MATH 3210-4, HW I
DUE WEDNESDAY SEP 04

In order to define multiplication of real numbers, it suffices to do so for positive numbers, and here it is convenient to use cuts of positive rational numbers. Thus a positive real number is defined a non-empty, bounded set $\alpha \subseteq \mathbb{Q}^+$ such that

- If $r \in \alpha$ and $s < r$ is a positive rational number, then $s \in \alpha$.
- α has no maximal element.

An example is "the cut of square root of 3": $\sqrt{3} = \{r \in \mathbb{Q}^+ | r^2 < 3\}$. The first bullet is clearly satisfied, for the second, note that the limit of $(r + \frac{1}{n})$ is r^2, as the positive integer n tends to infinity, hence for n large enough $(r + \frac{1}{n})^2 < 3$ if $r^2 < 3$.

1) For $i = 0, \ldots, 10$, construct the greatest rational number in the cut of square root of 3 in the form of a (non-reduced) fraction $a_i = x_i/2^i$, and the least rational number not in the cut in the same form $b_i = y_i/2^i$. For example, $a_0 = 1$ and $b_0 = 2$. Their average is $3/2$. Since $(3/2)^2 < 3$, it follows that $a_1 = 3/2$ and $b_1 = 4/2$ etc... Note: if you had a calculator that "computes" $\sqrt{3}$ and expresses the answer in binary digits, a_i would be what you get after chopping off all digits after the i-th place right of the point.

2) Let α and β be two cuts of $\mathbb{Q}^+$. Let $\alpha \cdot \beta = \{rs | r \in \alpha, s \in \beta\}$. Prove that $\alpha \cdot \beta$ is a cut of $\mathbb{Q}^+$.

3) Prove that $\sqrt{3} \cdot \sqrt{3} = 3^*$, where $\sqrt{3}$ is the cut defined above.

4) Let $1^* = \{r \in \mathbb{Q}^+ | r < 1\}$. Prove that $\alpha \cdot 1^* = \alpha$ for any cut α of $\mathbb{Q}^+$.

5) Let α be a cut of $\mathbb{Q}^+$. Construct a cut β such that $\alpha \cdot \beta = 1^*$.

6) The set of 2×2 matrices with real coefficients is a non-commutative ring with respect to the usual addition and multiplication of matrices. We can use this information to quickly construct complex numbers and prove that it is a field. Let $\mathbb{C}$ be the set of 2×2 matrices

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where a and b are any real numbers. If $A, B \in \mathbb{C}$, prove that $A + B$, $A - B$ and AB are in $\mathbb{C}$ and that $AB = BA$. This implies that $\mathbb{C}$ is a ring (why?). Finally, for every non-zero $A \in \mathbb{C}$ find $B \in \mathbb{C}$ such that $AB = 1$ i.e. $\mathbb{C}$ is a field. What fails if we instead consider $\mathbb{C}'$, the set of all 2×2 matrices

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where a and b are any real numbers?