MATH 3210-4, HW IV
DUE WEDNESDAY OCT 30

1) Let $f : X \to Y$ be a continuous map of metric spaces. Let $E \subset X$. Prove that $f(\overline{E}) \subseteq \overline{f(E)}$. Give an example where the inclusion is proper.

2) Let $F_1 \supseteq F_2 \supseteq \dots$ be a descending sequence of non-empty compact sets in a metric space X. Prove that $\cap_{n=1}^{\infty} F_n$ is not empty. Give an example of a descending sequence of closed sets such that the intersection is empty.

3) Let $f : X \to Y$ be a uniformly continuous map of metric spaces. If x_n is a Cauchy sequence in X, prove that $f(x_n)$ is a Cauchy sequence in Y. Give an example of a continuous function such that this fails.

4) Let $f : [0, 1] \to [0, 1]$ be a continuous function. Prove that there exists $x \in [0, 1]$ such that $f(x) = x$.

5) Let X be a metric space, and fix a non-empty subset $E \subset X$. The distance from $x \in X$ to E is defined by

$$d(x, E) = \inf_{y \in E} d(x, y).$$

Prove that $d(x, E) = 0$ if and only if $x \in \overline{E}$. Prove that

$$|d(x, E) - d(y, E)| \leq d(x, y)$$

i.e. $x \mapsto d(x, E)$ is a uniformly continuous function.