MATH 5210, HW II
DUE MARCH 04

1) A metric space X is separable if it contains a dense countable set S. Prove that any open set V in X is a union of balls centered at points in S and with rational radii. (Since the set of such balls is countable, it follows that any open set is a countable union of balls).

2) Let $X = [0,1]^2$. Choose the distance on X wisely, and use the previous exercise to prove that any open set in X is Lebesgue measurable.

3) Let $P = [0,1]^2$. If E and F are two elementary sets such that $E \cup F = P$ then $m(E \cap F) = m(E) + m(F) - 1$. Now assume $E = \cup_{i=1}^{\infty} E_i$ and $F = \cup_{i=1}^{\infty} F_i$, disjoint unions of elementary sets each, and $E \cup F = P$. Observe that $E \cap F$ is the disjoint union of $E_i \cap F_j$. Prove that

$$\sum_{i,j} m(E_i \cap F_j) = \sum_i m(E_i) + \sum_j m(F_j) - 1.$$

4) Let $\sum_{n=1}^{\infty} x_n$ be a series of non-negative real numbers. Show that its sum (which can be ∞) is equal to the supremum of the set of sums $\sum_{n \in S} x_n$ where S runs over all finite subsets of the set of natural numbers. Conclude that any sequence of non-negative numbers can be added in any order.

5) In the following exercises, $\mathcal{M}$ is a σ-algebra of a non-empty set X, that is, a family of subsets of X closed under complements and countable unions, and μ is a σ-measure. Let $A_1 \supseteq A_2 \supseteq \ldots$ be a sequence of sets in $\mathcal{M}$. Let $A = \cap_{i=1}^{\infty} A_i$. Prove that $\lim_{i \mapsto \infty} \mu(A_i) = \mu(A)$, assuming that $\mu(X) = 1$.

6) A subset of X is called measurable if it belongs to $\mathcal{M}$. Let $f : X \to \mathbb{R}$ prove that

$$\{x | f(x) < c\}$$

is measurable for every $c \in \mathbb{R}$ if and only if

$$\{x | f(x) \leq c\}$$

is measurable for every $c \in \mathbb{R}$.

7) Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions on X. Prove that

$$g(x) = \inf\{f_1(x), f_2(x), \ldots\} \text{ and } G(x) = \sup\{f_1(x), f_2(x), \ldots\}$$

are measurable functions.

8) Let f be an integrable function on X, such that $f(x) \geq 0$ for all $x \in X$. Prove that $\int_X f = 0$ if and only if the measure of $A = \{x \in X \mid f(x) > 0\}$ is 0, that is, $f = 0$ almost everywhere. Hint consider the sets $A_n = \{x \in X \mid f(x) > 1/n\}$ for $n = 1, 2, \ldots$.

9) Let $X = (0,1]$, with the usual measure, and let $f(x) = 1/\sqrt{x}$. Use the monotone convergence theorem to prove that f is integrable and compute its integral.