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Rings of Frobenius operators

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Abstract

Let R be a local ring of prime characteristic. We study the ring of Frobenius operators $\mathcal{F}(E)$, where E is the injective hull of the residue field of R. In particular, we examine the finite generation of $\mathcal{F}(E)$ over its degree zero component $\mathcal{F}^0(E)$, and show that $\mathcal{F}(E)$ need not be finitely generated when R is a determinantal ring; nonetheless, we obtain concrete descriptions of $\mathcal{F}(E)$ in good generality that we use, for example, to prove the discreteness of F-jumping numbers for arbitrary ideals in determinantal rings.

1. *Introduction*

Lyubeznik and Smith [**LS**] initiated the systematic study of rings of Frobenius operators and their applications to tight closure theory. Our focus here is on the Frobenius operators on the injective hull of $R/\mathfrak{m}$, when $(R, \mathfrak{m})$ is a complete local ring of prime characteristic.

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Definition 1·1. Let R be a ring of prime characteristic p, with Frobenius endomorphism F. Following [**LS**, section 3], we set $R\{F^e\}$ to be the ring extension of R obtained by adjoining a noncommutative variable χ subject to the relations $\chi r = r^{p^e}\chi$ for all $r \in R$.

Let M be an R-module. Extending the R-module structure on M to an $R\{F^e\}$-module structure is equivalent to specifying an additive map $\varphi\colon M \to M$ that satisfies

$$\varphi(rm) = r^{p^e}\varphi(m), \qquad \text{for each } r \in R \text{ and } m \in M.$$

Define $\mathcal{F}^e(M)$ to be the set of all such maps φ arising from $R\{F^e\}$-module structures on M; this is an Abelian group with a left R-module structure, where $r \in R$ acts on $\varphi \in \mathcal{F}^e(M)$ to give the composition $r \circ \varphi$. Given elements $\varphi \in \mathcal{F}^e(M)$ and $\varphi' \in \mathcal{F}^{e'}(M)$, the compositions $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are elements of the module $\mathcal{F}^{e+e'}(M)$. Thus,

$$\mathcal{F}(M) = \mathcal{F}^0(M) \oplus \mathcal{F}^1(M) \oplus \mathcal{F}^2(M) \oplus \cdots$$

has a ring structure; this is the *ring of Frobenius operators* on M.

Note that $\mathcal{F}(M)$ is an $\mathbb{N}$-graded ring; it is typically not commutative. The degree 0 component $\mathcal{F}^0(M) = \operatorname{End}_R(M)$ is a subring, with a natural R-algebra structure. Lyubeznik and Smith [**LS**, section 3] ask whether $\mathcal{F}(M)$ is a finitely generated ring extension of $\mathcal{F}^0(M)$. From the point of view of tight closure theory, the main cases of interest are where $(R, \mathfrak{m})$ is a complete local ring, and the module M is the local cohomology module $H^{\dim R}_{\mathfrak{m}}(R)$ or the injective hull of the residue field, $E_R(R/\mathfrak{m})$, abbreviated E in the following discussion. In the former case, the algebra $\mathcal{F}(M)$ is finitely generated under mild hypotheses, see Example 1·2.2; an investigation of the latter case is our main focus here.

It follows from Example 1·2.2 that for a Gorenstein complete local ring $(R, \mathfrak{m})$, the ring $\mathcal{F}(E)$ is a finitely generated extension of $\mathcal{F}^0(E) \cong R$. This need not be true when R is not Gorenstein: Katzman [**Ka**] constructed the first such examples. In Section 3 we study the finite generation of $\mathcal{F}(E)$, and provide descriptions of $\mathcal{F}(E)$ even when it is not finitely generated: this is in terms of a graded subgroup of the anticanonical cover of R, with a Frobenius-twisted multiplication structure, see Theorem 3·3.

Section 4 studies the case of $\mathbb{Q}$-Gorenstein rings. We show that $\mathcal{F}(E)$ is finitely generated (though not necessarily principally generated) if R is $\mathbb{Q}$-Gorenstein with index relatively prime to the characteristic, Proposition 4·1; the dual statement for the Cartier algebra was previously obtained by Schwede in [**Sc**, remark 4·5]. We also construct a $\mathbb{Q}$-Gorenstein ring for which the ring $\mathcal{F}(E)$ is *not* finitely generated over $\mathcal{F}^0(E)$; in fact, we conjecture that this is always the case for a $\mathbb{Q}$-Gorenstein ring whose index is a multiple of the characteristic, see Conjecture 4·2.

In Section 5 we show that $\mathcal{F}(E)$ need not be finitely generated for determinantal rings, specifically for the ring $\mathbb{F}[X]/I$, where X is a 2×3 matrix of variables, and I is the ideal generated by its size 2 minors; this proves a conjecture of Katzman, [**Ka**, conjecture 3·1]. The relevant calculations also extend a result of Fedder, [**Fe**, proposition 4·7].

One of the applications of our study of $\mathcal{F}(E)$ is the discreteness of F-jumping numbers; in Section 6 we use the description of $\mathcal{F}(E)$, combined with the notion of gauge boundedness, due to Blickle [**Bl2**], to obtain positive results on the discreteness of F-jumping numbers for new classes of rings including determinantal rings, see Theorem 6·4. In the last section, we obtain results on the linear growth of Castelnuovo-Mumford regularity for rings with finite Frobenius representation type; this is also with an eye towards the discreteness of F-jumping numbers.

To set the stage, we summarize some previous results on the rings $\mathcal{F}(M)$.

Example 1·2. Let R be a ring of prime characteristic.

(1) For each $e \geqslant 0$, the left R-module $\mathcal{F}^e(R)$ is free of rank one, spanned by F^e; this is [**LS**, example 3·6]. Hence, $\mathcal{F}(R) \cong R\{F\}$.

(2) Let $(R, \mathfrak{m})$ be a local ring of dimension d. The Frobenius endomorphism F of R induces, by functoriality, an additive map

$$F\colon H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(R),$$

which is the natural *Frobenius action* on $H^d_{\mathfrak{m}}(R)$. If the ring R is complete and S_2, then $\mathcal{F}^e(H^d_{\mathfrak{m}}(R))$ is a free left R-module of rank one, spanned by F^e; for a proof of this, see [**LS**, example 3·7]. It follows that

$$\mathcal{F}\big(H^d_{\mathfrak{m}}(R)\big) \;\cong\; R\{F\}.$$

In particular, $\mathcal{F}(H^d_{\mathfrak{m}}(R))$ is a finitely generated ring extension of $\mathcal{F}^0(H^d_{\mathfrak{m}}(R))$.

(3) Consider the local ring $R = \mathbb{F}[[x, y, z]]/(xy, yz)$ where $\mathbb{F}$ is a field, and set E to be the injective hull of the residue field of R. Katzman [**Ka**] proved that $\mathcal{F}(E)$ is not a finitely generated ring extension of $\mathcal{F}^0(E)$.

(4) Let $(R, \mathfrak{m})$ be the completion of a Stanley–Reisner ring at its homogeneous maximal ideal, and let E be the injective hull of $R/\mathfrak{m}$. In [**ABZ**] Àlvarez, Boix and Zarzuela obtain necessary and sufficient conditions for the finite generation of $\mathcal{F}(E)$. Their work yields, in particular, Cohen–Macaulay examples where $\mathcal{F}(E)$ is not finitely generated over $\mathcal{F}^0(E)$. By [**ABZ**, theorem 3·5], $\mathcal{F}(E)$ is either 1-generated or infinitely generated as a ring extension of $\mathcal{F}^0(E)$ in the Stanley–Reisner case.

Remark 1·3. Let $R^{(e)}$ denote the R-bimodule that agrees with R as a left R-module, and where the right module structure is given by

$$x \cdot r = r^{p^e} x, \quad \text{for all } r \in R \text{ and } x \in R^{(e)}.$$

For each R-module M, one then has a natural isomorphism

$$\mathcal{F}^e(M) \;\cong\; \operatorname{Hom}_R\big(R^{(e)} \otimes_R M,\ M\big)$$

where $\varphi \in \mathcal{F}^e(M)$ corresponds to $x \otimes m \mapsto x\varphi(m)$ and $\psi \in \operatorname{Hom}_R(R^{(e)} \otimes_R M,\ M)$ corresponds to $m \mapsto \psi(1 \otimes m)$; see [**LS**, remark 3·2].

Remark 1·4. Let R be a Noetherian ring of prime characteristic. If M is a Noetherian R-module, or if R is complete local and M is an Artinian R-module, then each graded component $\mathcal{F}^e(M)$ of $\mathcal{F}(M)$ is a finitely generated left R-module, and hence also a finitely generated left $\mathcal{F}^0(M)$-module; this is [**LS**, proposition 3·3].

Remark 1·5. Let R be a complete local ring of prime characteristic p; set E to be the injective hull of the residue field of R. Let A be a complete regular local ring with $R = A/I$. By [**Bl1**, proposition 3·36], one then has an isomorphism of R-modules

$$\mathcal{F}^e(E) \;\cong\; \frac{I^{[p^e]} :_A I}{I^{[p^e]}}.$$

2. *Twisted multiplication*

Let R be a complete local ring of prime characteristic; let E denote the injective hull of the residue field of R. In Theorem 3·3 we prove that $\mathcal{F}(E)$ is isomorphic to a subgroup of the anticanonical cover of R, with a twisted multiplication structure; in this section, we describe this twisted construction in broad generality:

Definition 2·1. Given an $\mathbb{N}$-graded commutative ring $\mathcal{R}$ of prime characteristic p, we define a new ring $\mathcal{T}(\mathcal{R})$ as follows: Consider the Abelian group

$$\mathcal{T}(\mathcal{R}) = \bigoplus_{e \geqslant 0} \mathcal{R}_{p^e - 1}$$

and define a multiplication $\circledast$ on $\mathcal{T}(\mathcal{R})$ by

$$a \circledast b = ab^{p^e}, \quad \text{for } a \in \mathcal{T}(\mathcal{R})_e \text{ and } b \in \mathcal{T}(\mathcal{R})_{e'}.$$

It is a straightforward verification that $\circledast$ is an associative binary operation; the prime characteristic assumption is used in verifying that $+$ and $\circledast$ are distributive. Moreover, for elements $a \in \mathcal{T}(\mathcal{R})_e$ and $b \in \mathcal{T}(\mathcal{R})_{e'}$ one has

$$ab^{p^e} \in \mathcal{R}_{p^e - 1 + p^e(p^{e'} - 1)} = \mathcal{R}_{p^{e+e'} - 1}$$

and hence

$$\mathcal{T}(\mathcal{R})_e \circledast \mathcal{T}(\mathcal{R})_{e'} \subseteq \mathcal{T}(\mathcal{R})_{e+e'}.$$

Thus, $\mathcal{T}(\mathcal{R})$ is an $\mathbb{N}$-graded ring; we abbreviate its degree e component $\mathcal{T}(\mathcal{R})_e$ as $\mathcal{T}_e$. The ring $\mathcal{T}(\mathcal{R})$ is typically not commutative, and need not be a finitely generated extension ring of $\mathcal{T}_0$ even when $\mathcal{R}$ is Noetherian:

Example 2·2. We examine $\mathcal{T}(\mathcal{R})$ when $\mathcal{R}$ is a standard graded polynomial ring over a field $\mathbb{F}$. We show that $\mathcal{T}(\mathcal{R})$ is a finitely generated ring extension of $\mathcal{T}_0 = \mathbb{F}$ if $\dim \mathcal{R} \leqslant 2$, and that $\mathcal{T}(\mathcal{R})$ is not finitely generated if $\dim \mathcal{R} \geqslant 3$.

(1) If $\mathcal{R}$ is a polynomial ring of dimension 1, then $\mathcal{T}(\mathcal{R})$ is commutative and finitely generated over $\mathbb{F}$: take $\mathcal{R} = \mathbb{F}[x]$, in which case $\mathcal{T}_e = \mathbb{F} \cdot x^{p^e - 1}$ and

$$x^{p^e - 1} \circledast x^{p^{e'} - 1} = x^{p^{e+e'} - 1} = x^{p^{e'} - 1} \circledast x^{p^e - 1}.$$

Thus, $\mathcal{T}(\mathcal{R})$ is a polynomial ring in one variable.

(2) When $\mathcal{R}$ is a polynomial ring of dimension 2, we verify that $\mathcal{T}(\mathcal{R})$ is a noncommutative finitely generated ring extension of $\mathbb{F}$. Let $\mathcal{R} = \mathbb{F}[x, y]$. Then

$$x^{p-1} \circledast y^{p-1} = x^{p-1} y^{p^2 - p} \quad \text{whereas} \quad y^{p-1} \circledast x^{p-1} = x^{p^2 - p} y^{p-1},$$

so $\mathcal{T}(\mathcal{R})$ is not commutative. For finite generation, it suffices to show that

$$\mathcal{T}_{e+1} = \mathcal{T}_1 \circledast \mathcal{T}_e, \quad \text{for each } e \geqslant 1.$$

Set $q = p^e$ and consider the elements

$$x^i y^{p-1-i} \in \mathcal{T}_1, \quad 0 \leqslant i \leqslant p-1 \quad \text{and} \quad x^j y^{q-1-j} \in \mathcal{T}_e, \quad 0 \leqslant j \leqslant q-1.$$

Then $\mathcal{T}_1 \circledast \mathcal{T}_e$ contains the elements

$$\left(x^i y^{p-1-i}\right) \circledast \left(x^j y^{q-1-j}\right) = x^{i+pj} y^{pq-pj-i-1},$$

for $0 \leqslant i \leqslant p-1$ and $0 \leqslant j \leqslant q-1$, and these are readily seen to span $\mathcal{T}_{e+1}$. Hence, the degree $p-1$ monomials in x and y generate $\mathcal{T}(\mathcal{R})$ as a ring extension of $\mathbb{F}$.

(3) For a polynomial ring $\mathcal{R}$ of dimension 3 or higher, the ring $\mathcal{T}(\mathcal{R})$ is noncommutative and not finitely generated over $\mathbb{F}$. The noncommutativity is immediate from (2); we give an argument that $\mathcal{T}(\mathcal{R})$ is not finitely generated for $\mathcal{R} = \mathbb{F}[x, y, z]$, and this carries over to polynomial rings $\mathcal{R}$ of higher dimension.
Set $q = p^e$ where $e \geqslant 2$. We claim that the element

$$xy^{q/p-1}z^{q-q/p-1} \in \mathcal{T}_e$$

does not belong to $\mathcal{T}_{e_1} * \mathcal{T}_{e_2}$ for integers $e_i < e$ with $e_1 + e_2 = e$. Indeed, $\mathcal{T}_{e_1} * \mathcal{T}_{e_2}$ is spanned by the monomials

$$\left(x^i y^j z^{q_1-i-j-1}\right) * \left(x^k y^l z^{q_2-k-l-1}\right) = x^{i+q_1k} y^{j+q_1l} z^{q-i-j-q_1k-q_1l-1}$$

where $q_i = p^{e_i}$ and

$$0 \leqslant i \leqslant q_1 - 1, \qquad 0 \leqslant j \leqslant q_1 - 1 - i,$$
$$0 \leqslant k \leqslant q_2 - 1, \qquad 0 \leqslant l \leqslant q_2 - 1 - k,$$

so it suffices to verify that the equations

$$i + q_1 k = 1 \quad \text{and} \quad j + q_1 l = q/p - 1$$

have no solution for integers i, j, k, l in the intervals displayed above. The first of the equations gives $i = 1$, which then implies that $0 \leqslant j \leqslant q_1 - 2$. Since q_1 divides q/p, the second equation gives $j \equiv -1 \bmod q_1$. But this has no solution with $0 \leqslant j \leqslant q_1 - 2$.

3. *The ring structure of $\mathcal{F}(E)$*

We describe the ring of Frobenius operators $\mathcal{F}(E)$ in terms of the symbolic Rees algebra $\mathcal{R}$ and the twisted multiplication structure $\mathcal{T}(\mathcal{R})$ of the previous section. First, a notational point: $\omega^{[p^e]}$ below denotes the iterated Frobenius power of an ideal ω, and $\omega^{(n)}$ its symbolic power, which coincides with reflexive power for divisorial ideals ω. We realize that the notation $\omega^{[n]}$ is sometimes used for the reflexive power, hence this note of caution. We start with the following observation:

LEMMA 3·1. *Let $(R, \mathfrak{m})$ be a normal local ring of characteristic $p > 0$. Let ω be a divisorial ideal of R, i.e., an ideal of pure height one. Then for each integer $e \geqslant 1$, the map*

$$H^{\dim R}_{\mathfrak{m}}\left(\omega^{[p^e]}\right) \longrightarrow H^{\dim R}_{\mathfrak{m}}\left(\omega^{(p^e)}\right)$$

induced by the inclusion $\omega^{[p^e]} \subseteq \omega^{(p^e)}$, is an isomorphism.

Proof. Set $d = \dim R$. Since R is normal and ω has pure height one, $\omega R_{\mathfrak{p}}$ is principal for each prime ideal $\mathfrak{p}$ of height one; hence $\left(\omega^{(p^e)}/\omega^{[p^e]}\right)R_{\mathfrak{p}} = 0$. It follows that

$$\dim\left(\omega^{(p^e)}/\omega^{[p^e]}\right) \leqslant d - 2,$$

which gives the vanishing of the outer terms of the exact sequence

$$H^{d-1}_{\mathfrak{m}}\left(\omega^{(p^e)}/\omega^{[p^e]}\right) \longrightarrow H^{d}_{\mathfrak{m}}\left(\omega^{[p^e]}\right) \longrightarrow H^{d}_{\mathfrak{m}}\left(\omega^{(p^e)}\right) \longrightarrow H^{d}_{\mathfrak{m}}\left(\omega^{(p^e)}/\omega^{[p^e]}\right),$$

and thus the desired isomorphism.

Definition 3·2. Let R be a normal ring that is either complete local, or $\mathbb{N}$-graded and finitely generated over R_0. Let ω denote the canonical module of R. The symbolic Rees algebra

$$\mathcal{R} = \bigoplus_{n \geqslant 0} \omega^{(-n)}$$

is the *anticanonical cover* of R; it has a natural $\mathbb{N}$-grading where $\mathcal{R}_n = \omega^{(-n)}$.

THEOREM 3·3. *Let $(R, \mathfrak{m})$ be a normal complete local ring of characteristic $p > 0$. Set d to be the dimension of R. Let ω denote the canonical module of R, and identify E, the injective hull of the $R/\mathfrak{m}$, with $H^d_{\mathfrak{m}}(\omega)$.*

(1) *Then $\mathcal{F}(E)$, the ring of Frobenius operators on E, may be identified with*

$$\bigoplus_{e \geqslant 0} \omega^{(1-p^e)} F^e,$$

where F^e denotes the map $H^d_{\mathfrak{m}}(\omega) \to H^d_{\mathfrak{m}}(\omega^{(p^e)})$ induced by $\omega \to \omega^{[p^e]}$.

(2) *Let $\mathcal{R}$ be the anticanonical cover of R. Then one has an isomorphism of graded rings*

$$\mathcal{F}(E) \cong \mathcal{T}(\mathcal{R}),$$

where $\mathcal{T}(\mathcal{R})$ is as in Definition 2·1.

Proof. By Remark 1·3, we have

$$\mathcal{F}^e\big(H^d_{\mathfrak{m}}(\omega)\big) \cong \operatorname{Hom}_R\big(R^{(e)} \otimes_R H^d_{\mathfrak{m}}(\omega),\ H^d_{\mathfrak{m}}(\omega)\big).$$

Moreover,

$$R^{(e)} \otimes_R H^d_{\mathfrak{m}}(\omega) \cong H^d_{\mathfrak{m}}\big(\omega^{[p^e]}\big) \cong H^d_{\mathfrak{m}}\big(\omega^{(p^e)}\big),$$

where the first isomorphism of by [**ILL**$^+$, exercise 9·7], and the second by Lemma 3·1. By similar arguments

$$\begin{aligned}\operatorname{Hom}_R\big(H^d_{\mathfrak{m}}\big(\omega^{(p^e)}\big),\ H^d_{\mathfrak{m}}(\omega)\big) &\cong \operatorname{Hom}_R\big(H^d_{\mathfrak{m}}\big(\omega \otimes_R \omega^{(p^e-1)}\big),\ H^d_{\mathfrak{m}}(\omega)\big)\\ &\cong \operatorname{Hom}_R\big(\omega^{(p^e-1)} \otimes_R H^d_{\mathfrak{m}}(\omega),\ H^d_{\mathfrak{m}}(\omega)\big)\\ &\cong \operatorname{Hom}_R\big(\omega^{(p^e-1)},\ \operatorname{Hom}_R\big(H^d_{\mathfrak{m}}(\omega),\ H^d_{\mathfrak{m}}(\omega)\big)\big),\end{aligned}$$

with the last isomorphism using the adjointness of Hom and tensor. Since R is complete, the module above is isomorphic to

$$\operatorname{Hom}_R\big(\omega^{(p^e-1)},\ R\big) \cong \omega^{(1-p^e)}.$$

Suppose $\varphi \in \mathcal{F}^e(M)$ and $\varphi' \in \mathcal{F}^{e'}(M)$ correspond respectively to aF^e and $a'F^{e'}$, for elements $a \in \omega^{(1-p^e)}$ and $a' \in \omega^{(1-p^{e'})}$. Then $\varphi \circ \varphi'$ corresponds to $aF^e \circ bF^{e'} = ab^{p^e} F^{e+e'}$, which agrees with the ring structure of $\mathcal{T}(\mathcal{R})$ since $a \ast b = ab^{p^e}$.

Remark 3·4. Let R be a normal complete local ring of prime characteristic p; let A be a complete regular local ring with $R = A/I$. Using Remark 1·5 and Theorem 3·3, it is now a straightforward verification that $\mathcal{F}(E)$ is isomorphic, as a graded ring, to

$$\bigoplus_{e \geqslant 0} \frac{I^{[p^e]} :_A I}{I^{[p^e]}},$$

where the multiplication on this latter ring is the twisted multiplication $\ast$. An example of the isomorphism is worked out in Proposition 5·1.

4. $\mathbb{Q}$-*Gorenstein rings*

We analyze the finite generation of $\mathcal{F}(E)$ when R is $\mathbb{Q}$-Gorenstein. The following result follows from the corresponding statement for Cartier algebras, [**Sc**, remark 4·5], but we include it here for the sake of completeness:

PROPOSITION 4·1. *Let* $(R, \mathfrak{m})$ *be a normal* $\mathbb{Q}$*-Gorenstein local ring of prime characteristic. Let* ω *denote the canonical module of* R*. If the order of* ω *is relatively prime to the characteristic of* R*, then* $\mathcal{F}(E)$ *is a finitely generated ring extension of* $\mathcal{F}^0(E)$*.*

Proof. Since $\mathcal{F}^0(E)$ is isomorphic to the $\mathfrak{m}$-adic completion of R, the proposition reduces to the case where the ring R is assumed to be complete.

Let m be the order of ω, and p the characteristic of R. Then p mod m is an element of the group $(\mathbb{Z}/m\mathbb{Z})^\times$, and hence there exists an integer e_0 with $p^{e_0} \equiv 1 \bmod m$. We claim that $\mathcal{F}(E)$ is generated over $\mathcal{F}^0(E)$ by $[\mathcal{F}(E)]_{\leqslant e_0}$.

We use the identification $\mathcal{F}(E) = \mathcal{T}(\mathcal{R})$ from Theorem 3·3. Since $\omega^{(m)}$ is a cyclic module, one has

$$\omega^{(n+km)} = \omega^{(n)}\omega^{(km)}, \quad \text{for all integers } k, n.$$

Thus, for each $e > e_0$, one has

$$\begin{aligned}
\mathcal{T}_{e-e_0} * \mathcal{T}_{e_0} &= \omega^{(1-p^{e-e_0})} * \omega^{(1-p^{e_0})} \\
&= \omega^{(1-p^{e-e_0})} \cdot \left(\omega^{(1-p^{e_0})}\right)^{[p^{e-e_0}]} \\
&= \omega^{(1-p^{e-e_0})} \cdot \omega^{(p^{e-e_0}(1-p^{e_0}))} \\
&= \omega^{(1-p^{e-e_0}+p^{e-e_0}-p^e)} \\
&= \omega^{(1-p^e)} \\
&= \mathcal{T}_e,
\end{aligned}$$

which proves the claim.

We conjecture that Proposition 4·1 has a converse in the following sense:

Conjecture 4·2. Let $(R, \mathfrak{m})$ be a normal $\mathbb{Q}$-Gorenstein ring of prime characteristic, such that the order of the canonical module in the divisor class group is a multiple of the characteristic of R. Then $\mathcal{F}(E)$ is not a finitely generated ring extension of $\mathcal{F}^0(E)$.

Veronese subrings. Let $\mathbb{F}$ be a field of characteristic $p > 0$, and $A = \mathbb{F}[x_1, \dots, x_d]$ a polynomial ring. Given a positive integer n, we denote the n-th Veronese subring of A by

$$A_{(n)} = \bigoplus_{k \geqslant 0} A_{nk}\,;$$

this differs from the standard notation, e.g., [**GW**], since we reserve superscripts $(\ )^{(n)}$ for symbolic powers. The cyclic module $x_1 \cdots x_d A$ is the graded canonical module for the polynomial ring A. By [**GW**, corollary 3·1·3], the Veronese submodule

$$\left(x_1 \cdots x_d A\right)_{(n)} = \bigoplus_{k \geqslant 0} \left[x_1 \cdots x_d A\right]_{nk}$$

is the graded canonical module for the subring $A_{(n)}$. Let $\mathfrak{m}$ denote the homogeneous maximal ideal of $A_{(n)}$. The injective hull of $A_{(n)}/\mathfrak{m}$ in the category of graded $A_{(n)}$-modules is

$$H^d_{\mathfrak{m}}\big((x_1\cdots x_d A)_{(n)}\big) = \big[H^d_{\mathfrak{m}}(x_1\cdots x_d A)\big]_{(n)} = \left[\frac{A_{x_1\cdots x_d}}{\sum_i x_1\cdots x_d A_{x_1\cdots\widehat{x_i}\cdots x_d}}\right]_{(n)},$$

see [**GW**, theorem 3·1·1]. By [**GW**, theorem 1·2·5], this is also the injective hull in the category of all $A_{(n)}$-modules.

Let R be the $\mathfrak{m}$-adic completion of $A_{(n)}$. As it is $\mathfrak{m}$-torsion, the module displayed above is also an R-module; it is the injective hull of $R/\mathfrak{m}R$ in the category of R-modules.

PROPOSITION 4·3. *Let $\mathbb{F}$ be a field of characteristic $p > 0$, and let $A = \mathbb{F}[x_1, \dots, x_d]$ be a polynomial ring of dimension d. Let n be a positive integer, and R be the completion of the n-th Veronese subring of A at its homogeneous maximal ideal. Set $E = M/N$ where*

$$M = R_{x_1^n\cdots x_d^n}$$

and N is the R-submodule spanned by elements $x_1^{i_1}\cdots x_d^{i_d} \in M$ with $i_k \geqslant 1$ for some k; the module E is the injective hull of the residue field of R.

Then $\mathcal{F}^e(E)$ is the left R-module generated by the elements

$$\frac{1}{x_1^{\alpha_1}\cdots x_d^{\alpha_d}}F^e,$$

where F is the pth power map, $\alpha_k \leqslant p^e - 1$ for each k, and $\sum \alpha_k \equiv 0 \bmod n$.

Remark 4·4. We use F for the Frobenius endomorphism of the ring M. The condition $\sum \alpha_k \equiv 0 \bmod n$, or equivalently $x_1^{\alpha_1}\cdots x_d^{\alpha_d} \in M$, implies that

$$\frac{1}{x_1^{\alpha_1}\cdots x_d^{\alpha_d}}F^e \in \mathcal{F}^e(M).$$

When $\alpha_k \leqslant p^e - 1$ for each k, the map displayed above stabilizes N and thus induces an element of $\mathcal{F}^e(M/N)$; we reuse F for the pth power map on M/N.

Proof of Proposition 4·3 In view of the above remark, it remains to establish that the given elements are indeed generators for $\mathcal{F}^e(E)$. The canonical module of R is

$$\omega_R = (x_1\cdots x_d A)_{(n)}R$$

and, indeed, $H^d_{\mathfrak{m}}(\omega_R) = E$. Thus, Theorem 3·3 implies that

$$\mathcal{F}^e(E) = \omega_R^{(1-q)}F^e,$$

where $q = p^e$. But $\omega_R^{(1-q)}$ is the completion of the $A_{(n)}$-module

$$\left[\frac{1}{x_1^{q-1}\cdots x_d^{q-1}}A\right]_{(n)} = \left(\frac{1}{x_1^{\alpha_1}\cdots x_d^{\alpha_d}} \mid \alpha_k \leqslant q-1 \text{ for each } k,\ \sum \alpha_k \equiv 0 \bmod n\right)A_{(n)},$$

which completes the proof.

Example 4·5. Consider $d = 2$ and $n = 3$ in Proposition 4·3, i.e.,

$$R = \mathbb{F}[[x^3,\ x^2y,\ xy^2,\ y^3]].$$

Then $\omega = (x^2y, xy^2)R$ has order 3 in the divisor class group of R; indeed,

$$\omega^{(2)} = (x^4y^2,\ x^3y^3,\ x^2y^4)R \quad \text{and} \quad \omega^{(3)} = (x^3y^3)R.$$

(1) If $p \equiv 1 \bmod 3$, then $\omega^{(1-q)} = (xy)^{1-q}R$ is cyclic for each $q = p^e$ and

$$\mathcal{F}^e(E) = \frac{1}{(xy)^{q-1}}F^e.$$

Since

$$\frac{1}{(xy)^{p-1}}F \circ \frac{1}{(xy)^{q-1}}F^e = \frac{1}{(xy)^{pq-1}}F^{e+1},$$

it follows that

$$\mathcal{F}(E) = R\left\{\frac{1}{(xy)^{p-1}}F\right\}.$$

(2) If $p \equiv 2 \bmod 3$ and $q = p^e$, then $\omega^{(1-q)} = (xy)^{1-q}R$ for e even and

$$\omega^{(1-q)} = \left(\frac{1}{x^{q-3}y^{q-1}},\ \frac{1}{x^{q-2}y^{q-2}},\ \frac{1}{x^{q-1}y^{q-3}}\right)R$$

for e odd. The proof of Proposition 4·1 shows that $\mathcal{F}(E)$ is generated by its elements of degree $\leqslant 2$ and hence

$$\mathcal{F}(E) = R\left\{\frac{1}{x^{p-3}y^{p-1}}F,\ \frac{1}{x^{p-2}y^{p-2}}F,\ \frac{1}{x^{p-1}y^{p-3}}F,\ \frac{1}{x^{p^2-1}y^{p^2-1}}F^2\right\}.$$

In the case $p = 2$, the above reads

$$\mathcal{F}(E) = R\left\{\frac{x}{y}F,\ F,\ \frac{y}{x}F,\ \frac{1}{x^3y^3}F^2\right\}.$$

(3) When $p = 3$, one has

$$\omega^{(1-q)} = \frac{1}{x^qy^q}(x^2y,\ xy^2)R = \left(\frac{1}{x^{q-2}y^{q-1}},\ \frac{1}{x^{q-1}y^{q-2}}\right)R$$

for each $q = p^e$. In this case,

$$\mathcal{F}(E) = R\left\{\frac{1}{xy^2}F,\ \frac{1}{x^2y}F,\ \frac{1}{x^7y^8}F^2,\ \frac{1}{x^8y^7}F^2,\ \frac{1}{x^{25}y^{26}}F^3,\ \frac{1}{x^{26}y^{25}}F^3,\ \dots\right\},$$

and $\mathcal{F}(E)$ is not a finitely generated extension ring of $\mathcal{F}^0(E) = R$; indeed,

$$\begin{aligned}
\omega^{(1-q)} \divideontimes \omega^{(1-q')} &= \frac{1}{x^qy^q}(x^2y,\ xy^2)R \divideontimes \frac{1}{x^{q'}y^{q'}}(x^2y,\ xy^2)R \\
&= \frac{1}{x^{qq'+q}y^{qq'+q}}(x^2y,\ xy^2)\cdot(x^{2q}y^q,\ x^qy^{2q})R \\
&= \frac{1}{x^{qq'}y^{qq'}}(x^{q+2}y,\ x^{q+1}y^2,\ x^2y^{q+1},\ xy^{q+2})R \\
&= \frac{1}{x^{qq'}y^{qq'}}(x^2y,\ xy^2)\cdot(x^q,\ y^q)R \\
&= (x^q,\ y^q)\,\omega^{(1-qq')}
\end{aligned}$$

for $q = p^e$ and $q' = p^{e'}$, where e and e' are positive integers.

5. *A determinantal ring*

Let R be the determinantal ring $\mathbb{F}[X]/I$, where X is a 2×3 matrix of variables over a field of characteristic $p > 0$, and I is the ideal generated by the size 2 minors of X. Set $\mathfrak{m}$ to be the homogeneous maximal ideal of R. We show that the algebra of Frobenius operators $\mathcal{F}(E)$ is not finitely generated over $\mathcal{F}^0(E) = \widehat{R}$; this proves [**Ka**, conjecture 3·1]. We also extend Fedder's calculation of the ideals $I^{[p]} : I$ to the ideals $I^{[q]} : I$ for all $q = p^e$.

The ring R is isomorphic to the affine semigroup ring

$$\mathbb{F}\begin{bmatrix} sx, & sy, & sz, \\ tx, & ty, & tz \end{bmatrix} \subseteq \mathbb{F}[s, t, x, y, z].$$

Using this identification, R is the Segre product $A\#B$ of the polynomial rings $A = \mathbb{F}[s, t]$ and $B = \mathbb{F}[x, y, z]$. By [**GW**, theorem 4·3·1], the canonical module of R is the Segre product of the graded canonical modules stA and $xyzB$ of the respective polynomial rings, i.e.,

$$\omega_R = stA \# xyzB = (s^2txyz,\ st^2xyz)R.$$

Let e be a nonnegative integer, and $q = p^e$. Then

$$\omega_R^{(1-q)} = \frac{1}{(st)^{q-1}} A \# \frac{1}{(xyz)^{q-1}} B$$

is the R module spanned by the elements

$$\frac{1}{(st)^{q-1} x^k y^l z^m}$$

with $k + l + m = 2q - 2$ and $k, l, m \leqslant q - 1$.

View E as M/N where $M = R_{s^2txyz}$, and N is the R-submodule spanned by the elements $s^i t^j x^k y^l z^m$ in M that have at least one positive exponent. Then $\mathcal{F}^e(E)$ is the left $\widehat{R}$-module generated by

$$\frac{1}{(st)^{q-1} x^k y^l z^m} F^e,$$

where F is the pth power map, $k + l + m = 2q - 2$, and $k, l, m \leqslant q - 1$. Using this description, it is an elementary—though somewhat tedious—verification that $\mathcal{F}(E)$ is not finitely generated over $\mathcal{F}^0(E)$; alternatively, note that the symbolic powers of the height one prime ideals $(sx, sy, sz)\widehat{R}$ and $(sx, tx)\widehat{R}$ agree with the ordinary powers by [**BV**, corollary 7·10]. Thus, the anticanonical cover of $\widehat{R}$ is the ring $\mathcal{R}$ with

$$\mathcal{R}_n = \frac{1}{(s^2txyz)^n} (sx, sy, sz)^n \widehat{R}$$

and so

$$\mathcal{T}_e = \frac{1}{(s^2txyz)^{q-1}} (sx, sy, sz)^{q-1} \widehat{R}.$$

Thus,

$$\begin{aligned}\mathcal{T}_{e_1} \circledast \mathcal{T}_{e_2} &= \frac{1}{(s^2txyz)^{q_1-1}}(sx, sy, sz)^{q_1-1} \circledast \frac{1}{(s^2txyz)^{q_2-1}}(sx, sy, sz)^{q_2-1}\\ &= \frac{1}{(s^2txyz)^{q_1q_2-1}}(sx, sy, sz)^{q_1-1} \cdot \left((sx, sy, sz)^{q_2-1}\right)^{[q_1]}\\ &= \frac{1}{(s^2txyz)^{q_1q_2-1}}(sx, sy, sz)^{q_1-1} \cdot \left((sx)^{q_1}, (sy)^{q_1}, (sz)^{q_1}\right)^{q_2-1}\end{aligned}$$

where $q_i = p^{e_i}$. We claim that

$$\mathcal{T}_e \not\subseteq \sum_{e_1=1}^{e-1} \mathcal{T}_{e_1} \circledast \mathcal{T}_{e-e_1}.$$

For this, it suffices to show that

$$\frac{1}{(s^2txyz)^{q-1}} sx(sy)^{q/p-1}(sz)^{q-q/p-1}$$

does not belong to $\mathcal{T}_{e_1} \circledast \mathcal{T}_{e_2}$ for integers $e_i < e$ with $e_1 + e_2 = e$. By the description of $\mathcal{T}_{e_1} \circledast \mathcal{T}_{e_2}$ above, this is tantamount to proving that

$$sx(sy)^{q/p-1}(sz)^{q-q/p-1} \notin (sx, sy, sz)^{q_1-1} \cdot \left((sx)^{q_1}, (sy)^{q_1}, (sz)^{q_1}\right)^{q_2-1},$$

but this is essentially Example 2·2·3.

Fedder's computation. Let A be the power series ring $\mathbb{F}[[u, v, w, x, y, z]]$ for $\mathbb{F}$ a field of characteristic $p > 0$, and let I be the ideal generated by the size 2 minors of the matrix

$$\begin{pmatrix} u & v & w \\ x & y & z \end{pmatrix},$$

In [**Fe**, proposition 4·7], Fedder shows that

$$I^{[p]} : I = I^{2p-2} + I^{[p]}.$$

We extend this next by calculating the ideals $I^{[q]} : I$ for each prime power $q = p^e$.

PROPOSITION 5·1. *Let A be the power series ring $\mathbb{F}[[u, v, w, x, y, z]]$ where K a field of characteristic $p > 0$. Let I be the ideal of A generated by $\Delta_1 = vz - wy$, $\Delta_2 = wx - uz$, and $\Delta_3 = uy - vx$.*

(1) *For $q = p^e$ and nonnegative integers s, t with $s + t \leqslant q - 1$, one has*

$$y^s z^t (\Delta_2 \Delta_3)^{q-1} \in I^{[q]} + x^{s+t} A.$$

(2) *For q, s, t as above, let $f_{s,t}$ be an element of A with*

$$y^s z^t (\Delta_2 \Delta_3)^{q-1} \equiv x^{s+t} f_{s,t} \bmod I^{[q]}.$$

Then $f_{s,t}$ is well-defined modulo $I^{[q]}$. Moreover, $f_{s,t} \in I^{[q]} :_A I$, and

$$I^{[q]} :_A I = I^{[q]} + \left(f_{s,t} \mid s + t \leqslant q - 1\right)A.$$

For $q = p$, the above recovers Fedder's computation that $I^{[p]} : I = I^{2p-2} + I^{[p]}$, though for $q > p$, the ideal $I^{[p]} : I$ is strictly bigger than $I^{2p-2} + I^{[p]}$.

Proof. (1) Note that the element

$$y^s z^t (\Delta_2\Delta_3)^{q-1} = y^s z^t (wx - uz)^{q-1}(uy - vx)^{q-1}$$

belongs to the ideals

$$(x, u)^{2q-2} \subseteq (x^{q-1}, u^q) \subseteq (x^{s+t}, u^q)$$

and also to

$$y^s z^t (x, z)^{q-1}(x, y)^{q-1} \subseteq y^s z^t (x^t, z^{q-t})(x^s, y^{q-s}) \subseteq (x^{s+t}, z^q, y^q).$$

Hence,

$$\begin{aligned} y^s z^t (\Delta_2\Delta_3)^{q-1} &\in \big(x^{s+t},\ u^q\big)A \cap \big(x^{s+t},\ z^q,\ y^q\big)A \\ &= \big(x^{s+t},\ u^q z^q,\ u^q y^q\big)A \\ &\subseteq \big(x^{s+t},\ \Delta_1^q,\ \Delta_2^q,\ \Delta_3^q\big)A. \end{aligned}$$

(2) The ideals I and $I^{[q]}$ have the same associated primes, [**ILL**+, corollary 21·11]. As I is prime, it is the only prime associated to $I^{[q]}$. Hence x^{s+t} is a nonzerodivisor modulo $I^{[q]}$, and it follows that $f_{s,t}$ mod $I^{[q]}$ is well-defined.

We next claim that

$$I^{2q-1} \subseteq I^{[q]}.$$

By the earlier observation on associated primes, it suffices to verify this in the local ring R_I. But R_I is a regular local ring of dimension 2, so IR_I is generated by two elements, and the claim follows from the pigeonhole principle. The claim implies that

$$x^{s+t} f_{s,t}\, I \in I^{[q]},$$

and using, again, that x^{s+t} is a nonzerodivisor modulo $I^{[q]}$, we see that $f_{s,t} I \subseteq I^{[q]}$, in other words, that $f_{s,t} \in I^{[q]} :_A I$ as desired.

By Theorem 3·3 and Remark 3·4, one has the R-module isomorphisms

$$\omega_R^{(1-q)} \cong \mathcal{F}^e(E) \cong \frac{I^{[q]} :_A I}{I^{[q]}}.$$

Choosing $\omega_R^{(-1)} = (x, y, z)R$, we claim that the map

$$\begin{aligned} (x, y, z)^{q-1}R &\longrightarrow \frac{I^{[q]} :_A I}{I^{[q]}} \\ x^{q-1-s-t} y^s z^t &\mapsto f_{s,t} \end{aligned}$$

is an isomorphism. Since the modules in question are reflexive R-modules of rank one, it suffices to verify that the map is an isomorphism in codimension 1. Upon inverting x, the above map induces

$$\begin{aligned} R_x &\longrightarrow \frac{I^{[q]}A_x :_{A_x} IA_x}{I^{[q]}A_x} \\ x^{q-1} &\mapsto (\Delta_2\Delta_3)^{q-1} \end{aligned}$$

which is readily seen to be an isomorphism since $IA_x = (\Delta_2, \Delta_3)A_x$.

6. *Cartier algebras and gauge boundedness*

For a ring R of prime characteristic $p > 0$, one can interpret $\mathcal{F}^e(E)$ in a dual way as a collection of p^{-e}-linear operators on R. This point of view was studied by Blickle [**Bl2**] and Schwede [**Sc**].

Definition 6·1. Let R be a ring of prime characteristic $p > 0$. For each $e \geqslant 0$, set $\mathcal{C}_e^R$ to be set of additive maps $\varphi\colon R \to R$ satisfying

$$\varphi(r^{p^e}x) = r\varphi(x), \quad \text{for } r, x \in R.$$

The *total Cartier algebra* is the direct sum

$$\mathcal{C}^R = \bigoplus_{e \geqslant 0} \mathcal{C}_e^R.$$

For $\varphi \in \mathcal{C}_e^R$ and $\varphi' \in \mathcal{C}_{e'}^R$, the compositions $\varphi \circ \varphi'$ and $\varphi' \circ \varphi$ are elements of $\mathcal{C}_{e+e'}^R$. This gives $\mathcal{C}^R$ the structure of an $\mathbb{N}$-graded ring; it is typically not a commutative ring. As pointed out in [**ABZ**, 2·2·1], if $(R, \mathfrak{m})$ is an F-finite complete local ring, then the ring of Frobenius operators $\mathcal{F}(E)$ is isomorphic to $\mathcal{C}^R$.

Each $\mathcal{C}_e^R$ has a left and a right R-module structure: for $\varphi \in \mathcal{C}_e^R$ and $r \in R$, we define $r \cdot \varphi$ to be the map $x \mapsto r\varphi(x)$, and $\varphi \cdot r$ to be the map $x \mapsto \varphi(rx)$.

Definition 6·2. Blickle [**Bl2**] introduced a notion of boundedness for Cartier algebras: Let $R = A/I$ for a polynomial ring $A = \mathbb{F}[x_1, \dots, x_d]$ over an F-finite field $\mathbb{F}$. Set R_n to be the finite dimensional $\mathbb{F}$-vector subspace of R spanned by the images of the monomials

$$x_1^{\lambda_1} \cdots x_d^{\lambda_d}, \quad \text{for } 0 \leqslant \lambda_j \leqslant n.$$

Following [**An**] and [**Bl2**], we define a map $\delta\colon R \longrightarrow \mathbb{Z}$ by $\delta(r) = n$ if $r \in R_n \smallsetminus R_{n-1}$; the map δ is a *gauge*. If $I = 0$, then $\delta(r) \leqslant \deg(r)$ for each $r \in R$. We recall some properties from [**An**, proposition 1] and [**Bl2**, lemma 4·2]:

$$\begin{aligned} \delta(r + r') &\leqslant \max\{\delta(r),\ \delta(r')\}, \\ \delta(r \cdot r') &\leqslant \delta(r) + \delta(r'). \end{aligned}$$

The ring $\mathcal{C}^R$ is *gauge bounded* if there exists a constant K, and elements $\varphi_{e,i}$ in $\mathcal{C}_e^R$ for each $e \geqslant 1$ generating $\mathcal{C}_e^R$ as a left R-module, such that

$$\delta(\varphi_{e,i}(x)) \leqslant \frac{\delta(x)}{p^e} + K, \quad \text{for each } e \text{ and } i.$$

Remark 6·3. We record two key facts that will be used in our proof of Theorem 6·4:

(1) If there exists a constant C such that $I^{[p^e]} :_A I$ is generated by elements of degree at most Cp^e for each $e \geqslant 1$, then $\mathcal{C}^R$ is gauge bounded; this is [**KZ**, lemma 2·2].

(2) If $\mathcal{C}^R$ is gauge bounded, then for each ideal $\mathfrak{a}$ of R, the F-jumping numbers of $\tau(R, \mathfrak{a}^t)$ are a subset of the real numbers with no limit points; in particular, they form a discrete set. This is [**Bl2**, theorem 4·18].

We now prove the main result of the section:

THEOREM 6·4. *Let R be a normal $\mathbb{N}$-graded that is finitely generated over an F-finite field R_0. (The ring R need not be standard graded.)*

Suppose that the anticanonical cover of R is finitely generated as an R-algebra. Then $\mathcal{C}^R$ is gauge bounded. Hence, for each ideal $\mathfrak{a}$ of R, the set of F-jumping numbers of $\tau(R, \mathfrak{a}^t)$ is a subset of the real numbers with no limit points.

Proof. Let A be a polynomial ring, with a possibly non-standard $\mathbb{N}$-grading, such that $R = A/I$. It suffices to obtain a constant C such that the ideals $I^{[p^e]} :_A I$ are generated by elements of degree at most Cp^e for each $e \geqslant 1$.

There exists a ring isomorphism $\bigoplus_{e\geqslant 0} \omega^{(1-p^e)} \cong \bigoplus_{e\geqslant 0}(I^{[p^e]} :_A I)/I^{[p^e]}$ by Remark 3·4 that respects the eth graded components. After replacing ω by an isomorphic R-module with a possible graded shift, we may assume that the isomorphism above induces degree preserving R-module isomorphisms $\omega^{(1-p^e)} \cong (I^{[p^e]} :_A I)/I^{[p^e]}$ for each $e \geqslant 0$. While ω is no longer canonically graded, we still have the finite generation of the anticanonical cover $\bigoplus_{n\geqslant 0} \omega^{(-n)}$. It suffices to check that there exists a constant C such that $\omega^{(1-p^e)}$ is generated, as an R-module, by elements of degree at most Cp^e.

Choose finitely many homogeneous R-algebra generators $z_1, \ldots, z_k$ for $\bigoplus_{n\geqslant 0} \omega^{(-n)}$, say with $z_i \in \omega^{(-j_i)}$. Set C to be the maximum of $\deg z_1, \ldots, \deg z_k$. Then the monomials

$$z^{\lambda} = z_1^{\lambda_1} z_2^{\lambda_2} \cdots z_k^{\lambda_k}, \quad \text{with } \sum \lambda_i j_i = p^e - 1$$

generate the R-module $\omega^{(1-p^e)}$, and it is readily seen that

$$\deg z^{\lambda} = \sum \lambda_i \deg z_i \leqslant C \sum \lambda_i \leqslant C(p^e - 1).$$

By [**KZ**, lemma 2·2], it follows that $\mathcal{C}^R$ is gauge bounded; the assertion now follows from [**Bl2**, theorem 4·18].

COROLLARY 6·5. *Let R be the determinantal ring $\mathbb{F}[X]/I$, where X is a matrix of indeterminates over an F-finite field $\mathbb{F}$ of prime characteristic, and I is the ideal generated by the minors of X of an arbitrary but fixed size. Then, for each ideal $\mathfrak{a}$ of R, the set of F-jumping numbers of $\tau(R, \mathfrak{a}^t)$ is a subset of the real numbers with no limit points.*

Proof. Since R is a determinantal ring, the symbolic powers of the ideal $\omega^{(-1)}$ agree with the ordinary powers by [**BV**, corollary 7·10]. Hence the anticanonical cover of R is finitely generated, and the result follows from Theorem 6·4.

Remark 6·6. It would be natural to remove the hypothesis that R is graded in Theorem 6·4. However, we do not know how to do this: when R is not graded, it is unclear if one can choose gauges that are compatible with the ring isomorphism

$$\bigoplus_{e\geqslant 0} \omega^{(1-p^e)} \cong \bigoplus_{e\geqslant 0}(I^{[p^e]} :_A I)/I^{[p^e]}.$$

7. *Linear growth of Castelnuovo–Mumford regularity for rings of finite Frobenius representation type*

Let A be a standard graded polynomial ring over a field $\mathbb{F}$, with homogeneous maximal ideal $\mathfrak{m}$. We recall the definition of the Castelnuovo-Mumford regularity of a graded module following [**Ei**, chapter 4]:

Definition 7·1. Let $M = \bigoplus_{d\in\mathbb{Q}} M_d$ be a graded A-module. If M is Artinian, we set

$$\operatorname{reg} M = \max\{d \mid M_d \neq 0\};$$

for an arbitrary graded module we define

$$\operatorname{reg} M = \max_{k\geqslant 0}\{\operatorname{reg} H^k_{\mathfrak{m}}(M) + k\}.$$

Definition 7·2. Let I and J be homogeneous ideals of A. We say that the regularity of $A/(I + J^{[p^e]})$ has *linear growth* with respect to p^e, if there is a constant C, such that

$$\operatorname{reg} A/(I + J^{[p^e]}) \leqslant Cp^e, \quad \text{for each } e \geqslant 0.$$

It follows from [**KZ**, corollary 2·4] that if $\operatorname{reg} A/(I + J^{[p^e]})$ has linear growth for each homogeneous ideal J, then $\mathcal{C}^{A/I}$ is gauge-bounded.

Remark 7·3. Let $R = A/I$ for a homogeneous ideal I. We define a grading on the bimodule $R^{(e)}$ introduced in Remark 1·3: when an element r of R is viewed as an element of $R^{(e)}$, we denote it by $r^{(e)}$. For a homogeneous element $r \in R$, we set

$$\deg' r^{(e)} = \frac{1}{p^e} \deg r.$$

For each ideal J of R, one has an isomorphism

$$R^{(e)} \otimes_R R/J \xrightarrow{\ \cong\ } R/J^{[p^e]}$$

under which $r^{(e)} \otimes \overline{s} \mapsto \overline{rs^{p^e}}$. To make this isomorphism degree-preserving for a homogeneous ideal J, we define a grading on $R/J^{[p^e]}$ as follows:

$$\deg' \overline{r} = \frac{1}{p^e} \deg \overline{r}, \quad \text{for a homogeneous element } r \text{ of } R.$$

The notion of finite Frobenius representation type was introduced by Smith and Van den Bergh [**SV**]; we recall the definition in the graded context:

Definition 7·4. Let R be an $\mathbb{N}$-graded Noetherian ring of prime characteristic p. Then R has *finite graded Frobenius-representation type* by finitely generated $\mathbb{Q}$-graded R-modules $M_1, \ldots, M_s$, if for every $e \in \mathbb{N}$, the $\mathbb{Q}$-graded R-module $R^{(e)}$ is isomorphic to a finite direct sum of the modules M_i with possible graded shifts, i.e., if there exist rational numbers $\alpha_{ij}^{(e)}$, such that there exists a $\mathbb{Q}$-graded isomorphism

$$R^{(e)} \cong \bigoplus_{i,j} M_i\big(\alpha_{ij}^{(e)}\big).$$

Remark 7·5. Suppose R has finite graded Frobenius-representation type. With the notation as above, there exists a constant C such that

$$\alpha_{ij}^{(e)} \leqslant C, \quad \text{for all } e, i, j\,;$$

see the proof of [**TT**, theorem 2·9].

We now prove the main result of this section; compare with [**TT**, theorem 4·8].

THEOREM 7·6. *Let A be a standard graded polynomial ring over an F-finite field of characteristic $p > 0$. Let I be a homogeneous ideal of A.*

Suppose $R = A/I$ has finite graded F-representation type. Then $\operatorname{reg} A/(I + J^{[p^e]})$ *has linear growth for each homogeneous ideal J. In particular, $\mathcal{C}^R$ is gauge bounded, and for each ideal $\mathfrak{a}$ of R, the set of F-jumping numbers of $\tau(R, \mathfrak{a}^t)$ is a subset of the real numbers with no limit points.*

Proof. We use J for the ideal of A, and also for its image in R. Let $a'(H^k_{\mathfrak{m}}(R/J^{[p^e]}))$ denote the largest degree of a nonzero element of $H^k_{\mathfrak{m}}(R/J^{[p^e]})$ under the $\deg'$-grading, i.e.,

$$a'\big(H^k_{\mathfrak{m}}(R/J^{[p^e]})\big) = \frac{1}{p^e} \operatorname{reg} H^k_{\mathfrak{m}}\big(R/J^{[p^e]}\big).$$

Since we have degree-preserving isomorphisms $R^{(e)} \otimes_R R/J \cong R/J^{[p^e]}$, and

$$R^{(e)} \;\cong\; \bigoplus_{i,j} M_i\big(\alpha_{ij}^{(e)}\big),$$

it follows that

$$\begin{aligned} H^k_{\mathfrak{m}}(R/J^{[p^e]}) \;&\cong\; H^k_{\mathfrak{m}}(R^{(e)} \otimes_R R/J) \\ &\cong\; \bigoplus_{i,j} H^k_{\mathfrak{m}}\big(M_i\big(\alpha_{ij}^{(e)}\big) \otimes_R R/J\big) \\ &\cong\; \bigoplus_{i,j} H^k_{\mathfrak{m}}(M_i/JM_i)\big(\alpha_{ij}^{(e)}\big). \end{aligned}$$

The numbers $\alpha_{ij}^{(e)}$ are bounded by Remark 7·5; thus,

$$a'\big(H^k_{\mathfrak{m}}(R/J^{[p^e]})\big) \leqslant \max_i \big\{a'(H^k_{\mathfrak{m}}(M_i/JM_i)) + C\big\}.$$

Since there are only finitely many modules M_i and finitely many homological indices k, it follows that $a'(H^k_{\mathfrak{m}}(R/J^{[p^e]})) \leqslant C'$, where C' is a constant independent of e and k. Hence

$$\operatorname{reg} H^k_{\mathfrak{m}}(R/J^{[p^e]}) \leqslant C' p^e, \quad \text{for all } e, k,$$

and so

$$\operatorname{reg} A/(I + J^{[p^e]}) = \max_k \big\{ \operatorname{reg} H^k_{\mathfrak{m}}(R/J^{[p^e]}) + k \big\} \leqslant C' p^e + \dim A.$$

This proves that reg $A/J^{[p^e]}$ has linear growth; [**KZ**, corollary 2·4] implies that $\mathcal{C}^R$ is gauge bounded, and the discreetness assertion follows from [**Bl2**, theorem 4·18].

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