Transformation Groups
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Transformation Groups

Local Cohomology of Modular Invariant Rings

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Abstract
For K a field, consider a finite subgroup G of $\mathrm{GL}_n(K)$ with its natural action on the polynomial ring $R := K[x_1, \ldots, x_n]$. Let $\mathfrak{n}$ denote the homogeneous maximal ideal of the ring of invariants R^G. We study how the local cohomology module $H^n_{\mathfrak{n}}(R^G)$ compares with $H^n_{\mathfrak{n}}(R)^G$. Various results on the a-invariant and on the Hilbert series of $H^n_{\mathfrak{n}}(R^G)$ are obtained as a consequence.

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1 Introduction

Let K be a field. Consider a finite group G acting on a polynomial ring $R := K[x_1, \ldots, x_n]$ via degree-preserving K-algebra automorphisms; the action of G on R is completely determined by its action on one-forms, so there is little loss of generality in taking G to be a finite subgroup of $\mathrm{GL}_n(K)$, with the action given by

$$M \colon X \mapsto MX,$$

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where X is a column vector of the indeterminates; this is the action of G on R considered throughout the present paper. In the *nonmodular* case—when the order of G is invertible in K—there is a wealth of results relating properties of the invariant ring R^G to properties of the group action; several of these fail in the *modular* case, i.e., when the order of G is a multiple of the characteristic of K. For instance, in the nonmodular case, the functor $(-)^G$ is exact, yielding an R^G-isomorphism of local cohomology modules

$$H^n_{\mathfrak{m}}(R)^G \;\cong\; H^n_{\mathfrak{n}}(R^G),$$

where $\mathfrak{m}$ and $\mathfrak{n}$ denote the respective homogeneous maximal ideals of R and R^G. This isomorphism no longer holds in the modular case; indeed, one of our goals is to study the failure of this isomorphism. Quite generally, the transfer map provides a surjection $H^n_{\mathfrak{m}}(R) \to H^n_{\mathfrak{n}}(R^G)$; when G contains no transvections, we explicitly describe the kernel in Theorem 3.1. This result may be viewed as a dual formulation of a theorem of Peskin [19], that relates the canonical modules of R and of R^G (see Remark 3.2).

We apply Theorem 3.1 to study the local cohomology a-invariant of R^G in Section 4, proving that the a-invariant of R^G equals that of R if and only if G is a subgroup of the special linear group with no pseudoreflections (see Theorem 4.4). In Section 5, we record a surprising consequence of our main theorem towards comparing the ranks of the graded components of the local cohomology modules $H^n_{\mathfrak{n}}(R^G)$ and $H^n_{\mathfrak{m}}(R)^G$, proving that they coincide when G is cyclic with no transvections. The study of local cohomology modules of invariant rings of finite groups goes back at least to the work of Ellingsrud and Skjelbred [7], where they use spectral sequences relating local cohomology and group cohomology to give upper bounds on the depth of modular invariant rings.

The article [20] by Stanley provides an excellent account of the theory in the nonmodular case; for sources that include the modular case as well, we refer the reader to Benson [1] and Campbell and Wehlau [6]. We have attempted to keep this paper largely self-contained, and accessible to the reader familiar with the basics of local cohomology; some preliminary results are reviewed or proved in Section 2, towards simplifying later arguments. Our study is closely related to earlier work on the canonical module and the Gorenstein property of invariant rings, e.g., [3, 5, 9, 12, 19, 21, 22]; these are discussed briefly in Section 2.

2 Preliminary Remarks

We begin with some standard facts about finite group actions:

Pseudoreflections An element $g \in \mathrm{GL}_n(K)$ of finite order is a *pseudoreflection* if it fixes a hyperplane; by convention, the group identity is not a pseudoreflection. It follows that g is a pseudoreflection precisely if the matrix $g - I$, with I the identity matrix, has rank one. An equivalent formulation is that the Jordan form of g, after extending scalars, is

$$\left[\begin{array}{c|cccc} \zeta & 0 & 0 & \cdots & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cc|ccc} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{array}\right].$$

Since g has finite order, the element ζ in the first case is a root of unity. The second case only occurs when K has characteristic $p > 0$; such an element is a *transvection*.

Remark 2.1 Fix $g \in G$. We use $(1-g)R$ to denote the ideal of $R := K[x_1, \ldots, x_n]$ generated by all elements of the form $r - g(r)$ for $r \in R$. Since

$$(1-g)(r_1 r_2) \;=\; r_2(1-g)(r_1) + g(r_1)(1-g)(r_2),$$

the ideal $(1-g)R$ is generated by the elements $(1-g)(x_i)$ for $1 \le i \le n$. Note that g is a pseudoreflection if and only if the ideal $(1-g)R$ has height one.

Transfer Let G be a finite subgroup acting on a ring R. For a subgroup H, the *transfer map* $\operatorname{Tr}_H^G \colon R^H \to R^G$ is defined as

$$\operatorname{Tr}_H^G(r) := \sum_{gH \in G/H} g(r),$$

where the sum is over a set of left coset representatives. It is straightforward to see that Tr_H^G is an R^G-linear map, independent of the coset representatives. Precomposing with the inclusion $R^G \subseteq R^H$, the composition

$$R^G \longrightarrow R^H \xrightarrow{\operatorname{Tr}_H^G} R^G$$

is multiplication by the integer $[G : H]$, i.e., by the index of H in G. It follows that Tr_H^G is surjective if $[G : H]$ is invertible in R.

When H is the subgroup consisting only of the identity element, we use Tr^G or Tr to denote the transfer map $R \longrightarrow R^G$.

The following lemma appears in various forms in the literature, e.g., [8, Theorem 2.4], [3, Proposition 3.7], and [18, Theorem 2.4.5]; we include a self-contained proof:

Lemma 2.2 *Let G be a finite subgroup of $GL_n(K)$, without transvections, acting on the polynomial ring $R := K[x_1, \ldots, x_n]$. Then, the image of the transfer map $\operatorname{Tr}\colon R \longrightarrow R^G$ is an ideal of R^G of height at least two.*

Proof The transfer map is surjective in the nonmodular case, so assume that K has positive characteristic p. The claim reduces to the case where K is algebraically closed, as we now assume. Let $\mathfrak{p}$ be a prime ideal of R^G height one, and $\mathfrak{q}$ a height one prime of R containing $\mathfrak{p}$. It suffices to show that there is a maximal ideal $\mathfrak{m}$ of R, containing $\mathfrak{q}$, such that $\operatorname{Tr}(R) \nsubseteq \mathfrak{m}$.

By Remark 2.1, the prime $\mathfrak{q}$ does not contain an ideal of the form $(1-g)R$ for any group element g of order p, since such an element would then be a transvection. Let $\mathfrak{a}$ denote the product of the ideals $(1-g)R$, taken over group elements g of order p. Then, $\mathfrak{a} \nsubseteq \mathfrak{q}$, so there exists a point $(a_1, \dots, a_n) \in \mathbb{A}^n_K$ that lies in the algebraic set $V(\mathfrak{q})$ but not in $V(\mathfrak{a})$. Set $\mathfrak{m} := (x_1 - a_1, \dots, x_n - a_n)R$. We claim that $g(\mathfrak{m}) \neq \mathfrak{m}$ for each $g \in G$ of order p.

If the claim is false, there exists an element g of order p such that

$$g(x_i - a_i) \;=\; g(x_i) - a_i \;\in\; \mathfrak{m} \qquad \text{for each } 1 \leq i \leq n.$$

But $x_i - a_i \in \mathfrak{m}$ as well, so $x_i - g(x_i) \in \mathfrak{m}$ for each i. These generate $(1-g)R$, yielding a contradiction. This proves the claim.

Consider the action of G on the set of maximal ideals of R. Since the stabilizer H of $\mathfrak{m}$ has no elements of order p, the order of H is invertible in K. The transfer map $R \to R^G$ factors as

$$R \xrightarrow{\operatorname{Tr}^H} R^H \xrightarrow{\operatorname{Tr}^G_H} R^G,$$

where the first map is surjective, so it suffices to show that the image of Tr^G_H is not contained in $\mathfrak{m}$. Let $\{g_1, \dots, g_\ell\}$ be coset representatives for G/H, where $g_1 H = H$. Then,

$$\mathfrak{m} = g_1^{-1}(\mathfrak{m}),\; g_2^{-1}(\mathfrak{m}),\; \dots,\; g_\ell^{-1}(\mathfrak{m})$$

are distinct maximal ideals of R, so there exists an element $r \in R$ with $r \in g_i^{-1}(\mathfrak{m})$ for each $i \leq 2 \leq \ell$, and $r \notin \mathfrak{m}$. These conditions are preserved when r is replaced by its orbit product under H, so we may assume $r \in R^H$. But then

$$\begin{aligned} \operatorname{Tr}^G_H(r) &= g_1(r) + g_2(r) \cdots + g_\ell(r) \\ &\equiv r \mod \mathfrak{m}. \end{aligned}$$

It follows that $\operatorname{Tr}^G_H(R^H)$ is not contained in $\mathfrak{m}$. □

Local Cohomology and the Canonical Module

Let S be an $\mathbb{N}$-graded ring that is finitely generated over a field $S_0 = K$. Let $\mathfrak{n}$ denote the homogeneous maximal ideal of S, and set $n := \dim S$. Let $y_1, \dots, y_n$ be a *homogeneous system of parameters* for S, i.e., a sequence of n homogeneous elements that generate an ideal with radical $\mathfrak{n}$. For an S-module M and an integer $k \geq 0$, the *local cohomology* module $H^k_{\mathfrak{n}}(M)$ is defined as

$$H^k_{\mathfrak{n}}(M) = \varinjlim_i \operatorname{Ext}^k_S(S/\mathfrak{n}^i, M),$$

and may be identified with the Čech cohomology module $\check{H}^k(y_1, \ldots, y_n;\ S)$, i.e., the k-th cohomology of the Čech complex

$$0 \longrightarrow M \longrightarrow \bigoplus_i M_{y_i} \longrightarrow \bigoplus_{i<j} M_{y_i y_j} \longrightarrow \cdots \longrightarrow M_{y_1 \cdots y_n} \longrightarrow 0.$$

In particular, this identifies $H^n_{\mathfrak{n}}(M)$ with

$$\frac{M_{y_1 \cdots y_n}}{\sum_i M_{y_1 \cdots \hat{y}_i \cdots y_n}}.$$

Under this identification, a local cohomology class

$$\left[\frac{m}{y_1^d \cdots y_n^d}\right] \in H^n_{\mathfrak{n}}(M),$$

for $m \in M$, is zero if and only if there exists an integer $\ell \geq 0$ such that

$$m(y_1 \cdots y_n)^\ell \in \left(y_1^{d+\ell}, \ \ldots, \ y_n^{d+\ell}\right)M.$$

When M is a $\mathbb{Z}$-graded S-module, each $H^k_{\mathfrak{n}}(M)$ acquires a natural $\mathbb{Z}$-grading. Following Goto and Watanabe [10], the *a-invariant* of the ring S, denoted $a(S)$, is the largest integer a such that the graded component $[H^n_{\mathfrak{n}}(S)]_a$ is nonzero.

Let M be a $\mathbb{Z}$-graded S-module. We use $M(i)$ to denote the module with the shifted grading $[M(i)]_j = [M]_{i+j}$ for each $j \in \mathbb{Z}$. The *graded K-dual* of M, denoted M^*, is the S-module with graded components

$$[M^*]_i = \operatorname{Hom}_K(M, K(i)),$$

where $\operatorname{Hom}_K(M, K(i))$ is the vector space of degree-preserving K-linear maps $M \to K(i)$. Assume now that S is normal; the *canonical module* of S is

$$\omega_S := H^n_{\mathfrak{n}}(S)^*.$$

When the ring S is Gorenstein, one has a degree-preserving isomorphism

$$\omega_S \cong S(a),$$

where $a = a(S)$. A normal $\mathbb{N}$-graded ring S is Gorenstein precisely if it is Cohen-Macaulay and ω_S is a cyclic S-module; dropping the Cohen-Macaulay condition, a normal $\mathbb{N}$-graded ring S is *quasi-Gorenstein* if ω_S is a cyclic S-module.

Let G be a finite subgroup of $\mathrm{GL}_n(K)$, acting on a polynomial ring R. In the nonmodular case, the invariant ring R^G is Cohen-Macaulay by [13], though it need not be Cohen-Macaulay in the modular case; this leads to interest in the quasi-Gorenstein property. We summarize some of the work in this direction:

Suppose first that the order of G is invertible in the field K; this is the nonmodular case. Watanabe proved that if $G \subseteq \mathrm{SL}_n(K)$, then R^G is Gorenstein [21], and that if G contains no pseudoreflections, then the converse holds as well, i.e., if R^G is Gorenstein, then $G \subseteq \mathrm{SL}_n(K)$ (see [22]). Braun [3] proved analogues of these in the modular case when G contains no pseudoreflections: the ring R^G is quasi-Gorenstein if and only if G is contained in $\mathrm{SL}_n(K)$. Some of these results are extended in [9] and [12].

It was conjectured that if R^G is Cohen-Macaulay and $G \subseteq \mathrm{SL}_n(K)$, then R^G is Gorenstein [15, Conjecture 5]; while this is true in the nonmodular case by [21], the conjecture was shown to be false by Braun [4], with the simplest example being the subgroup G of $\mathrm{SL}_2(\mathbb{F}_9)$ generated by

$$\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

where ζ is a primitive 4-th root of unity. Note that G contains a transvection—as it must!

The Group Action on Local Cohomology

Let G be a finite subgroup of $\mathrm{GL}_n(K)$, acting on a polynomial ring $R := K[x_1, \dots, x_n]$. The action of G on $H^n_{\mathfrak{m}}(R)$ may be interpreted in several equivalent ways: for $g \in G$, the automorphism $g\colon R \longrightarrow R$ induces a map

$$H^n_{\mathfrak{m}}(R) \xrightarrow{\ g\ } H^n_{g(\mathfrak{m})}(R) = H^n_{\mathfrak{m}}(R),$$

where the equality is simply because $g(\mathfrak{m}) = \mathfrak{m}$.

Alternatively, let $y_1, \dots, y_n$ be a homogeneous system of parameters for R^G, and use the identification of $H^n_{\mathfrak{m}}(R)$ with Čech cohomology $\check{H}^n(y_1, \dots, y_n;\ R)$. Under this identification, for $g \in G$ and $r \in R$, one has

$$\eta := \left[\frac{r}{y_1^d \cdots y_n^d} \right] \mapsto \left[\frac{g(r)}{y_1^d \cdots y_n^d} \right] = g(\eta).$$

Note that η is fixed by g precisely if there exists an integer $\ell \geq 0$ such that

$$\big(g(r) - r\big)(y_1 \cdots y_n)^\ell \ \in\ \big(y_1^{d+\ell},\ \dots,\ y_n^{d+\ell}\big)R.$$

Since $y_1, \dots, y_n$ is a regular sequence on R, this is equivalent to

$$g(r) - r \ \in\ \big(y_1^d,\ \dots,\ y_n^d\big)R.$$

It follows that η as above if fixed by g precisely if the image of r in the Artinian ring

$$A := R/(y_1^d, \dots, y_n^d)R$$

is fixed by g under the induced action. More generally, A is isomorphic as a G-module to the submodule of $H^n_{\mathfrak{m}}(R)$ consisting of elements of the form

$$\left[\frac{r}{y_1^d \cdots y_n^d}\right], \qquad \text{for } r \in R.$$

Yet another point of view may be obtained from the ideas surrounding Remark 4.3; we leave this to the interested reader.

Recall that the transfer map $\mathrm{Tr}\colon R \longrightarrow R^G$ is a homomorphism of R^G-modules, and hence induces a map

$$H^n_{\mathfrak{n}}(R) \xrightarrow{\mathrm{Tr}} H^n_{\mathfrak{n}}(R^G), \tag{2.2.1}$$

where $\mathfrak{n}$ is the homogeneous maximal ideal of R^G. Since $\mathfrak{n}R$ has radical $\mathfrak{m}$, one may identify the modules $H^n_{\mathfrak{n}}(R)$ and $H^n_{\mathfrak{m}}(R)$. The transfer map (2.2.1) is then precisely the map $H^n_{\mathfrak{m}}(R) \to H^n_{\mathfrak{n}}(R^G)$ with

$$\left[\frac{r}{y_1^d \cdots y_n^d}\right] \mapsto \left[\frac{\mathrm{Tr}(r)}{y_1^d \cdots y_n^d}\right],$$

where $r \in R$, and $y_1, \ldots, y_n$ is a homogeneous system of parameters for R^G, as above.

Maps on Local Cohomology

For a local ring $(S, \mathfrak{n})$, and M a finitely generated S-module, the local cohomology modules $H^k_{\mathfrak{n}}(M)$ vanish for $k > \dim M$. It follows that the functor $H^{\dim S}_{\mathfrak{n}}(-)$ is right-exact. More generally:

Lemma 2.3 *Let $(S, \mathfrak{n})$ be a local ring and set $n := \dim S$. Let*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be a complex of finitely generated S-modules.

(1) *If $B_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}}$ is surjective for each prime ideal $\mathfrak{p}$ with $\dim S/\mathfrak{p} = n$, then the induced map $H^n_{\mathfrak{n}}(B) \longrightarrow H^n_{\mathfrak{n}}(C)$ is surjective.*
(2) *If $B_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}}$ is injective for each prime ideal $\mathfrak{p}$ with $\dim S/\mathfrak{p} = n$, and surjective for each $\mathfrak{p}$ with $\dim S/\mathfrak{p} = n - 1$, then $H^n_{\mathfrak{n}}(B) \longrightarrow H^n_{\mathfrak{n}}(C)$ is an isomorphism.*
(3) *If $B_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}}$ is surjective for each $\mathfrak{p}$ with $\dim S/\mathfrak{p} = n-1$, and $A_{\mathfrak{p}} \to B_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}}$ is exact for each $\mathfrak{p}$ with $\dim S/\mathfrak{p} = n$, then the induced sequence*

$$H^n_{\mathfrak{n}}(A) \longrightarrow H^n_{\mathfrak{n}}(B) \longrightarrow H^n_{\mathfrak{n}}(C) \longrightarrow 0$$

is exact.

Proof The exact sequence $B \longrightarrow C \longrightarrow \operatorname{coker}\beta \longrightarrow 0$ induces

$$H^n_{\mathfrak{n}}(B) \longrightarrow H^n_{\mathfrak{n}}(C) \longrightarrow H^n_{\mathfrak{n}}(\operatorname{coker}\beta) \longrightarrow 0.$$

Since $(\operatorname{coker}\beta)_{\mathfrak{p}}$ vanishes for each prime $\mathfrak{p}$ with $\dim S/\mathfrak{p} = n$, one has $\dim(\operatorname{coker}\beta) < n$. But then $H^n_{\mathfrak{n}}(\operatorname{coker}\beta) = 0$, proving (1).

For (2), consider the exact sequences

$$0 \longrightarrow \ker\beta \longrightarrow B \longrightarrow \operatorname{im}\beta \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im}\beta \longrightarrow C \longrightarrow \operatorname{coker}\beta \longrightarrow 0.$$

The hypothesis $(\ker\beta)_{\mathfrak{p}} = 0$ for each $\mathfrak{p}$ with $\dim S/\mathfrak{p} = n$ implies that $\dim(\ker\beta) < n$, so $H^n_{\mathfrak{n}}(\ker\beta) = 0$. Using the first sequence, $H^n_{\mathfrak{n}}(B) \longrightarrow H^n_{\mathfrak{n}}(\operatorname{im}\beta)$ is an isomorphism.

Similarly, since $(\operatorname{coker}\beta)_{\mathfrak{p}} = 0$ for each prime $\mathfrak{p}$ with $\dim S/\mathfrak{p} = n-1$, it follows that $\dim(\operatorname{coker}\beta) < n-1$, so $H^{n-1}_{\mathfrak{n}}(\operatorname{coker}\beta) = 0 = H^n_{\mathfrak{n}}(\operatorname{coker}\beta)$. Passing to local cohomology, the second displayed sequence yields the isomorphism $H^n_{\mathfrak{n}}(\operatorname{im}\beta) \longrightarrow H^n_{\mathfrak{n}}(C)$.

For (3), we may replace A by its image in B, and then apply (2) to $B/A \longrightarrow C$ to obtain the isomorphism $H^n_{\mathfrak{n}}(B/A) \longrightarrow H^n_{\mathfrak{n}}(C)$. Combine this with the exact sequence

$$H^n_{\mathfrak{n}}(A) \longrightarrow H^n_{\mathfrak{n}}(B) \longrightarrow H^n_{\mathfrak{n}}(B/A) \longrightarrow 0.$$

□

3 Comparing Local Cohomology

Theorem 3.1 *For K a field, let G be a finite subgroup of $GL_n(K)$, without transvections, acting on the polynomial ring $R := K[x_1, \dots, x_n]$. Then, there is an exact sequence*

$$\bigoplus_{g \in G} H^n_{\mathfrak{m}}(R) \xrightarrow{\alpha} H^n_{\mathfrak{m}}(R) \xrightarrow{\operatorname{Tr}} H^n_{\mathfrak{n}}(R^G) \longrightarrow 0,$$

where $\mathfrak{m}$ and $\mathfrak{n}$ denote the respective homogeneous maximal ideals of R and R^G, and

$$\alpha \colon (\eta_g)_{g \in G} \mapsto \sum_{g \in G} \big(\eta_g - g(\eta_g)\big).$$

Proof Note that the ideal $\mathfrak{n}R$ is $\mathfrak{m}$-primary, so $H^n_{\mathfrak{m}}(R) = H^n_{\mathfrak{n}}(R)$. In view of Lemma 2.3 (3), it suffices to consider the complex of R^G-modules

$$\bigoplus_{g \in G} R \xrightarrow{\alpha} R \xrightarrow{\operatorname{Tr}} R^G \longrightarrow 0, \tag{3.1.1}$$

where

$$\alpha\colon (r_g)_{g\in G} \mapsto \sum_{g\in G} \big(r_g - g(r_g)\big),$$

and verify that $\mathrm{Tr}\colon R \longrightarrow R^G$ is surjective after localizing at each height one prime $\mathfrak{p}$ of R^G, and that the sequence (3.1.1) is exact upon tensoring with the fraction field of R^G. The surjectivity of $\mathrm{Tr}\colon R \longrightarrow R^G$ at height one primes comes from Lemma 2.2. For the second verification, let L denote the fraction field of R, in which case $L^G = \mathrm{frac}(R^G)$ as G is finite. We then need to verify the exactness of the sequence

$$\bigoplus_{g\in G} L \xrightarrow{\alpha} L \xrightarrow{\mathrm{Tr}} L^G \longrightarrow 0. \tag{3.1.2}$$

But $\mathrm{Tr}\colon L \longrightarrow L^G$ is a surjective map of L^G-vector spaces, so its kernel is an L^G-vector space of rank $|G|-1$. By the normal basis theorem, there exists $\lambda \in L$ such that

$$\{g(\lambda) | g \in G\}$$

is an L^G-basis for L. But then the image of α in (3.1.2) contains the $|G|-1$ linearly independent elements $\lambda - g(\lambda)$, as g varies over the nonidentity elements of G. □

Remark 3.2 Theorem 3.1 admits a dual formulation that extends [19, Theorem 2.7] as follows. Suppose G contains no transvections. Using $(-)^*$ for the graded K-dual, one has $H^n_{\mathfrak{m}}(R)^* = \omega_R$ and $H^n_{\mathfrak{n}}(R^G)^* = \omega_{R^G}$, so Theorem 3.1 yields the exact sequence

$$0 \longrightarrow \omega_{R^G} \longrightarrow \omega_R \xrightarrow{\alpha^*} \bigoplus_{g\in G} \omega_R.$$

Endowing ω_R with the G-action induced by the identification $\omega_R = H^n_{\mathfrak{m}}(R)^*$, one has $\ker \alpha^* \cong (\omega_R)^G$. The exact sequence above then gives

$$\omega_{R^G} \;\cong\; (\omega_R)^G.$$

This does not require the hypothesis that R^G is Cohen-Macaulay, assumed in [19].

Remark 3.3 In the statement of Theorem 3.1, one may replace $\bigoplus_{g\in G} H^n_{\mathfrak{m}}(R)$ by the direct sum over a generating set for G, and α by its restriction: if $g, h \in G$, then

$$(1-hg)(\eta) \;=\; (1-g)(\eta) + (1-h)(g(\eta)).$$

The hypothesis that G does not contain transvections is indeed required in Theorem 3.1:

Example 3.4 Consider the symmetric group $S_2 = \langle g\rangle$ acting on $R := K[x,y]$ by permuting the variables. Then $R^{S_2} = K[e_1, e_2]$, where $e_1 := x+y$ and $e_2 := xy$.

While g is a pseudoreflection independent of the characteristic of K, it is a transvection if and only if K has characteristic two. We examine the complex

$$H^2_{\mathfrak{m}}(R) \xrightarrow{1-g} H^2_{\mathfrak{m}}(R) \xrightarrow{\text{Tr}} H^2_{\mathfrak{n}}(R^{S_2}) \longrightarrow 0 \tag{2.2.3}$$

in degree -2. Note that $[H^2_{\mathfrak{n}}(R^{S_2})]_{-2} = 0$, while $[H^2_{\mathfrak{m}}(R)]_{-2}$, computed via the Čech complex on e_1, e_2, is the rank one K-vector space spanned by

$$\eta := \left[\frac{x}{e_1 e_2}\right].$$

Since

$$(1-g)(\eta) = \left[\frac{x-y}{e_1 e_2}\right] = \left[\frac{2x}{e_1 e_2}\right] = 2\eta,$$

the degree -2 strand of (2.2.3) takes the form

$$K \xrightarrow{2} K \longrightarrow 0 \longrightarrow 0,$$

which is exact precisely when the characteristic of K is other than two, i.e., precisely when the group contains no transvections.

4 When is the *a*-invariant Invariant?

We record in this section when the a-invariant of a ring of invariants coincides with that of the ambient polynomial ring. The following proposition is likely well known to experts, for example, it is an extension of [14, Lemma 2.17] (see also [17, Theorem 1.1]).

Proposition 4.1 *Let G be a finite subgroup of $GL_n(K)$, acting on a polynomial ring R. Then, for each subgroup H of G, one has $a(R^G) \leq a(R^H)$.*

Proof Consider the transfer map $\text{Tr}^G_H \colon R^H \longrightarrow R^G$ given by

$$\text{Tr}^G_H(r) := \sum_{gH \in G/H} g(r). \tag{4.1.1}$$

Let L denote the fraction field of R. Since G and H are finite, one has $L^G = \text{frac}(R^G)$ and $L^H = \text{frac}(R^H)$. Distinct cosets gH induce distinct automorphisms $g \colon L^H \to L^H$, so Dedekind's theorem implies that the corresponding characters $(L^H)^\times \to (L^H)^\times$ are linearly independent over L^H, and hence over L^G. It follows that their sum

$$\sum g \colon (L^H)^\times \longrightarrow L^H,$$

taken over coset representatives, is a nonzero map, and hence that the transfer map (4.1.1) is nonzero. As the transfer is R^G-linear, one has an exact sequence of R^G-modules

$$R^H \xrightarrow{\mathrm{Tr}_H^G} R^G \longrightarrow R^G/\mathrm{im}(\mathrm{Tr}_H^G) \longrightarrow 0.$$

Applying the functor $H^n_{\mathfrak{n}}(-)$, one obtains the surjection

$$H^n_{\mathfrak{n}}(R^H) \xrightarrow{\mathrm{Tr}_H^G} H^n_{\mathfrak{n}}(R^G),$$

since $R^G/\mathrm{im}(\mathrm{Tr}_H^G)$ has smaller dimension. The homogeneous maximal ideals of R^H and R^G agree up to radical, so the assertion follows. □

The following is [14, Theorem 2.18], and also related to work of Broer [5]:

Corollary 4.2 *Let K be a field of characteristic $p > 0$, and G a finite subgroup of $GL_n(K)$ acting on a polynomial ring $R := K[x_1, \dots, x_n]$. If $a(R^G) = a(R)$, and p divides the order of G, then the inclusion $R^G \subseteq R$ is not R^G-split.*

Proof Consider the maps of rank one K-vector spaces

$$[H^n_{\mathfrak{n}}(R^G)]_{-n} \xrightarrow{i} [H^n_{\mathfrak{m}}(R)]_{-n} \xrightarrow{\mathrm{Tr}} [H^n_{\mathfrak{n}}(R^G)]_{-n},$$

where i is induced by the inclusion $R^G \subseteq R$. The composition is then multiplication by $|G|$, which equals zero in K. As Tr above is surjective, the map i must be zero. But then the inclusion $R^G \subseteq R$ is not R^G-split. □

Remark 4.3 Let G be a finite subgroup of $\mathrm{GL}_n(K)$, acting on $R := K[x_1, \dots, x_n]$. We claim that for each $g \in G$ and $\eta \in [H^n_{\mathfrak{m}}(R)]_{-n}$, one has

$$g \cdot \eta \;=\; (\det g)^{-1}\eta.$$

Since $[H^n_{\mathfrak{m}}(R)]_{-n}$ has rank one, without loss of generality, take η to be

$$\left[\frac{1}{x_1 \cdots x_n}\right].$$

If $f_1, \dots, f_n$ is a homogeneous system of parameters for R, the natural isomorphism between Čech and local cohomology induces a natural isomorphism between the Čech cohomology modules $\check{H}^n(x_1, \dots, x_n;\ R)$ and $\check{H}^n(f_1, \dots, f_n;\ R)$. To make this explicit, following [16, Theorem 4.18], let A be a matrix over R, such that

$$\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Then, under the isomorphism $\check{H}^n(x_1, \dots, x_n;\ R) \to \check{H}^n(f_1, \dots, f_n;\ R)$, one has

$$\left[\frac{1}{x_1 \cdots x_n}\right] \mapsto \left[\frac{\det A}{f_1 \cdots f_n}\right].$$

It follows that

$$g \cdot \left[\frac{1}{x_1 \cdots x_n}\right] = \left[\frac{1}{g(x_1) \cdots g(x_n)}\right],$$

viewed as an element of $\check{H}^n(g(x_1), \dots, g(x_n);\ R)$, corresponds to

$$\left[\frac{(\det g)^{-1}}{x_1 \cdots x_n}\right] = (\det g)^{-1}\eta$$

in $\check{H}^n(x_1, \dots, x_n;\ R)$.

The following theorem has been obtained independently by Hashimoto [12]:

Theorem 4.4 *For K a field, let G be a finite subgroup of $GL_n(K)$ acting on the polynomial ring $R := K[x_1, \dots, x_n]$. Then, $a(R^G) = a(R)$ if and only if G is a subgroup of $SL_n(K)$ that contains no pseudoreflections.*

Proof We first show that if G contains a pseudoreflection, then $a(R^G) < a(R)$. In view of Proposition 4.1, it suffices to consider the case where G is a cyclic group, generated by a pseudoreflection g. After extending scalars, we may assume that g takes the form

$$\left[\begin{array}{c|cccc} \zeta & 0 & 0 & \cdots & 0 \\ \hline 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{cc|ccc} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & 1 \end{array}\right],$$

where ζ is a primitive k-th root of unity. In the first case, $R^G = K[x_1^k,\ x_2, \dots, x_n]$, and in the second $R^G = K[x_1^p - x_1x_2^{p-1},\ x_2, \dots, x_n]$, where $p > 0$ is the characteristic of K. In each case, R^G is a polynomial ring, with $a(R^G)$ strictly less than $a(R)$.

It remains to verify that if G has no pseudoreflections, then $a(R^G) = a(R)$ if and only if G is a subgroup of $\mathrm{SL}_n(K)$. The exact sequence from Theorem 3.1, when restricted to the degree $-n$ strand, gives an exact sequence of K-vector spaces

$$\bigoplus_{g \in G} [H^n_{\mathfrak{m}}(R)]_{-n} \xrightarrow{\alpha} [H^n_{\mathfrak{m}}(R)]_{-n} \xrightarrow{\mathrm{Tr}} [H^n_{\mathfrak{n}}(R^G)]_{-n} \longrightarrow 0.$$

Since $[H^n_{\mathfrak{m}}(R)]_{-n}$ is a rank one vector space, it follows that $a(R^G) = -n$ if and only if the map α above is identically zero, i.e., if and only if the map

$$[H^n_{\mathfrak{m}}(R)]_{-n} \xrightarrow{1-g} [H^n_{\mathfrak{m}}(R)]_{-n}$$

is zero for each $g \in G$. Taking

$$\eta := \left[\frac{1}{x_1 \cdots x_n}\right]$$

as in Remark 4.3, this is equivalent to the condition that

$$\eta - g(\eta) \;=\; \eta - (\det g)^{-1}\eta$$

is zero for each g, i.e., that $\det g = 1$ for each $g \in G$. □

5 Hilbert Series of Local Cohomology

Theorem 3.1 has an amusing consequence for the Hilbert series of local cohomology:

Corollary 5.1 *For K a field, let G be a finite cyclic subgroup of $GL_n(K)$, without transvections, acting on the polynomial ring $R := K[x_1, \ldots, x_n]$. Then, the Hilbert series of $H^n_{\mathfrak{n}}(R^G)$ and $H^n_{\mathfrak{m}}(R)^G$ coincide, i.e., for each integer k, one has*

$$\operatorname{rank}_K [H^n_{\mathfrak{n}}(R^G)]_k \;=\; \operatorname{rank}_K [H^n_{\mathfrak{m}}(R)^G]_k.$$

Proof Let $G = \langle g \rangle$. Then, by Theorem 3.1 and Remark 3.3, one has an exact sequence

$$H^n_{\mathfrak{m}}(R) \xrightarrow{1-g} H^n_{\mathfrak{m}}(R) \xrightarrow{\operatorname{Tr}} H^n_{\mathfrak{n}}(R^G) \longrightarrow 0.$$

But the kernel of the first map is precisely $H^n_{\mathfrak{m}}(R)^G$, so

$$0 \longrightarrow H^n_{\mathfrak{m}}(R)^G \longrightarrow H^n_{\mathfrak{m}}(R) \xrightarrow{1-g} H^n_{\mathfrak{m}}(R) \xrightarrow{\operatorname{Tr}} H^n_{\mathfrak{n}}(R^G) \longrightarrow 0$$

is exact. Taking the degree k strand, the alternating sum of the ranks is zero. □

We will see in Example 5.3 that the equality of Hilbert series need not hold when G is not cyclic; however, before that, it is worth emphasizing that both $H^n_{\mathfrak{n}}(R^G)$ and $H^n_{\mathfrak{m}}(R)^G$ are graded R^G-modules, and Corollary 5.1 says precisely that they are isomorphic as graded K-vector spaces. They need not be isomorphic as R^G-modules:

Example 5.2 Consider the alternating group A_3 acting on $R := \mathbb{F}_3[x, y, z]$ by permuting the variables. The ring of invariants R^{A_3} is then generated by the elements

$$e_1 := x + y + z, \quad e_2 := xy + yz + zx, \quad e_3 := xyz, \quad \Delta := x^2y + y^2z + z^2x.$$

It follows that R^{A_3} is a hypersurface; the defining equation is readily seen to be

$$\Delta^2 - e_1e_2\Delta + e_2^3 + e_1^3e_3.$$

Taking a Čech complex on e_1, e_2, e_3, the socle of the R^{A_3}-module $H^3_{\mathfrak{n}}(R^{A_3})$ is the rank one vector space spanned by the cohomology class

$$\eta := \left[\frac{\Delta}{e_1 e_2 e_3}\right].$$

Note that η belongs to the kernel of the natural map $H^3_{\mathfrak{n}}(R^{A_3}) \to H^3_{\mathfrak{n}}(R)$ since $R^{A_3} \to R$ is not R^{A_3}-split; alternatively, it is a routine verification that

$$\Delta \in (e_1, e_2, e_3)R.$$

We claim that, in contrast with $H^3_{\mathfrak{n}}(R^{A_3})$, the socle of $H^3_{\mathfrak{n}}(R)^{A_3}$, as an R^{A_3}-module, has larger rank: for this, one may verify that the elements

$$\left[\frac{x\Delta}{e_1^2 e_2 e_3}\right], \quad \left[\frac{\Delta}{e_1^2 e_2 e_3}\right], \quad \left[\frac{\Delta}{e_1 e_2^2 e_3}\right], \quad \left[\frac{1}{e_1 e_2 e_3}\right],$$

are all nonzero in $H^3_{\mathfrak{n}}(R)$, that they are A_3-invariant, and that they are annihilated by the ideal $(e_1, e_2, e_3, \Delta)R^{A_3}$. Note that they have degrees -3, -4, -5, and -6, respectively.

The equality of Hilbert series, Corollary 5.1, fails for an action of the Klein-4 group:

Example 5.3 The following matrices over $\mathbb{F}_2$ generate the Klein-4 group:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Each of these is a transvection; the invariant ring for this action of the Klein-4 group on $\mathbb{F}_2[x, y, z]$ is the polynomial ring

$$\mathbb{F}_2[z,\ x^2 + xz,\ y^2 + yz].$$

The situation is more interesting if we take the 2-fold diagonal embedding, i.e., if we consider the representation of the Klein-4 group, over $\mathbb{F}_2$, determined by the matrices:

$$g := \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \quad \text{and} \quad h := \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Under the action of this group G on the polynomial ring $R := \mathbb{F}_2[u, v, w, x, y, z]$, the following elements are readily seen to be invariant:

$$w, \quad z, \quad u^2 + uw, \quad v^2 + vw, \quad x^2 + xz, \quad y^2 + yz, \quad uz + wx, \quad vz + wy.$$

Indeed, the invariant ring R^G is generated by these elements and is a complete intersection ring with defining relations

$$(uz + wx)^2 + (uz + wx)wz + (u^2 + uw)z^2 + (x^2 + xz)w^2$$

and

$$(vz + wy)^2 + (vz + wy)wz + (v^2 + vw)z^2 + (y^2 + yz)w^2.$$

It follows that R^G has Hilbert series

$$\frac{(1-t^4)^2}{(1-t)^2(1-t^2)^6} = \frac{(1+t^2)^2}{(1-t)^2(1-t^2)^4} = 1 + 2t + 9t^2 + \cdots .$$

Set $\mathfrak{n}$ to be the ideal of R^G generated by the homogeneous system of parameters

$$w^2, \quad z^2, \quad u^2 + uw, \quad v^2 + vw, \quad x^2 + xz, \quad y^2 + yz.$$

Since R^G is Gorenstein with $a(R^G) = -6$, the Hilbert series above yields

$$\operatorname{rank}[H^6_{\mathfrak{n}}(R^G)]_{-6} = 1 \qquad \text{and} \qquad \operatorname{rank}[H^6_{\mathfrak{n}}(R^G)]_{-7} = 2.$$

We claim that, on the other hand,

$$\operatorname{rank}[H^6_{\mathfrak{n}}(R)^G]_{-7} = 4.$$

Consider the Artinian ring $A := R/\mathfrak{n}R$; we identify $[H^6_{\mathfrak{n}}(R)]_{-6}$ with $[A]_6$, and $[H^6_{\mathfrak{n}}(R)]_{-7}$ with $[A]_5$ as G-modules.

The rank one space $[A]_6$ has basis $uvwxyz$, which is fixed by g and h, (as it must!) since

$$g\colon uvwxyz \mapsto u(v + w)wx(y + z)z \equiv uvwxyz$$

in A, and

$$h\colon uvwxyz \mapsto (u + w)vw(x + z)yz \equiv uvwxyz.$$

For $[A]_5$, we work with the basis $vwxyz$, $uwxyz$, $uvxyz$, $uvwyz$, $uvwxz$, and $uvwxy$. The first of these elements is fixed since

$$g\colon vwxyz \mapsto (v + w)wx(y + z)z \equiv vwxyz$$

and

$$h\colon vwxyz \mapsto vw(x + z)yz \equiv vwxyz.$$

Similar calculations show that $uwxyz$, $uvwyz$, $uvwxz$ are fixed by g and h. On the other hand,

$$g\colon uvxyz \;\mapsto\; u(v+w)x(y+z)z \;\equiv\; (uv+uw)xyz$$

and

$$g\colon uvwxy \;\mapsto\; u(v+w)wx(y+z) \;\equiv\; uvw(xy+xz),$$

so g fixes no nonzero $\mathbb{F}_2$-linear combination of $uvxyz$ and $uvwxy$. It follows that the subspace of $[A]_5$ fixed by G has basis $vwxyz$, $uwxyz$, $uvwyz$, and $uvwxz$.

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Declarations

Conflict of Interest The authors declare no competing interests.

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