MATH 5210, LECTURE 10 - FOURIER EXPANSION CONTINUED APRIL 08

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic, $f(t+1) = f(t)$ for all $t \in \mathbb{R}$, and continuous function. Such f is determined by restriction to the interval

$$X = \left(-\frac{1}{2}, \frac{1}{2}\right]$$

and f can be considered an element of $L^2(X)$, since continuous and bounded functions on $(-1/2, 1/2]$ are clearly square integrable. We have an identity in $L^2(X)$

$$f(t) = a_0 + \sum_{n \neq 0} a_n \cos(2\pi n t) + \sum_{m \neq 0} b_n \sin(2\pi m t)$$

where

$$a_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) dt,$$

$$a_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \cos(2\pi n t)\ dt \text{ if } n \neq 0,$$

$$b_m = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \sin(2\pi m t)\ dt \text{ if } m \neq 0.$$

Since elements of $L^2(X)$ are not functions, this is not an identity between functions. On the other hand, f, a periodic continuous function, has a well defined sup norm (why?)

$$\sup_{t \in \mathbb{R}} |f(t)|$$

Theorem 0.1. *Assume f is a periodic, twice differentiable function such that f'' is continuous. Then the Fourier series converges to $f(t)$ uniformly for all $t \in \mathbb{R}$.*

Proof. We start with the following

Exercise. Assume f and g are two differentiable periodic functions with continuous derivatives. Then we have the following simplified variant of integration by parts formula

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} fg' = -\int_{-\frac{1}{2}}^{\frac{1}{2}} f'g.$$

Solution. By the Newton-Leibniz formula and the fundamental theorem of calculus,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} fg' + f'g = \int_{-\frac{1}{2}}^{\frac{1}{2}} (fg)' = f(1/2)g(1/2) - f(-1/2)g(-1/2) = 0$$

where vanishing follows from periodicity of f and g.

Applying integration by parts twice,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} fg'' = \int_{-\frac{1}{2}}^{\frac{1}{2}} f''g.$$

Put $g(t) = \cos(2\pi nt)$ into the above formula. Observe that $g''(t) = -(2\pi n)^2 \cos(2\pi nt)$. After dividing by $-(2\pi n)^2$ we get the following identity between the Fourier coefficients a_n of f and those of f'':

$$a_n = -\frac{1}{(2\pi n)^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} f''(t) \cos(2\pi nt)\ dt.$$

Since f'' is periodic and continuous

$$M = \sup_{t \in \mathbb{R}} |f''(t)|$$

is finite. Hence $|f''(t) \cos(2\pi nt)| \leq M$ and

$$|a_n| \leq \frac{M}{(2\pi n)^2}.$$

We have the same identity for the coefficients b_n. Since the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent, it follows at once that the Fourier series is a uniformly convergent series of continuous functions. Thus it converges, uniformly, to a continuous periodic function g. From a homework exercise, the series converges to g in $L^2(X)$. Hence $f = g$ as elements in $L^2(X)$, but we still need to show that $f = g$, as functions, i.e $f(t) = g(t)$ for all $t \in \mathbb{R}$. Since $||f - g|| = 0$,

$$0 = ||f - g||^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} (f - g)^2\ dt.$$

In one of the homework problems we have shown that a non-negative continuous function whose integral is 0 must be 0. Hence $f = g$ as functions.

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