Deformation of F-Injectivity and Local Cohomology

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ABSTRACT. We give a sufficient condition for F-injectivity to deform in terms of local cohomology. We show this condition is met in two geometrically interesting settings: namely, when the special fiber has isolated non-CM locus or is F-split.

1. INTRODUCTION

A central and interesting question in the study of singularities is how they behave under deformation. Given a local ring of positive characteristic, let us view this ring as the total space of a fibration. The special fiber of this fibration is a hypersurface in R, that is, a variety with coordinate ring R/xR where $x \in R$ is a regular element. An important question is whether or not the singularity type of the total space R is no worse than the singularity type as the special fiber. This deformation question has been studied in detail for singularities defined by Frobenius [Fed83, Sin99b], where it is noted that F-rationality always deforms, and that both F-purity and F-regularity fail to deform in general. An important and outstanding conjecture asserts that F-injectivity deforms in general. Recall that a local ring $(R, \mathfrak{m})$ of prime characteristic $p > 0$ is F-*injective* provided the Frobenius action on the local cohomology $H^i_{\mathfrak{m}}(R)$ induced by the Frobenius map on R is injective for all $i \geqslant 0$. The general conjecture is supported by recent work showing that the characteristic 0 analogue of F-injective singularities (called Du Bois singularities) deforms [KS11]. When R is Cohen-Macaualy, it is known that F-injectivity deforms [Fed83]. Our main theorem describes a condition sufficient to guarantee F-injectivity to deform which only requires information about the special fiber and not the total space.

Main Theorem (cf. Theorem 3.7). *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$, and $x \in \mathfrak{m}$ a regular element. If R/xR is F-injective, and if for each $\ell > 0$ and $i \geqslant 0$ the homomorphism $H^i_{\mathfrak{m}}(R/x^\ell R) \to H^i_{\mathfrak{m}}(R/xR)$ induced by the natural surjection $R/x^\ell R \to R/xR$, is surjective, then R is F-injective.*

We show in particular that this hypothesis is satisfied when the length of the local cohomology modules $H^i_{\mathfrak{m}}(R/xR)$ is finite for $i < \dim R - 1$, a condition called *finite-length cohomology* (FLC). Geometrically, this is the condition that the non–Cohen-Macaualy locus of R/xR is isolated, and this combination shows that F-injectivity deforms under mild geometric criteria in low dimensions (see Corollary 4.8).

Main Theorem (cf. Corollary 4.7). *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ with perfect residue field, and $x \in \mathfrak{m}$ a regular element. If R/xR has* FLC *and is F-injective, then R is F-injective.*

Utilizing a sharper study of Frobenius actions on local cohomology, we can state our condition in terms of the condition *anti-nilpotency*. Using results of L. Ma, we demonstrate a deformation-theoretic relationship between F-injectivity and F-splitting.

Main Theorem (Corollary 4.13). *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ and $x \in \mathfrak{m}$ a regular element. If R/xR is F-split, then R is F-injective.*

Convention. Unless otherwise stated, all rings are Noetherian and of characteristic $p > 0$ where p is a prime integer.

2. Preliminaries and Notation

2.1. Notation. For a ring R of characteristic $p > 0$, the Frobenius map is the map $F\colon R \to R$ sending an element to its p-th power. For an R-module M, denote $F_*M = \{F_*m \mid m \in M\}$. This module is called the *Frobenius pushforward* of M. As abelian groups $M \cong F_*M$, but its R-module structure is twisted by Frobenius. In particular, if $r \in R$ and $F_*m \in F_*M$, then $r \cdot F_*m = F_*(r^p m)$. We also denote the e-th iterate of the Frobenius pushforward of M by F^e_*M. The functor F^e_* is exact and commutes with localization.

2.2. Local cohomology. For a more complete introduction, see [ILL]. Fix a ring R and an ideal I. Let M be an R-module, not necessarily Noetherian. The local cohomology module supported at I is $H^i_I(M) = \varinjlim_t \operatorname{Ext}^i_R(R/I^t, M)$. When I is generated up to radical by $g_1, \dots, g_n$, one may compute $H^i_I(M)$ as the i-th cohomology of the Čech complex with respect to I, denoted as $\check{C}^\bullet(M; I)$:

$$0 \to M \to \bigoplus_i M_{g_i} \to \bigoplus_{i<j} M_{g_i g_j} \to \cdots \to M_{g_1 \cdots g_n} \to 0.$$

We briefly discuss iterated local cohomology as it plays a role in the proof of Theorem 3.7. For more detail, see [Har67]. Given two ideals I and J in R, and an R-module M, let $\check{C}^\bullet(M; I)$ (respectively, $\check{C}^\bullet(M; J)$) be the Čech complex of M with respect to I (respectively, with respect to J). Considering $\check{C}^\bullet(M; I)$ as the horizontal complex and $\check{C}^\bullet(M;J)$ as the vertical complex, one obtains a double complex $C^{\bullet\bullet} = \check{C}^\bullet(M;I) \otimes_R \check{C}^\bullet(M;J)$. This double complex is the first page of a spectral sequence $E_0^{p,q}$, called the *local cohomology spectral sequence*. For more on spectral sequences, see [Wei94]. The convergence of this spectral sequence is known.

Theorem 2.1 (***Convergence of local cohomology spectral sequence*** [Har67, Proposition 1.4]). *For I and J ideals in a ring R and M an R-module,*

$$E_2^{p,q} = H_J^p(H_I^q(M)) \Rightarrow E_\infty^{p,q} = H_{I+J}^{p+q}(M).$$

Using this theorem, it is easy to compute an isomorphism that we need.

Lemma 2.2. *Let $(R, \mathfrak{m}, k)$ be a local ring and M an R-module. If $x \in \mathfrak{m}$ is a regular element, then, for all $i \geqslant 0$, $H_\mathfrak{m}^i(H_{(x)}^1(M)) \cong H_\mathfrak{m}^{i+1}(M)$.*

Proof. First, note that $H_{(x)}^q(M)$ is nonzero only when $q = 1$. Thus, the $E_2^{p,q}$ page of the spectral sequence computing the double complex $H_\mathfrak{m}^\bullet(H_{(x)}^\bullet(-))$ degenerates. By Theorem 2.1, $E_2^{p,q} = H_\mathfrak{m}^p(H_{(x)}^q(M))$ and $E_\infty^{p,q} = H_\mathfrak{m}^{p+q}(M)$ for all $p \geqslant 0$ and $q \geqslant 0$. Since the sequence degenerates at the $E_2^{p,q}$ page, we have $H_\mathfrak{m}^p(H_{(x)}^q(M)) = E_2^{p,q} = E_\infty^{p,q} = H_\mathfrak{m}^{p+q}(M)$ for all $p \geqslant 0$ and $q \geqslant 0$. Applying this with $p = i$ and $q = 1$ gives the result. □

We also offer a second proof of Lemma 2.2 free of spectral sequences due to Alberto Boix.

Proof of Lemma 2.2 with thanks to Alberto Boix. Note the following exact sequence of R-modules:

$$0 \to \Gamma_{(x)}(M) \to M \to M_x \to H_{(x)}^1(M) \to 0.$$

Since x is not a zero divisor, $\Gamma_{(x)}(M) = 0$, and so we have a short exact sequence

$$0 \to M \to M_x \to H_{(x)}^1(M) \to 0.$$

Taking $H_\mathfrak{m}^i$ induces a long exact sequence

$$H_\mathfrak{m}^i(M_x) \to H_\mathfrak{m}^i(H_{(x)}^1(M)) \to H_\mathfrak{m}^{i+1}(M) \to H_\mathfrak{m}^{i+1}(M_x).$$

One may then apply flat base change to check for all $j \geqslant 0$ that $H_\mathfrak{m}^j(M_x) \cong H_{\mathfrak{m}R_x}^j(M_x)$, and since $x \in \mathfrak{m}$, the extension $\mathfrak{m}R_x = R_x$. Thus, we have $H_\mathfrak{m}^j(M_x) \cong H_{R_x}^j(M_x) = 0$. This gives the desired result. □

Remark 2.3. Neither proof of Lemma 2.2 depends on the local cohomology's being supported in the maximal ideal; rather, each depends only on having the regular element x be a member of the ideal of support for the local cohomology modules in question.

It is often easier to study spectral sequences as compositions of derived functors; see [Lip02] for explicit details about derived categories and local cohomology. We summarize what we need. For an abelian category $\mathcal{A}$, denote by $K(\mathcal{A})$ the category of complexes in $\mathcal{A}$ up to homotopic equivalence, and by $\mathbf{D}(\mathcal{A})$ its derived category. For R a ring, denote by R–mod the category of R-modules. Let $I \subseteq R$ an ideal and $\mathcal{A} = R$–mod. One realizes the i-th local cohomology module with support in I as a functor $H_I^i \colon K(R\text{–mod}) \to R\text{–mod}$ which takes quasi-isomorphisms in $K(R\text{–mod})$ to isomorphisms in R–mod, and so it can be regarded as a functor on $\mathbf{D}(R\text{–mod})$. Denote by Γ_I the I-torsion functor. The right derived functor $\mathbf{R}\Gamma_I \colon \mathbf{D}(R\text{–mod}) \to \mathbf{D}(R\text{–mod})$ has the information of taking all of the local cohomology modules H_I^i at once, and each H_I^i can be recovered in a functorial way from $\mathbf{D}(R\text{–mod})$ by taking the i-th cohomology of the image of $\mathbf{R}\Gamma_I$. The spectral sequence in Theorem 2.1 can be understood as a consequence of the Grothendieck spectral sequence theorem [Wei94, Corollary 10.8.3] stating that $\mathbf{R}\Gamma_I \circ \mathbf{R}\Gamma_J \cong \mathbf{R}\Gamma_{I+J}$. This equivalence will be used in Theorem 3.7.

2.3. Frobenius linear maps. Frobenius linear maps are a central tool in our approach. These are thoroughly explored in [HS77] under the name p-linear maps. We review the topic.

Definition 2.4. Let R be a commutative ring of characteristic p. For R-modules M and N, a *Frobenius linear map* is an element of $\operatorname{Hom}_R(M, F_*N)$. More specifically, it is an additive map $\rho \colon M \to F_*M$ such that $\rho(ra) = r^p\rho(a)$ for any $r \in R$ and $a \in M$. If $M = N$, we call $\rho \colon M \to F_*M$ a *Frobenius action* on M.

Since F_* commutes with localization, given a Frobenius linear map between M and N, there is an induced Frobenius linear map $H_\mathfrak{m}^i(M) \to F_*H_\mathfrak{m}^i(N)$ for each $i \geqslant 0$. One can make this explicit using Čech resolutions as in Example 2.5. Any Frobenius linear map $\rho \colon M \to F_*N$ induces a morphism $\mathbf{R}\Gamma_I(\rho) \colon \mathbf{R}\Gamma_I(M) \to \mathbf{R}\Gamma_I(F_*N) \cong F_*\mathbf{R}\Gamma_I(N)$ where $I \subseteq R$ is an ideal, and the last isomorphism follows as F_* is exact. In particular, the Frobenius map on R, thought of as a Frobenius action $\rho_F \colon R \to F_*R$, induces a natural Frobenius action on the local cohomology

$$\mathbf{R}\Gamma_I(\rho_R) \colon \mathbf{R}\Gamma_I(R) \to F_*\mathbf{R}\Gamma_I(R).$$

This Frobenius action can be computed explicitly using Čech complexes.

Example 2.5. Consider $(R, \mathfrak{m}, k)$ a local ring with $x \in \mathfrak{m}$ a regular element. Each term of the Čech complex $0 \to R \to R_x \to 0$ has a Frobenius linear map induced from the Frobenius map on R. Therefore, we have a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & R & \longrightarrow & R_x & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \rho_F} & & \downarrow{\scriptstyle \rho_F} & & \downarrow \\
0 & \longrightarrow & F_*R & \longrightarrow & F_*(R_x) & \longrightarrow & 0.
\end{array}$$

Of course, $H^0_{(x)}(R) = 0$ and $H^1_{(x)}(R) = R_x/R$. Since F_* commutes with localization, it also commutes with local cohomology. Therefore, we have a natural Frobenius action ρ on the R-module $H^1_{(x)}(R) = R_x/R$. In particular, $\rho\colon H^i_{(x)}(R) \to F_*H^i_{(x)}(R)$ is just the natural Frobenius $R_x/R \to F_*(R_x/R)$.

We see immediately the benefit of studying Frobenius linear maps on finite length modules when the residue field is perfect.

Lemma 2.6. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic $p > 0$ with perfect residue field, and let M be an R-module admitting an injective Frobenius action ρ. If M has finite length, then M is a finite-dimensional k-vector space, and ρ is a bijection.*

Proof. Since M has finite length, there exists $\ell > 0$ such that $\mathfrak{m}^\ell \cdot M = 0$. Fix $c \in \mathfrak{m}$. One has $\rho^e(c \cdot M) = c^{p^e} \cdot \rho(M) = 0$ for $p^e \geqslant \ell$. Since ρ is injective, $c \cdot M = 0$. Therefore, M is a finite-dimensional k-vector space, and ρ descends to an additive map on $M = M/\mathfrak{m}M$. Now, since k is perfect and M is finite dimensional, as M has finite length and ρ is injective, ρ must be bijective. ◻

Remark 2.7. It is necessary to assume that the residue field in Lemma 2.6 is perfect. In the case $R = k$, the natural Frobenius action on the simple k-module k is bijective if and only if k is perfect. See also [Ene12, Corollary 7.7 and Proposition 7.12] for a similar discussion.

3. Proof of the Main Theorem

We start with the following notation defining the key property about a regular element that we need to guarantee that F-injectivity deforms.

Definition 3.1. Let $(R, \mathfrak{m})$ be a local ring with $x \in \mathfrak{m}$ a regular element. We say that x is a *surjective element* if the map on local cohomology $H^i_\mathfrak{m}(R/x^\ell R) \to H^i_\mathfrak{m}(R/xR)$, which is induced by the natural surjection $R/x^\ell R \to R/xR$, is surjective for all $\ell > 0$ and $i \geqslant 0$.

We immediately see that surjective elements induce *injections* between specific local cohomology modules.

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a local ring of arbitrary characteristic. Assume that $x \in \mathfrak{m}$ is a surjective element. For each $\ell > 0$ and $j \geqslant \ell$, the multiplication map*

$$R/x^\ell R \xrightarrow{x^{j-\ell}} R/x^j R$$

induces an injection $H^i_\mathfrak{m}(R/x^\ell R) \to H^i_\mathfrak{m}(R/x^j R)$ for each $i \geqslant 0$.

Proof. Note that $R/x^\ell R \xrightarrow{x^{j-\ell}} R/x^j R$ is injective, and that it suffices by induction to prove the lemma when $j = \ell + 1$. The short exact sequence

$$0 \to R/x^\ell R \xrightarrow{\cdot x} R/x^{\ell+1} R \to R/xR \to 0$$

induces the following exact portion of the long exact sequence:

$$H^{i-1}_{\mathfrak{m}}(R/x^{\ell+1}R) \xrightarrow{\beta_1} H^{i-1}_{\mathfrak{m}}(R/xR) \xrightarrow{\delta} H^{i}_{\mathfrak{m}}(R/x^{\ell}R) \xrightarrow{\beta_2} H^{i}_{\mathfrak{m}}(R/x^{\ell+1}R).$$

Since x is a surjective element, β_1 is surjective, and hence δ is the zero map. This makes β_2 injective as desired. □

Theorem 3.3. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$, and let $x \in \mathfrak{m}$ be a surjective element. Assume that R/xR is F-injective, and denote by*

$$\rho_{\ell,i}\colon H^{i}_{\mathfrak{m}}(R/x^{\ell}R) \to F_*H^{i}_{\mathfrak{m}}(R/x^{p\ell}R)$$

the Frobenius linear map induced by the natural Frobenius map $\rho_F : R/x^{\ell}R \to F_*(R/x^{p\ell}R)$. For each $\ell > 0$ and $i \geqslant 0$, the map $\rho_{\ell,i}$ is injective.

Proof. For every $\ell > 0$, the natural Frobenius map on $R/x^{\ell}R$ is a composition of ρ_F and a natural surjection π, that is,

$$R/x^{\ell}R \xrightarrow{\rho_F} F_*(R/x^{p\ell}R) \xrightarrow{\pi} F_*(R/x^{\ell}R).$$

Denote by $\rho_{\ell,i}\colon H^{i}_{\mathfrak{m}}(R/x^{\ell}R) \to F_*H^{i}_{\mathfrak{m}}(R/x^{p\ell}R)$ the Frobenius linear map induced by ρ_F. We proceed by induction on ℓ to show that $\rho_{\ell,i}$ is injective for all $\ell > 0$. The case $\ell = 1$ is assured by hypothesis.

Assume $\ell > 1$ and consider the commutative diagram of R-modules with exact rows

$$(3.1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & R/x^{\ell-1}R & \xrightarrow{\cdot x} & R/x^{\ell}R & \longrightarrow & R/xR & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_*(R/x^{p(\ell-1)}R) & \xrightarrow{\cdot x^p} & F_*(R/x^{p\ell}R) & \longrightarrow & F_*(R/x^{p}R) & \longrightarrow & 0 \end{array}$$

where all vertical maps are the natural Frobenius linear maps. This induces the following commutative diagram of R-modules:

$$(3.2)\qquad \begin{array}{ccccc} H^{i-1}_{\mathfrak{m}}(R/xR) & \longrightarrow & H^{i}_{\mathfrak{m}}(R/x^{\ell-1}R) & \longrightarrow & \\ \downarrow{\scriptstyle \rho_{1,i-1}} & & \downarrow{\scriptstyle \rho_{\ell-1,i}} & & \\ F_*H^{i-1}_{\mathfrak{m}}(R/x^{p}R) & \xrightarrow{F_*\delta_{i-1}} & F_*H^{i}_{\mathfrak{m}}(R/x^{p(\ell-1)}R) & \xrightarrow{F_*\beta} & \end{array}$$

$$\begin{array}{ccccc} \longrightarrow & H^{i}_{\mathfrak{m}}(R/x^{\ell}R) & \xrightarrow{\alpha} & H^{i}_{\mathfrak{m}}(R/xR) \\ & \downarrow{\scriptstyle \rho_{\ell,i}} & & \downarrow{\scriptstyle \rho_{1,i}} \\ \longrightarrow & F_*H^{i}_{\mathfrak{m}}(R/x^{p\ell}R) & \longrightarrow & F_*H^{i}_{\mathfrak{m}}(R/x^{p}R). \end{array}$$

The map $\alpha : H^i_{\mathfrak{m}}(R/x^\ell R) \to H^i_{\mathfrak{m}}(R/xR)$ is surjective, since x is a surjective element by assumption. From Lemma 3.2 and the fact that F_* is exact, the map $F_*\beta$ is injective. Hence, $F_*\delta_{i-1}$ is the zero map. Thus, we have a commutative diagram
(3.3)

$$\begin{array}{ccccccc} & & H^i_{\mathfrak{m}}(R/x^{\ell-1}R) & \longrightarrow & H^i_{\mathfrak{m}}(R/x^\ell R) & \longrightarrow & H^i_{\mathfrak{m}}(R/xR) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho_{\ell-1,i}} & & \downarrow{\scriptstyle \rho_{\ell,i}} & & \downarrow{\scriptstyle \rho_{1,i}} & & \\ 0 & \longrightarrow & F_*H^i_{\mathfrak{m}}(R/x^{p(\ell-1)}R) & \longrightarrow & F_*H^i_{\mathfrak{m}}(R/x^{p\ell}R) & \longrightarrow & F_*H^i_{\mathfrak{m}}(R/x^pR). & & \end{array}$$

To complete the argument, apply the snake lemma to Diagram (3.3). This gives an exact sequence $\ker \rho_{\ell-1,i} \to \ker \rho_{\ell,i} \to \ker \rho_{1,i}$. Since $\rho_{1,i}$ is injective by F-injectivity of R/xR, and $\rho_{\ell-1,i}$ is injective by induction, we have $\ker \rho_{\ell,i} = 0$. Hence, $\rho_{\ell,i}$ is injective. □

Remark 3.4. The specific point where the fact that x is a surjective element was used was to obtain that $F_*\beta$ in Diagram (3.2) is injective. The fact that α is surjective is not really required, as one can do a straightforward chase on Diagram (3.3), similar to how one proves the snake lemma to conclude the result.

We record a lemma used in the proof of the main theorem whose proof is left to the reader.

Lemma 3.5. *For a directed system $\{N_i, \tau_{i,j}\}_{i\in\Lambda}$ of R-modules, the system*

$$\{F_*N_i, F_*\tau_{i,j}\}_{i\in\Lambda}$$

is also directed, and $F_* \varinjlim N_i \cong \varinjlim F_*N_i$.

The next lemma explains the basic isomorphisms needed in the proof of the main theorem.

Lemma 3.6. *Let $(R, \mathfrak{m})$ be a local ring with $x \in \mathfrak{m}$ a regular element. For each $i > 0$, we have isomorphisms*

$$H^i_{\mathfrak{m}}(H^1_{(x)}(R)) \cong H^{i+1}_{\mathfrak{m}}(R) \cong \varinjlim_{\ell} H^i_{\mathfrak{m}}(R/x^\ell R) \cong \varinjlim_{\ell} H^i_{\mathfrak{m}}(R/x^{p\ell}R).$$

Proof. We show this by showing that R-modules $H^{i+1}_{\mathfrak{m}}(R)$, $\varinjlim_\ell H^i_{\mathfrak{m}}(R/x^\ell R)$, and $\varinjlim_\ell H^i_{\mathfrak{m}}(R/x^{p\ell}R)$ are all isomorphic to the iterated local cohomology module $H^i_{\mathfrak{m}}(H^1_{(x)}(R))$. Computing $H^1_{(x)}(R)$ as

$$\varinjlim\{R/xR \xrightarrow{x} R/x^2R \xrightarrow{x} R/x^3R \xrightarrow{x} \cdots\},$$

and noting that local cohomology commutes with direct limits, one has

$$\varinjlim_{\ell} H^i_{\mathfrak{m}}(R/x^\ell R) \cong H^i_{\mathfrak{m}}(\varinjlim_{\ell} R/x^\ell R) \cong H^i_{\mathfrak{m}}(H^1_{(x)}(R)).$$

By Lemma 2.2,

$$\varinjlim_{\ell} H^i_{\mathfrak{m}}(R/x^{\ell}R) \cong H^i_{\mathfrak{m}}(H^1_{(x)}(R)) \cong H^{i+1}_{\mathfrak{m}}(R).$$

Since $\{x^{p\ell}\}_{\ell\in\mathbf{N}}$ is cofinal in $\{x^{\ell}\}_{\ell\in\mathbf{N}}$, one can compute $H^1_{(x)}(R)$ as the limit

$$\varinjlim\{R/x^pR \xrightarrow{x^p} R/x^{2p}R \xrightarrow{x^p} R/x^{3p}R \xrightarrow{x^p} \cdots\},$$

and as before, we have $\varinjlim_{\ell} H^i_{\mathfrak{m}}(R/x^{p\ell}R) \cong H^i_{\mathfrak{m}}(H^1_{(x)}(R))$. ◻

We now prove the main theorem of this article.

Theorem 3.7. *Let $(R,\mathfrak{m},k)$ be a local ring of prime characteristic $p>0$, and $x\in\mathfrak{m}$ a regular surjective element. If R/xR is F-injective, then R is also F-injective.*

Proof. Since R has a regular element x, we have $H^0_{\mathfrak{m}}(R)=0$, and there is nothing to prove in the case $i=0$. Fix $i>0$, and consider the following commutative diagram of R-modules, where ρ_F denotes the natural Frobenius map:

(3.4)
$$\begin{array}{ccccc}
R/xR & \longrightarrow & R/x^2R & \xrightarrow{\cdot x} & \cdots \\
\downarrow{\scriptstyle \rho_F} & & \downarrow{\scriptstyle \rho_F} & & \\
F_*(R/x^pR) & \xrightarrow{\cdot x^p} & F_*(R/x^{2p}R) & \xrightarrow{\cdot x^p} & \cdots
\end{array}$$

Taking direct limits on the rows of Diagram (3.4), and applying $H^i_{\mathfrak{m}}(-)$, we get two directed systems $\{H^i_{\mathfrak{m}}(R/x^{\ell}R)\}_{\ell>0}$ and $\{H^i_{\mathfrak{m}}(R/x^{p\ell}R)\}_{\ell>0}$ with Frobenius linear maps

$$\rho_{\ell,i}\colon H^i_{\mathfrak{m}}(R/x^{\ell}R) \to F_*H^i_{\mathfrak{m}}(R/x^{p\ell}R)$$

which are injective for each $\ell>0$ by Theorem 3.3. Thus, the collection of injective Frobenius linear maps $\rho_{\ell,i}\colon H^i_{\mathfrak{m}}(R/x^{\ell}R) \to F_*H^i_{\mathfrak{m}}(R/x^{p\ell}R)$ induces an injective Frobenius linear map

$$\rho_1 = \varinjlim_{\ell}\rho_{\ell,i} : \varinjlim_{\ell} H^i_{\mathfrak{m}}(R/x^{\ell}R) \to F_* \varinjlim_{\ell} H^i_{\mathfrak{m}}(R/x^{p\ell}R),$$

since F_* commutes with $\varinjlim$ by Lemma 3.5. The module $H^1_{(x)}(R)$ has a natural Frobenius action induced from the Frobenius on R, which in turn induces a Frobenius action ρ_2 on $H^i_{\mathfrak{m}}(H^1_{(x)}(R))$. Let ρ_3 denote the natural Frobenius action on $H^{i+1}_{\mathfrak{m}}(R)$.

It suffices to show that the following diagram commutes for each $i \geqslant 0$:

$$
\begin{array}{ccccc}
\varinjlim_{\ell} H^i_{\mathfrak{m}}(R/x^{\ell}R) & \xrightarrow{\alpha_1} & H^i_{\mathfrak{m}}(H^1_{(x)}(R)) & \xrightarrow{\beta_1} & H^{i+1}_{\mathfrak{m}}(R) \\
\downarrow{\scriptstyle \rho_1} & & \downarrow{\scriptstyle \rho_2} & & \downarrow{\scriptstyle \rho_3} \\
\varinjlim_{\ell} F_* H^i_{\mathfrak{m}}(R/x^{p\ell}R) & \xrightarrow{F_*\alpha_2} & F_* H^i_{\mathfrak{m}}(H^1_{(x)}(R)) & \xrightarrow{F_*\beta_2} & F_* H^{i+1}_{\mathfrak{m}}(R),
\end{array}
\tag{3.5}
$$

where α_1 and $F_*\alpha_2$ are the isomorphisms coming from Lemma 3.6, and β_1 and $F_*\beta_2$ are the isomorphisms coming from Lemma 2.2. Since ρ_1 is injective, it follows from this commutativity that ρ_3 is injective. We show that Diagram (3.5) commutes by splitting it into two commuting squares.

To show that the first square in Diagram (3.5) commutes, note that this square is simply applying $H^i_{\mathfrak{m}}(-)$ to the following square, where the vertical Frobenius linear maps are those induced by the natural Frobenius on R:

$$
\begin{array}{ccc}
\varinjlim_{\ell} R/x^{\ell} & \xrightarrow{\cong} & H^1_{(x)}(R) \\
\downarrow & & \downarrow \\
\varinjlim_{\ell} F_*(R/x^{p\ell}) & \xrightarrow{\cong} & F_* H^1_{(x)}(R).
\end{array}
$$

The second square in Diagram (3.5) commutes since $\mathbf{R}\Gamma_{\mathfrak{m}} \circ \mathbf{R}\Gamma_{(x)} \cong \mathbf{R}\Gamma_{\mathfrak{m}}$ in the derived category by [Wei94, Corollary 10.8.3], and we are simply applying each functor to the natural Frobenius action $\rho_F \colon R \to F_*R$. In other words, $\mathbf{R}\Gamma_{\mathfrak{m}}(\mathbf{R}\Gamma_{(x)}(\rho_F)) = \mathbf{R}\Gamma_{\mathfrak{m}}(\rho_F)$. □

3.1. Deforming surjectivity of Frobenius linear maps. Clearly, Frobenius linear maps are not generally surjective. However, they are often surjective "up to Frobenius". To make this clear, we start with a simple example.

Example 3.8. Let k be a perfect field of characteristic $p > 0$. The natural Frobenius action $k[x] \to F_*k[x]$ is $k[x]$-linear and has image $F_*k[x^p]$ in $F_*k[x]$. It is thus not surjective. However, it is surjective up to $F_*k[x]$-span in the sense that the singleton set $\{F_*1\}$ forms a $F_*k[x]$-basis of $F_*k[x]$.

Definition 3.9. Let R be a ring of characteristic p, and M and N be R-modules. Call an e-th iterated Frobenius linear map $\rho : M \to F^e_*N$ *surjective up to F^e_*R-span*, when the F^e_*R-span of $\operatorname{Im}(\rho)$ is equal to F^e_*N.

The condition in Definition 3.9 is equivalent to having a set $\{a_i\}_{i\in\Lambda}$ of generators for M for which F^e_*N is the F^e_*R-submodule of F^e_*N spanned by $\{\rho(a_i)\}_{i\in\Lambda}$. This section investigates how this property deforms.

We leave it to the reader to check for a directed system of R-modules $\{M_i\}_{i\in I}$ and Frobenius actions $\phi_i\colon M_i \to F_*M_i$ for each $i \in I$ with each ϕ_i surjective up to F_*R-span; the natural induced map $\phi = \varinjlim_i \phi_i\colon \varinjlim_i M_i \to \varinjlim_i F_*M_i$ is also surjective up to F_*R-span.

Lemma 3.10. *Let R be a commutative ring of characteristic $p > 0$, and assume that*

$$\begin{array}{ccccc} L & \xrightarrow{\alpha_1} & M & \xrightarrow{\alpha_2} & N \\ {\scriptstyle\rho_1}\downarrow & & {\scriptstyle\rho_2}\downarrow & & {\scriptstyle\rho_3}\downarrow \\ F_*L' & \xrightarrow{F_*\alpha_1'} & F_*M' & \xrightarrow{F_*\alpha_2'} & F_*N' \end{array}$$

is a commutative diagram where L, M, N, L', M', and N' are R-modules such that the top row is R-linear and exact and the bottom row is F_*R-linear and exact, and such that each ρ_i is a Frobenius linear map for $i = 1, 2, 3$. If ρ_1 and ρ_3 are surjective up to F_*R-span and α_2 is surjective, then ρ_2 is also surjective up to F_*R-span.

Proof. Choose sets of generators of R-modules L, M, and N, say $\{x_i\}$, $\{y_j\}$, and $\{z_k\}$, respectively. Without loss of generality, we may assume $\{\alpha_1(x_i)\} \subseteq \{y_j\}$ and $\alpha_2(\{y_j\}\backslash\{\alpha_1(x_i)\}) = \{z_k\}$. It suffices to show each element of F_*M' can be presented as an F_*R-linear combination of $\{\rho_2(y_j)\}$. Pick $F_*m \in F_*M'$, and consider $F_*\alpha_2'(F_*m) \in F_*N'$. By hypothesis, we can write

$$F_*\alpha_2'(F_*m) = \sum_i F_*c_i\rho_3(z_i) \tag{3.6}$$

with $F_*c_i \in F_*R$. Now, let $y_i' \in M$ be the inverse image of each $z_i \in N$ appearing in the equation (3.6). By our setup, we have $y_i' \in \{y_j\}$. By commutativity of the diagram, we also have

$$F_*m - \sum_i F_*c_i\rho_2(y_i') \in \ker F_*\alpha_2'.$$

Since the bottom row is exact, one has

$$F_*m - \sum_i F_*c_i\rho_2(y_i') = \sum_j F_*a_jF_*\alpha_1'(\rho_1(x_j))$$

for some $F_*a_j \in F_*R$, and thus

$$\begin{aligned} F_*m &= \sum_j F_*a_jF_*\alpha_1'(\rho_1(x_j)) + \sum_i F_*c_i\rho_2(y_i') \\ &= \sum_j F_*a_j(F_*\alpha_1'(\rho_1(x_j))) + \sum_i F_*c_i\rho_2(y_i'), \end{aligned}$$

which proves the lemma, since each $F_*\alpha_1'(\rho_1(x_j)) \in \{\rho_2(y_i)\}$. □

As a corollary, we obtain the following result.

Theorem 3.11. *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ with $x \in \mathfrak{m}$ a surjective element. If the Frobenius linear map $H^i_\mathfrak{m}(R/xR) \to F_* H^i_\mathfrak{m}(R/x^pR)$ is surjective up to F_*R-span for all $i \geqslant 0$, then the Frobenius action $H^i_\mathfrak{m}(R) \to F_* H^i_\mathfrak{m}(R)$ is also surjective up to F_*R-span for all $i \geqslant 0$.*

Proof. We use the notation and setup from the proof of Theorem 3.3. We start by showing that $\rho_1 := \varinjlim_\ell \rho_{\ell,i}$ is surjective up to F_*R-span; to do so, it suffices to check that each $\rho_{\ell,i}$ is surjective up to F_*R-span. Proceed by induction on $\ell > 0$ (defined in the proof of Theorem 3.3), where the base case (i.e., that $\rho_{1,i}$ is surjective up to F_*R-span) is guaranteed by hypothesis. We assume $\rho_{\ell-1,i}$ is surjective up to F_*R-span. Note that Diagram (3.3) of Theorem 3.3 has exact rows, and by Lemma 3.10, $\rho_{\ell,i}$ is surjective up to F_*R-span for all $\ell > 0$.

Now, proceed as in the proof of Theorem 3.7. Here, $\rho_1 = \varinjlim_\ell \rho_{\ell,i}$ is surjective up to F_*R-span, and $\beta_1 \circ \alpha_1$ and $F_*\beta_2 \circ F_*\alpha_2$ are isomorphisms. From Diagram (3.5), we see that ρ_3 is surjective up to F_*R-span as well. Putting this together, we have shown that the Frobenius action

$$H^{i+1}_\mathfrak{m}(R) \to F_* H^{i+1}_\mathfrak{m}(R)$$

is surjective up to F_*R-span for $i \geqslant 0$, as desired. ◻

4. Applications

Using Theorem 3.7, we now describe two conditions for when F-injectivity deforms. One is a finite length condition on local cohomology modules; the other is F-purity. Both can be stated in terms of Frobenius actions on local cohomology using the notion of anti-nilpotent modules.

4.1. Finite-length cohomology.

The first case that we can apply our main theorem to is one utilizing a finiteness condition on local cohomology modules.

Definition 4.1. For a local ring $(R, \mathfrak{m})$, we say an R-module M has *finite local cohomology* (FLC) provided the local cohomology module $H^i_\mathfrak{m}(M)$ has finite length for all $i \leqslant \dim M - 1$.

Remark 4.2. Sometimes when a local ring R has FLC it is called a *generalized Cohen-Macaulay ring*. When R has a dualizing complex, this means exactly that the non-CM locus of R is isolated [Sch75].

In the setting of a local ring $(R, \mathfrak{m})$ with $x \in \mathfrak{m}$ a regular element, we are most concerned with the R-modules R and $R/x^\ell R$, that is, an infinitesimal neighborhood of the special fiber. We now show that FLC extends to such neighborhoods when imposed on the special fiber.

Lemma 4.3. *Let $(R, \mathfrak{m}, k)$ be a local ring with $x \in R$ a regular element such that $\mathfrak{m}^s \cdot H^i_\mathfrak{m}(R/xR) = 0$ for some $s \geqslant 0$. For each $\ell > 0$, we have $\mathfrak{m}^{s\ell} \cdot H^i_\mathfrak{m}(R/x^\ell R) = 0$. In particular, if R/xR has* FLC*, so does $R/x^\ell R$.*

Proof. We show this by induction on ℓ. If $\ell = 1$, then this is just the hypothesis. Assume $\ell > 1$ and $\mathfrak{m}^{sj} \cdot H^i_{\mathfrak{m}}(R/x^jR) = 0$ for all $j < \ell$. The short exact sequence

$$0 \to R/x^{\ell-1}R \xrightarrow{x} R/x^{\ell}R \to R/xR \to 0,$$

induces a long exact sequence in local cohomology. We only need the portion

$$H^i_{\mathfrak{m}}(R/x^{\ell-1}R) \xrightarrow{\alpha} H^i_{\mathfrak{m}}(R/x^{\ell}R) \xrightarrow{\beta} H^i_{\mathfrak{m}}(R/xR),$$

which is an exact sequence of R-modules. Take an element $\eta \in H^i_{\mathfrak{m}}(R/x^{\ell}R)$ and $c \in \mathfrak{m}^s$. One has $\beta(c\eta) = c\beta(\eta) = 0$, which implies that $c\eta$ has a preimage $\theta \in H^i_{\mathfrak{m}}(R/x^{\ell-1}R)$ along α. By induction, we have $m\cdot\theta = 0$ for any $m \in \mathfrak{m}^{s(\ell-1)}$. Therefore, $\alpha(m\cdot\theta) = 0$ and $m\cdot c\eta = (mc)\cdot\eta = 0$. Since c and m were arbitrarily chosen, we have that $\mathfrak{m}^{s\ell} \cdot H^i_{\mathfrak{m}}(R/x^{\ell}R) = 0$. □

Remark 4.4. We note that there was no restriction on the characteristic of the rings in Lemma 4.3.

An easy consequence of the FLC property is a result on surjective maps of local cohomology.

Lemma 4.5. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ with perfect residue field k and $x \in \mathfrak{m}$ a regular element. Assume that R/xR is F-injective and* FLC. *For each $\ell > 0$, the surjection $R/x^{\ell}R \to R/xR$ induces a surjection*

$$H^i_{\mathfrak{m}}(R/x^{\ell}R) \to H^i_{\mathfrak{m}}(R/xR)$$

for each $0 \leqslant i \leqslant \dim R - 2$.

Proof. By Lemma 2.6, since R/xR has FLC and is F-injective with perfect residue field, for i in the interval $[0, \dim R - 2]$, the e-th iterated Frobenius action

$$H^i_{\mathfrak{m}}(R/xR) \to F^e_* H^i_{\mathfrak{m}}(R/xR)$$

induced by Frobenius on R/xR is surjective. For $\ell > 0$, choose $e \gg 0$ so that the surjection $R/x^{p^e}R \twoheadrightarrow R/xR$ factors as $R/x^{p^e}R \twoheadrightarrow R/x^{\ell}R \twoheadrightarrow R/xR$. This induces a composition of maps:

$$H^i_{\mathfrak{m}}(R/xR) \to F^e_* H^i_{\mathfrak{m}}(R/x^{p^e}R) \to F^e_* H^i_{\mathfrak{m}}(R/x^{\ell}R) \to F^e_* H^i_{\mathfrak{m}}(R/xR).$$

The composition is surjective, and so $H^i_{\mathfrak{m}}(R/x^{\ell}R) \to H^i_{\mathfrak{m}}(R/xR)$ must be also. □

Remark 4.6. The assumption that the residue field of R is perfect is necessary in the proof of Lemma 4.5. If R is F-injective and contains a non-perfect field K, it is not necessarily true that $R \otimes_K K^{1/p}$ is F-injective. For example, set $K := \mathbf{F}_p(x)$. Note that $R := K[t]/(t^p - x)$ is a field; however, $R \otimes_K K^{1/p} \cong K^{1/p}[t]/(t - x^{1/p})^p$ is not reduced, and hence not F-injective.

Corollary 4.7. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic p with perfect residue field and $x \in \mathfrak{m}$ a regular element. If R/xR has* FLC *and is F-injective, then R is F-injective.*

Proof. We use the same notation as Theorem 3.3 and Theorem 3.7. Applying Lemma 4.5, we see that $H^i_{\mathfrak{m}}(R/x^\ell R) \to H^i_{\mathfrak{m}}(R/xR)$ is surjective for all $i \in [0, \dim R - 2]$ and $\ell > 0$. Now, following the proof of Lemma 3.2, $H^i_{\mathfrak{m}}(R/x^\ell R) \to H^i_{\mathfrak{m}}(R/x^{\ell+1}R)$ is injective for all $\ell > 0$ and $i \in [0, \dim R - 1]$. This suffices in the proof of Theorem 3.3 to conclude that $\rho_{i,\ell} \colon H^i_{\mathfrak{m}}(R/x^\ell R) \to F_* H^i_{\mathfrak{m}}(R/x^{p\ell} R)$ is injective for $i \in [0, \dim R - 1]$ and all $\ell > 0$. Finally, this is sufficient to apply the proof of Theorem 3.7 to conclude that R is F-injective. □

Immediately, this shows that potential counterexamples to the deformation of F-injectivity in nice geometric settings must have dimension at least 4.

Corollary 4.8. *If $(R, \mathfrak{m}, k)$ is a complete local ring of characteristic $p > 0$, with perfect residue field and dimension at most 4, and $x \in \mathfrak{m}$ is a regular element with R/xR normal and F-injective, then R is F-injective.*

Proof. Since R/xR is a local normal domain and $x \in \mathfrak{m}$ is a regular element, R is also normal by [Gro65, 5.12.7]. In particular, R is a domain and equidimensional. Since $\dim R \leqslant 4$, one has $\dim R/xR \leqslant 3$. By normality of R/xR, this satisfies Serre's condition S_2, and therefore the non-CM locus is isolated. Hence, R/xR has FLC, and by Corollary 4.7, R must be F-injective. □

Example 4.9. For any ring A which is not Cohen Macaulay, has FLC, and is F-split, the ring $R := A[\![x]\!]$ does not have FLC. However, R/xR is F-injective and has FLC. In particular, consider

$$A = \mathbf{F}_p[\![a, b, c, d]\!]/(a, b) \cap (c, d).$$

Note that A has FLC and is even Buchsbaum (see [GO83]). It is also not Cohen-Macaulay, but is F-split by Fedder's criterion [Fed83]. Thus, $A[\![x]\!]$ is F-injective, and the non-CM locus of R is defined by the non-maximal ideal $\mathfrak{n}R$ where $\mathfrak{n}$ is the maximal ideal $\mathfrak{n}$ of A.

4.2. F-splitting and F-injectivity.

The second application concerns F-purity. We use work of L. Ma [Ma] building on work by Enescu and Hochster [EH]. The language used in [EH] is in terms of $R\{F\}$-modules which are modules over a ring R with a specified Frobenius action. For such a module M with a distinguished Frobenius action $\rho \colon M \to F_* M$, a submodule $N \subset M$ is called *F-compatible*, provided that $\rho(N) \subseteq F_* N$. Ma showed that F-split local rings have local cohomology modules, which when equipped with the natural Frobenius action satisfy an interesting condition, originally introduced in [EH].

Definition 4.10 ([EH, **Definition 4.6**]). Let $(R, \mathfrak{m})$ be a local ring. An R-module M with a Frobenius action ρ is called *anti-nilpotent*, provided that, for any

F-compatible submodule N (i.e., $\rho(N) \subseteq F_*N$), the induced action of ρ on M/N is injective.

Theorem 4.11. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ and $x \in \mathfrak{m}$ a regular element. If $H^i_{\mathfrak{m}}(R/xR)$ is anti-nilpotent for all $i \geqslant 0$, then x is a surjective element.*

Proof. By definition, we must check that the map $H^i_{\mathfrak{m}}(R/x^\ell R) \to H^i_{\mathfrak{m}}(R/xR)$, which is induced by the surjection $R/x^\ell R \to R/xR$, is surjective. Denote, therefore, its cokernel by C. It suffices to show that $C = 0$. Consider the exact sequence $H^i_{\mathfrak{m}}(R/x^\ell R) \to H^i_{\mathfrak{m}}(R/xR) \to C \to 0$. Denote by $\rho^e_{\ell,i} \colon H^i_{\mathfrak{m}}(R/x^\ell R) \to F^e_* H^i_{\mathfrak{m}}(R/x^\ell R)$ the Frobenius linear map induced naturally by the Frobenius on R composed with the natural surjection.

The map $\rho^e_{1,i}$ induces a Frobenius linear map $C \to F^e_* C$; denote this by ρ^e_C. These Frobenius linear maps fit together to give a commutative diagram with exact rows, since F^e_* is exact for all e:

$$\begin{array}{ccccccc}
H^i_{\mathfrak{m}}(R/x^\ell R) & \longrightarrow & H^i_{\mathfrak{m}}(R/xR) & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow{\scriptstyle \rho^e_{\ell,i}} & & \downarrow{\scriptstyle \rho^e_{\ell,i}} & & \downarrow{\scriptstyle \rho^e_C} & & \downarrow \\
F^e_* H^i_{\mathfrak{m}}(R/x^\ell R) & \longrightarrow & F^e_* H^i_{\mathfrak{m}}(R/xR) & \longrightarrow & F^e_* C & \longrightarrow & 0.
\end{array}$$

The image of $H^i_{\mathfrak{m}}(R/x^\ell R)$ in $H^i_{\mathfrak{m}}(R/xR)$ is certainly F-compatible. Since we assume $H^i_{\mathfrak{m}}(R/xR)$ is anti-nilpotent, the Frobenius action ρ^e_C on C is injective. Note also that, when $e \gg 0$, the map $\rho^e_{1,i}$ factors as

$$H^i_{\mathfrak{m}}(R/xR) \to F^e_* H^i_{\mathfrak{m}}(R/x^{p^e} R) \to F^e_* H^i_{\mathfrak{m}}(R/x^\ell R) \to F^e_* H^i_{\mathfrak{m}}(R/xR).$$

We may thus define the map φ, making the following diagram commute:

(4.1)
$$\begin{array}{ccccccc}
H^i_{\mathfrak{m}}(R/x^\ell R) & \longrightarrow & H^i_{\mathfrak{m}}(R/xR) & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow{\scriptstyle \rho^e_{\ell,i}} & \swarrow{\scriptstyle \varphi} & \downarrow{\scriptstyle \rho^e_{\ell,i}} & & \downarrow{\scriptstyle \rho^e_C} & & \downarrow \\
F^e_* H^i_{\mathfrak{m}}(R/x^\ell R) & \longrightarrow & F^e_* H^i_{\mathfrak{m}}(R/xR) & \longrightarrow & F^e_* C & \longrightarrow & 0.
\end{array}$$

We show that $C = 0$ by employing a diagram chase on (4.1). Let $z \in C$. As such, it has a preimage $z' \in H^i_{\mathfrak{m}}(R/xR)$. By commutativity of the diagram, it follows that $\rho^e_{1,i}(z')$ has preimage $z'' = \varphi(z')$. As the bottom row is exact, z'' maps to $\rho^e_C(z)$, which is zero. However, ρ^e_C was shown to be injective, and this implies that $z = 0$, and therefore $C = 0$, as desired. ∎

Remark 4.12. The proof of Theorem 4.11 can be modified to show that when the natural Frobenius linear map $H^i_{\mathfrak{m}}(R/xR) \to F_* H^i_{\mathfrak{m}}(R/xR)$ is surjective up to F_*R-span, for each $\ell > 0$ the map $H^i_{\mathfrak{m}}(R/x^\ell R) \to H^i_{\mathfrak{m}}(R/xR)$ is surjective.

Corollary 4.13. *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$ and $x \in \mathfrak{m}$ a regular element. If R/xR is F-split, then R is F-injective.*

Proof. Since R/xR is F-split, the module $H^i_{\mathfrak{m}}(R/xR)$ is anti-nilpotent for all $i \geqslant 0$ by [Ma, Theorem 3.7], and so Theorem 4.11 gives that x is a surjective element. The rest follows by Theorem 3.7. ∎

Remark 4.14. We note that when the residue field is perfect and R/xR is F-injective and has FLC, then $H^i_{\mathfrak{m}}(R/xR)$ is anti-nilpotent for all $i < \dim R/xR$ (since $H^i_{\mathfrak{m}}(R/xR)$ is a finite-dimensional k-vector space, and thus Frobenius acts injectively). Thus, one may use Theorem 4.11 to replace the role of Lemma 4.5 in the proof of Corollary 4.7.

Remark 4.15. Under an F-finite assumption, Theorem 4.11 says that F-purity deforms to F-injectivity. Enescu obtained some results on this finiteness property on local cohomology modules of finite length [Ene12, Theorem 7.14].

Example 4.16. A particularly well-known example where F-purity fails to deform was introduced by Fedder [Fed83]; see also [Sin99a, Example 3.2]. In particular, the ring

$$R := \mathbf{F}_p[\![X, Y, Z, W]\!]/(XY, XW, W(Y - Z^2))$$

is not F-pure, but $R/ZR = \mathbf{F}_p[\![X, Y, W]\!]/(XY, XW, WY)$ is known to be F-pure [HR76, Proposition 5.38]. This also means that our main result serves as a way for checking F-injectivity by taking specialization, that is, by checking that R/ZR is F-pure.

Appendix A. F-Injectivity and Depth

by Karl Schwede and Anurag K. Singh

Our goal here is to prove a prime characteristic analog of a result of Kollár and Kovács, [KK10, Theorem 7.12]: if $X \to B$ is a flat family with Du Bois fibers, such that the generic fiber is Cohen-Macaulay (respectively S_k), then all fibers of the map $X \to B$ are Cohen-Macaulay (respectively S_k). The prime characteristic version of this is Theorem A.3 below. As applications of this theorem, we extend a result of Fedder and Watanabe [FW89, Proposition 2.13] to the case where R is not *a priori* assumed to be Cohen-Macaulay (see Corollary A.4), and also obtain a new result on the deformation of F-injectivity, Corollary A.5.

We begin with some preliminary observations.

Lemma A.1. *Let $(R, \mathfrak{m})$ be a local ring, and set d to be the depth of R. Suppose there exists a regular element f in R such that the Frobenius action on $H^{d-1}_{\mathfrak{m}}(R/fR)$ is injective. Then, the map $H^d_{\mathfrak{m}}(R) \xrightarrow{f^{p-1}F} H^d_{\mathfrak{m}}(R)$ is injective; in particular, the Frobenius action on $H^d_{\mathfrak{m}}(R)$ is injective.*

Proof. Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & R/fR & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & & \\ 0 & \longrightarrow & R & \xrightarrow{f} & R & \longrightarrow & R/fR & \longrightarrow & 0. \end{array}$$

Since R/fR has depth $d-1$, applying the functor $H^\bullet_\mathfrak{m}(\ )$ yields the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & H^{d-1}_\mathfrak{m}(R/fR) & \longrightarrow & H^d_\mathfrak{m}(R) & \xrightarrow{f} & H^d_\mathfrak{m}(R) & \longrightarrow & H^d_\mathfrak{m}(R/fR) \\ & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f^{p-1}F} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\ 0 \longrightarrow & H^{d-1}_\mathfrak{m}(R/fR) & \longrightarrow & H^d_\mathfrak{m}(R) & \xrightarrow{f} & H^d_\mathfrak{m}(R) & \longrightarrow & H^d_\mathfrak{m}(R/fR). \end{array}$$

The map $f^{p-1}F$ is injective if and only if it is injective when restricted to the socle of $H^d_\mathfrak{m}(R)$. The socle is annihilated by f, and thus lies in the image of $H^{d-1}_\mathfrak{m}(R/fR)$. However, the Frobenius action on $H^{d-1}_\mathfrak{m}(R/fR)$ is injective by assumption. □

The next lemma is the main ingredient in the proof of Theorem A.3. For a local ring $(R, \mathfrak{m})$, we use $\operatorname{Spec}^\circ R$ to denote the *punctured spectrum* of R, that is, the set $\operatorname{Spec} R \setminus \{\mathfrak{m}\}$. The F-finite hypothesis in the sequel ensures the existence of a dualizing complex by Gabber ([Ga04, Remark 13.6]). By Kawasaki [Kaw02, Corollary 1.4], local rings possessing dualizing complexes are precisely those that are homomorphic images of Gorenstein local rings.

Lemma A.2. *Let $(R, \mathfrak{m})$ be an F-finite local ring. Suppose there exists a regular element f in R such that R/fR is F-injective.*

If $R_\mathfrak{p}$ satisfies the Serre condition S_k for each $\mathfrak{p}$ in $\operatorname{Spec}^\circ R$, then R satisfies S_k.

Proof. Let d be the depth of R. If R does not satisfy S_k, then $d < k$.

The module $H^d_\mathfrak{m}(R)$ is nonzero, but has finite length since $R_\mathfrak{p}$ satisfies S_k for each prime ideal $\mathfrak{p}$ in $\operatorname{Spec}^\circ R$. We claim that $\mathfrak{m} H^d_\mathfrak{m}(R) = 0$. Because it has finite length, the module $H^d_\mathfrak{m}(R)$ is annihilated by $\mathfrak{m}^q$ for some $q = p^e$. For each $x \in \mathfrak{m}$ and $\eta \in H^d_\mathfrak{m}(R)$, it follows that $x^q F^e(\eta) = 0$. But the Frobenius action on $H^d_\mathfrak{m}(R)$ is injective by Lemma A.1, and so $x\eta = 0$, which proves the claim.

But then $f^{p-1}FH^d_\mathfrak{m}(R) = 0$. Since $f^{p-1}F\colon H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(R)$ is injective by Lemma A.1, we must have $H^d_\mathfrak{m}(R) = 0$, which is a contradiction. □

Theorem A.3. *Let R be an F-finite local ring. Suppose there exists a regular element f in R such that R/fR is F-injective.*

If the localization $R_f = R[f^{-1}]$ satisfies the Serre condition S_k for a positive integer k, then R satisfies condition S_k. In particular, if R_f is Cohen-Macaulay, then R is Cohen-Macaulay.

Proof. If the ring R is not S_k, take a prime $\mathfrak{q}$ that is minimal with respect to the property that $R_\mathfrak{q}$ does not satisfy S_k. As R_f is S_k by assumption, it follows that $f \in \mathfrak{q}$. Since it is a localization of an F-injective ring, the ring $(R/fR)_\mathfrak{q} = R_\mathfrak{q}/fR_\mathfrak{q}$ is F-injective (see, e.g., [Sch09, Proposition 4.3]). But $(R_\mathfrak{q})_\mathfrak{p}$ satisfies condition S_k for each prime ideal $\mathfrak{p}$ in $\operatorname{Spec}^\circ R_\mathfrak{q}$, and so $R_\mathfrak{q}$ satisfies S_k by Lemma A.2. This is a contradiction. ∎

The following corollary was proved as [FW89, Proposition 2.13] under the additional hypothesis that R is Cohen-Macaulay.

Corollary A.4. *Let R be an F-finite local ring. Suppose there exists a regular element f in R such that R/fR is F-injective. If R_f is F-rational, then R is F-rational.*

Proof. Theorem A.3 implies that R is Cohen-Macaulay. But then R is F-rational by [FW89, Proposition 2.13]; Fedder and Watanabe require R_f to be regular in the statement of the proposition, but their proof works verbatim if some power of f is a parameter test element, and this is indeed the case by [Ve95, Theorem 1.13]. ∎

Fedder [Fed83, Theorem 3.4 (1)] proved that F-injectivity deforms in the case of Cohen-Macaulay rings; we extend this as follows.

Corollary A.5. *Let R be an F-finite local ring. If $f \in R$ is a regular element such that R/fR is F-injective, and R_f is Cohen-Macaulay, then R is F-injective.*

Proof. Theorem A.3 implies that the ring R is Cohen-Macaulay; we may then use [Fed83, Theorem 3.4.1]. ∎

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References

[BH98] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956 (95h:13020).

[EH] F. Enescu and M. Hochster, *The Frobenius structure of local cohomology*, Algebra Number Theory **2** (2008), no. 7, 721–754. http://dx.doi.org/10.2140/ant.2008.2.721. MR2460693 (2009i:13009).

[Ene12] F. ENESCU, *Finite-dimensional vector spaces with Frobenius action*, Progress in Commutative Algebra 2, Walter de Gruyter, Berlin, 2012, pp. 101–128. MR2932592.

[Fed83] R. FEDDER, *F-purity and rational singularity*, Trans. Amer. Math. Soc. **278** (1983), no. 2, 461–480. http://dx.doi.org/10.2307/1999165. MR701505 (84h:13031).

[FW89] R. FEDDER AND K. WATANABE, *A characterization of F-regularity in terms of F-purity*, Commutative Algebra (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 15, Springer, New York, 1989, pp. 227–245. http://dx.doi.org/10.1007/978-1-4612-3660-3_11. MR1015520 (91k:13009).

[Ga04] O. GABBER, *Notes on some t-structures*, Geometric Aspects of Dwork Theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 711–734. MR2099084 (2005m:14025).

[GO83] S. GOTO AND T. OGAWA, *A note on rings with finite local cohomology*, Tokyo J. Math. **6** (1983), no. 2, 403–411. http://dx.doi.org/10.3836/tjm/1270213880. MR732093 (85j:13020).

[Gro65] A. GROTHENDIECK, *Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965), 231 (French). MR0199181 (33 #7330).

[Har67] R. HARTSHORNE, *Local Cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin, 1967. MR0224620 (37 #219).

[HS77] R. HARTSHORNE AND R. SPEISER, *Local cohomological dimension in characteristic p*, Ann. of Math. (2) **105** (1977), no. 1, 45–79. http://dx.doi.org/10.2307/1971025. MR0441962 (56 #353).

[HR76] M. HOCHSTER AND J. L. ROBERTS, *The purity of the Frobenius and local cohomology*, Advances in Math. **21** (1976), no. 2, 117–172. http://dx.doi.org/10.1016/0001-8708(76)90073-6. MR0417172 (54 #5230).

[Hun96] C. HUNEKE, *Tight Closure and its Applications*, CBMS Regional Conference Series in Mathematics, vol. 88, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. With an appendix by Melvin Hochster. MR1377268 (96m:13001).

[ILL] S. B. IYENGAR, G. J. LEUSCHKE, A. LEYKIN, C. MILLER, E. MILLER, A. K. SINGH, AND U. WALTHER, *Twenty-four Hours of Local Cohomology*, Graduate Studies in Mathematics, vol. 87, American Mathematical Society, Providence, RI, 2007. MR2355715 (2009a:13025).

[Kaw02] T. KAWASAKI, *On arithmetic Macaulayfication of Noetherian rings*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 123–149 (electronic). http://dx.doi.org/10.1090/S0002-9947-01-02817-3. MR1859029 (2002i:13001).

[KK10] J. KOLLÁR AND S. J. KOVÁCS, *Log canonical singularities are Du Bois*, J. Amer. Math. Soc. **23** (2010), no. 3, 791–813. http://dx.doi.org/10.1090/S0894-0347-10-00663-6. MR2629988 (2011m:14061).

[KS11] S. J. KOVÁCS AND K. E. SCHWEDE, *Hodge theory meets the minimal model program: A survey of log canonical and Du Bois singularities*, Topology of Stratified Spaces, Math. Sci. Res. Inst. Publ., vol. 58, Cambridge Univ. Press, Cambridge, 2011, pp. 51–94. MR2796408 (2012k:14003).

[KS] ———, *Du Bois singularities deform*, available at http://arxiv.org/abs/arXiv:1107.2349.

[Kun69] E. KUNZ, *Characterizations of regular local rings for characteristic p*, Amer. J. Math. **91** (1969), 772–784. http://dx.doi.org/10.2307/2373351. MR0252389 (40 #5609).

[Lip02] J. LIPMAN, *Lectures on local cohomology and duality*, Local Cohomology and its Applications (Guanajuato, 1999), Lecture Notes in Pure and Appl. Math., vol. 226, Dekker, New York, 2002, pp. 39–89. MR1888195 (2003b:13027).

[Ma] L. MA, *Finiteness property of local cohomology for F-pure local rings*, available at http://arxiv.org/abs/arXiv:1204:1539.

[Sch75] P. SCHENZEL, *Einige Anwendungen der lokalen Dualität und verallgemeinerte Cohen-Macaulay-Moduln*, Math. Nachr. **69** (1975), 227–242. http://dx.doi.org/10.1002/mana.19750690121. MR0399089 (53 #2940).

[Sch09] K. E. SCHWEDE, *F-injective singularities are Du Bois*, Amer. J. Math. **131** (2009), no. 2, 445–473. http://dx.doi.org/10.1353/ajm.0.0049. MR2503989 (2010d:14016).

[Sin99a] A. K. SINGH, *Deformation of F-purity and F-regularity*, J. Pure Appl. Algebra **140** (1999), no. 2, 137–148. http://dx.doi.org/10.1016/S0022-4049(98)00014-0. MR1693967 (2000f:13004).

[Sin99b] ———, *F-regularity does not deform*, Amer. J. Math. **121** (1999), no. 4, 919–929. http://dx.doi.org/10.1353/ajm.1999.0029. MR1704481 (2000e:13006).

[Wei94] CH. A. WEIBEL, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324 (95f:18001).

[Ve95] J. D. VÉLEZ, *Openness of the F-rational locus and smooth base change*, J. Algebra **172** (1995), no. 2, 425–453. http://dx.doi.org/10.1016/S0021-8693(05)80010-9. MR1322412 (96g:13003).

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