ring of dimension $(m-k)(n-k)$, with maximal ideal $I_{k+1} R_{I_{k+1}}$, it follows that

$$I_t^{N+l} \ \subseteq\ I_{k+1}^{[q]} R_{I_{k+1}}$$

for positive integers l and $q = p^e$ satisfying

$$(t-k)l \ >\ (q-1)(m-k)(n-k)\,.$$

Returning to the polynomial ring R, the ideal I_{k+1} is the unique associated prime of $I_{k+1}^{[q]}$; this follows from the flatness of the Frobenius endomorphism, see for example, [14, Corollary 21.11]. Hence, in the ring R, we have

$$I_t^{N+l} \subseteq I_{k+1}^{[q]}$$

for all integers q, l satisfying the above inequality. This implies that

$$\nu_{I_t}(q) \leqslant N + \frac{(q-1)(m-k)(n-k)}{t-k}.$$

Dividing by q and passing to the limit, one obtains

$$\operatorname{fpt}(I_t) \leqslant \frac{(m-k)(n-k)}{t-k}.$$

Next, fix k and u be as in Lemma 2.2, and consider Δ_k and Δ'_k as in Notation 2.3; the latter is defined only in the case $k \geqslant 1$. Set

$$\Delta = \begin{cases} \Delta_0^t & \text{if} \quad k = 0, \\ \Delta_k^u \cdot (\Delta'_k)^{t-k-u} & \text{if} \quad k \geqslant 1 \text{ and } u \geqslant 0, \\ (\Delta'_k)^{t-k+u} \cdot \Delta_{k-1}^{-u} & \text{if} \quad k \geqslant 1 \text{ and } u < 0, \end{cases}$$

bearing in mind that $t - k - u \geqslant 0$ by Lemma 2.2.

We claim that Δ belongs to the integral closure of the ideal $I_t^{(m-k)(n-k)}$. This holds by Theorem 2.1, since, in each case,

$$\deg \Delta = t(m-k)(n-k),$$

and Δ is a product of at most $(m-k)(n-k)$ minors: if $k \geqslant 1$, then Δ is a product of exactly $(m-k)(n-k)$ minors, whereas if $k = 0$ then Δ is a product of $t(m+n-1)$ minors and, by Lemma 2.2, one has $t(m+n-1) \leqslant mn$.

Let $\mathfrak{m}$ be the homogeneous maximal ideal of R. For a positive integer s that is not necessarily a power of p, set

$$\mathfrak{m}^{[s]} = (x_{ij}^s \mid i = 1, \ldots, m,\ j = 1, \ldots, n).$$

Using the lexicographical term order from Notation 2.3, the initial forms $\operatorname{in}(\Delta_k)$ and $\operatorname{in}(\Delta'_k)$ are square-free monomials, and

$$\operatorname{in}(\Delta) = \begin{cases} \operatorname{in}(\Delta_0)^t & \text{if} \quad k = 0, \\ \operatorname{in}(\Delta_k)^u \cdot \operatorname{in}(\Delta'_k)^{t-k-u} & \text{if} \quad k \geqslant 1 \text{ and } u \geqslant 0, \\ \operatorname{in}(\Delta'_k)^{t-k+u} \cdot \operatorname{in}(\Delta_{k-1})^{-u} & \text{if} \quad k \geqslant 1 \text{ and } u < 0. \end{cases}$$

Thus, each variable x_{ij} occurs in the monomial $\mathrm{in}(\Delta)$ with exponent at most $t-k$. It follows that

$$\Delta \notin \mathfrak{m}^{[t-k+1]}.$$

As Δ belongs to the integral closure of $I_t^{(m-k)(n-k)}$, there exists a nonzero homogeneous polynomial $f \in R$ such that

$$f\Delta^l \in I_t^{(m-k)(n-k)l} \qquad \text{for all integers } l \geqslant 1\,.$$

But then

$$f\Delta^l \in I_t^{(m-k)(n-k)l} \setminus \mathfrak{m}^{[q]}$$

for all integers l with $\deg f + l(t-k) \leqslant q-1$. Hence,

$$\nu_{I_t}(q) \geqslant (m-k)(n-k)l \qquad \text{for all integers } l \text{ with } l \leqslant \frac{q-1-\deg f}{t-k}\,.$$

Thus,

$$\nu_{I_t}(q) \geqslant (m-k)(n-k)\left(\frac{q-1-\deg f}{t-k} - 1\right),$$

and dividing by q and passing to the limit, one obtains

$$\mathrm{fpt}(I_t) \geqslant \frac{(m-k)(n-k)}{t-k},$$

which completes the proof. □

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