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Frobenius representation type for invariant rings of finite groups ☆

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Let V be a finite rank vector space over a perfect field of characteristic $p > 0$, and let G be a finite subgroup of $\mathrm{GL}(V)$. If V is a permutation representation of G, or more generally a monomial representation, we prove that the ring of invariants $(\operatorname{Sym} V)^G$ has finite Frobenius representation type. We also construct an example with V a finite rank vector space over the algebraic closure of the function field $\mathbb{F}_3(t)$, and G an elementary abelian subgroup of $\mathrm{GL}(V)$, such that the invariant ring $(\operatorname{Sym} V)^G$ does not have finite Frobenius representation type.

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1. Introduction

The study of rings of finite Frobenius representation type was initiated by Smith and Van den Bergh [29], as part of an attack on the conjectured simplicity of rings of differential operators on invariant rings; indeed, using this notion, they proved that if R is a graded direct summand of a polynomial ring over a perfect field k of positive characteristic, e.g., if R is the ring of invariants for a linearly reductive group acting linearly on the polynomial ring, then the ring of k-linear differential operators on R is a simple ring [29, Theorem 1.3].

A reduced ring R of prime characteristic $p > 0$, satisfying the Krull-Schmidt theorem, has *finite Frobenius representation type* (FFRT) if there exists a finite set $\mathcal{S}$ of R-modules such that for each integer $e \geqslant 0$, each indecomposable R-module summand of R^{1/p^e} is isomorphic to an element of $\mathcal{S}$; the FFRT property and its variations are reviewed in §2. Examples of rings with FFRT include Cohen-Macaulay rings of finite representation type, graded direct summands of polynomial rings [29, Proposition 3.1.6], and Stanley-Reisner rings [20, Example 2.3.6]. More recently, Raedschelders, Špenko, and Van den Bergh proved that over an algebraically closed field of characteristic $p \geqslant \max\{n-2, 3\}$, the Plücker homogeneous coordinate ring of the Grassmannian $G(2, n)$ has FFRT [23]. In another direction, work of Hara and Ohkawa [8] investigates the FFRT property for two-dimensional normal graded rings in terms of $\mathbb{Q}$-divisors.

In addition to the original motivation, the FFRT property has found several applications. Suppose a ring R has FFRT. Then Hilbert-Kunz multiplicities over R are rational numbers by [24]; tight closure and localization commute in R, [31]; local cohomology modules of the form $H^k_{\mathfrak{a}}(R)$ have finitely many associated primes, [30,18,5]. For more on the FFRT property, we point the reader towards [1,20,22,25,26,28].

Our goal here is to investigate the FFRT property for rings of invariants of finite groups. Let V be a finite rank vector space over a perfect field k of characteristic $p > 0$, and let G be a finite subgroup of $\mathrm{GL}(V)$. In the nonmodular case, that is, when the order of G is not divisible by p, the invariant ring S^G is a direct summand of the polynomial ring $S := \operatorname{Sym} V$ via the Reynolds operator; it follows by [29, Proposition 3.1.4] that S^G has FFRT. The question becomes more interesting in the modular case, i.e., when p divides $|G|$. We prove that if V is a monomial representation of G, then the ring of invariants S^G has FFRT, Theorem 4.1; this includes the case of a subgroup G of the symmetric group $\mathfrak{S}_n$, acting on a polynomial ring $S := k[x_1, \dots, x_n]$ by permuting the indeterminates. On the other hand, while it had been expected that rings of invariants of reductive groups have FFRT (see for example the abstract of [23]), we prove that this is not the case:

Theorem 1.1. *Set k to be the algebraic closure of the function field $\mathbb{F}_3(t)$. Then there is an order 9 subgroup G of $\mathrm{GL}_3(k)$, such that $k[x_1, x_2, x_3]^G$ does not have FFRT.*

This is proved as Theorem 3.1; the reader will find that a similar construction may be performed over any algebraically closed field k that is not algebraic over $\mathbb{F}_p$. However, we do not know if $(\operatorname{Sym} V)^G$ always has FFRT when V is a finite rank vector space over $\overline{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$.

Returning to the nonmodular case, let k be an algebraically closed field of characteristic $p > 0$, and V a finite rank k-vector space. Set $S := \operatorname{Sym} V$ and $R := S^G$, for G a finite subgroup of $\mathrm{GL}(V)$ of order coprime to p. The rings $S^{1/q}$ and $R^{1/q}$ admit $\mathbb{Q}$-gradings extending the standard $\mathbb{N}$-grading on the polynomial ring S. Let M be a $\mathbb{Q}$-graded finitely generated indecomposable R-module. By [29, Proposition 3.2.1], the module $M(d)$ is a direct summand of $R^{1/q}$ for some $d \in \mathbb{Q}$ if and only if

$$M \;\cong\; (S \otimes_k L)^G$$

for some irreducible representation L of G. Let $V_1, \dots, V_\ell$ be a complete set of representatives of the isomorphism classes of irreducible representations of G, and set

$$M_i := (S \otimes_k V_i)^G$$

for $i = 1, \dots, \ell$. Then, for each integer $e \geqslant 0$, the decomposition of R^{1/p^e} into indecomposable R-modules takes the form

$$R^{1/p^e} \;\cong\; \bigoplus_{i=1}^{\ell} \bigoplus_{j=1}^{c_{ie}} M_i(d_{ij}),$$

where $d_{ij} \in \mathbb{Q}$ and $c_{ie} \in \mathbb{N}$. Suppose additionally that G does not contain any pseudoreflections; by [12, Theorem 3.4], the *generalized F-signature*

$$s(R, M_i) := \lim_{e \longrightarrow \infty} \frac{c_{ie}}{p^{e(\dim R)}}$$

then agrees with

$$(\operatorname{rank}_k V_i)/|G|.$$

By [13, Theorem 5.1], this description of the asymptotic behavior of R^{1/p^e} remains valid in the modular case. It follows that for the invariant ring $R := k[x_1, x_2, x_3]^G$ in Theorem 1.1, while there exist infinitely many nonisomorphic indecomposable R-modules that are direct summands of some R^{1/p^e} up to a degree shift, almost all are "asymptotically negligible."

In §2, we review some basics on the FFRT property and on equivariant modules; these are used in §3 in the proof of Theorem 1.1. In §4, we prove that if V is a monomial representation then $(\operatorname{Sym} V)^G$ has FFRT, and also that $(\operatorname{Sym} V)^G$ is F-pure in this case; the latter extends a result of Hochster and Huneke [16, page 77] that $(\operatorname{Sym} V)^G$ is F-pure when V is a permutation representation. Lastly, in §5, we construct a family of examples that are not F-regular or F-pure, but nonetheless have the FFRT property.

2. Preliminaries

We collect some definitions and results that are used in the sequel.

Krull-Schmidt category. Let k be a perfect field of characteristic $p > 0$, and R a finitely generated *positively graded* commutative k-algebra, i.e., R is $\mathbb{N}$-graded with $[R]_0 = k$. Let $R\mathbb{Q}\,\mathrm{grmod}$ denote the category of finitely generated $\mathbb{Q}$-graded R-modules. For modules M, N in $R\mathbb{Q}\,\mathrm{grmod}$, the module $\mathrm{Hom}_R(M, N)$ again lies in $R\mathbb{Q}\,\mathrm{grmod}$; in particular,

$$\mathrm{Hom}_{R\mathbb{Q}\,\mathrm{grmod}}(M, N) \ = \ [\mathrm{Hom}_R(M, N)]_0$$

is a finite rank k-vector space. Since $\mathrm{Hom}_{R\mathbb{Q}\,\mathrm{grmod}}(M, M) = [\mathrm{Hom}_R(M, M)]_0$ has finite rank for each M in $R\mathbb{Q}\,\mathrm{grmod}$, the category $R\mathbb{Q}\,\mathrm{grmod}$ is Krull-Schmidt; see [14, §3].

Frobenius twist. Let e be a nonnegative integer. For a k-vector space V, we use ${}^eV$ to denote the k-vector space that coincides with V as an abelian group, but has the left k-action $\alpha \cdot v = \alpha^{p^e} v$ for $\alpha \in k$ and $v \in V$, with the right action unchanged. An element $v \in V$, when viewed as an element of ${}^eV$, will be denoted ${}^ev$, so

$${}^eV \ = \ \{{}^ev \mid v \in V\}.$$

The map $v \longmapsto {}^ev$ is an isomorphism of abelian groups, but not an isomorphism of k-vector spaces in general. Note that $\alpha \cdot {}^ev = {}^e(\alpha^{p^e} v)$. When V is $\mathbb{Q}$-graded, we define a $\mathbb{Q}$-grading on ${}^eV$ as follows: for a homogeneous element $v \in V$, set

$$\deg {}^ev := (\deg v)/p^e.$$

Let V and W be k-vector spaces. For $f \in \mathrm{Hom}_k(V, W)$, we define ${}^ef\colon {}^eV \longrightarrow {}^eW$ by ${}^ef({}^ev) = {}^e(fv)$. It is easy to see that ${}^ef \in \mathrm{Hom}_k({}^eV, {}^eW)$. This makes ${}^e(-)$ an auto-equivalence of the category of k-vector spaces. Note that the map

$${}^eV \otimes_k {}^eW \longrightarrow {}^e(V \otimes_k W)$$

with ${}^ev \otimes {}^ew \longmapsto {}^e(v \otimes w)$ is well-defined, and an isomorphism. It is easy to check that ${}^e(-)$ is a monoidal functor; the composition ${}^e(-) \circ {}^{e'}(-)$ is canonically isomorphic to ${}^{e+e'}(-)$, and ${}^0(-)$ is the identity.

For a k-vector space V, the map ${}^e(-)\colon\ \mathrm{GL}(V) \longrightarrow \mathrm{GL}({}^eV)$ given by $f \longmapsto {}^ef$ is an isomorphism of abstract groups. If V is a G-module, then the composition

$$G \longrightarrow \mathrm{GL}(V) \longrightarrow \mathrm{GL}({}^eV)$$

gives ${}^eV$ a G-module structure. Thus, $g({}^ev) = {}^e(gv)$ for $g \in G$ and $v \in V$. Suppose $x_1, \ldots, x_n$ is a k-basis of V. Then for each integer $e \geqslant 0$, the elements ${}^ex_1, \ldots, {}^ex_n$ form a k-basis for ${}^eV$. If $f \in \mathrm{GL}(V)$ has matrix (m_{ij}) with respect to the basis $x_1, \dots, x_n$, then the matrix for ${}^ef$ with respect to ${}^ex_1, \dots, {}^ex_n$ is (m_{ij}^{1/p^e}). Indeed,

$$ {}^ef({}^ex_j) \;=\; {}^e(fx_j) \;=\; {}^e(\sum_i m_{ij}x_i) \;=\; \sum_i {}^e(m_{ij}x_i) \;=\; \sum_i m_{ij}^{1/p^e} \cdot {}^ex_i. $$

When R is a k-algebra, the k-algebra ${}^eR$ has multiplication defined by $({}^er)({}^es) := {}^e(rs)$. For R a commutative k-algebra, the iterated Frobenius map $F^e\colon R \longrightarrow {}^eR$ with

$$ r \longmapsto {}^e(r^{p^e}) $$

is a homomorphism of k-algebras. When R is a positively graded finitely generated commutative k-algebra, the ring ${}^eR$ admits a $\mathbb{Q}$-grading where for homogeneous $r \in R$,

$$ \deg {}^er := (\deg r)/p^e. $$

The ring ${}^eR$ is then positively graded in the sense that $[{}^eR]_j = 0$ for $j < 0$, and $[{}^eR]_0 = k$. The iterated Frobenius map $F^e\colon R \longrightarrow {}^eR$ is degree-preserving and module-finite. Moreover,

$$ {}^e(-)\colon R\mathbb{Q}\,\mathrm{grmod} \longrightarrow R\mathbb{Q}\,\mathrm{grmod} $$

is an exact functor. If $M \in R\mathbb{Q}\,\mathrm{grmod}$, then the graded k-vector space ${}^eM$ is equipped with the R-action $r \cdot {}^em = {}^e(r^{p^e}m)$, so ${}^eM$ is the graded ${}^eR$-module with the action ${}^er \cdot {}^em = {}^e(rm)$, and the action of R on ${}^eM$ is induced via $F^e\colon R \longrightarrow {}^eR$.

When R is reduced, it is sometimes more transparent to use the notation r^{1/p^e} in place of ${}^er$, and R^{1/p^e} in place of ${}^eR$.

Graded FFRT. When the equivalent conditions in the following lemma are satisfied, the ring R is said to have finite Frobenius representation type (FFRT) in the graded sense:

Lemma 2.1. *Let R be a positively graded finitely generated commutative k-algebra. Then the following are equivalent:*

(1) *There exist $M_1, \dots, M_\ell \in R\mathbb{Q}\,\mathrm{grmod}$ such that for any $e \geqslant 1$, one has*

$$ {}^eR \;\cong\; M_1^{\oplus c_{1e}} \oplus \cdots \oplus M_\ell^{\oplus c_{\ell e}} $$

as (non-graded) R-modules.

(2) *There exist $M_1, \dots, M_\ell \in R\mathbb{Q}\,\mathrm{grmod}$ such that for any $e \geqslant 1$, the R-module ${}^eR$ is isomorphic, as a $\mathbb{Q}$-graded R-module, to a finite direct sum of copies of modules of the form $M_i(d)$ with $1 \leqslant i \leqslant \ell$ and $d \in \mathbb{Q}$.*

Proof. The direction (2) $\Longrightarrow$ (1) is obvious; we prove the converse. Fix $e \geqslant 1$. For a positive integer c, set $M^{\langle c \rangle}$ to be M with the grading $[M^{\langle c \rangle}]_{cj} = [M]_j$. Then $M^{\langle c \rangle}$ is a $\mathbb{Q}$-graded module over the graded ring $R^{\langle c \rangle}$. Taking c sufficiently divisible, we may assume that $R^{\langle c \rangle}$ is $p^e\mathbb{Z}$-graded and each $M_i^{\langle c \rangle}$ is $\mathbb{Z}$-graded. By [14, Corollary 3.9], ${}^eR^{\langle c \rangle}$ is isomorphic to a finite direct sum of modules of the form $(M_i^{\langle c \rangle})(d)$ with $1 \leqslant i \leqslant \ell$ and $d \in \mathbb{Z}$. It follows that ${}^eR$ is a finite direct sum of modules of the form $M_i(d/c)$. $\square$

It follows from [14, Corollary 3.9] that R has FFRT in the graded sense if and only if the $\mathfrak{m}$-adic completion $\widehat{R}$ has FFRT, for $\mathfrak{m}$ the homogeneous maximal ideal of R.

Pseudoreflections. Let V be a finite rank k-vector space. An element $g \in \mathrm{GL}(V)$ is a *pseudoreflection* if $\operatorname{rank}(1_V - g) = 1$. Let G be a finite group and V a G-module. The action of G on V is *small* if $\rho\colon G \longrightarrow \mathrm{GL}(V)$ is injective, and $\rho(G)$ does not contain a pseudoreflection. If in addition $G \subset \mathrm{GL}(V)$, then G is a *small subgroup of* $\mathrm{GL}(V)$.

The twisted group algebra. Let V be a finite rank k-vector space. Let G be a subgroup of $\mathrm{GL}(V)$, and set $S := \operatorname{Sym} V$. If $x_1, \dots, x_n$ is a basis for V, then $\operatorname{Sym} V = k[x_1, \dots, x_n]$ is a polynomial ring in n variables. The action of G on V induces an action of G on the polynomial ring S by degree preserving k-algebra automorphisms.

We say that M is a $\mathbb{Q}$-graded (G, S)-module if M is a G-module as well as a $\mathbb{Q}$-graded S-module such that the underlying k-vector space structures agree, each graded component $[M]_i$ is a G-submodule of M, and $g(sm) = (gs)(gm)$ for all $g \in G$, $s \in S$, and $m \in M$.

We recall the *twisted group algebra* construction $S * G$ from [2]. Set $S * G$ to be $S \otimes_k kG$ as a k-vector space, with kG the group algebra, and define

$$(s \otimes g)(s' \otimes g') := s(gs') \otimes gg'.$$

For $s \in S$ homogeneous, set the degree of $s \otimes g$ to be that of s; this gives $S * G$ a graded k-algebra structure. A $\mathbb{Q}$-graded $S * G$-module M is a $\mathbb{Q}$-graded (G, S)-module where

$$sm := (s \otimes 1)m \qquad \text{and} \qquad gm := (1 \otimes g)m.$$

Conversely, if M is a $\mathbb{Q}$-graded (G, S)-module, then $(s \otimes g)m := sgm$, gives M the structure of a $\mathbb{Q}$-graded $S * G$-module. Thus, a $\mathbb{Q}$-graded $S * G$-module and a $\mathbb{Q}$-graded (G, S)-module are one and the same thing. Similarly, a homogeneous (i.e., degree-preserving) map of $\mathbb{Q}$-graded (G, S)-modules is precisely a homomorphism of graded $S * G$-modules.

With this setup, one has the following equivalence of categories:

Lemma 2.2. *Let V be a finite rank k-vector space, and G a small subgroup of* $\mathrm{GL}(V)$. *Set $S := \operatorname{Sym} V$ and $T := S * G$. Let $T\mathbb{Q}\,\mathrm{grmod}$ denote the category of finitely generated $\mathbb{Q}$-graded left T-modules, and ${}^*\operatorname{Ref}(G, S)$ denote the full subcategory of $T\mathbb{Q}\,\mathrm{grmod}$ con-*

sisting of those that are reflexive as S-modules; let ${}^*\operatorname{Ref} S^G$ denote the full subcategory of $S^G\mathbb{Q}\operatorname{grmod}$ consisting of modules that are reflexive as S^G-modules.

Then one has an equivalence of categories

$$ {}^*\operatorname{Ref}(G,S) \longrightarrow {}^*\operatorname{Ref} S^G, \qquad \textit{where} \qquad M \longmapsto M^G, $$

with quasi-inverse $N \longmapsto (N \otimes_{S^G} S)^{**}$, where $(-)^* := \operatorname{Hom}_S(-,S)$.

For the proof, see [11, Lemma 2.6]; an extension to group schemes may be found in [9]. Note that under the functor displayed above, one has ${}^eS \longmapsto ({}^eS)^G = {}^e(S^G)$.

3. An invariant ring without FFRT

We construct the counterexample promised in Theorem 1.1; more precisely, we prove:

Theorem 3.1. *Let k be the algebraic closure of $\mathbb{F}_3(t)$, the rational function field in one indeterminate over $\mathbb{F}_3$. Let G be the subgroup of $\operatorname{GL}(k^3)$ generated by the matrices*

$$ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textit{and} \qquad \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}. $$

Then G is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. The invariant ring for the natural action of G on the polynomial ring $\operatorname{Sym}(k^3)$ does not have FFRT.

Lemma 3.2. *Let $k := \overline{\mathbb{F}_3(t)}$ as above. Let $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = \langle \sigma, \tau \rangle$, where $\sigma^3 = \mathrm{id} = \tau^3$, and $\sigma\tau = \tau\sigma$. Then the group algebra kG equals the commutative ring $k[a,b]/(a^3,b^3)$, where $a := \sigma - 1$ and $b := \tau - 1$. For $\alpha \in k$, set $V(\alpha)$ to be k^3 with the G-action determined by the homomorphism $G \longrightarrow \operatorname{GL}_3(k)$ with*

$$ \sigma \longmapsto \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textit{and} \qquad \tau \longmapsto \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{bmatrix}. $$

Then:

1. *(1) If $\alpha \notin \mathbb{F}_3$, then the action of G on $V(\alpha)$ is small.*
2. *(2) For $\alpha \neq \beta$ in k, the G-modules $V(\alpha)$ and $V(\beta)$ are nonisomorphic.*
3. *(3) The Frobenius twist ${}^e(V(\alpha))$ is isomorphic to $V(\alpha^{1/3^e})$ as a G-module.*
4. *(4) For each $\alpha \in k$, the G-module $V(\alpha)$ is indecomposable.*

Proof. Setting

$$ N := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} $$

and taking I to be the identity matrix, one has

$$\sigma^i\tau^j \;=\; (I+N)^i(I+\alpha N)^j \;=\; \left[I+iN+\binom{i}{2}N^2\right]\left[I+j\alpha N+\binom{j}{2}\alpha^2N^2\right]$$
$$=\; I+(i+j\alpha)N+\left[\binom{i}{2}+ij\alpha+\binom{j}{2}\alpha^2\right]N^2,$$

so $\sigma^i\tau^j - I$ has rank 2 unless $\alpha\in\mathbb{F}_3$ or $(i,j)=(0,0)$ in $\mathbb{F}_3^2$. This proves (1).

For (2), note that the annihilators of $V(\alpha)$ and $V(\beta)$ are the ideals $(b-\alpha a)$ and $(b-\beta a)$ respectively in $kG = k[a,b]/(a^3,b^3)$. These ideals are distinct when $\alpha\neq\beta$.

The representation matrices for σ and τ in $\mathrm{GL}({}^e(V(\alpha)))$ are

$$^e(I+N) = I+N \qquad \text{and} \qquad {}^e(I+\alpha N) = I+\alpha^{1/3^e}N$$

respectively, so ${}^eV(\alpha)\cong V(\alpha^{1/3^e})$ as G-modules, proving (3).

For (4), note that kG is an artinian local ring, so each nonzero kG-module has a nonzero socle. The socle of $V(\alpha)$ is spanned by the vector $(1,0,0)^{\mathrm{tr}}$, and hence has rank one. It follows that $V(\alpha)$ is an indecomposable kG-module. □

Proof of Theorem 3.1. Set S to be the polynomial ring $\mathrm{Sym}(k^3)$, and $T := S*G$. For M a nonzero module in $T\mathbb{Q}\,\mathrm{grmod}$, set

$$\mathrm{LD}(M) := \min\{i\in\mathbb{Q} \mid [M]_i\neq 0\} \qquad\text{and}\qquad \mathrm{LRep}(M) := [M]_{\mathrm{LD}(M)},$$

i.e., $\mathrm{LRep}(M)$ is the nonzero $\mathbb{Q}$-graded component of M of least degree. Note that for d a rational number, $\mathrm{LRep}(M(d))$ and $\mathrm{LRep}(M)$ are isomorphic as G-modules.

As $T\mathbb{Q}\,\mathrm{grmod}$ is Krull-Schmidt, there is a unique decomposition $M = N_1\oplus\cdots\oplus N_r$ of M into indecomposable objects. Setting $d := \mathrm{LD}(M)$, we have

$$\mathrm{LRep}(M) \;=\; [M]_d \;=\; [N_1]_d\oplus\cdots\oplus[N_r]_d.$$

Suppose $\mathrm{LRep}(M)$ is an indecomposable G-module. After a possible change of indices, we may assume that $\mathrm{LRep}(M) = [N_1]_d$ and that $[N_j]_d = 0$ for $j>1$. Note that, up to isomorphism, N_1 is the unique indecomposable direct summand of M with $\mathrm{LD}(N_1) = \mathrm{LD}(M)$. We define $\mathrm{LInd}(M) := N_1$. Note that we have $\mathrm{LRep}(N_1)\cong\mathrm{LRep}(M)$.

For M as above, and $d\in\mathbb{Q}$, define

$$M_{\langle d\rangle} := \bigoplus_{i\equiv d \bmod \mathbb{Z}} [M]_i,$$

which is also an element of $T\mathbb{Q}\,\mathrm{grmod}$.

Since the degree $1/3^e$-component of ${}^eS$ is ${}^eV(t) = V(t^{1/3^e})$, one has

$$\mathrm{LRep}\left({}^eS_{\langle 1/3^e\rangle}\right) \;=\; V(t^{1/3^e}),$$

which is indecomposable by Lemma 3.2 (4). The G-modules $V(t)$, $V(t^{1/3})$, $V(t^{1/3^2})$, ... are nonisomorphic by Lemma 3.2 (2), so the isomorphism classes of the indecomposable T-modules

$$\mathrm{LInd}\left(S_{\langle 1\rangle}\right), \quad \mathrm{LInd}\left({}^1S_{\langle 1/3\rangle}\right), \quad \mathrm{LInd}\left({}^2S_{\langle 1/3^2\rangle}\right), \quad \ldots$$

are distinct; specifically, any two of these indecomposable objects of $\mathbb{Q}\,\mathrm{grmod}\,T$ are nonisomorphic even after a degree shift. By Lemma 2.2, it follows that the indecomposable $\mathbb{Q}$-graded S^G-modules

$$\Big(\mathrm{LInd}\left(S_{\langle 1\rangle}\right)\Big)^G, \quad \Big(\mathrm{LInd}\left({}^1S_{\langle 1/3\rangle}\right)\Big)^G, \quad \Big(\mathrm{LInd}\left({}^2S_{\langle 1/3^2\rangle}\right)\Big)^G, \quad \ldots$$

are nonisomorphic. These occur as indecomposable summands of ${}^e(S^G)$ for $e \geqslant 1$, so the ring S^G does not have FFRT. $\square$

Remark 3.3. For the interested reader, we give a presentation of the invariant ring S^G in Theorem 3.1. This was obtained using `Magma` [4], though one may verify all claims by hand, after the fact. Take $S := \mathrm{Sym}\,V$ to be the polynomial ring $k[x_1, x_2, x_3]$, where the indeterminates x_1, x_2, x_3 are viewed as the standard basis vectors in $V := k^3$. Then the invariant ring S^G is generated by the polynomials

$$\begin{aligned}
f_1 &:= x_1,\\
f_3 &:= tx_1^2x_2 - (t+1)x_1^2x_3 - (t+1)x_1x_2^2 + x_2^3,\\
f_5 &:= t(t-1)^2x_1^4x_3 + t(t^2+1)x_1^3x_2^2 - t(t+1)x_1^3x_2x_3 - (t+1)^2x_1^3x_3^2\\
&\quad - (t+1)(t-1)^2x_1^2x_2^3 + (t+1)^2x_1^2x_2^2x_3 + x_1^2x_3^3 - (t-1)^2x_1x_2^4\\
&\quad - (t+1)x_1x_2^3x_3 - (t+1)x_2^5,\\
f_9 &:= x_3(x_2+x_3)(x_1-x_2+x_3)(tx_2+x_3)(tx_1+x_2+tx_2+x_3)\\
&\quad \times (x_1 - tx_1 - x_2 + tx_2 + x_3)(t^2x_1 - tx_2 + x_3)(t^2x_1 - tx_1 + x_2 - tx_2 + x_3)\\
&\quad \times (x_1 + tx_1 + t^2x_1 - x_2 - tx_2 + x_3),
\end{aligned}$$

where f_9 is the product over the orbit of x_3. These four polynomials satisfy the relation

$$t(t-1)^2(t^2+1)f_1^3f_3^4 - t^2(t-1)^2f_1^4f_3^2f_5 + (t^3+1)f_3^5 + (t^3+1)f_1f_3^3f_5 - f_1^6f_9 + f_5^3$$

that defines a normal hypersurface. Using this defining equation, one may see that S^G is not F-pure. The defining equation also confirms that the a-invariant is $a(S^G) = -3$, as follows from [10, Theorem 3.6] or [6, Theorem 4.4] since G is a subgroup of $\mathrm{SL}(V)$ without pseudoreflections.

4. Ring of invariants of monomial actions

Let k be a field of positive characteristic, and let G be a finite group. Consider a finite rank k-vector space V that is a G-module. A k-basis Γ of V is a *monomial basis* for the action of G if for each $g \in G$ and $\gamma \in \Gamma$, one has $g\gamma \in k\gamma'$ for some $\gamma' \in \Gamma$. We say that V is a *monomial representation* of G if V admits a monomial basis.

A monomial representation V as above is a *permutation representation* of G if V admits a k-basis Γ such that each $g \in G$ permutes the elements of Γ.

Theorem 4.1. *Let k be a perfect field of positive characteristic, G a finite group, and V a monomial representation of G over k. Then the ring of invariants $(\operatorname{Sym} V)^G$ has FFRT.*

Proof. Set $q := p^e$, where k has characteristic p and $e \in \mathbb{N}$. The action of G on $S := \operatorname{Sym} V$ extends uniquely to an action of G on ${}^eS = S^{1/q}$; note that

$$(S^{1/q})^G \;=\; (S^G)^{1/q}.$$

Let $\{x_1, \dots, x_n\}$ be a monomial basis for V. The ring $S^{1/q}$ then has an S-basis

$$B_e := \Big\{ x_1^{\lambda_1/q} \cdots x_n^{\lambda_n/q} \;\mid\; \lambda_i \in \mathbb{Z}, \quad 0 \leqslant \lambda_i \leqslant q-1 \Big\}. \tag{4.1.1}$$

For $\mu \in B_e$, set γ_μ to be the k-vector space spanned by the elements $g\mu$ for all $g \in G$. Then $S^{1/q}$ is a direct sum of modules of the form $S\gamma_\mu$, and the action of G on $S^{1/q}$ restricts to an action on each $S\gamma_\mu$. To prove that S^G has FFRT, it suffices to show that there are only finitely many isomorphism classes of S^G-modules of the form

$$(S\gamma_\mu)^G \;=\; \Big(\sum_{g \in G} Sg\mu\Big)^G$$

as e varies. Fix $\mu \in B_e$, and consider the rank one k-vector space $k\mu$. Set

$$H := \{g \in G \;\mid\; g\mu \in k\mu\}.$$

Let $g_1, \dots, g_m$ be a set of left coset representatives for G/H, where g_1 is the group identity. We claim that the map

$$\sum_{i=1}^m g_i \colon (S\mu)^H \;\longrightarrow\; (S\gamma_\mu)^G \tag{4.1.2}$$

is an isomorphism of $\mathbb{Q}$-graded S^G-modules. Assuming the claim, $(S\mu)^H = (S \otimes_k k\mu)^H$ is isomorphic, up to a degree shift, with a module of the form $(S \otimes_k \chi)^H$, where χ is a rank one representation of H. Since there are only finitely many subgroups H of G, only finitely many rank one representations χ of H, and only finitely many isomorphism classes of indecomposable $\mathbb{Q}$-graded S^G-summands of $(S \otimes_k \chi)^H$ by the Krull-Schmidt theorem, the claim indeed completes the proof.

It remains to verify the isomorphism (4.1.2). Given $g \in G$, there exists a permutation $\sigma \in \mathfrak{S}_m$ such that $gg_i = g_{\sigma i}h_i$ for each i, with $h_i \in H$. Given $s\mu \in (S\mu)^H$, one has

$$g\Big(\sum_i g_i(s\mu)\Big) \ = \ \sum_i g_{\sigma i}h_i(s\mu) \ = \ \sum_i g_{\sigma i}(s\mu) \ = \ \sum_i g_i(s\mu),$$

so $\sum_i g_i(s\mu)$ indeed lies in $(S\gamma_\mu)^G$. Since each g_i is S^G-linear and preserves degrees, the same holds for their sum. As to the injectivity, if

$$\sum_i g_i(s\mu) \ = \ \sum_i (g_i s)(g_i \mu) \ = \ 0,$$

then $g_i s = 0$ for each i, since $g_1\mu, \dots, g_m\mu$ are distinct elements of the basis B_e as in (4.1.1), and hence linearly independent over S. But then $s = 0$. For the surjectivity, first note that an element of $S\gamma_\mu$ may be written as $\sum_i s_i g_i \mu$. Consider

$$f := s_1 g_1 \mu + s_2 g_2 \mu + \cdots + s_m g_m \mu \ \in \ (S\gamma_\mu)^G.$$

Apply g_i to the above; since $g_i f = f$, and $g_1\mu, \dots, g_m\mu$ are linearly independent over S, it follows that $g_i s_1 = s_i$. But then

$$f \ = \ \sum_i g_i(s_1\mu),$$

so it remains to show that $s_1\mu \in (S\mu)^H$. Fix $h \in H$. Since $hf = f$, one has

$$\sum_i hg_i(s_1\mu) \ = \ \sum_i g_i(s_1\mu).$$

As $hg_1 \in H$ and $hg_i \notin H$ for $i \geqslant 2$, the linear independence of $g_1\mu, \dots, g_m\mu$ over S implies that $h(s_1\mu) = s_1\mu$. $\square$

Remark 4.2. For k a field of positive characteristic, and V a finite rank permutation representation of G, Hochster and Huneke showed that the invariant ring $(\operatorname{Sym} V)^G$ is F-pure [16, page 77]; the same holds more generally when V is a monomial representation:

It suffices to prove the F-purity in the case where the field k is perfect. With the notation as in the proof of Theorem 4.1, $(S^G)^{1/q}$ is a direct sum of S^G-modules of the form $(S\gamma_\mu)^G$, where γ_μ is the k-vector space spanned by $g\mu$ for $g \in G$. When $\mu := 1$ one has $\gamma_\mu = k$, so S^G indeed splits from $(S^G)^{1/q}$.

Remark 4.3. In Theorem 4.1 suppose, moreover, that V is a permutation representation of G. Then one may choose a basis $\{x_1, \dots, x_n\}$ for V whose elements are permuted by G. In this case, each $g \in G$ permutes the elements of B_e for $e \in \mathbb{N}$, and each rank one representation $\chi\colon H \longrightarrow k^*$ is trivial; it follows that $(S^G)^{1/q}$ is a direct sum of S^G-modules of the form S^H, for subgroups H of G.

Example 4.4. Let p be a prime integer. Set $S := \mathbb{F}_p[x_1, \dots, x_p]$, and let $G := \langle\sigma\rangle$ be the cyclic group of order p acting on S by cyclically permuting the variables. The ring S^G has FFRT by Theorem 4.1. Let $q = p^e$ be a varying power of p.

If $p = 2$, then S^G is a polynomial ring, and each $(S^G)^{1/q}$ is a free S^G-module; thus, up to isomorphism and degree shift, the only indecomposable summand of $(S^G)^{1/q}$ is S^G.

Suppose $p \geqslant 3$. For $\mu \in B_e$, consider the kG-submodule $\gamma_\mu = kg\mu$ of $S^{1/q}$. If the stabilizer of μ is G, then $\gamma_\mu = k\mu$ is an indecomposable kG module, and $(S\mu)^G = S^G\mu \cong S^G$ is an indecomposable S^G-summand of $(S^G)^{1/q}$. Since the only subgroups of G are $\{\mathrm{id}\}$ and G, the only other possibility for the stabilizer of an element μ of B_e is $\{\mathrm{id}\}$, in which case the orbit is a *free orbit*, i.e., an orbit of size $|G|$, and $\gamma_\mu \cong kG$. We claim that

$$(S \otimes_k kG)^G \cong S$$

is an indecomposable S^G-module. Since the group G contains no pseudoreflections in the case $p \geqslant 3$, Lemma 2.2 is applicable, and it suffices to verify that $S \otimes_k kG$ is an indecomposable graded (G, S)-module. Note that $kG = k[\sigma]/(1-\sigma)^p$ is an indecomposable kG-module. Suppose one has a decomposition as graded (G, S)-modules

$$S \otimes_k kG \ \cong\ P_1 \oplus P_2,$$

apply $(-) \otimes_S S/\mathfrak{m}$ where $\mathfrak{m}$ is the homogeneous maximal ideal of S. Then

$$kG \ \cong\ P_1/\mathfrak{m}P_1 \oplus P_2/\mathfrak{m}P_2.$$

The indecomposability of kG implies that $P_i/\mathfrak{m}P_i = 0$ for some i. But then Nakayama's lemma, in its graded form, gives $P_i = 0$, which proves the claim. Lastly, it is easy to see that both of these types of G-orbits appear in B_e if $e \geqslant 1$ so, up to isomorphism and degree shift, the indecomposable S^G-summands of $(S^G)^{1/q}$ are indeed S^G and S.

Example 4.5. As a specific example of the above, consider the alternating group A_3 with its natural permutation action on the polynomial ring $S := \mathbb{F}_3[x_1, x_2, x_3]$. For $q = 3^e$, consider the S-basis (4.1.1) for $S^{1/q}$. It is readily seen that the monomials

$$(x_1x_2x_3)^{\lambda/q} \qquad \text{where } \lambda \in \mathbb{Z}, \quad 0 \leqslant \lambda \leqslant q-1$$

are fixed by A_3, whereas every other monomial in B_e has a free orbit. It follows that, ignoring the grading, the decomposition of $(S^{A_3})^{1/q}$ into indecomposable S^{A_3}-modules is

$$(S^{A_3})^{1/q} \;\cong\; (S^{A_3})^q \oplus S^{(q^3-q)/3}.$$

Example 4.6. Let k be a perfect field of characteristic 2 that contains a primitive third root ω of unity. Let G be the group generated by

$$\sigma := \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$$

acting on $S := k[x_1, x_2]$. The invariant ring S^G is the Veronese subring

$$k[x_1, x_2]^{(3)} \;=\; k[x_1^3,\ x_1^2x_2,\ x_1x_2^2,\ x_2^3].$$

The action of G on S extends to an action on $S^{1/q}$ where $\sigma(x_i^{1/q}) = \omega^q x_i^{1/q}$. For B_e as in (4.1.1), consider

$$S^{1/q} \;=\; \bigoplus_{\mu \in B_e} S\mu.$$

Suppose $\mu = x_1^{\lambda_1/q} x_2^{\lambda_2/q}$, where λ_i are integers with $0 \leqslant \lambda_i \leqslant q-1$. Then

$$(S\mu)^G \;=\; \begin{cases} S^G\mu & \text{if} \quad \lambda_1+\lambda_2 \equiv 0 \mod 3, \\ S^Gx_1\mu + S^Gx_2\mu & \text{if} \quad \lambda_1+\lambda_2 \equiv 2q \mod 3, \\ S^Gx_1^2\mu + S^Gx_1x_2\mu + S^Gx_2^2\mu & \text{if} \quad \lambda_1+\lambda_2 \equiv q \mod 3. \end{cases}$$

The S^G-modules that occur in the three cases above are, respectively, isomorphic to the ideals S^G, $(x_1^3, x_1^2x_2)S^G$, and $(x_1^3, x_1^2x_2, x_1x_2^2)S^G$, that constitute the indecomposable summands of $S^{1/q}$. The number of copies of each of these is *asymptotically* $q^2/3$.

This extends readily to Veronese subrings of the form $k[x_1, x_2]^{(n)}$, for k a perfect field of characteristic p that contains a primitive nth root of unity; see [19, Example 17].

Example 4.7. Let $G := \langle\sigma\rangle$ be a cyclic group of order 4, acting on $S := \mathbb{F}_2[x_1, x_2, x_3, x_4]$ by cyclically permuting the variables. In view of [3], the invariant ring S^G is a UFD that is not Cohen-Macaulay; S^G has FFRT by Theorem 4.1.

We describe the indecomposable summands that occur in an S^G-module decomposition of $(S^G)^{1/q}$ for $q = 2^e$. The group G contains no pseudoreflections, so Lemma 2.2 applies. Consider the S-basis B_e for $S^{1/q}$, as in (4.1.1). The monomials

$$(x_1x_2x_3x_4)^{\lambda/q} \qquad \text{where } 0 \leqslant \lambda \leqslant q-1$$

are fixed by G; each such monomial μ gives an indecomposable kG module $\gamma_\mu = k\mu$, and an indecomposable S^G-summand $(S\mu)^G \cong S^G$ of $(S^G)^{1/q}$. The monomials μ of the form

$$(x_1x_3)^{\lambda_1/q}(x_2x_4)^{\lambda_2/q} \qquad \text{with } 0 \leqslant \lambda_i \leqslant q-1, \quad \lambda_1 \neq \lambda_2$$

have stabilizer $H := \langle \sigma^2 \rangle$. In this case, $\gamma_\mu \cong k[\sigma]/(1-\sigma)^2$ is an indecomposable kG module, corresponding to an indecomposable S^G-summand $(S \otimes_k \gamma_\mu)^G \cong S^H$. Any other monomial in B_e has a free orbit that corresponds to a copy of $(S \otimes_k kG)^G \cong S$.

Ignoring the grading, the decomposition of $(S^G)^{1/q}$ into indecomposable S^G-modules is

$$(S^G)^{1/q} \;\cong\; (S^G)^q \oplus (S^H)^{(q^2-q)/2} \oplus S^{(q^4-q^2)/4}.$$

5. Examples that are FFRT but not F-regular

The notion of F-regular rings is central to Hochster and Huneke's theory of tight closure, introduced in [15]; while there are different notions of F-regularity, they coincide in the graded case under consideration here by [21, Corollary 4.3], so we downplay the distinction. The FFRT property and F-regularity are intimately related, though neither implies the other: The hypersurface

$$\mathbb{F}_p[x,y,z]/(x^2+y^3+z^5)$$

has FFRT for each prime integer p, though it is not F-regular if $p \in \{2,3,5\}$; Stanley-Reisner rings have FFRT by [20, Example 2.3.6], though they are F-regular only if they are polynomial rings. For the other direction, the hypersurface

$$R := \mathbb{F}_p[s,t,u,v,w,x,y,z]/(su^2x^2+sv^2y^2+tuvxy+tw^2z^2)$$

is F-regular for each prime integer p, but admits a local cohomology module $H^3_{(x,y,z)}(R)$ with infinitely many associated prime ideals, [27, Theorem 5.1], and hence does not have FFRT by [30, Corollary 3.3] or [18, Theorem 1.2]. Nonetheless, for the invariant rings of finite groups that are our focus here, F-regularity implies FFRT; this follows readily from well-known results, but is recorded here for the convenience of the reader:

Proposition 5.1. *Let k be a perfect field, G a finite group, and V a finite rank k-vector space that is a G-module. If the invariant ring $(\operatorname{Sym} V)^G$ is F-regular, then it has FFRT.*

Proof. An F-regular ring is *splinter* by [17, Theorem 5.25], i.e., it is a direct summand of each module-finite extension ring. Hence, if $(\operatorname{Sym} V)^G$ is F-regular, then it is a direct summand of $\operatorname{Sym} V$. But then it has FFRT by [29, Proposition 3.1.4]. □

We next present a family of examples where $(\operatorname{Sym} V)^G$ is not F-regular or even F-pure, but has FFRT:

Example 5.2. Let p be a prime integer, $V := \mathbb{F}_p^4$, and G the subgroup of $\mathrm{GL}(V)$ generated by the matrices

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is readily seen that the matrices commute, and that the group G has order p^3. Consider the action of G on the polynomial ring $S := \operatorname{Sym} V = \mathbb{F}_p[x_1, x_2, x_3, x_4]$, where x_1, x_2, x_3, x_4 are viewed as the standard basis vectors in V. While x_1 and x_2 are fixed under the action, the orbits of x_3 and x_4 respectively consist of all linear forms

$$x_3 + \alpha x_1 + \gamma x_2 \qquad \text{and} \qquad x_4 + \beta x_1 + \alpha x_2,$$

where α, β, γ are in $\mathbb{F}_p$. Using Moore determinants as in [7, Chapter 1.3], the respective orbit products may be expressed as

$$u := \frac{\det \begin{bmatrix} x_1 & x_2 & x_3 \\ x_1^p & x_2^p & x_3^p \\ x_1^{p^2} & x_2^{p^2} & x_3^{p^2} \end{bmatrix}}{\det \begin{bmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{bmatrix}} \qquad \text{and} \qquad v := \frac{\det \begin{bmatrix} x_1 & x_2 & x_4 \\ x_1^p & x_2^p & x_4^p \\ x_1^{p^2} & x_2^{p^2} & x_4^{p^2} \end{bmatrix}}{\det \begin{bmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{bmatrix}}.$$

In addition to these, it is readily seen that the polynomial $t := x_1 x_4^p - x_1^p x_4 + x_2 x_3^p - x_2^p x_3$ is invariant. These provide us with a *candidate* for the invariant ring, namely

$$C := \mathbb{F}_p[x_1, x_2, t, u, v].$$

Note that S is integral over C as x_3 and x_4 are, respectively, roots of the monic polynomials

$$\prod_{\alpha, \gamma \in \mathbb{F}_p} (T + \alpha x_1 + \gamma x_2) - u \qquad \text{and} \qquad \prod_{\beta, \alpha \in \mathbb{F}_p} (T + \beta x_1 + \alpha x_2) - v$$

that have coefficients in C. Using the first of these polynomials, one also sees that

$$[\operatorname{frac}(C)(x_3) : \operatorname{frac}(C)] \leqslant p^2.$$

Bearing in mind that $t \in C$, one then has $[\operatorname{frac}(C)(x_3, x_4) : \operatorname{frac}(C)(x_3)] \leqslant p$, and hence

$$[\operatorname{frac}(S) : \operatorname{frac}(C)] \leqslant p^3.$$

Since $C \subseteq S^G \subseteq S$ and $|G| = p^3$, it follows that $\operatorname{frac}(C) = \operatorname{frac}(S^G)$. To prove that $C = S^G$, it suffices to verify that C is normal. Note that C must be a hypersurface; we arrive at its defining equation as follows: One readily verifies the identity

$$\det\begin{bmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{bmatrix}\left(\det\begin{bmatrix} x_1 & x_4 \\ x_1^p & x_4^p \end{bmatrix} + \det\begin{bmatrix} x_2 & x_3 \\ x_2^p & x_3^p \end{bmatrix}\right)^p$$
$$- x_1^p \det\begin{bmatrix} x_1 & x_2 & x_4 \\ x_1^p & x_2^p & x_4^p \\ x_1^{p^2} & x_2^{p^2} & x_4^{p^2} \end{bmatrix} - x_2^p \det\begin{bmatrix} x_1 & x_2 & x_3 \\ x_1^p & x_2^p & x_3^p \\ x_1^{p^2} & x_2^{p^2} & x_3^{p^2} \end{bmatrix}$$
$$= \left(\det\begin{bmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{bmatrix}\right)^p \left(\det\begin{bmatrix} x_1 & x_4 \\ x_1^p & x_4^p \end{bmatrix} + \det\begin{bmatrix} x_2 & x_3 \\ x_2^p & x_3^p \end{bmatrix}\right),$$

which may be rewritten as

$$t^p \det\begin{bmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{bmatrix} \;-\; v x_1^p \det\begin{bmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{bmatrix} \;-\; u x_2^p \det\begin{bmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{bmatrix} \;=\; t\left(\det\begin{bmatrix} x_1 & x_2 \\ x_1^p & x_2^p \end{bmatrix}\right)^p.$$

Dividing by the determinant that occurs on the left, one then has

$$t^p - v x_1^p - u x_2^p \;=\; t(x_1 x_2^p - x_1^p x_2)^{p-1}. \tag{5.2.1}$$

The Jacobian criterion shows that a hypersurface with (5.2.1) as its defining equation must be normal; it follows that C is indeed a normal hypersurface, with defining equation (5.2.1), and hence that C is precisely the invariant ring S^G. Equation (5.2.1) shows that S^G is not F-pure: t is in the Frobenius closure of $(x_1, x_2)S^G$, though it does not belong to this ideal.

It remains to prove that the ring $C = S^G$ has FFRT. For this, note that after a change of variables, one has

$$S^G \;\cong\; \mathbb{F}_p[x_1, x_2, t, \widetilde{u}, \widetilde{v}]/(t^p - \widetilde{v} x_1^p - \widetilde{u} x_2^p).$$

But then S^G has FFRT by [25, Observation 3.7, Theorem 3.10]: Set $A := \mathbb{F}_p[x_1, x_2, \widetilde{u}, \widetilde{v}]$, and note that

$$A \;\subseteq\; S^G \;\subseteq\; A^{1/p},$$

where A is a polynomial ring.

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