CALCULUS
Table of Contents

Calculus I, First Semester

Calculus II, Second Semester

Calculus III, Third Semester

Preface

As I neared the end of a second decade of teaching Calculus at the University of Utah, I became aware of a student type that persisted in all my classes, comprising between 10 and 20 percent of the class. These students came to class irregularly, often just to ask a question about one of the more demanding problems. They took all the exams, and, almost uniformly ended up with grades in the top quartile. Looking at their transcripts, and talking with several of them, I discovered that they were among the better prepared of the Calculus students, having taken Calculus in high school, either in an AP class, or from another country. Now, the University of Utah has an AP Calculus course for such students, but as that course emphasizes theory, these students preferred to be in the regular engineering sequence. As these students were well grounded in Algebra, and had already seen some of the basics of Calculus, I concluded that what was appropriate for them was a course in Calculus that emphasized a deep intuitive understanding of Calculus and problems sets that depended on, and extended that understanding.

So, with the collusion of my chair, I started an intensive summer calculus course: all three semesters in ten weeks. After all, if this population was ten percent of the total Calculus cohort, and if half of those took the summer course, that would provide a class of about 50 students enough to make the experiment economically feasible. We met in this class for five hours a day, five days a week. For about half the time the instructor developed the basic ideas, illustrating them through problems, and for the rest of the time, the students do their homework in the presence of graduate assistants. This worked, and that course continues still with about the same numbers of students.

I taught that course for three years, and then turned to developing an online course, based on the same approach. The first few semesters were difficult: there was not a good fit between the text and the online problem sets I had created. I spent many hours responding to email inquiries, started to write and post supplementary notes, and decided that it was a good idea to keep past years' problems with solutions posted. As new email inquiries and responses were integrated into the supplementary notes, it occurred to me that I had almost a complete first draft of a text on Calculus. I spent about a semester organizing the material and completing it to a full text that still exists as a supplementary text for the online course.

That is the genesis of this text. As a result, it is thin on drill exercises, informal and intuitive on theoretical issues, approaches the ideas of the subject in the context of the problems they solve, and is rich in examples that illustrate those ideas. As such, it has served well that class of students with sufficient technical competence to follow the ideas in an argument.

Conceptual Apporach of the Text

There is no chapter 0: survey of algebra, trigonometry and pre-calculus. It is assumed that students have sufficient grasp of the concept of function to be able to get right into that which the Calculus is about. Ideas and techniques from the pre-calculus are reviewed in context as the need for them arises.

The approach in Chapter 1 is intuitive and informal: the fundamental issue is to see how understanding of rules of change leads to qualitative comprehension of processes and predictions of future behavior. The first two sections start with the Newton and Liebniz approaches to Differential Calculus. The Newtonian approach is presented as one focusing on rates of change of functions of a given independent variable (usually time), while that of Liebniz deals with variables and how they change with respect to each other. In this way, the two threads of derivatives (Newton) and differentials (Liebniz) are introduced and used throughout the text to develop the various ways of looking at a particular process.

I've always been of two minds concerning theoretical issues. I feel that they are too deep for the first calculus course, but find it difficult to be vague and intuitive about them. So, the approach I've adopted in this text is that of Newton: position and velocity are measurable attributes of moving bodies, and the limit idea of calculus is the tool for solving problems about movement. Similarly, area is a measurable attribute of planar figures, and the idea of accumulation of the Calculus is the tool for calculating area from the algebraic expressions delimiting the figure. The problems of existence of limits and area are thus avoided.

On the other hand, there are real issues in relating velocity with change in position, and in defining area, and I can't allow myself to sneak through the calculus without pointing it out. These discussions are collected in sections entitled something like "theoretical considerations."

As a result this text has no preliminary section on limits. I feel that this would be misplaced: students get the erroneous impression that a limit is calculated by substitution if the function is given by a formula, and otherwise one should look at the graph. So, at the beginning, the calculus of polynomials is developed, and the calculation of the derivative is done algebraically. Derivation of the trigonometric functions is presented using Pascal's idea. Here the issue of existence of limits is joined: we must know that $\lim \sin x/x = 1$ as x approaches zero. This is discussed in chapter 2, in which a wide variety of questions of limits is discussed, including l'Hôpital's rule, which is usually postponed to second semester.

Similarly, integration is introduced as the problem of finding functions with given derivatives using the idea of differentials. It is essential that students understand that it is a differential, not a function, that is integrated. By showing that the differential of the area underneath a curve $y = f(x)$ is $f(x)dx$, we see that that area is given by integration. The technigue of integration by substitution is a direct consequence of the chain rule. From there, we move directly into solving separable differential equations (of the type $f(x)dx = g(y)dy$) and from there to the introduction of the exponential function as the solution of $dy/y = rdx$. Then, in the subsequent chapter the issue of existence of area is taken up; we turn to the integral as a limit of an accumulation process and the Fundamental Theorem of the Calculus is now seen as the binding together of the Newton and Liebniz approaches.

New techniques and ideas are introduced (where ever possible) in the context of a problem to be solved. Thus the chapter on sequences and series is viewed as developing a technique for approximating values of transcendental functions. Conics are introduced as a tool for understanding quadratic relations, and this leads directly into the study of second order equations.

The Chapter on Vector Calculus is based on intuition provided by fluid flows. The motivation is that it is better to emphasize the understanding of the physical content of the fundamental theorems of vector calculus, rather than providing proofs of these theorems. After all, the correcrt proofs of these theorems reduce them to the one-variable fundamental theorem of the calculus by judicious choice of coordinates. The hard part, best deferred to the course in Advanced Calculus, is to show that the statement of the theorems is independent of the coordinates in which they are calculated.

Outline of the Text

First Semester

In Chapter 1 the basics of the differential calculus are introduced, mostly in the context of polynomial functions and relations. In section four the derivatives of the trigonometric functions are found using the differential triangle of Pascal.

Chapter 2 discusses the concept of limit, techniques for calculating limits, including l'Hôpital's rule. We show that the sine function is differentiable (a fact postponed from chapter 1). Limits at infinity and asymptotes are discussed. The point of the last section is the affirmation that a function with derivative 0 everywhere in an interval is constant. This is always a difficult passage in the first calculus course, for the students see this as the tautology that a function with no change throughout an interval ends up at the same value with which it started. So, the instructor has to first convince the student that there is a real issue here. In class, I'll do that by exhibiting the Cantor function, which is almost always constant, but somehow gets from 0 to 1. Then the instructor must confess that the issue will not be joined, and instead this fundamental fact will be shown to follow from statements (intermediate value, existence of maxima for continuous functions) that, for some reason, are more intuitively obvious. I leave the strategy for doing this to the instructor, and instead just lay down the minimal exposition.

Chapter 3 covers the geometric significance of the first and second derivatives, optimization and graph sketching,

Chapter 4 introduces integration as the process reverse to differentiation, and extends this to separable differential equations. This leads to the introduction of the exponential function as the solution of $dy/y = rdx$. I do not feel that it is necessary to discuss the general concept of inverse functions to get from the exponential to the logarithm; the logarithm is simply what we use to find a if we know e^a.

Chapter 5 turns to Liebniz' concept of integration as a method of accumulation, and shows (through the Fundamental theorem) that the Calculi of Newton and Liebniz are the same subject. This is followed by the application of integration to a variety of accumulation processes.

Covering these first 5 chapters in one semester is a tall order. It may be that the course will end with the fundamental theorem, and that the second semester starts with the applications of integration. Alternatively, some sections of these first five chapters may be omitted.

Second Semester

We are now in a position to abstractly discuss inverse functions and the fact that a function with nonzero derivative in an interval has an inverse, leading to the introduction of many useful transcendental functions. This, together with the solution of general first order differential equations, is covered in Chapter 6.

Chapter 7 covers the techniques of integration necessary to make use of a table of integrals: substitution, integration by parts and partial fractions. It is the intent of this text that from this point on, students will use a table of integrals.

Chapter 8 begins with a brief review of l'Hôpital's rule, as some instructors may prefer to cover this in the second semester. Then Improper Integrals are covered as this material will be necessary for the next chapter.

We begin Chapter 9 (Sequences and Series) with a more formal discussion of the Principle of Mathematical Induction (although it has already been used informally in several places in the text), in connection with the definition of sequences by recursion. As the point of this chapter is to develop ways to approximate values of transcendental functions, the focus is to get to Taylor series quickly. It also works to cover section 10.1 (on the Taylor formula with estimate) before Chapter 9, to better motivate the discussion of series. The important thing is to emphasize that a calculation is not an approximation without some estimate of the error.

Chapter 10 is a brief introduction to the simplest numerical methods; enough to give an idea of how the basic computer algorithms work.

Chapter 11 covers materials on the conics and polar coordinates. Although this material is traditionally pre-calculus, it is necessary to review it before launching into mulit-variate calculus. The string properties of the conics are derived, and their optical properties are derived as differential versions of the string properties.

Chapter 12 covers the usual material on second order differential equations and basic harmonics. Many students will have already covered this in the physics classes, and in many cases this is done at the beginning of the course on differential equations. Thus, this chapter may be omitted, particularly if Calculus is three quarters rather than three semesters.

Third Semester

The introduction to Vector Algebra (chapter 13) is motivated by geometry, the point being to characterize linear subspaces in two and three dimensions in a variety of ways.

One of the central themes of this text is the study of particles in motion; we discuss the calculus of vector-valued functions of a single variable (Chapter 14)in this context. Sections 2 and three cover the differential geometry of curves in two and three dimensions, and section 4 provides a derivation of Kepler's Laws of Planetary Motion from Newton's Laws of Motion.

Chapter 15 (Coordinates and Surfaces) covers the basic geometric tools of the several variable calculus: change of coordinates in two and three variables, surfaces, and in particular, the normal forms of quadrics.

Chapters 16 and 17 cover differentiation and integration in several variables. The differential of a function is introduced as the linear function which best fits the given function near a base point. Applications of integrals are discussed in detail in two dimensions.

Chapter 18 (Vector Calculus) is motivated as the study of fluids in motion, paralleling the theme of particles in motion for the single variable calculus. The basic forms (curl and divergence) are introduced in this context. This leads to an interpretation that makes the fundamental theorems (Green, Stokes, Gauss) plausible. Here the emphasis is on the examples which show how to use and apply these theorems.

CALCULUS I, First Semester

I. Rate of Change, Tangent Line and Differentiation

1.1 Newton's Calculus

Early in his career, Isaac Newton wrote, but did not publish, a paper referred to as the *tract of october 1666*. This was his sole work purely on mathematics, and contained the fundamental ideas and techniques of the calculus. While writing (1684-87) the *Principia Mathematica*, the fundamental exposition of his mathematical physics of "the system of the world", he reworked and expanded these ideas and included them as part of this treatise. Newton's central conception was that of objects in motion. To Newton motion is described by the position and velocity of the particle relative to a fixed coordinate system, as functions of time. These then are the fundamental variables: x, y, z, etc., and their velocities are denoted by $\dot{x}, \dot{y}, \dot{z}$, etc. Now, velocity is a measure of the rate of change of position (and acceleration, denoted $\ddot{x}$, etc., is a rate of change of velocity), and what was needed was a means to express this relationship, and a process of deriving relations among the various velocities and accelerations from given relations among the variables. This is the Calculus of Newton.

Calculus, as it is presented today starts in the context of two variables, or measurable quantities, x, y, which are related in the sense that values of one of the variables determine values of the other. A *function* $y = f(x)$ is a rule which specifies this relation between the *input* or *independent* variable x and the *output* or *dependent* variable y. This may be given by a formula, a table, or a computer algorithm; in fact, any set of rules which uniquely determine outputs for given inputs. The set of allowable values for the input variable is called the *domain* of the function, and the set of outputs, the *range*. In our context both the domain and the range are sets of real numbers.

An *interval* is the set of all real numbers between specified real numbers c and d. This is denoted by

$$(c, d) = \{x : c < x < d\} \tag{1.1}$$

if neither endpoint is included, and by

$$[c, d] = \{x : c \leq x \leq d\} \tag{1.2}$$

if both are included. We shall say "I is an interval about a" to mean that a is between the endpoints of the interval I.

Now, suppose that $y = f(x)$ is a function defined for all x in an interval $I = (c, d)$. The *graph* of f is the set of all points (x, y) in the plane where x is in the interval (c, d) and $y = f(x)$. Calculus studies the behavior of y as a function of x in terms of the way y changes as x changes. For $x_0,\ x_1$ points in the interval, and $y_0 = f(x_0),\ y_1 = f(x_1)$, the ratio

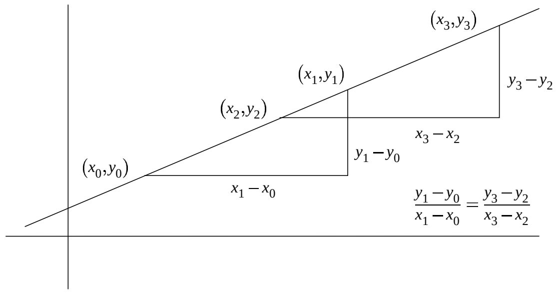
$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} \tag{1.3}$$

is called the *average rate of change* of y with respect to x in the interval between x_0 and x_1. This is the ratio of the change in y (denoted Δy) with the change in x (denoted Δx).

Linear Functions

If the ratio (1.3) is the same for all points (x_0, y_0), (x_1, y_1) on the graph, we say that $y = f(x)$ is a *linear function* of x.This is because that condition is equivalent to saying that the graph of $y = f(x)$ is a straight line (which is easy to see using similar triangles; see figure (1.1)).

Figure 1.1



For $y = f(x)$ a linear function, the ratio (1.3) is called the **slope** of the line, denoted m. Then another point (x, y) is on the line if and only if the calculation of (1.3) gives the slope. Thus, the condition for (x, y) to be on the line is

$$\frac{y - y_0}{x - x_0} = m \qquad \text{or} \qquad y - y_0 = m(x - x_0)\ . \tag{1.4}$$

This is called the *point-slope* equation of the line. If we wish to find the equation of the line through two points (x_0, y_0) and (x_1, y_1), we use those points to find the slope and then use the above equation. Thus, the condition for (x, y) to be on the line through these points is

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}\ . \tag{1.5}$$

This is the *two-point equation* for the line.

Example 1.1. Find the equation of the line through the points (2,-1), (3,7). Then find the line through (6,-1) of the same slope.

Using (1.5) with $(x_0, y_0) = (2, -1)$ and $(x_1, y_1) = (3, 7)$, we have that (x, y) is on the line when

$$\frac{y - (-1)}{x - 2} = \frac{7 - (-1)}{3 - 2} \quad \text{or} \quad \frac{y + 1}{x - 2} = \frac{8}{1}\ ,$$

giving us the equation $y = 8x - 17$. This line has slope $m = 8$. Thus the line through (6,-1) of the same slope has the equation

$$\frac{y - (-1)}{x - 6} = 8 \quad \text{or} \quad y = 8x - 49\ .$$

Example 1.2. Is $P(5,12)$ on the line joining $Q(2,7)$ and $R(8,15)$?

The slope of the line through Q and R is $(15-7)/(8-2) = 4/3$. The slope of the line through P and Q is $(12-7)/(5-2) = 5/3$. Since these two lines do not have the same slope, they cannot be the same line. Thus P is not on the line through Q and R.

Here are some facts about lines which will be useful when studying more general curves.

a. If L is a line of slope m, then

$$m = \tan\theta \tag{1.6}$$

where θ is the angle that L makes with a horizontal line. If the line is vertical, then L has *infinite* slope.

Suppose we are given two lines: L_1 of slope m_1, and L_2 of slope m_2. Then

b. L_1 and L_2 are *parallel* if and only if $m_1 = m_2$.

c. L_1 and L_2 are *perpendicular* if and only if $m_1 m_2 = -1$.

d. The length of of the line segment between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ (the *distance* between the two points) is

$$|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\ . \tag{1.7}$$

Example 1.3 Find the equation of the line through (2,3) which is perpendicular to the line $L:\ 2x + 3y = 11$.

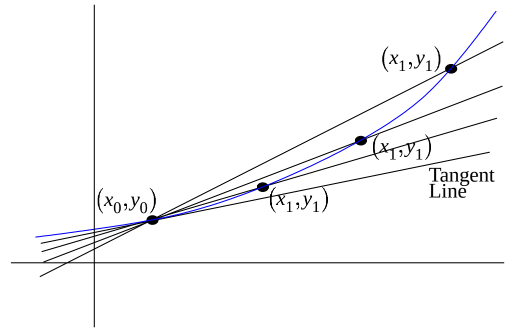
The line L has slope $m = -2/3$. Thus the line perpendicular to L has slope $-1/(-2/3) = 3/2$. Thus the equation of the line we seek is

$$\frac{y-3}{x-2} = \frac{3}{2} \quad \text{or} \quad y = \frac{3}{2}x\ .$$

Polynomial functions

For the general curve given by the equation $y = f(x)$, the ratio (1.3): $\Delta y/\Delta x$ is the slope of the line joining the two points (x_0, y_0) and (x_1, y_1) on the graph of f. But now, if the graph is not a line, this ratio changes as the point (x_1, y_1) moves. As x_1 approaches x_0, this ratio may approach a specific number. If it does, this number is called the *derivative* of y with respect to x, evaluated at the point x_0. It is the *instantaneous rate of change* of y with respect to x at x_0, and also the slope of the line which best approximates the curve at (x_0, y_0), called the *tangent line* to the curve (see Figure 1.2).

Figure 1.2



In this chapter we shall concentrate on finding the derivative of functions given by a formula; this process is called *differentiation*. It turns out to be quite simple for polynomial functions. But first, we make these definitions explicit.

Definition 1.1. Let $y = f(x)$ be a function defined for all values of x in an interval about the point a. If the difference quotient

$$\frac{f(x) - f(a)}{x - a}$$

approaches a specific number L, then we say that f is **differentiable** at a, and the number L is called the **derivative** of f at a, denoted $f'(a)$. It is the slope of the **tangent line** of $y = f(x)$ at a.

For now, we will rely on an intuitive understanding of the phrase "approaches a specific number L"; the meaning will be made clear through the examples. After these examples, we introduce an explicit definition of the idea of limit, and in the next chapter we will discuss this definition in greater depth.

Example 1.4. Consider $f(x) = x^2$, Find the tangent line to this curve at the point $(a, f(a))$.

We take a point $(x, f(x))$ near $(a, f(a))$ and calculate the difference quotient

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - a^2}{x - a} = \frac{(x - a)(x + a)}{x - a} = x + a\ .)$$

Clearly, as x approaches a, $x + a$ approaches $2a$, so we get $f'(a) = 2a$.

Since this is true for any value a, we can conclude that if $f(x) = x^2$, its derivative is $f'(x) = 2x$.

Example 1.5. Find the equation of the line tangent to the curve $y = x^2$ at the point (3,9).

For $x = 3$, the derivative is $f'(3) = 2(3) = 6$. Thus the tangent line has the equation

$$\frac{y - 9}{x - 3} = 6 \quad \text{or} \quad y = 6x - 9\ .$$

The ease with which we calculated the derivative for $y = x^2$ followed from simple algebraic facts. We shall see that this works in general for polynomials; but first, one more example:

Example 1.6. If $f(x) = x^3$, $f'(x) = 3x^2$. Fix a point (a, a^3) on the graph, and let (x, x^3) be a nearby point. We look at the slope of the line joining these points:

$$\frac{\Delta y}{\Delta x} = \frac{x^3 - a^3}{x - a} \ .$$

Since the quotient of $x^3 - a^3$ by $x - a$ is $x^2 + ax + a^2$ this can be rewritten as

$$\frac{x^3 - a^3}{x - a} = \frac{(x - a)(x^2 + ax + a^2)}{x - a} = x^2 + ax + a^2 \ ,$$

and evaluating this at a, we get $f'(a) = 3a^2$.

Now, for any polynomial $y = f(x)$, this process will work: divide $f(x) - f(a)$ by $x - a$, and evaluate the quotient at $x = a$ to calculate the derivative. Let's spell this out, starting with the division theorem of algebra:

Theorem 1.2. Let f be a polynomial of degree d. Then, for any number a when we divide $f(x)$ by $x - a$, we get a quotient which is a polynomial of degree $d - 1$ and a remainder of $f(a)$:

$$\frac{f(x)}{x - a} = q(x) + \frac{f(a)}{x - a} \ . \tag{1.8}$$

Now, to apply this to the calculation of instantaneous rate of change, move the second term on the right to the left:

$$\frac{f(x) - f(a)}{x - a} = q(x) \ .$$

Aas we let x approach a, the difference quotient $q(x)$ approaches $q(a)$, so the polynomial is differentiable at a, and its derivative is $q'(a)$.

Theorem 1.3. A polynomial $y = f(x)$ is everywhere differentiable. Its derivative at $x = a$ is $q(a)$, where q is the quotient of $f(x) - f(a)$ by $x - a$.

Newton realized that using long division would be a tedious way to calculate derivatives, and with them instantaneous rates of change, so he had the genius to take a slightly more abstract approach to lead to an automatic way of calculating derivatives. First, we must make explicit what we mean by the phrase "the expression approaches a specific number" by introducing the notion of *limit.* Suppose that $y = g(x)$ defines a function in an interval about x_0. We say that g has the limit L as x approaches x_0 if we can make the difference $|g(x) - L|$ as small as we need it to be by taking x as close to x_0 as we have to. More precisely, but less intuitively,

Definition 1.4. $\lim_{x \to x_0} g(x) = L$ if, for any $\epsilon > 0$ there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|g(x) - L| < \epsilon$.

We now observe that limits behave well under algebraic operations.

Proposition 1.5. Suppose that f and g are functions defined in an interval about x_0 and that

$$\lim_{x \to x_0} f(x) = L \ , \quad \lim_{x \to x_0} g(x) = M \ .$$

Then

a)
$$\lim_{x \to x_0} (f(x) + g(x)) = L + M$$

b)
$$\lim_{x \to x_0} (f(x) \cdot g(x)) = L \cdot M$$

c) If $M \neq 0$, then
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{L}{M} \ .$$

Applying this proposition to the calculation of derivatives, we see how differentiation behaves under algebraic operations:

Proposition 1.6. Suppose that f and g are functions defined and differentiable in an interval I. Then

a) $f + g$ is differentiable in I, and $(f + g)' = f' + g'$.

b) fg is differentiable in I, and $(fg)' = f'g + fg'$.

We give a brief justification of these rules, which follow from the corresponding rules for limits (proposition 1.5). This is straighforward for part a), but for the product, the argument requires some preliminary algebraic manipulation. Suppose then, that f and g are differentiable at a, and let $h = fg$. Then to see that h is differentiable, we must take the limit, as x approaches a of

$$\frac{h(x) - h(a)}{x - a} = \frac{f(x)g(x) - f(a)g(a)}{x - a} \ .)$$

The product rule for limits does not apply directly, for this is not a product. However, if we add and multiply $f(a)g(x)$, we get

$$f(x)g(x) - f(a)g(a) = [(f(x) - f(a))g(x)] + [f(a)(g(x) - g(a))] \ ,$$

which leads to a sum of products

$$\frac{f(x)g(x) - f(a)g(a)}{x - a} = g(x)\frac{f(x) - f(a)}{x - a} + f(a)\frac{g(x) - g(a)}{x - a} \ .$$

Now, we can take the limits using proposition 1.5. We have to note that, since g is differentiable, we also have $\lim_{x \to a} g(x) = g(a)$.

This brings us to the rule for differentiating polynomials.

Proposition 1.7.

a) If f is constant, then $f' = 0$.

b) If $f(x) = ax^n$ for some positive integer n, then $f'(x) = anx^{n-1}$.

c) A polynomial is differentiated term by term, using b) for each term.

To verify a), we only have to note that a constant function is unchanging; its graph is a horizontal line, so has slope 0. c) follows from the fact that the limit of a sum is the sum of the limits. b) follows by a bootstrap method. We have already seen this for $n = 0, 1, 2, 3$. To proceed, we use the product rule. For example, take $n = 4$:

$$(x^4)' = (x^3x)' = (x^3)'x + x^3x' = (3x^2)x + x^3(1) = 4x^3 \ .$$

If we have the proposition for all integers up to $n-1$, then we have it for n by the same method:

$$(x^n)' = (x^{n-1}x)' = ((n-1)x^{n-2})x + x^{n-1}(1) = nx^{n-1} \ .$$

Example 1.7. Let $f(x) = 2x^2 - 3x + 3$. Find $f'(x)$. What is the equation of the line tangent to the curve given by $y = f(x)$, at the point (1,2)?

Using proposition 1.7, we have

$$f'(x) = 2(2x) - 3(1) + 0 = 4x - 3 \ .$$

This gives the slope of the tangent line at (1,2) by evaluating at $x = 1$: $f'(1) = 4(1) - 3 = 1$. Thus the equation of the tangent line is

$$\frac{y-2}{x-1} = 1 \quad \text{or} \quad y = x + 1$$

Example 1.8. If

$$f(x) = 2x^5 - x^4 + 8x^3 + 2x - 5 \ ,$$

then

$$f'(x) = 2(5x^4) - 4x^3 + 8(3x^2) + 2x^0 - 0 = 10x^4 - 4x^3 + 24x^2 + 2 \ .$$

If a function f is differentiable at every point of an interval I, then the derivative is defined at every point in the interval I, and thus is a function on I. This function, denoted f', is defined by the rule: for all x in I,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \ . \tag{1.9}$$

In particular, since f' is now a function on I, it too may be differentiable. If so, its derivative is denoted $f''(x)$, and is called the *second derivative of* f with respect to x. Proceeding, we can define third and fourth derivatives and so forth.

So far, we have been interpreting the derivative as the instantaneous rate of change of y with respect to x, or the slope of the tangent line. Another fundamental interpretation is in terms of motion. Consider an object moving along a straight line. Let the variable t represent time, and x the displacement of the object from a fixed point, 0, on the line. Then the position of the object at time t is given by a function $x = x(t)$. The *velocity* (denoted at time t as $v(t)$) of the object is the instantaneous rate of change of x with respect to t. The *acceleration* of this object (denoted $a(t)$) is the instantaneous rate of change of v with respect to t. Thus, if $v(t)$ is the velocity of the object at time t, and $a(t)$ its acceleration, we have

$$v(t) = x'(t) \ , \qquad a(t) = v'(t) = x''(t) \ . \tag{1.10}$$

Example 1.9 Suppose an object is moving in a straight line so that its displacement at time t is given by $x(t) = 4t^2 + 12t$. Find the formulas for the velocity and acceleration of this object. What are the velocity and acceleration at time $t = 5$?

Differentiating, we find that $v(t) = x'(t) = 8t + 12$, $a(t) = x''(t) = 8$. Thus the velocity at time $t = 5$ is $v(5) = 8(5) + 12 = 52$, and the acceleration is $a(5) = 8$.

In many dynamical problems, an object is moving at constant acceleration. For example, an object falls near the surface of the earth at an acceleration of 32 ft/sec^2 downward (or 9.8 m/sec^2 downward, in the metric system). If the acceleration is constant, that tells us that ratio of the change in velocity over the change in time is constant: that is, the velocity is a linear function of time. Similarly, since the velocity is a linear function, the distance travelled must be given by a quadratic function; all we have to do is to use the given data to find the coefficients. We conclude:

Proposition 1.8. Suppose an object moves at constant acceleration a. If at time $t = 0$ the object is at position x_0 and has velocity v_0, then at any time we have

$$x(t) = \frac{a}{2}t^2 + v_0 t + x_0 \ , \qquad v(t) = at + v_0 \ . \tag{1.11}$$

It is easy to check that these functions do have the desired properties, that is, that $x'(t) = v(t)$ and $v'(t) = a$, and that their values at $t = 0$ are as given. Furthermore, we can argue intuitively, as we have done above, that these are the precise formulas for distance and velocity. For, since the rate of change of velocity is constant, velocity must be linear, and since the rate of change of distance is linear, it must be quadratic. This was the way Newton argued; but there are loose ends as was pointed out very articulately by Newton's contemporary, George Berkeley. Why indeed, are these the only formulas with the desired properties? How do we know that there does not exist some as yet unknown mysterious functions which have the same values at $t = 0$, and the given acceleration? The third book of Newton's Principia gives formidable evidence that no such mysteries exist, and that work, together with much subsequent experimental evidence, carried the day. But Berkeley's objections were valid on logical grounds, and the issue was not satisfactorily resolved until the nineteenth century.

Example 1.10. An object is projected upward at an initial velocity of 48 ft/sec. How high does it go?

We measure distance upward from the starting point, so that $x_0 = 0$ and $v_0 = 48$. The acceleration due to gravity is $a =$ -32 ft/sec, so (by proposition 1.8), the equations of motion (1.11) are

$$x(t) = -16t^2 + 48t \ , \qquad v(t) = -32t + 48 \ .$$

If we complete the square for $x(t)$, we have

$$x(t) = -16(t - 3/2)^2 + 36 \ .$$

Thus the greatest value of x is achieved at $t = 3/2$, and is $x(3/2) = 36$ feet. Note that at this highest point, $v(3/2) = 0$, confirming our intuition that at the moment the object turns around its velocity must be zero.

Example 1.11. An automobile is traveling at 60 mph. At what rate must it decelerate so as to stop in 100 yards?

Converting everything to feet and seconds, we have an initial velocity of 88 ft/sec, and we can take $s(0) = 0$. At some future time T, we have $s(T) = 300$ feet, $v(T) = 0$. Call the rate of deceleration a. The equations of motion (1.11) are

$$x(t) = -\frac{a}{2}t^2 + 88t \ , \qquad v(t) = -at + 88 \ .$$

At time T we have $300 = -(a/2)T^2 + 88T$, $0 = -aT + 88$. From the second we get $T = 88/a$; putting that in the first we get

$$300 = -\frac{a}{2}\frac{88}{a}^2 + 88\frac{88}{a} \quad \text{or} \quad 300 = 88^2(-\frac{1}{2a} + \frac{1}{a}) \quad \text{or} \quad 300 = \frac{88^2}{2a} \ ,$$

so $a = 12.91$ ft/sec^2.

More rules of differentiation

Eventually we will develop a full set of rules for finding the derivative of any function given by a formula. We turn now to the *quotient rule* to handle quotients of polynomials (called *rational functions*).

Proposition 1.9. Suppose that f and g are differentiable at a point a, and $g(a) \neq 0$. Then $1/g$ and $h = f/g$ are differentiable at a, and

$$(1.12) \qquad (\frac{1}{g})' = -\frac{g'}{g^2}, \qquad h' = \frac{gf' - fg'}{g^2} \ .$$

To show that $1/g$ is differentiable, we must calculate the limit as $x \to a$ of

$$\frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x - a} \ .)$$

Once again, a little algebra helps us. Simplifying the compound fraction, we get

$$\frac{1}{x-a} \cdot \frac{g(a) - g(x)}{g(x)g(a)} = \frac{-1}{g(x)g(a)}\frac{g(x) - g(a)}{x - a} \ ,$$

which has as its limit

$$(\frac{1}{g})'(a) = -\frac{g'(a)}{(g(a))^2} \ .$$

Now the second equation of (1.38) follows from this and the product rule appied to f/g considered as $f \cdot (1/g)$.

$$(\frac{f}{g})' = f(\frac{1}{g})' + f'(\frac{1}{g}) = f(\frac{-g'}{g^2}) + f'(\frac{1}{g}) = \frac{gf' - fg'}{g^2} \ .$$

In particular, we have

$$\frac{d}{dx}(\frac{1}{x}) = -\frac{1}{x^2} \ .$$

Proposition 1.10. Let n be any integer, positive, zero, or negative. Then

$$\text{(1.13)} \qquad \text{for} \quad f(x) = x^n \qquad \text{we have} \qquad f'(x) = nx^{n-1} \ .$$

By proposition 1.7b, this is true for n positive or zero. For negative exponents, we apply the quotient rule to $f(x) = 1/x^n$ with n positive:

$$f'(x) = -\frac{nx^{n-1}}{(x^n)^2} = (-n)x^{-n-1} \ ,$$

which is just (1.13) for the negative exponent $-n$.

Example 1.12. Find the derivative of

$$f(x) = x^2 - 2x + \frac{3}{x} - \frac{5}{x^2} \ .$$

Rewrite the function in exponential notation: $f(x) = x^2 - 2x + 3x^{-1} - 5x^{-2}$. Now use (1.13): $f'(x) = 2x - 2 + 3(-x^{-2}) - 5(-2x^{-3})$, which can be rewritten as

$$f(x) = 2x - 2 - \frac{3}{x^2} + \frac{10}{x^3} \ .$$

Example 1.13. Let $f(x) = 30x + 2x^{-1}$. For what value of x is $f'(x) = 0$?

Differentiate: $f'(x) = 30 - 2x^{-2}$. Now solve $f'(x) = 0$:

$$0 = 30 - \frac{2}{x^2} \quad \text{so that} \quad x^2 = 15$$

and the answer is $x = \pm\sqrt{15}$.

As a last observation, we return to the definition of the derivative to differentiate the square root function:

Proposition 1.11. If $f(x) = \sqrt{x}$ for $x > 0$, then $f'(x) = 1/(2\sqrt{x})$.

Here we use the fact that $x - a = (\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})$. Thus

$$\frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{\sqrt{x} - \sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}} \to \frac{1}{2\sqrt{a}}$$

as $x \to a$, for $a \neq 0$.

Problems 1.1

1. Find the equation of the line which goes through the point (2,-1) and is perpendicular to the line given by the equation $2x - y = 1$.

2. a) Let $f(x) = x^2 + 3x - 1$. Find the slope of the secant line joining the points $(2,9)$ and $(x, f(x))$.
   b) Find the slope of the tangent line to the curve $y = f(x)$ at the point $(2,9)$.
   c) What is the equation of this tangent line?

3. Let

$$f(x) = \frac{1}{x^2} .$$

a) Find the slope of the secant line through the points $(x, \frac{1}{x^2})$ and $(x+h, \frac{1}{(x+h)^2})$.

b) Find the slope of the line tangent to the graph of $y = f(x)$ at the point $(3, \frac{1}{9})$.

4. Find the derivatives of the following functions:

a)
$$f(x) = x^3 - x^2 + 1$$

b)
$$g(x) = x^2 + \frac{1}{x^3}$$

c)
$$h(x) = (x^2 + \frac{1}{x^3})(x^3 - x^2 + 1)$$

5. Find the derivatives of the given functions:

a)
$$f(x) = 3x^{-1} + x^3$$

b)
$$f(x) = (x^2 + \frac{1}{x^3})(x^3 - x^2 + 1)$$

6. Find the derivative of the given functions:

a)
$$f(x) = \frac{x^2 + 1}{x + 1}$$

b)
$$f(x) = x^2 + \frac{1}{x^3}$$

7. Find the derivative of

$$f(x) = \frac{x^2 + 1}{x + 1}$$

8. Differentiate : $$h(t) = \frac{1-t^2}{1+t^3}$$

10. Sketch the graph of a function with these properties:
a) $f(0) = 2$ and $f(1) = 0$;
b) $f'(x) < 0$ for $0 < x < 2$;
c) $f'(x) > 0$ for $x < 0$ or $x > 2$.

11. Sketch the graph of a function with these properties:
a) $f(0) = 1$ and $f'(0) = 0$;
b) $f(-1) = 0,\ f(1) = 0$,
c) $f'(x) < 0$ for $0x < -1$ and $0 < x < 1$,
c) otherwise, $f'(x) > 0$.

12. Find the value of x where the graphs of these two functions have parallel tangent lines:

$$f(x) = x^2 - 3x + 2\ , \qquad g(x) = 5x^2 - 11x - 17\ .$$

13. Find the points on the curve $y = 3x^2 - 3x + 1$ whose tangent line is perpendicular to the line $x + 2y = 7$.

14. Let C_1 and C_2 be curves given by the equations $C_1 : y = x^3 + x^2,\ C_2 : y = x^2 + x$. For what values of x do these curves have parallel tangent lines?

15. Find the derivative of $f(x) = (x+1)(\frac{1}{x} + 1)$.

16. Find the slope of the line tangent to the curve

$$y = x^2 - 3x + 1/x$$

at the point (3,1/3).

17. Let $y = x^3 - 48x + 1$. Find the x coordinate of the points at which the graph has a horizontal tangent line.

18. A man standing at the edge of the roof of a building 120 feet high throws a ball directly upwards at a velocity of 48 ft/sec. a) How high does the ball go? b) Assuming that it proceeds to fall along the side of the building, how long does it take to hit ground level?

19. Another man standing on ground level throws the ball back to his friend on the roof. At what initial velocity must he throw it in order to reach the roof?

20. On the planet Garbanzo in the Weirdoxus solar system, the equation of motion of a falling body is

$$s = s_0 + v_0 t - 10t^3$$

where s_0 is the initial height above ground level and v_0 is the initial velocity. Distance is measured in garbanzofeet. If a ball is thrown upwards from ground level at an initial velocity of 120 garbanzofeet/second, how high does the ball rise?

1.2 Liebniz' Calculus of Differentials

Up to this point we have been following the development of the Calculus according to Newton. We have been considering variables y, z, u, v, etc. as functions of a particular variable (called the "independent variable") x, and discovering how to find rates of change of the dependent variables relative to the independent variable.

The ideas of Liebniz follow a different, but equivalent, set of ideas. Liebniz is concerned with a collection of variables $x, y, z, u, v, etc.$ and their "infinitesimal increments". This is a hard concept to get a hold on, but we can think of it this way. When we actually make measurements, we always have in mind, even if unspecified, an "error bar"; that is, a largest allowable error. Thus, our calculators display numbers to 8 decimal points, allowing for a "negligible" error of at most 10^{-8}. A more efficient computer has a smaller error bar, perhaps 10^{-32}, or 2^{-128}. Instruments of measurement, no matter how delicate, have to allow for such an error bar. So, if u is a measurable variable, it comes equipped with an error bar: an allowable increment in a measurement which does not change the accepted value of the measurement. It is this which we should call the "infintesimal increment" in u, called by Liebniz the *differential* of u, denoted du. However, the important feature of this concept is that it is not tied down to the level of accuracy of today's instruments, but it represents the error bar for all time: du stands for the smallest measurable increment for all ways of measuring ever to come.

We get a more concrete interpretation of the differential by relating it to the linear approximation to the variables. More precisely, suppose the variables x, y are related by $y = f(x)$. The tangent line at a point (x_0, y_0) is the line which best approximates the curve. We have used the symbols $\Delta x,\ \Delta y$ to represent changes in the variables x and y along the curve; now we let $dx,\ dy$ represent changes in the variables along the tangent line. Since the slope of the tangent line at (x_0, y_0) is $f'(x_0)$, we obtain this important relationship between the differentials: at the point $(x_0, f(x_0))$,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)\ , \tag{1.14}$$

or, without specifying the particular point, $dy = f'(x)dx$. This we can interpret as the equation of the tangent line by replacing dx and dy by $x - x_0,\ y - y_0$.

Example 1.14. Find the equation of the tangent line to the curve $y = x^3 - 2x + 5$ at the point $(2, 9)$. First we calculate the relation between the differentials:

$$dy = (3x^2 - 2)dx\ .$$

At $x = 2$, this gives $dy = 10dx$. Now we interpret this as the equation of the tangent line by replacing dy by $y - 9$ and dx by $x - 2$. The equation of the tangent line is thus

$$y - 9 = 10(x - 2) \qquad or \qquad y = 10x - 11\ .$$

Finally, considering the equation $dy = f'(x)dx$ as the linear approximation to the equation $y = f(x)$ (at a particular point), we can make preliminary estimates of the change in y, given a change in x.

Example 1.15. The volume of a sphere of radius r is $V = (4/3)\pi r^3$. Suppose the surface of a sphere of radius 6 feet is covered by a 1 inch coat of paint. About how much paint will be needed?

From the defining equation we have $dV = 4\pi r^2 dr$; so letting $r = 6$ feet and $dr = 1/12$ feet, we can estimate the change in volume to be

$$dV = 4\pi(6)^2(\frac{1}{12}) = 37.7 \quad \text{cu. ft.}$$

Thinking of the derivative as the ratio of two quantities which eventually become zero has its philosophical problems and was also subjected to the scathing criticism of Berkeley. This concept of "evanescent quantities" (as Berkeley sarcastically identified them) was controversial in the days of Newton and Liebniz and remained so for the following 200 years. Note, by the way, that one can object to Newton's methods on the same ground: when we write

$$\frac{f(x) - f(a)}{x - a} = \frac{x^2 - a^2}{x - a} = \frac{(x - a)(x + a)}{x - a} = x + a \ ,$$

by what right are we now able to let x become a? If we did so one step sooner, we'd be dividing by zero, which is forbidden. So, in this set of equations x cannot be a. But in the next line we say, "let x be a"! These philosophical obstacles were eventually overcome; we shall proceed without resolution, as did Newton, Liebniz and their successors to enormous effect. Suffice it to say that this can all be put on a logical footing, while at the same time, the concept of differential as "smallest possible increment" is a powerful intuitive tool throughout mathematics and its applications. To illustrate, in the next section, we shall give a heuristic derivation of the law of differentiation for composite functions.

Problems 1.2

1. Let $\qquad y = \dfrac{x}{x^2 + 1} \ .$

Find the equation of the tangent line to the graph at the point (2,0.4).

2. Find the equation of the tangent line to $y = x^2(x^3 - 1)$ at (2,28).

3. Find the equation of the tangent line to the curve $y = x \cos x$ at $(\pi/4, \pi\sqrt{2}/8)$.

4. Find the the equation of the tangent line to the curve $y = x - x^{-2}$ at (2,7/4).

5. Let $y = x + 25x^{-1}$. Find an approximate value of y when $x = 3.2$.

1.3 The Chain Rule

Suppose that y, u and x are variables such that u is a function of x: $u = f(x)$, and y is a function of u: $y = g(u)$. Then y can be viewed as a function of x by writing $y = h(x) = f(g(x))$. How do we find the rate of change of y with respect to x? Using differentials, we have: $dy = f'(u)du$, and $du = g'(x)dx$, so that $dy = f'(u)g'(x)dx$, it being understood that in this formula u is to be expressed in terms of x. A shorthand for this is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \ .$$

Example 1.16. Let $y = (4x+1)^3$. We introduce the intermediate variable $u = 4x+1$, so that $y = u^3$. Then $dy = 3u^2 du$, and $du = 4dx$, so that

$$\frac{dy}{dx} = 3u^2(4) = 12(4x+1)^2 \ .$$

Example 1.17. If $y = x^{-n}$ with n positive , we introduce $u = x^n$, so that $y = u^{-1}$. Then

$$\frac{d}{dx}(x^{-n}) = \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{u^2}nx^{n-1} = -\frac{nx^{n-1}}{(x^n)^2} = -nx^{-n-1} \ ,$$

giving another derivation of proposition 1.11.

Example 1.18. Let $y = ((2x+1)^3+5)^2$. Here we need to use the chain rule more than once. We think of $y = v^2$, where $v = u^3+5$, and $u = 2x+1$. Then

$$\frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{du}\frac{du}{dx} = 2v(3u^2)(2) \ .$$

Now replace v and u by their expressions in x:

$$\frac{dy}{dx} == 12((2x+1)^3+5)(2x+1)^2 \ .$$

The statement of the chain rule is as follows.

Proposition 1.12. Suppose that g is differentiable at the point a, and f is differentiable at $g(a)$. Then the composed function $h(x) = f(g(x))$ is differentiable at a and

$$h'(a) = f'(g(a))g'(a) \ . \tag{1.15}$$

In particular,

Proposition 1.13. If f is differentiable at a, and n is any positive or negative integer, $h(x) = (f(x))^n$ is also differentiable at a and

$$h'(x) = n(f(x))^{n-1}f'(x) \ . \tag{1.16}$$

Of course, the Liebniz formulation (1.55) is easier to remember and apply than proposition 1.12. For that reason we shall begin to adopt the Liebniz notation for differentiation: if $y = f(x)$ is differentiable in an interval I, we write

$$f'(x) = \frac{dy}{dx} \tag{1.17}$$

and use $f'(x)$ and dy/dx interchangeably. The notation for higher derivatives is:

$$f''(x) = \frac{d^2y}{dx^2} \ , \qquad f'''(x) = \frac{d^3y}{dx^3} \ , \qquad \text{etc.} \tag{1.18}$$

In this notation, Proposition I.13 becomes simply

$$\frac{d}{dx}(y^n) = ny^{n-1}\frac{dy}{dx} \ .$$

Problems 1.3

1. Find the derivative of $g(x) = (x^3 + 1)^4$.

2. Find the first and second derivatives of $f(x) = x\sqrt{1 - x^2}$

3. Differentiate: $f(x) = \sqrt{2x^2 - 3x + 1}$.

4. Find $f'(x)$: $\qquad f(x) = \dfrac{(x+1)^2}{(x-1)^2}$

5. Find $g'(x), g''(x)$: $\quad g(x) = (x^3 + 1)^4$.

1.4 Trigonometric Functions

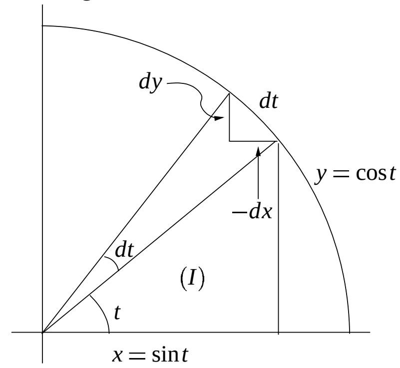
Consider a particle moving in the counterclockwise direction around the circle of radius 1 with constant angular velocity of 1 radian/second such that at time $t = 0$ it is at the point $(1, 0)$. Then its position at time t is $(\cos t, \sin t)$. These functions are defined for all values of t, and are periodic of period 2π since in time 2π the particle will make one full circuit of the circle.

There are four other trigonometric functions defined by the equations

$$\tan t = \frac{\sin t}{\cos t} \ , \quad \cot t = \frac{\cos t}{\sin t} \ , \quad \sec t = \frac{1}{\cos t} \ , \quad \csc t = \frac{1}{\sin t} \ .$$

Assuming that these functions are differentiable, we can calculate the derivatives by an argument using differentials due to Blaise Pascal.

Figure 1.3: Pascal's derivation.



In figure 1.3 we have located the moving point at $P = (\cos t, \sin t)$ at time t, and its position Q after an infinitesimal increment dt. Since we are on the unit circle t also measures arc length along the circle. The triangle with sides dx, dy, dt is called the "differential triangle". It may be of concern that dt represents an arc of the circle, but, remember - at the differential level an arc and a straight line are indistinguishable. Since the tangent line to the circle is perpendicular to the radius at the point P, the differential triangle is similar to the triangle (I) . Thus

$$\frac{-dx}{\sin t} = \frac{dt}{1} \ , \quad \frac{dy}{\cos t} = \frac{dt}{1} \ ,$$

so

$$\frac{dx}{dt} = -\sin t \ , \quad \frac{dy}{dt} = \cos t \ .$$

Since $x = \cos t, \ y = \sin t$, we obtain the first part of the following.

Proposition 1.14.

a)
$$\frac{d}{dt}(\sin t) = \cos t \ , \qquad \frac{d}{dt}(\cos t) = -\sin t \ ,$$

b)
$$\frac{d}{dt}(\tan t) = \sec^2 t \ , \qquad \frac{d}{dt}(\cot t) = -\csc^2 t \ ,$$

c)
$$\frac{d}{dt}(\sec t) = \sec t \tan t \ , \qquad \frac{d}{dt}(\csc t) = -\csc t \cot t \ .$$

b) anc c) follow from the quotient rule. For example, b):

$$\frac{d}{dt}(\tan t) = \frac{d}{dt}\frac{\sin t}{\cos t} = \frac{\cos t \cos t - \sin t(-\sin t)}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t \ .$$

Remember: in the the above discussion we have assumed that the trigonometric functions were differentiable, and it was that assumption that allowed us to consider an arc of a (differential) circle as a straight line. These formulae imply the following limit results, just by the definition of the derivative.

Proposition 1.15.

(1.19)
$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \ , \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0 \ .$$

We remind the reader that we started out assuming that the functions $\sin x$ and $\cos x$ are differentiable, so this problem tells us what the derivatives are at $x = 0$ under those assumptions. In the next chapter we wil prove proposition 1.15 directly by geometric methods, from which we conclude that the sine and cosine functions are indeed differentiable.

Problems 1.4

1. From a point 1000 feet away from the base of a building, the angle of elevation of its roof is 17 degrees. How tall is the building?

2. A marker is rotating counterclockwise around a circle of radius 4 centered at the origin at the rate of 7 revolutions per minute. a) What is its position after 2.3 minutes? b) How soon after 2.3 minutes will it cross the x-axis again?

3. If $\tan\alpha = -\sqrt{3}$, what are the possible values of $\sin\alpha$?

4. Express as a function of $2x$:

$$\frac{\sin x - \cos x}{\sin x + \cos x}$$

5. Find the derivative:

$$f(x) = \sin x \cos x$$

6. Find the derivatives of the following functions:

a)

$$f(x) = \cos^2 x$$

b)

$$g(x) = \frac{\sin^2 x}{\cos x}$$

7. Find the derivative:

$$g(x) = \frac{\sin x}{\cos x}$$

8. Differentiate: $y = (x^2 - 1)\sin(x^2 + 1)$.

9. Find the derivative:

$$h(x) = (\cos(2x) + 1)\sin(3x)$$

10. Differentiate: $g(x) = (\sin(3x) + 1)^3$.

11. A point moves around the unit circle so that the angle it makes with the x-axis at time t is $\theta(t) = (t^2 + t)\pi$. Let $(x(t), y(t))$ be the cartesian coordinates of the point at time t. What is dy/dt when $t = 3$?

12. Let $f(x) = x\sin x$. Find the equation of the tangent line to the graph $y = f(x)$ at the points $x = (2n + 1/2)\pi$ for any integer n.

13. Find the derivatives of these functions:

(*a*)

$$h(x) = (\cos(2x) + 1)\sin(3x)$$

b)

$$g(x) = (\tan(3x) - 1)^2$$

14. Consider the curves $C_1 : y = \sin x$ and $C_2 : y = \cos x$.
   a) At which points x between $-\pi/2$ and $\pi/2$ do the curves have parallel tangent lines?
   b) At which such points do they have perpendicular tangent lines?

15. Differentiate:

$$f(x) = \frac{1 + \tan x}{1 - \tan x}$$

1.5 Implicit Differentiation and Related Rates

Suppose that x and y are variables which are related by a functional equation: $F(x, y) = c$, a constant. We say that this relation defines y *implicitly* as a function of x. For, in principle, given a value of x, say $x = a$, we can solve the equation $F(a, y) = c$ for y, giving the "rule" defining y in terms of x. However, to find dy/dx we need not solve this equation. When y and x are so related, their differentials are related as well, and the chain rule can be used to find that relationship, as a function of both x and y. The idea is to think of $z = F(x, y)$ as another variable which, because of the relation $F(x, y) = c$ is constant, so $dz = 0$. Now, apply the chain rule to the expression for z.

Example 1.19. Suppose that the variables x and y satisfy the relation

$$x^2 - xy + 2y^2 = 4 \ . \tag{1.20}$$

Letting z represent this defining relation, we have $dz = 0$. Now, using (1.15) and the rules for differentiation,

$$dz = 2xdx - (xdy + ydx) + 4ydy = 0 \ ,$$

giving us

$$(-x + 4y)dy = (-2x + y)dx \ , \tag{1.21}$$

leading to this expression for the derivative:

$$\frac{dy}{dx} = \frac{2x - y}{4y - x} \ .$$

Example 1.20. What is the equation of the tangent line to the curve given by (1.68) at the point (2,1)? We find the slope by substituting the values $x = 2, y = 1$ in equation (1.21):

$$m = \frac{dy}{dx} = \frac{2(2) - 1}{4(1) - 2} = \frac{3}{2} \ .$$

Then the equation of the line is

$$y - 1 = \frac{3}{2}(x - 2) \qquad or \qquad y = \frac{3}{2}x - 2 \ .$$

Notice that, if we substitute $x = 2, y = 1, dy = y - 1, dx = x - 2$ into equation (1.21) we get the same result. That is because equation (1.21) is the linear approximation to the relation between x and y, which is of course the same as the equation of the tangent line.

Example 1.21 . Find the equation of the tangent line to the curve $y^3 + 2\cos^2 x = 0$ at the point $(\pi/4, -1)$. We differentiate implicitly:

$$3y^2dy - 2\cos x \sin x dx = 0 \ .$$

Now, at the point $(\pi/4, -1)$, $y^2 = 1$, $\cos x = \sin x = \sqrt{2}/2$, so this becomes $3dy - dx = 0$. Replacing dx by $x - \pi/4$ and dy by $y - (-1)$ gives the equation of the tangent line:

$$3(y + 1) - (x - \frac{\pi}{4}) = 0 \ , \qquad or \quad 3y - x = \frac{\pi}{4} - 3 \ .$$

Related Rates

Suppose we are in a situation where one or more variables are related, and the variables are functions of time. For example, if a spherical balloon is being inflated, then during this process the volume (V), area (A) and radius (r) are increasing with time. Since these are all related, we are able, by differentiation to relate the rates of growth. For example, suppose the balloon is being inflated by putting gas in at a steady rate of 3 cc/sec. We may ask "at what rate is the radius changing?" We start with the formula relating volume with radius: $V = (4/3)\pi r^3$. V and r are functions of time, so, differentiating with respect to time we obtain

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2\frac{dr}{dt}) = 4\pi r^2\frac{dr}{dt} \ .$$

Putting in the datum $dV/dt = 3$, we find

$$\frac{dr}{dt} = \frac{3}{4\pi r^2} \quad cm/sec \ ,$$

so depends upon the radius at the time.

Here is a protocol for attacking such problems.

Step 1. Draw a picture (if appropriate), and identify the relevant variables: those things which can change. State the problem in terms of the variables.

Step 2. Find a relationship among the variables.

Step 3. Differentiate, to obtain a relationship among the variables and their rates of change.

Step 4. Put in the values of the variables at the time in question, and solve the resulting equation.

Example 1.22. Suppose as above a balloon is being inflated with gas at a rate of 3 cc/sec. At what rate is the area increasing when the radius is 14 cm?

First, we identify the variables as volume: V, area: A, and radius: r. Now, these are related by the equations

$$V = \frac{4}{3}\pi r^3 \ , \qquad A = 4\pi r^2 \ .$$

Now, we differentiate these equations:

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \ , \qquad \frac{dA}{dt} = 8\pi r\frac{dr}{dt} \ .$$

Now, at the specific time of interest, $r = 14$ cm, and $dV/dt = 3$ cc/sec. Substituting these values, we have:

$$3 = 4\pi(14)^2\frac{dr}{dt} \ , \qquad \frac{dA}{dt} = 8\pi(14)\frac{dr}{dt} \ .$$

Then $dr/dt = 3/[4(14)^2\pi]$, so the second equation gives

$$\frac{dA}{dt} = 8\pi(14)\frac{3}{4(14)^2\pi} = \frac{3}{7} \quad cm^2/sec \ .$$

Example 1.23. A ship leaves port at noon heading north at 25 knots (nautical miles per hour), and 2 hours later another ship leaves heading west at 30 knots. Assuming the ships travel in straight lines, at what rate is the distance between the ships increasing after an additional 3 hours?

First, the variables are: t, the time elapsed since noon, N, the distance travelled in that time by the ship heading north, W, the distance travelled by the ship heading west, and Z, the distance between them. The relations among these variables are:

$$Z^2 = N^2 + W^2 \ ,$$

from the Pythagorean theorem. In t hours after noon, the first ship has travelled $25t$ nautical miles: $N = 25t$, and, since the second ship started two hours later, it has travelled $30(t-2)$ nautical miles (notice, we are assuming that $t \geq 2$). Now since we have been given the rates of change of N and W, and want to find that of Z, we differentiate the first equation with respect to t to relate these rates:

$$2Z\frac{dZ}{dt} = 2N\frac{dN}{dt} + 2W\frac{dW}{dt} \ .$$

Now, at $t = 5$, we have $N = 125$, $W = 75$, $dN/dt = 25$, $dW/dt = 30$, giving

$$Z\frac{dZ}{dt} = 125(25) + 75(30) = 5375 \ .$$

We find Z by the first relation $Z^2 = 125^2 + 75^2 = (25)^2(34)$, so $Z = 25\sqrt{34}$. Finally,

$$\frac{dZ}{dt} = \frac{5375}{25\sqrt{34}} = 36.87 \quad \text{nautical miles/hour} \ .$$

Example 1.24. Suppose that x and y are functions of t which satisfy the relation $x^3y^2 + 2y = 8$. Suppose that at the point $(1, 2)$, the velocity of x is 3 in/sec. What is the velocity of y? Differentiating the relation implicitly, we get

$$3x^2\frac{dx}{dt}y^2 + x^3(2y\frac{dy}{dt}) + 2\frac{dy}{dt} = 0 \ .$$

Now substituting $x = 1$, $y = 2$, $dx/dt = 3$, in this equation:

$$3(1)^2(3)(2^2) + (1)^3(2(2)\frac{dy}{dt}) + 2\frac{dy}{dt} = 0 \ .$$

Solving for dy/dt, we find $36 + 6dy/dt = 0$, or, the velocity of y is -6 in/sec.

Definition 1.16. For integers p and q, the function

$$y = x^{p/q} \tag{1.23}$$

is defined, for all positive x, as the positive solution of the equation $y^q = x^p$.

So, for example, $\sqrt{x}$ can be written as $x^{1/2}$, the cube root of x as $x^{1/3}$, etc. Using implicit differentiation we verify:

Proposition 1.17.

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \textit{for all rational numbers } \ n \ . \tag{1.24}$$

A rational number is a quotient p/q of integers. Differentiate the equation $y^q = x^p$ implicitly:

$$qy^{q-1}dy = px^{p-1}dx \qquad or \qquad \frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} \ .)$$

Replacing y by $x^{p/q}$, we get

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q}x^{p-1-p+p/q} = \frac{p}{q}x^{(p/q)-1}$$

which is the desired result, since $n = p/q$.

Problems 1.5

1. A curve is given by the equation $x^2 - xy + y^2 = 7$. Find the equation of the line tangent to this curve at the point (2,-1).

2. Find the slope of the curve defined by the relation

$$4(x^2 + xy) = 2y^3 - y^2$$

at the point $(1, 2)$.

3. Variables x and y are related by the formula

$$x \sin y + y \sin x = \pi \ .$$

If $dy/dt = 3$ when $x = 3\pi/2$ and $y = \pi/2$, what is dx/dt?

4. The relation $\cos y + x = \sin y$ determines a curve in the x-y plane. Find the slope of the line tangent to the curve at the point $(1, \pi/2)$.

5. Consider the curve given by the equation: $y^2 + xy + x^2 = 1$. At what points does this curve have a horizontal tangent line?

6. Consider the curve given by the equation: $x^2y - y^3 = 1$. At what points does this curve have a vertical tangent line?

7. A ship is travelling in a circle of radius 6 nautical miles around an island at a speed of 10 knots (nautical miles per hour). A lighthouse is 10 miles due east of the island. At what rate is the distance between ship and lighthouse increasing when the ship is exactly due north of the island?

8. A new stadium, built like a cylinder capped with a hemispherical dome is proposed to have a diameter of 500 feet. To include another 2000 seats, the diameter must be increased by 10 feet. By approximately how much will the area of the dome be increased? (Note: the area of a sphere of radius r is $4\pi r^2$.)

9. A cat is walking toward a telelphone pole of height 30 feet. She is walking at a steady rate of 4 ft/sec. A bird is perched on top of the tellephone pole. When the cat is 45 feet from the base of the pole, at what rate is the distance between bird and cat decreasing?

10. Water is flowing into a conical tank through an opening at the vertex at the top at the rate of 12 cu. ft./min. The base of the tank is a circle of radius 12 ft. and the height of the cone is 20 ft. At what rate is the water level rising when the water level is 4 ft. from the top? The formula for the volume of a cone of base radius r and height h is $V = (1/3)\pi r^2 h$.

11. Let $\mathcal{P}$ be an upward-opening parabola whose axis is the y-axis and whose vertex is the origin. Suppose the line $y = C$ intersects the parabola in two points. Show that the tangent lines at these points intersect on the y-axis of the parabola.

12. Suppose that a point moves along the x-axis according to the formula $x(t) = 1/(t^2 + 1)$. Let A(t) be the area of the circle with diameter joining the origin to the point $x(t)$. Find $A'(t)$ when $t = 3$.

II. Theoretical Considerations

2.1 Limit Operations

In this section we shall go more deeply into the concept of limits than we did in chapter 1. Suppose that $y = f(x)$ is a function defined in an interval about the point x_0. Each value of x determines a value y using the rule represented by the function f. We say that y approaches a number L as x approaches x_0 if we can be sure that y is as close as we please to L just by taking x close enough to x_0. A little more precisely, if we allow an error $\epsilon > 0$ in the calculation of L, we can find an error $\delta > 0$ for x_0 such that if x is within δ of x_0, then y is within ϵ of L. If the limit L is the number y_0 calculated from x_0 by f, then we say that f is *continuous* at x_0. That is the content of the following two definitions.

Definition 2.1. $\lim_{x\to x_0} f(x) = L$ if, for any $\epsilon > 0$ there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - L| < \epsilon$.

If the limit L is $f(x_0)$, then we say that f is continuous at x_0:

Definition 2.2. A function f, defined in an interval about x_0 is *continuous* at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$. A function is said to be *continuous* if it is continuous at every point where it is defined.

Example 2.1. Let $f(x) = x/|x|$. Then, $f(0) = 0$, for $x > 0$, $f(x) = 1$, and for $x < 0$, $f(x) = -1$. Thus, in any interval about 0, there are values of x for which $f(x) = 1$ and other values of x for which $f(x) = -1$. There is thus no number L such that both 1 and -1 are within .5 of L, so there can be no $\lim_{x\to 0} f(x)$.

Example 2.2. Let $f(x) = \cos(1/x)$ for $x \neq 0$. There is no value to assign to $f(0)$ to make this function continuous. For if $x = (2\pi n)^{-1}$, $f(x) = 1$ for n even, and $f(x) = -1$ for n odd, so we are in the same situation as that of example 1. However, for the function $g(x) = x\cos(1/x)$, we can define $g(0) = 0$ to get a continuous function. For $|g(x)| \leq |x|$ for every x, since the cosine is bounded by 1. Thus, for any $\epsilon > 0$, if $|x| < \epsilon$, we also have $|g(x)| < \epsilon$.

Now we state the basic facts describing how limits behave under algebraic operations.

Proposition 2.1. Suppose that f and g are functions defined in an interval about x_0 and that

$$\lim_{x\to x_0} f(x) = L \ , \quad \lim_{x\to x_0} g(x) = M \ .$$

Then

a)

$$\lim_{x\to x_0} (f(x) + g(x)) = L + M$$

b)

$$\lim_{x\to x_0} (f(x) \cdot g(x)) = L \cdot M$$

c) If $M \neq 0$, then

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M} \ .$$

This proposition then tells us the following about continuous functions:

Proposition 2.2 Suppose that f and g are defined in an interval about x_0, and are continuous at x_0. Then the sum $f+g$ and product $f \cdot g$ are also continuous at x_0. If $g(x_0) \neq 0$, the quotient f/g is also continuous at x_0.

Now it is clear that a constant function is continuous: if $f(x) = C$ for all x, then the difference $|f(x) - C| = 0$ no matter what x is. Also, the function $f(x) = x$ is continuous everywhere: we can make $|f(x) - f(x_0)| < \epsilon$ just by taking $|x - x_0| < \epsilon$. Thus, by proposition 2.2, any function formed from constants and tne function $f(x) = x$ by taking products and sums is continuous. But these are the polynomials.

Proposition 2.3. All polynomials are continuous everywhere. A rational function (that is, a quotient of polynomials) is continuous everywhere where its denominator is non-zero.

Example 2.3. That is not to say that a rational function is *not* continuous where the denominator is zero; perhaps it can be defined at those points so as to be continuous, For example, consider

$$f(x) = \frac{x^2 - 4x - 5}{x - 5} \ .$$

Since we cannot divide by zero, $f(x)$ is not defined for $x = 5$. But, can we define $f(5)$ so that the function is continuous? Noting that $x^2 - 4x - 5 = (x-5)(x+1)$, we see that for $x \neq 5$, $f(x) = x + 1$. Thus by defining $f(5) = 6$, we get a continuous function.

Suppose now that g is defined in an interval around x_0 and f is a function defined on the range (set of values) of g. Then we can form the *composition* of the two functions, $f \circ g$, just by applying the rule defining f to the value of g : $f \circ g(x) = f(g(x))$.

Proposition 2.4. Suppose that g is defined in an interval about the point x_0, $g(x_0) = y_0$ and f is defined in an interval about y_0. If g is continuous at x_0, and f is continuous at y_0, then $h = f \circ g$ is also continuous at x_0.

To show this, we have to show that we can insure that $h(x)$ is within ϵ of $h(x_0)$ by taking x close enough to x_0. By the continuity of f we can be sure that $f(y)$ is within ϵ of $f(y_0)$ by taking y within some small number, η of y_0. But then, by the continuity of g, there is a δ such that, if x is within δ of x_0, $g(x)$ is within η of $g(x_0) = y_0$, and finally, $f(g(x))$ is within ϵ of $f(y_0) = f(g(x_0))$.

A useful technique is what is called the squeeze theorem. Suppose, in some interval containing the point a, the values of f lie between those of two other functions g and h. Suppose also that g and h have the same limit as x approaches a, then f also has that limit.

Proposition 2.5 (Squeeze Theorem). Suppose that $f,\ g,\ h$ are defined in a interval containing a and that $g(x) \leq f(x) \leq h(x)$. If

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L \ ,$$

we also have $\lim_{x \to a} f(x) = L$.

Suppose an allowable error $\epsilon > 0$ is specified. From the hypothesis, we know that there is a $\delta_1 > 0$ such that if x is within δ_1 of x_0, then $g(x) \geq L - \epsilon$, and there is a $\delta_2 > 0$ such that if x is within δ_2 of x_0, then $h(x) \leq L + \epsilon$. Then, so long as δ is less than both δ_1 and δ_2, we have

$$L - \epsilon \leq g(x) \leq f(x) \leq h(x) \leq L + \epsilon$$

which is to say that $f(x)$ is within ϵ of L.

Now suppose again that f is defined in a neighborhood of x_0 and continuous there. We now turn to the question of the differentiability of f at x_0.

Definition 2.3. Let f be defined in a neighborhood of x_0. If the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, it is denoted $f'(x_0)$, and is called the **derivative** of f at x_0. f is said to be **differentiable** at x_0.

Proposition 2.1c suggests that the limit does not exist since the denominator approaches 0. But we have to be careful: the numerator is also going to zero (as in example 2.3). In fact, as we saw by the division theorem of chapter 1, If f is a polynomial, then so is this difference quotient, and the limit is the value of that quotient at x_0. In fact, in general it is a necessary condition for differentiability that the limit of the numerator is zero - a fact we already used several times in chapter 1.

Proposition 2.6. Let f be defined in a neighborhood of x_0. If f is differentiable at x_0, then it is continuous at x_0.

Let $L = f'(x_0)$. The hypothesis tells us that we can be sure the difference quotient is within ϵ of L by taking x close enough to x_0. So, taking, for example, $\epsilon = 1$, then if x is close enough to x_0,

$$-1 < \frac{f(x) - f(x_0)}{x - x_0} - L < +1,$$

from which we conclude that

$$(L - 1)(x - x_0) < f(x) - f(x_0) < (L + 1)(x - x_0) \ .$$

Now, the left and right hand sides tend to 0 as x approaches x_0, so, by the squeeze theorem, $\lim(f(x) - f(x_0)) = 0$. But that is the same as $\lim f(x) = f(x_0)$; that is, f is continuous at x_0.

Now, in section 1 of chapter 1, no problems arose in calculating limits, since we were there dealing with polynomials (even in calculating derivatives). However, more generally questions about limits can become real issues. For example, when we turned to trigonometric functions and the square root function, we tacitly assumed their continuity. Since the continuity is intuitively clear (if we envision the graph of these functions), this was not an obstacle to finding derivatives. However, in more general contexts, the continuity is not at all clear. As preparation for this, we shall here reconsider the assumptions of continuity made in chapter 1. First, the square root.

Example 2.4. For $a \geq 0$,

$$\lim_{x \to a} \sqrt{x} = \sqrt{a} \ .$$

We have to distinguish the cases $a \neq 0$ and $a = 0$. First, the case $a \neq 0$. We start with the identity

$$(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a}) = x - a \ ,$$

which, for our purposes should be written as

$$\sqrt{x} - \sqrt{a} = \frac{x - a}{\sqrt{x} + \sqrt{a}} \ ,$$

since it is the expression on the left we need to make small. Given $\epsilon > 0$, choose $\delta > 0$ so that $\delta/\sqrt{a} < \epsilon$. Then if $|x - a| < \delta$,

$$|\sqrt{x} - \sqrt{a}| = \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \frac{|x - a|}{\sqrt{a}} < \frac{\delta}{\sqrt{a}} < \epsilon \ .$$

Since this argument fails if $a = 0$, we need another idea.

Proposition 2.7 (Archimedean principle). For any positive real number M, there is an integer n such that $n > M$.

Now, given $\epsilon > 0$, choose the integer n so that $n > 1/\epsilon^2$. Then $\sqrt{n} > 1/\epsilon$, so $1/\sqrt{n} < \epsilon$ which is what we need. For $x < 1/n$,

$$\sqrt{x} < \frac{1}{\sqrt{n}} < \epsilon \ .$$

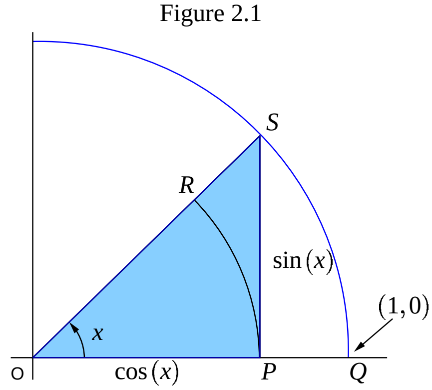
Now, in section section 1.4 we derived the formulae for the derivatives of the sine and cosine functions, assuming that they were differentiable. Here we would like to justify that assumption. The crux of the matter is the following proposition (which we derived in section 1.4 from the formulae for differentiation).

Proposition 2.8.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \ , \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \ .$$

This is just the assertion that the sine and cosine functions are differentiable at $x = 0$. Then, using the addition formulae for these functions, we can prove their differentiability everywhere. Here is a geometric argument for proposition 2.8.

Figure 2.1



In figure 2.1, let A be the area of the sector OPR, B the area of triangle OPS, and C the area of sector OQS. Then $A \leq B \leq C$. Using the formulae for these areas (measuring the angle x in radians), this gives us

$$\frac{1}{2}x\cos^2 x \leq \frac{1}{2}\cos x \sin x \leq \frac{1}{2}x(1)^2 \ .$$

Dividing by $x\cos x/2$, this gives us

$$\cos x \leq \frac{\sin x}{x} \leq \frac{1}{\cos x} \ .$$

But now, since $\lim_{x\to 0}\cos x = 1$, as is obvious from the figure, the first part of proposition I.12 follows from the squeeze theorem. The second now follows from the first using the equalities:

$$\frac{\cos x - 1}{x} = \frac{\cos x - 1}{x}\frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 - 1}{x(\cos x + 1)} = \frac{\sin x}{x}\sin x\frac{1}{1+\cos x} \to 0 \ ,$$

since $\lim_{x\to 0}\sin x = 0$, as is clear from the figure.

Example 2.5. Find the limit as $x \to 0$ of $\sin(3x)/\sin(4x)$.

$$\lim_{x\to 0}\frac{\sin(3x)}{\sin(4x)} = \frac{3}{4}\lim_{x\to 0}\frac{\sin(3x)}{3x}\frac{4x}{\sin(4x)} = \frac{3}{4}\lim_{3x\to 0}\frac{\sin(3x)}{3x}\lim_{4x\to 0}\frac{4x}{\sin(4x)} = \frac{3}{4}(1)(1) = \frac{3}{4}$$

Example 2.6. Find the limit as $x \to \pi/2$ of $\cos x/(x - \pi/2)$. Let $t = x - \pi/2$. Then $\cos x = \sin(\pi/2 - x) = -\sin(x - \pi/2) = -\sin t$, and $t \to 0$ as $x \to \pi/2$. Thus

$$\lim_{x\to\pi/2}\frac{\cos x}{x - \pi/2} = -\lim_{t\to 0}\frac{\sin t}{t} = -1 \ .$$

Problems 2.1

1. If, in definition 2.1 we just restrict attention to those $x > x_0$, we call the limit the limit from the right, denoted $\lim_{x\to x_0^+} f(x)$, and if we restrict to those $x < x_0$ we call it the limit from the left, denoted $\lim_{x\to x_0^-} f(x)$. Suppose that f is defined in an interval about x_0, and both the limit from the left and the limit from the right exist. Show that if they are both equal to L, then

$$\lim_{x\to x_0} f(x) = L \ .$$

2. Show that if f and g are functions defined in an interval near x_0 and

$$\lim_{x\to x_0} f(x) = L \ , \qquad \lim_{x\to x_0} g(x) = M \ ,$$

then

$$\lim_{x\to x_0} f(x)g(x) = LM \ .$$

Hint: Write $f(x) = L + a(x)$, $g(x) = M + b(x)$, noting that by the hypothesis we can ensure that $a(x), b(x)$ can be made as small as we please by taking x sufficiently close to x_0.

3. $$\lim_{x\to 2}\frac{x^2 - 4}{x^2 - 3x + 2} =$$

Hint: Factor numerator and denominator.

4. $$\lim_{x\to 0} \frac{\cos x - 1}{x \sin x} =$$

Hint: multiply and divide by $\cos x + 1$.

5. Suppose that f is defined in an interval about 0, and that $|f(x)| \le |x|^2$ in that interval. Show that f is differentiable at 0 and $f'(0) = 0$.

6. Show that the function f defined by $f(0) = 0$ and for $x \neq 0$, $f(x) = x^2 \sin(1/x)$ is differentiable at 0 and has derivative zero.

7. Show that the function f defined by $f(0) = 0$ and for $x \neq 0$, $f(x) = x \sin(1/x)$ is not differentiable at 0.

2.2 Limits at Infinity

Suppose that f is defined for all positive numbers. We say that f has the limit L as $x \to +\infty$ if we can make f as close as we please to L by taking x large enough. For example

$$\lim_{x\to+\infty} \frac{1}{x} = 0 \ ,$$

since we can make $1/x < \epsilon$ just by taking $x > 1/\epsilon$.

Definition 2.4. Suppose that $f(x)$ is defined for all $x > M_0$. We say that

$$\lim_{x\to+\infty} f(x) = L$$

if, for every $\epsilon > 0$, we can find an $M \ge M_0$ such that if $x > M$, then $|f(x) - L| < \epsilon$.

Suppose that $f(x)$ is defined for all $x < M_0$. We say that

$$\lim_{x\to-\infty} f(x) = L$$

if, for every $\epsilon > 0$, we can find an $M \le M_0$ such that if $x < M$, then $|f(x) - L| < \epsilon$.

Example 2.7.

$$\lim_{x\to+\infty} \frac{x}{x+1} = 1 \ .$$

For, given $\epsilon > 0$, choose $M = 1/\epsilon$. Then, for $x > M$, we have

$$\left|\frac{x}{x+1} - 1\right| = \left|\frac{x-(x+1)}{x+1}\right| = \left|\frac{1}{x+1}\right| < \left|\frac{1}{x}\right| < \frac{1}{M} = \epsilon \ .$$

Now, we define what it means to have $\pm\infty$ as a limit.

Definition 2.5. Let f be defined for all x in an interval about a, except perhaps at a. We write

$$\lim_{x\to a} f(x) = +\infty$$

if, for any $M > 0$, there is an $\epsilon > 0$ such that for $|x-a| < \epsilon$, we have $f(x) > M$.

Similarly,

$$\lim_{x\to a} f(x) = -\infty$$

if, for any $M > 0$, there is an $\epsilon > 0$ such that for $|x-a| < \epsilon$, we have $f(x) < -M$.

We will also say that $\lim_{x\to+\infty} f(x) = +\infty$ if we can make $f(x)$ as large as we please by taking x sufficiently large, and similarly, we define $\lim_{x\to+\infty} f(x) = -\infty$, $\lim_{x\to-\infty} f(x) = +\infty$, and so forth.

Proposition 2.9. Let p be a polynomial of degree $n > 1$, with leading coefficient 1 .
a) If n is even,

$$\lim_{x\to\pm\infty} p(x) = +\infty \ .$$

b) If n is odd

$$\lim_{x\to+\infty} p(x) = +\infty \ , \quad \lim_{x\to-\infty} p(x) = -\infty \ .$$

To see this, write $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$. Then, by factoring out the highest power of x:

$$p(x) = x^n\big(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\big) \ .$$

The term in parenthesis goes to 1 as $|x|$ becomes infinite. Now, since $|x^n| \geq |x|$ so long as $|x| \geq 1$, The term x^n approaches $+\infty$ as $|x|$ becomes large, except when n is odd, and $x \to -\infty$, in which case $x^n \to -\infty$.

Now, we can make the same kind of qualitative statements about quotients of polynomials (rational functions). Let $f(x) = p(x)/q(x)$ where p and q are polynomials with no common factors. Then f is defined and continuous at all points except those points a such that $q(a) = 0$. At such an a, the graph of $y = f(x)$ will go off the graph paper, either upwards or downwards, since the denominator is 0 at a. In this case we say that the graph has a *vertical asymptote* at $x = a$. To determine the behavior, we write $q(x) = (x-a)^n q_0(x)$ for some positive integer n and some polynomial q_0 such that $q_0(a) \neq 0$. Since p has no factor in common with q, $p(a) \neq 0$. Then

$$\lim_{x\to a} \frac{p(x)}{q(x)} = \lim_{x\to a} \frac{1}{(x-a)^n} \frac{p(x)}{q_0(x)} \ .$$

Since the second factor converges to $p(a)/q_0(a)$, the behavior of p/q near a is determined by the behavior of the first factor. For this, if n is odd, it depends upon whether we approach a from the right or the left, since $(x-a)^n$ is negative if $x < a$, and is positive if $x > a$. We summarize the result as

Proposition 2.10. a) If n is even,

$$\lim_{x\to a} \frac{1}{(x-a)^n} = +\infty \ .$$

b) If n is odd,

$$\lim_{x\to a^-} \frac{1}{(x-a)^n} = -\infty \ , \quad \lim_{x\to a^+} \frac{1}{(x-a)^n} = +\infty \ .$$

Finally, we summarize the limits for rational functions as $x \to \pm\infty$.

Proposition 2.11. Let $f(x) = p(x)/q(x)$, where p and q are polynomials of degree n and m respectively.

a) If $n < m$,

$$\lim_{x\to\pm\infty} f(x) = 0 \ .$$

b) If $n = m$,

$$\lim_{x\to\pm\infty} f(x) = \frac{a_n}{b_n} \ ,$$

where $a_n, \ b_n$ are the leading coefficients of p and q respectively.

c) If $n = m + d$,

$$\lim_{x\to\pm\infty} |f(x) - Q(x)| = 0$$

where Q is the polynomial of degree d obtained by dividing the polynomial p by the polynomial q.

The conclusion of part c) is to be understood this way: the graph of $y = f(x)$ approaches the graph of $y = Q(x)$ as $x \to \pm\infty$ so that for $|x|$ large enough the curves are indistinguishable. We say that he latter curve is an *asymptote* for $y = f(x)$.

Example 2.8. Let $f(x) = (x^2 + x)/((x + 2)$. f is not defined at $x = -2$. To see how f behaves at -2 and at infinity, we do the division:

$$\frac{x^2 + x}{x + 2} = x - 1 + \frac{2}{x + 2}$$

Thus, for x very close to, but to the left of -2, $f(x)$ is negative; but for x very close to, but to the right of -2, $f(x)$ is positive. Thus

$$\lim_{x\to-2^-} \frac{x^2 + x}{x + 2} = -\infty \ , \qquad \lim_{x\to-2^+} \frac{x^2 + x}{x + 2} = \infty \ .$$

Finally, $y = f(x)$has the asymptote $y = x - 1$ as x goes to infinity, since the difference

$$\left|\frac{x^2 + x}{x + 2} - (x - 1)\right| = \left|\frac{2}{x + 2}\right|$$

goes to zero as $|x|$ goes to infinity.

Example 2.9. Let

$$f(x) = \frac{x}{x - 1}$$

If we write

$$f(x) = x\frac{1}{x - 1}$$

after observing that x is positive near 1, we see from proposition 2.10 that

$$\lim_{x\to 1^-} f(x) = -\infty \ , \qquad \lim_{x\to 1^+} f(x) = \infty \ .$$

Finally, to see what happens as x goes to infinity, write

$$f(x) = \frac{1}{1-\frac{1}{x}} \quad \text{or} \quad f(x) = 1 + \frac{1}{x-1}$$

so that $y = 1$ is the asymptote.

Example 2.10. Let
$$f(x) = \frac{x}{(x-1)^2}$$

If we write
$$f(x) = x\frac{1}{(x-1)^2}$$
after observing that x is positive near 1, we see from proposition 2.10 that $\lim_{x\to 1} f(x) = \infty$ from both sides. Since the degree of the denominator is greater than the degree of the numerator, the asymptote is $y = 0$, with $y = f(x)$ below the x-axis to the left, and above the x-axis to the right.

Example 2.11. Let
$$f(x) = \frac{x^3 + 2x^2}{x^2 - 3x + 2} \ .$$
First, we factor numerator and denominator as much as possible:

$$f(x) = \frac{x^2(x+2)}{(x-1)(x-2)} \ .$$

From this we see that f is defined except for the points $x = 1, 2$. We know that at these points $f(x)$ becomes infinite; we need only to determine whether the limit is $+\infty$ or $-\infty$.

First we look near $x = 1$. To the left of 1, the negative terms are $x - 1$ and $x - 2$; since all others are positive, $f(x) > 0$. We conclude

$$\lim_{x\to 1^-} f(x) = +\infty \ .$$

But to the right of $x = 1$, the term $x - 1$ changes sign, so now $f(x) < 0$, and we conclude

$$\lim_{x\to 1^+} f(x) = -\infty \ .$$

As we pass through $x = 2$, the only change is in from $x - 2 < 0$ to $x - 2 > 0$, so we conclude

$$\lim_{x\to 2^-} f(x) = -\infty \ ,$$

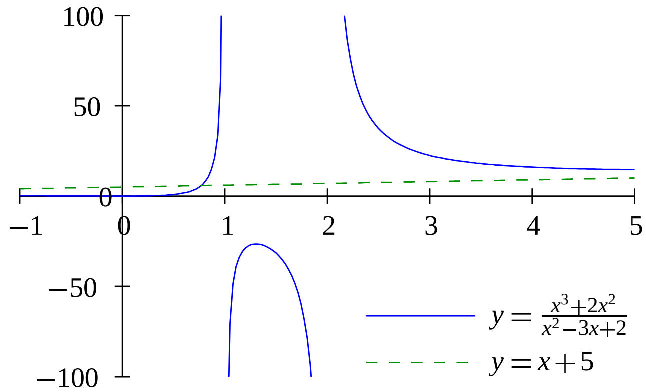
$$\lim_{x\to 2^+} f(x) = +\infty \ .$$

Finally we look for asymptotes as $x \to \pm\infty$. Long division gives

$$\frac{x^3 + 2x^2}{x^2 - 3x + 2} = (x + 5) + \frac{15x - 10}{x^2 - 3x + 2} ,$$

so, as x approaches infinity, the graph approaches the asymptote $y = x + 5$ (see the figure below).

Figure 2.2



Problems 2.2

1. Let $$f(x) = \frac{x^2 - 3x + 2}{x^2 + 3x - 4} .$$

Find $\lim_{x\to 1} f(x),\ \lim_{x\to\infty} f(x)$.

2. Let $$f(x) = \frac{x^2 + x}{(x + 2)^2} .$$

Find $\lim_{x\to -2} f(x),\ \lim_{x\to\infty} f(x)$.

3. Let $$f(x) = \frac{x^2 + x}{x - 1} .$$

Find $\lim_{x\to 1} f(x),\ \lim_{x\to\infty} f(x)$.

4. Let $$f(x) = \frac{3x^2 + 6x + 1}{x^3} .$$

Find $\lim_{x\to 0} f(x),\ \lim_{x\to\infty} f(x)$.

5. Suppose that $p(x)$ is a polynomial and that, for some $K > 0, M > 0$, we have $|p(x)| \le K|x|^n$ for $|x| \ge M$. Show that the degree of p is at most n.

6. Suppose that $p(x)$ is a polynomial and that, for some $K > 0, M > 0$, we have $|p(x)| \leq K|x|^2$ for $|x| \leq M$. Show that x^2 divides $p(x)$.

2.3 Some Basic Theorems

The preceding sections discussed the behavior of functions *locally*, that is, for x varying in a neighborhood of a particular point a. In this section we summarize more *global* results; that is the behavior of the function as x varies over an interval $[a, b]$. Most of these results are intuitively clear, and were taken as such by the founders of the Calculus.

Theorem 2.1 (Intermediate Value Theorem). Suppose that f is a continuous function on the interval $[a.b]$. Then, for every number w between $f(a)$ nd $f(b)$, there is a c between a and b such that $f(c) = w$.

Intuitively, this says that as you draw the graph of the function $y = f(x)$, your pencil point never leaves the paper.

Theorem 2.2 (Maxima and Minima). Suppose that f is a continuous function on the interval $[a.b]$. There are points c, C in $[a, b]$ such that $f(c)$ is the minimum value of f on the interval, and $f(C)$ is the maximum value of f on the interval.

Theorem 2.3 (Rolle's theorem). Let f be continuous on $[a, b]$ and differentiable in (a, b), and suppose that $f(a) = f(b)$. Then there is a point c in (a, b) at which $f'(c) = 0$.

We can derive Rolle's theorem from theorems 2.1 and 2.2. First of all, if f is constant, then $f'(c) = 0$ for all c. If f is nonconstant, there is a point c in the interval (a, b) at which f has either a maximum or a minimum. Suppose it is a maximum. Then, for all other x in (a, b), $f(x) \leq f(c)$. In particular, for $x < c$, $f(x) - f(c)$ and $x - c$ have the same sign, so

$$\frac{f(x) - f(c)}{x - c} \geq 0 \ .$$

Taking the limit as $x \to c$, we conclude $f'(c) \geq 0$. But now if $x > c$, the denominator changes sign, but the numerator doesn't, so, in this case

$$\frac{f(x) - f(c)}{x - c} \leq 0 \ ,$$

from which we conclude $f'(c) \leq 0$.Thus $f'(c) = 0$.

Notice, that the above argument incidentally shows that at a maximum or minimum point x_0 of a differentiable function, we must have $f'(x_0) = 0$. In the next chapter we shall see that this provides a method for finding maxima and minima.

Theorem 2.4 (The Mean Value Theorem). Let f be continuous on $[a, b]$ and differentiable in (a, b). There is a c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \ . \tag{2.1}$$

To see why this is true , start with figure 2.3.

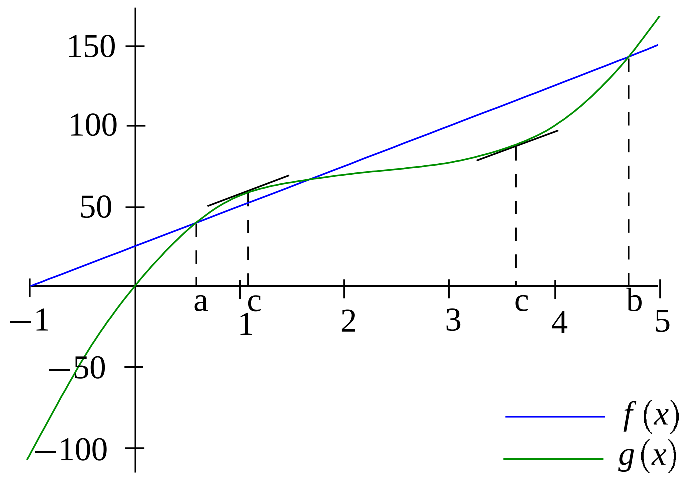


Figure 2.3

Here, $y = g(x)$ is the line joining $(a, f(a))$ to $(b, (f(b))$. The slope of this line is the right hand side of (1), which is also $g'(x)$ for any x in (a, b). Now the function $f(x) - g(x)$ satisfies the hypotheses of Rolle's theorem, so there is a c in (a, b) at which the derivative is zero, that is $f'(c) = g'(c)$. But this is the same as equation (2.1).

The point of this section is to demonstrate that Newton's uniqueness hypothesis (that a function with derivative zero everywhere is constant) follows from basic intuitive facts.

Theorem 2.5. Suppose that f is differentiable in an interval, and has derivative zero everywhere. Then f is constant.

Let $a,\ b$ be different points in the interval. By the Mean Value Theorem, there is a point c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\ .$$

But the hypothesis is that $f'(c) = 0$, so we must have $f(b) - f(a) = 0$, or $f(a) = f(b)$. This for any two points a and b, so f is constant.

Problems 2.3

1. Suppose that f is a continuous function on the interval $[0, 1]$ and $f(0) = 0,\ f(1) = 0$. By Rolle's theorem, if there is no point c between 0 and 1 for which $f'(c) = 0$, this must be because there are points in the interval at which f is not differentiable. Find such a function with just one point of nondifferentiability.

2. Suppose that f is a function differentiable on an interval (a, b) and that $f'(x) = c$, for some constant c. Show that there is a constant d such that $f(x) = cx+d$. Hint: consider $g(x) = f(x)-ax$.

3. Suppose that f is twice differentiable in an interval containing 0, and that $f(0) = 3$, $f'(0) = 4$ and $f''(x) = 2$ for all x in the interval. Show that $f(x) = x^2 + 4x + 3$.

4. Show by example that Theorem 2.1 is false for a discontinuous function.

5. Show by example that a function defined on an open interval (a, b) and continuous there need not have a maximum on the interval.

2.4 L'Hôpital's Rule

The student may have noticed that in almost all of the examples involving limits, the given expression was a quotient. Indeed, the very definition of the derivative is that of a limit of a quotient:

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \ . \tag{2.2}$$

In this section we introduce a quick way of calculating the limit of a quotient $f(x)/g(x)$ as x approaches a in the case where both $f(a) = 0$ and $g(a) = 0$. This is, of course, the interesting case; otherwise we can obtain the limit by substitution. This technique, called *l'Hôpital's rule*, will not work if either $f(a) \neq 0$ or $g(a) \neq 0$ as we shall see. So it is important to verify those conditions. l'Hôpital's rule also should not be confused with the rule for differentiating a quotient, which is quite a bit more complicated.

Proposition 2.12 (l'Hôpital's Rule). If f and g have continuous derivatives at a and $f(a) = 0$ and $g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \ .$$

To see this we use the Mean Value Theorem, theorem 2.4. According to that theorem, we can write $f(x) - f(a) = f'(c)(x - a)$ for some c between x and a, and $gx) - g(a) = g'(d)(x - a)$ for some d between x and a. Since $f(a) = 0$ and $g(a) = 0$, we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c)(x - a)}{g'(d)(x - a)} = \lim_{x \to a} \frac{f'(c)}{g'(d)} \ .$$

But now, by assumption the derivatives f' and g' are continuous. So, since c and d lie between x and a, $f'(c)$ and $g'(d)$ have the same limits as $f'(x)$ and $g'(x)$ as $x \to a$.

Example 2.12. $\lim_{x \to 0} \frac{\sin x}{x} =$

Here the functions are differentiable and both zero at $x = 0$, so l'Hôpital's rule applies:

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = \cos(0) = 1 \ .$$

Of course this example is a fake, since we needed to validate this limit just to show the differentiability of $\sin x$.

Example 2.13. $\lim_{x \to 0} \frac{\sin(3x)}{4x} =$

Both numerator and denominator are 0 at $x = 0$, so we can apply l'H (a convenient abbreviation for l'Hôpital's rule):

$$\lim_{x\to 0} \frac{\sin(3x)}{4x} =^{l'H} \lim_{x\to 0} \frac{3\cos(3x)}{4} = \frac{3}{4} \ .$$

Example 2.14. $\displaystyle \lim_{x\to 5} \frac{x^2 - 4x + 5}{x - 5} =$

Here both numerator and denominator are zero, so l'H applies:

$$\lim_{x\to 5} \frac{x^2 - 4x + 5}{x - 5} =^{l'H} \lim_{x\to 5} \frac{2x - 4}{1} = 6 \ .$$

Note that we could also have divided the numerator by the denominator, getting

$$\frac{x^2 - 4x + 5}{x - 5} = x + 1$$

whose value at $x = 5$ is 6.

Example 2.15. $\displaystyle \lim_{x\to 0} \frac{x + 2}{3x + 1} =$

Since neither the numerator nor denominator is zero at $x = 0$, we can just substitute 0 for x, obtaining 2 as the limit. Note that if we blindly apply l'Hôpital's rule, we get the wrong answer, 1/3.

Example 2.16. $\displaystyle \lim_{x\to 2} \frac{x^3 - 3x + 2}{\tan(\pi x)} =$

After checking that the hypotheses are satisfied, we get

$$\lim_{x\to 2} \frac{x^3 - 3x + 2}{\tan(\pi x)} =^{l'H} \lim_{x\to 2} \frac{3x^2 - 3}{\pi \sec^2(\pi x)} = \frac{12 - 3}{\pi} = \frac{9}{\pi} \ .$$

The second limit can be evaluated since both functions are continuous and the denominator nonzero.

Example 2.17. $\displaystyle \lim_{x\to 0} \frac{\sin^2(2x)}{\cos x - 1} =$

Both numerator and denominator are zero at $x = 0$, so l'Hôpital's rule applies:

$$\lim_{x\to 0} \frac{\sin^2(2x)}{\cos x - 1} =^{l'H} \lim_{x\to 0} \frac{4\sin(2x)\cos(2x)}{-\sin x} \ .$$

Now, numerator and denominator are still zero at $x = 0$, so we can apply l'Hôpital's rule again:

$$=^{l'H} \lim_{x\to 0} \frac{8\cos^2(2x) - 8\sin^2(2x)}{-\cos x} = -8 \ ,$$

for now we can take the limit by evaluating the functions.

l'Hôpital's rule also works when taking the limit as x goes to infinity.

Proposition 2.13. If f and g are differentiable functions, and

$$\lim_{x\to\infty} f(x) = 0 \qquad \lim_{x\to\infty} g(x) = 0 \ ,$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} \ .$$

We see that this is true by the substitution $t = 1/x$, which leads us back to proposition 8.1:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0} \frac{f(1/t)}{g(1/t)} =^{l'H} \lim_{t\to 0} \frac{\frac{-1}{t^2} f'(1/t)}{\frac{-1}{t^2} g'(1/t)} \ ,$$

by l'Hôpital's rule and the chain rule. But the factors introduced cancel, so, changing back to $x = 1/t$, we get the proposition.

Finally, we remark, without verification, that l'Hôpital's rule also works if the limits are infinite:

Proposition 2.14. If f and g are differentiable functions, and $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = a$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \ .$$

Here the limit point a may also be infinity.

Example 2.18. $$\lim_{x\to\infty} \frac{2x^2 - x + 3}{x^2 + 5x} =$$

By proposition 2.14, we can replace the numerator and denominator by their derivatives:

$$\lim_{x\to\infty} \frac{2x^2 - x + 3)}{x^2 + 5x} =^{l'H} \lim_{x\to\infty} \frac{4x - 1}{2x + 5} =^{l'H} \lim_{x\to\infty} \frac{4}{2} = 2 \ .$$

Of course, this could also be shown by direct application of proposition 2.11.

Problems 2.4

1. Find the limit of example 2.5 using l'Hôpital's rule.

2. Find the limit of example 2.6 using l'Hôpital's rule.

3. $$\lim_{x\to 2} \frac{x^2 - 4}{x^2 - 3x + 2} =$$

4. $$\lim_{x\to 3} \frac{x^2 - 3x}{x^2 + 2x - 15} =$$

5. $$\lim_{x\to 1^-} \frac{x - 1}{\sqrt{1 - x^2}} =$$

6. $$\lim_{x\to 0} \frac{x}{\sqrt{x^2}} =$$

7. $$\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x} =$$

8. $$\lim_{x\to\pi} \frac{\sin^2 x}{x^2 - 2\pi x + \pi^2} =$$

9. $$\lim_{x\to\pi} \frac{\sin^2 x}{x^2 - 3\pi x + 2\pi^2} =$$

10. $$\lim_{x\to\pi} \frac{\sin^2 x}{(x-\pi)^2} =$$

11. $$\lim_{x\to\pi/3} \frac{2\cos x - 1}{3x - \pi} =$$

III. Extrema, Concavity, and Graphs

3.1 Monotonicity and the First Derivative

In this chaqpter we will be studying the behavior of differentiable functions in terms of their derivatives. Thus, whenever a function f is introduced, it is to be understood that it is defined and has first and second derivatives on an interval I.

Definition 3.1. We say
a) f is *increasing* on the interval I if, for $a \leq b$, we have $f(a) \leq f(b)$.
b) f is *strictly increasing* on the interval I if, for $a < b$, we have $f(a) < f(b)$.
c) f is *decreasing* on the interval I if, for $a \leq b$, we have $f(a) \geq f(b)$.
b) f is *strictly decreasing* on the interval I if, for $a < b$, we have $f(a) > f(b)$.

Proposition 3.1
a) If f' is always positive on I, then f is strictly increasing on the interval I.
b) If f' is always negative on I, then f is strictly decreasing on the interval I.

For $a < b$, we have $b - a > 0$. By the mean value theorem (see theorem 2.4 of chapter 2), there is a c between a and b such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \ .$$

Now in case the hypothesis of a) holds, this is positive. Since the denominator of the expression on the left is positive, that implies that $f(b) - f(a) > 0$. On the other hand, if b) holds, so $f'(c) < 0$, we conclude that the numerator is negative, so $f(b) - f(a) < 0$.

Definition 3.2. a) Let a be a point in I. We say that f has a *local maximum* at a if, for all x sufficiently close to a, $f(a) \leq f(x)$.
b) We say that f has a *local minimum* at a if, for all x sufficiently close to a, $f(a) \geq f(x)$.

If we want to graph the function $y = f(x)$, it is important to calculate f', and determine the intervals in which it is positive or negative. Then we know that the graph must "go up" in an interval where f' is positive, and "go down" where f' is negative. Clearly, at points at which the sign of of f' changes, there must be either a local maximum or a minimum. For example, if f' is negative to the left of a, and positive to the right of a, then f is decreasing to the left of a and increasing to the right of a, so has a local minimum at a.

Proposition 3.2 (First Derivative Test). a) If $f'(x) > 0$ for $x < a$ and $f'(x) < 0$ for $x > a$, then f has a local maximum at a.
b) If $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > a$, then f has a local minimum at a.

If the function has a continuous second derivative, the following test may be computationally easier:

Proposition 3.3 (Second Derivative Test). a) If $f'(a) = 0$ and $f''(a) < 0$, then f has a local maximum at a.
b) If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at a.

Take case (a). We must have f'' negative in a neighborhood of a, so f' is decreasing in that neighborhood. Since $f'(a) = 0$, we must have f' positive to the left of a and negative to the right. So, by the first derivative test, f has a local maximum at a.

Example 3.1. Let $f(x) = 2x^3 - 15x^2 + 24x + 10$. Then

$$f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x-1)(x-4) \ ,$$

$$f''(x) = 12x - 30 = 6(2x - 5) \ .$$

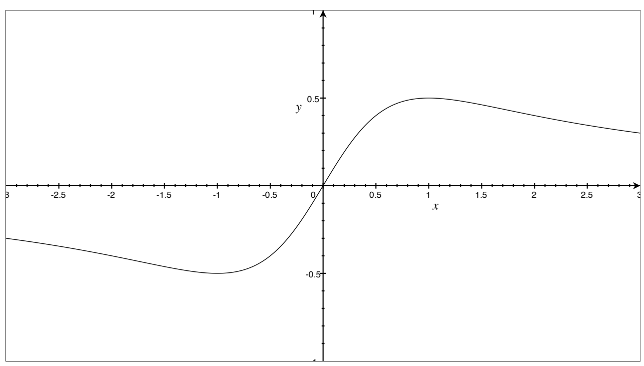
We see that $f'(1) = 0$, and $f'(4) = 0$, and $f' > 0$ for $x < 1$, $f' < 0$ for x between 1 and 4, and $f' > 0$ for $x > 4$. Thus the graph of $y = f(x)$ is increasing until $x = 1$, decreases from 1 to 4, increases again for $x > 4$, and $x = 1$ is a local maximum, $x = 4$ is a local minimum. We can confirm this by calculating the second derivative: $f''(1) = -18 < 0$, and $f''(4) = 18 > 0$. To see the graph of this function, go to example 3.8 of section 3.

Example 3.2. Find the intervals in which $f(x) = x/(x^2 + 1)$ is increasing and decreasing.

First, we calculate the derivative:

$$f'(x) = \frac{x^2 + 1 - x(2x)}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2} \ .$$

Since the denominator is always positive, the sign is determined by the numerator $1 - x^2$. This is negative for $x < -1$ and $x > 1$ and positive for $-1 < x < 1$. Thus the function is increasing between -1 and 1 and otherwise decreasing. Noticing that $f(x) < 0$ for $x < 0$, $f(x) > 0$ for $x > 0$, and that $f(x) \to 0$ as $|x| \to \infty$, we have enough information to sketch the graph:



Problems 3.1

1. Find all points of local maxima and minima of the function $f(x) = x(4 + x^{-2})$.

2. For what number x between 0 and 1 is $x^{1/3} - x$ a maximum?

3. Let $f(x) = 2x^3 - 24x^2 + 72x + 12$. Find the intervals in which $f(x)$ is increasing; in which $f(x)$ is decreasing.

4. Let

$$y = \frac{x}{x^2 - 4x + 3} \ .$$

Find the intervals in which y is increasing; in which y is decreasing.

5. Let $h(x) = \sec x + \tan x$. Find out where h is increasing or decreasing in the interval $(0, 2\pi)$.

6. Let $y = x^4 - x^3 - x + 1$. Find the value of x where y has its minimum.

7. Graph a function with these conditions: a) $f(0) = 2$, b) $f(2) = 4$, c) $f'(x) < 0$ for $x < 0$, d) f has a local maximum between $x = 0$ and $x = 2$, e) f has a local minimum at some point c with $c > 2$.

8. Show that the equation $2x^{12} - 3x^6 + x = 0$ has a root strictly between 0 and 1.

3.2 Optimization

The knowledge about a function that we obtain from the first derivative often suffices to solve practical optimization problems. Typically, the given situation leads to a function $y = f(x)$ defined on a closed interval $a \le x \le b$, and the problem is to find the maximum or minimum value of the function on the interval. This can occur only at one of the following points: the endpoints, a, b, any point in the interval at which f does not have a derivative, or any point c on the interval at which $f'(c) = 0$. These are the *critical points* of the function. Evaluate the function at all of the critical points; the largest of these values is the maximum on the interval, and the smallest, the minimum.

Example 3.3. Find the maximum and the minimum of $y = x\sqrt{1-x^2}$ on the interval $-1 \le x \le 1$.

Differentiate:

$$\frac{dy}{dx} = \sqrt{1-x^2} + x\frac{-2x}{2\sqrt{1-x^2}} = \frac{1-2x^2}{(1-x^2)^{3/2}}$$

This is zero when $x = \pm 1/\sqrt{2}$. The critical values are $-1, -1/\sqrt{2}, 1/\sqrt{2}, 1$, and the corresponding values of y are 0, $-[(1/2)\sqrt{3/2}]$, $[(1/2)\sqrt{3/2}]$, 0, so the nonzero values are the minimum and maximum repsectively.

Example 3.4. Let $y = \sin^2 x + \cos x$, for x in the interval $[-\pi, \pi]$. Find the absolute maximum and minimum of y.

Differentiate:

$$y' = 2\sin x \cos x - \sin x = \sin x(2\cos x - 1)\ .$$

This is zero at $x = -\pi, 0, \pi$ and $x = \pm\pi/3$. The values of y at these points are

x	$-\pi$	$-\pi/3$	0	$\pi/3$	π
y	-1	$\frac{5}{4}$	1	$\frac{5}{4}$	-1

Thus the absolute maximum is 5/4, and the absolute minimum is -1. Note that at $x = 0$ we have a local minimum.

Many problems involve finding the values of certain variables which make a related variable a maximum or a minimum. The method of attack on such probems is as follows:

Step 1. Draw a picture (if appropriate), and identify the relevant variables: those things which can change. State the problem in terms of the variables. Here it is important to distinguish the variable to be optimized: call it the *objective variable*.

Step 2. Express the objective variable in terms of the other relevant variables. If there is more than one such variable, look for relations among them: these are called the *constraints*.

Step 3. Use the constraints to express the objective as a function of only one of the other variables.

Step 4. Differentiate and set the derivative equal to zero.

Step 5. Calculate the values of the objective variable at each of the points found in step 4 (as well as the endpoints of the range of that variable), and choose the one which is the desired maximum (or minimum).

Example 3.5. A triangle in the first quadrant has vertices at the points $(0,0),(t,0),(t,9-t^2)$. Find the triangle of of maximum area.

Here the variables are area, A, and value t determining the triangle. Area is the objective variable; in terms of t this area is

$$A(t) = \frac{1}{2}t(9-t^2) = \frac{1}{2}(9t-t^3) \ .$$

Since the triangle is in the first quadrant, we must have $t \geq 0$ and $9-t^2 \geq 0$, so the interval on which the area is defined is $0 \leq t \leq 3$. Now

$$f'(t) = \frac{9-3t^2}{2}$$

so $f'(t) = 0$ when $t = \pm\sqrt{3}$. The critical points on the interval in question are $0, \sqrt{3}, 3$. The values of f at these points are $0, 3\sqrt{3}, 0$, so the maximum value is $3\sqrt{3}$.

Example 3.6. A box with a square bottom is to be made so as to contain 150 in^3.The cost of the material to make the sides is \$2 per in^2 and the material for the top and bottom is \$3 per in^2. What are the dimensions of the box of minimal cost?

Here the variables are: x, the length of a side of the base of the box, h, the height, the area A_b of the base (which is the same as the area of the top), the area A_s of a side (of which there are 4), and the cost, C. Since we want to minimize C, this is the objective variable.

According to our data, the cost is given by $C = 3(2A_b) + 2(4A_s)$. Now $A_b = x^2$, and $A_s = xh$, so we obtain

$$C = 6x^2 + 8xh \ .$$

Now, we need a relationship between x and h: that is provided by the constraint that the box must contain 150 in^3. Thus $x^2h = 150$, or $h = 150x^{-2}$. The objective equation becomes

$$C = 6x^2 + 8x(150x^{-2}) = 6x^2 + 1200x^{-1} \ . \tag{3.1}$$

In principle, the variable x ranges over all positive numbers, but we note from equation (3.1) that C becomes arbitrarily large as x becomes very large or very small. Thus there is a minimum somewhere between: at a place where $C'(x) = 0$. Differentiating:

$$C'(x) = 12x - 1200x^{-2} \ ,$$

so $C'(x) = 0$ when $12x^3 = 1200$, or $x = 4.64$ in. Since this is the only point at which $C' = 0$, it is the value of x which minimizes the cost. Using $x^2 h = 150$, we find $h = 6.96$ in. Thus the box should be (approximately) 4.5 in square on the base, and of height 7 in.

Example 3.7. An automobile maufacturer sells, on average, 8000 cars per month at twenty-five thousand dollars. The marketing department has determined that for every one thousand dollar reduction in price, the company can sell an additional 500 cars per month. At what price should the car be sold so as to maximize revenue?

Let p be the price reduction (in thousands of dollars) at which the automobile is to be sold. With this reduction, the manufacturer sells $8 + p/2$ thousand cars, and the price is $25 - p$ thousand. Thus the total revenue is $R = (8+p/2)(25-p)$. To maximize revenue we differentiate, and set the derivative equal to zero:

$$R' = \frac{1}{2}(25 - p) - (8 + p/2) = 4.5 - p \ .$$

The desired reduction is $p = \$4500$ and the selling price should be \$20,500.

Problems 3.2

1. Find the absolute maxima and minima of the function

$$f(w) = w\sqrt{w+1}$$

on the interval [-1.4].

2. Find the maximum and the minimum of $y = x\sqrt{1-x^2}$ on the interval $-1 \leq x \leq 1$.

3. Let $y = \sin x + \cos^2 x$, for x in the interval $[-\pi, \pi]$. Find the absolute maximum and minimum of y.

4. Find the dimensions of the right triangle with one vertex at the origin, another on the positive x-axis, and the third on the curve $y = 4 + x^{-2}$ which is of minimum area.

5. We are asked to make an open-topped box out of a rectangular sheet of tin 24 in. wide and 45. in long. This is to be done by cutting congruent squares out of each corner of the sheet and then bending sides upward to from the sides of the box. What are the dimensions of the box of greatest volume?

6. A particle travels at 2 ft/sec in the upper half-plane, and at 3 ft/sec on the x-axis. The object starts at one foot up the y-axis (at (0,1)), and will travel to a point 3 ft down the x axis (at (3,0)) by heading straight for some point (x,0) and then along the x axis. Find the value of x which minimizes the time it takes.

7. A commuter train carries 600 passengers each day from a suburb to a city. It costs \$ 1.50 per person to ride the train. It is found that 40 more people will ride the train for each 5 cent decrease in the fare. What fare should be charged to make the largest possible revenue?

8. A rectangular racecourse is to be made so the diagonal measures 5 furlongs, and so we can place 20 spectators per furlong along the horizontal sides, and 30 spectators per furlong along the

vertical sides. What should the dimensions of the course be so that number of spectators is the maximum?

9. The heat produced at a point x feet from a heating source is proportional to I/x^2 where I is the intensity of the source. Suppose that two heaters, one three times as intense as the other, are place 60 feet apart. At what point between the heaters is the temperature a minimum?

10. Farmer Brown wants to build a right triangular chicken coop containing 100 square feet. The hypotenuse will lie on an existing wall, but the other two sides are to be built. What should the dimensions of these sides be so as to minimize the sum of their lengths?

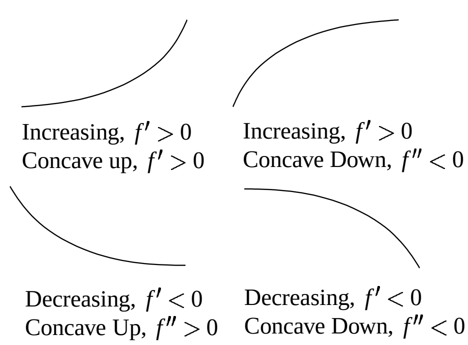
11. The Jones Jumpersuit Company can sell $400 - 8p$ jumpersuits each month at a price of $120 + p$ dollars. Jones has fixed costs of \$ 8000 per month, and the cost in labor and material for each suit is \$ 25. At what price will Jones maximize profit?

12. Tickets for the Giulia Opera sell at \$65 each. However, for groups, this deal is offered: for every 10 tickets over 100 purchased there will be a 2% discount on all tickets. Of course, as we can see there has to be an upper limit: for a group of 600 people, the tickets will be free. Giulia decides to set a limit at that number which maximizes the total revenue. What is that number?

3.3. Concavity and the Second Derivative

Definition 3.3. a) If, at every point a in I, the graph of $y = f(x)$ always lies above the tangent line at a, we say that f is *concave up*. ((somehow the original text has $-f$, not f)) b) If, at every point a in I, the graph of $y = f(x)$ always lies below the tangent line at a, we say that f is *concave down*. (See figure 3.1).

Figure 3.1



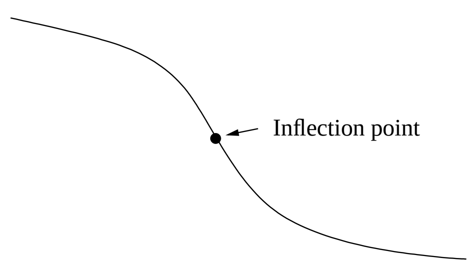
Proposition 3.4.
a) If f'' is always positive in the interval I, then f is concave up in that interval.
b) If f'' is always negative in the interval I, then f is concave down in that interval.

The reasoning is this: where $f'' > 0$ the slope of the tangent line is increasing, so the graph of the function is curving counterclockwise. Another way to put this: under the condition $f`''(a) > 0$, near a the graph of $y = f(x)$ lies above the tangent line at the point $(a, f(a)$. The equation of the tangent line is $y = g(x) = f(a) + f'(a)(x - a)$; we want to show that $f(x) > g(x)$ for x near a, $x \neq$. That is, we want to show that the function $h(x) = f(x) - g(x)$ has a minimum at $x = a$. Now, $h'(x) = f'(x) - f'(a)$, so $h'(a) = 0$. Furthermore $h''(a) = f''(a) > 0$, so, by the second derivative test, h has a minimum at a.

By including this information given by the second derivative, we can accurately sketch the graph of the function $y = f(x)$, by determining the intervals in which f' and f'' have given signs. The four possibilities are illustrated in figure 1. Points where the second derivative changes sign are points at which the concavity changes: these are called *points of inflection* (see figure 3.2). Clearly

Proposition 3.5. If a is a point of inflection of f, and $f''(a)$ exists, then it is zero.

Figure 3.2

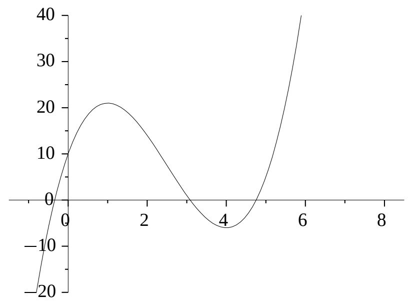


Example 3.8. For $f(x) = 2x^3 - 15x^2 + 24x + 10$, we calculated (in example 3.1) $f''(x) = 6(2x - 5)$. Thus f'' is negative for $x < 5/2$, and positive for $x > 5/2$, so $x = 5/2$ is a point of inflection. Putting together all the information we have obtained on this function we can draw its graph showing all important features. First we tabulate the data obtained:

x	f	f'	f''
$x < 1$	*incr*	> 0	*conc dwn*
1	21	0	-6
$1 < x < 2.5$	*decr*	< 0	*conc dwn*
2.5	7.5	-13.5	0
$2.5 < x < 4$	*decr*	< 0	*conc up*
4	-6	0	18
$x > 1$	*incr*	> 0	*conc up*

Based on these data, we draw the graph:

Figure 3.3: Plot of $2x^3 - 15x^2 + 24x + 10$



Problems 3.3

1. Let $y = (x-2)^2 + \frac{1}{3}(x-1)^3$. Find the intervals in which the function is increasing and decreasing, and where it is concave up and concave down. Sketch the graph.

2. Consider the function $y = 2x^3 + 3x^2 - 12x + 11$. Find the intervals in which the function is concave up, and in which it is concave down. Sketch the graph.

3. For the following function, find a) all critical values, b) intervals in which the function is increasing and where it is decreasing, c) intervals in which the function is concave up or concave down:

$$g(x) = x^4 - 4x^3 + 4x^2 + 2 \ .$$

Sketch the graph.

4. Consider the function

$$y = \frac{1}{x} - \frac{5}{2x^2} + \frac{4}{3x^3} \ ,$$

defined for $x > 0$. a) Find the intervals in which the function is increasing, and the intervals in which it is decreasing. b) Find the intervals in which the function is concave up, concave down. Sketch the graph.

5. Consider the function

$$f(x) = \sin x + \cos^2 x$$

as defined on the interval $-\pi, \pi]$ (see problem 3 of section 3.2). Find a) all critical values, b) all points of inflection, c) the value at which the function takes its maximum. Sketch the graph.

3.4 Graphing Functions

Sketching Graphs of Rational Functions

To sketch a graph of a function $y = f(x)$ follow these steps.

1. Determine where the function is not defined.

2. Determine the points x where $f'(x) = 0$ and where $f''(x) = 0$.

3. Make a table of the values of $f,\ f',\ f''$ at the points found in steps 1 and 2.

4. Determine the signs of f' and f'' in the intervals separated by the points found in steps 1 and 2.

5. Determine the vertical asymptotes of the function at the points found in step 1 (see Chapter 2, section 2).

6. Determine the horizontal asymptotes of the function as $x \to \pm\infty$ (see Chapter 2, section 2).

7. Using all this information and the concavity templates of figure 3.1, sketch the graph.

Example 3.9. Let $y = x^4 - 18x^2 + 2$. We shall follow the steps above:

1. The function is defined everywhere.

2. Calculate the first and second derivatives:

$$f'(x) = 4x^3 - 36x^2 = 4x(x^2 - 9)\ ,$$

$$f''(x) = 12x^2 - 36 = 12(x^2 - 3)\ .$$

$f'(x) = 0$ for $x = 0, \pm 3$ and $f''(x) = 0$ for $x = \pm\sqrt{3}$.

3. We create the table of relevant values at these points. In making this table, leave spaces for additional values of x which may be relevant.

x	f	f'	f''
-3	-79	0	
$-\sqrt{3}$	-43		0
0	2	0	
$\sqrt{3}$	-43		0
3	-79	0	

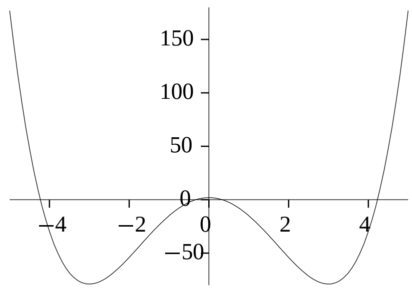
4. We still have to determine the signs of f'' at $x = 0, \pm 3$, and the signs of f' at $x = \pm\sqrt{3}$, and finally at points before -3 and after 3. When that is done, we shall have completed the table:

x	f	f'	f''
< -3	$\to +\infty$	$-$	$+$
-3	-79	0	$+$
$-\sqrt{3}$	-43	$+$	0
0	2	0	$-$
$\sqrt{3}$	-43	$-$	0
3	-79	0	$+$
> 3	$\to +\infty$	$+$	$+$

5 and 6. The asymptotes as $x \to \pm\infty$ are read off the first and last lines of the table.

7. To graph, first locate the points $(x, f(x))$ for x on the table, then connect these points using the knowledge about f' and f'', and the templates of figure 3.2. The result is shown in figure 3.4.

Figure 3.4: Plot of $f(x) = x^4 - 18x^2 2 + 2$



Example 3.10. Graph

$$y = \frac{x}{x-1} \ . \tag{3.2}$$

1. The function is not defined at $x = 1$.
2. We can rewrite the function (using long division) as

$$f(x) = 1 + (x-1)^{-1} \ .$$

Then

$$f'(x) = -(x-1)^{-2} \ , \qquad f''(x) = 2(x-1)^{-3} \ .$$

Thus there are no points at which these are zero, so the only important data will be the signs of f, f' and f'' in the various intervals.

4. Calculation of f and its derivatives at sample points yields the table

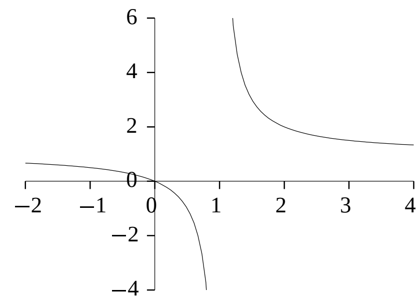
x	f	f'	f''
< 0	$+$	$-$	$-$
0	0	$-$	$-$
$0 < x < 1$	$-$	$-$	$-$
1	ND	ND	ND
> 1	$+$	$-$	$+$

5. The vertical assymptotes are at $x = 1$. Line 3 of the table tells us that as $x \to 1^-$, $f(x)$ is negative, so $f(x) \to -\infty$, and line 5 of the table tells us that as $x \to 1^+$,$f(x)$ is positive, so $f(x) \to +\infty$

6. The form (3.2) of the function tells us that as $x \to -\infty$, $f(x) \to 1^-$ and as $x \to +\infty$, $f(x) \to 1^+$.

7. The graph can now be drawn as in figure 3.5.

Figure 3.5



Example 3.11. Graph

$$y = \frac{x}{x^2 - 1} \ .$$

1. The function is not defined at the points $1, -1$.
2.

$$f'(x) = -\frac{x^2 + 1}{(x^2 - 1)^2} \ , \qquad f''(x) = \frac{2x}{(x^2 - 1)^3} \ .$$

3. We note that f' is never zero; in fact, it is always negative (except where undefined), and f'' is zero just at $x = 0$. This gives the preliminary table

x	f	f'	f''
< -1		$-$	
-1	ND	ND	ND
$-1 < x < 0$		$-$	
0	0	$-$	0
$0 < x < 1$		$-$	
1	ND	ND	ND
$x > 1$		$-$	

4. A few calculations at intermediate points fills in the table:

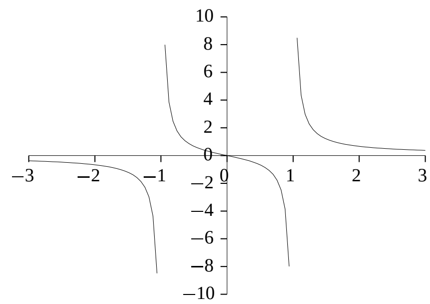
x	f	f'	f''
< -1	$-$	$-$	$-$
-1	ND	ND	ND
$-1 < x < 0$	$+$	$-$	$+$
0	0	$-$	0
$0 < x < 1$	$-$	$-$	$-$
1	ND	ND	ND
$x > 1$	$+$	$-$	$+$

5. We determine the nature of the asymptotes on each side of -1 using lines 1 and 3 of the table, and on each side of 1 using lines 5 and 7.

6. Since the degree of the denominator is greater than the degree of the numerator, the line $y = 0$ is the horizontal asymptote on each side. The approach as $x \to -\infty$ is from below (by line 1), and from above (by line 7) as $x \to +\infty$.

7. Putting these data together, we obtain the graph in figure 3.6.

Figure 3.6: Plot of $\frac{x}{x^2-1}$



Other Sketches

To graph more complicated functions, we follow the same steps, but may have to do some additional investigations.

Sometimes the asymptotic directions give us sufficient information, and it is not really necessary to calculate the first and second derivative (which for a rational function could be quite tedious). To illustrate this, we return to example 2.11.

Example 3.12. Graph

$$f(x) = \frac{x^3 + 2x^2}{x^2 - 3x + 2} \ .$$

In section 2.2 we found the factorization

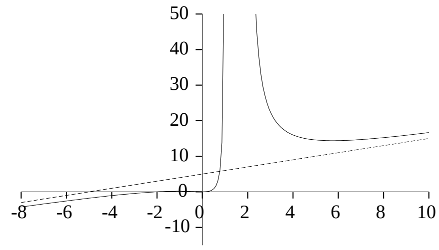
$$f(x) = \frac{x^2(x+2)}{(x-1)(x-2)} \ ,$$

and the asymptotic behavior as $x = 1, 2$. Also, by long division, we saw that the horizontal asymptote is $y = x + 5$. Noting that the function is zero at the points $x = 0, -2$, we can tabulate the results just for the values of the function.

x	< -2	-2	$-2 < x < 0$	0	$0 < x < 1$	1	$1 < x < 2$	2	$x > 2$
$f(x)$	$-$	0	$+$	0	$-$	ND	$-$	ND	$+$

The simplest graph with these properties and the horizontal asymptote $y = x+5$ is shown in figure 3.7. We'd have to calculate the derivatives to be sure that there are no other ripples, but as a first sketch, this is satisfactory. (In fact, figure 3.7 does accurately show the local maxima and minima and the correct concavity).

Figure 3.7: Plot of $\frac{x^3+2x^2}{x^2-3x+2}$ and the asymptote $y = x + 5$



Example 3.13. Graph $y = \sqrt{x^2 - 1}$.

1. Unless otherwise specified, the symbol $\sqrt{}$ refers to the nonnegative square root. Since we cannot take the square root of a negative number, this function is undefined for x between -1 and 1.

2. We calculate:

$$f'(x) = x(x^2 - 1)^{-1/2} \ , \qquad f''(x) = -(x^2 - 1)^{-3/2} \ .$$

3 and 4. We obtain the following table of relevant values:

x	f	f'	f''
< -1	$+$	$-$	$-$
-1	0	∞	∞
$-1 < x < 1$	ND	ND	ND
1	0	∞	∞
> 1	$+$	$+$	$-$

Thus the graph is decreasing for $x \leq 1$ and increasing for $x \geq 1$ and is everywhere concave down.

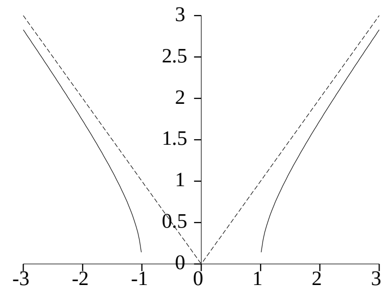
5 and 6. There are no vertical asymptotes. To find the horizontal asymptotes, we rewrite the function as

$$y = |x|\sqrt{1 - \frac{1}{x^2}} \ .$$

Since the second factor tends to 1 as $x \to \infty$ we conclude that the graph approaches that of $y = |x|$ asymptotically from below.

7.

Figure 3.8: Plot of $y = x^2 - 1$ and the asymptote $y = |x|$.



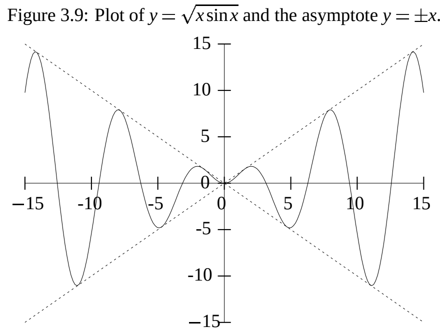
Example 3.14. Graph $y = x \sin x$.

The function is defined everywhere. Since $\sin x$ oscillates, this function oscillates, but grows in height as $|x|$ grows. In fact the graph will hit the lines $y = \pm x$ at every point where $|\sin x| = 1$; that is, at all points $x = n\pi + \pi/2$. This is enough information to graph the function, except around zero. But, since

$$f'(x) = \sin x + x \cos x$$

we see that $f'(0) = 0$, and the graph has a horizontal tangent at the origin. The graph now can be drawn as a sine curve "damped" by the lines $y = \pm x$ (see figure 3.9).

Figure 3.9: Plot of $y = \sqrt{x \sin x}$ and the asymptote $y = \pm x$.



Problems 3.4

1. Let

$$y = \frac{1}{x^2 + 1}.$$

Find the x coordinate of the points of inflection of the curve so defined. Sketch the graph.

In each of the following, derive enough information about the curve to sketch the graph, showing clearly all asymptotes.

2. Sketch the graph : $$y = \frac{x}{x^2+1}.$$

3. Sketch the graph : $$y = \frac{x}{(x-1)(x-2)}.$$

4. Sketch the graph : $$y = \frac{x^2}{(x-1)(x-2)}.$$

5. Sketch the graph : $$y = \frac{x^3}{(x-1)(x-2)}.$$

6. Sketch the graph : $$y = \frac{x^2-3x+2}{x^2-7x+12}.$$

7. Sketch the graph : $$y = \sec x + \tan x \quad 0 \le x \le 2\pi\ .$$

To get started, recall problem 5 of section 3.1.

8. Sketch the graph : $$y = \sqrt{1-\cos x} \quad 2\pi \le x \le 2\pi\ .$$

IV. Integration: A Differential Equations Approach

4.1 Antiderivatives

Newton's basic idea of dynamics is this: if we know the state of a system at a particular time, and we know the law of change, then we can predict the state at any future time. The law of change will ordinarily be an equation involving the function, its derivative, and the independent variable: this is called a *differential equation*. Newton's first law of motion expresses this idea: in the absence of external forces, an object in motion will continue its motion in the same direction with the same speed. Put another way, if acceleration is zero, then velocity is constant; and yet another way: if $dv/dt = 0$, then $v(t) = v(0)$ for all t. We have already seen this in Chapter II (as theorem 2.5):

Proposition 4.1. Suppose that f is differentiable in an interval, and has derivative zero everywhere. Then f is constant.

As a consequence, we have

Proposition 4.2. If two functions have the same derivative, they differ by a constant.

For suppose f and g are the two functions, and $f' = g'$. We can apply proposition 4.1 to $h = f - g$: $h' = f' - g' = 0$, so $h(x) = C$, some constant. Then $f(x) = g(x) + C$.

Definition 4.1. Given a function f, any function F such that $F' = f$ is an *antiderivative* or *indefinite integral*, or just *integral* of f. Any integral is denoted

$$\int f(x)dx \ .$$

We emphasize that any two integrals of a given function differ by a constant. So, for example, we know that if $f(x) = x^2$, then $f'(x) = 2x$, so x^2 is an integral of $2x$, and therefore any integral of $2x$ is of the form $x^2 + C$, for some constant C. We indicate this by writing $\int 2xdx = x^2 + C$. This process of finding integrals is called *integration*. Now, the formulas of the differential calculus lead to the formulas for finding integrals, although not always so easily, as we shall see. For example, since the derivative of a sum is the sum of the derivatives, then the integral of a sum is the sum of the integrals. Since we differentiate x^n by multiplying by the exponent and reducing the exponent by 1, we integrate x^n by reversing the process: increase the exponent by 1, and divide by the new exponent. To summarize:

Proposition 4.3.

a) Let f be a given function, and a a number. Then $\int af(x)dx = a\int f(x)dx$.

b) Let f and g be given functions. Then $\int(f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$.

c) $\int x^n dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1$.

Of course, the exclusion $n = -1$ is necessary, for in this case the right hand side doesn't make sense. However, as we'll see in section 4.5, $\int x^{-1}dx$ does make sense and defines an important function.

Example 4.1. Find the integral of $f(x) = x^4 - 3x^2 + x - 1$. We integrate term by term, using Proposition 4.3c for each term:

$$\int f(x)dx = \frac{x^5}{5} - 3\frac{x^3}{3} + \frac{x^2}{2} - x + C$$

$$= \frac{1}{5}x^5 - x^3 + \frac{1}{2}x^2 - x + C$$

Example 4.2.

$$\int (4x^{-3} + x^2)dx = 4(\frac{x^{-2}}{-2}) + \frac{x^3}{3} + C = -2x^{-2} + \frac{x^3}{3} + C$$

Example 4.3. A function $f(x)$ has the derivative $f'(x) = x^2 + x^{-2}$, and the value at $x = 2$ is 5. What is the function?

First we find the general function with the given derivative by integrating term by term:

$$f(x) = \frac{1}{3}x^3 - x^{-1} + C$$

Now we substitute the given values, and solve for C:

$$5 = \frac{1}{3}2^3 - \frac{1}{2} + C$$

giving $C = 17/6$. Thus the desired function is

$$f(x) = \frac{1}{3}x^3 - x^{-1} + \frac{17}{6}$$

Proposition 4.3 shows us how to find integrals of polynomials. The differentiation formulas for the trigonometric functions also lead to integration formulas for these functions.

Proposition 4.4.

a) $$\int \cos x dx = \sin x + C$$

b) $$\int \sin x dx = -\cos x + C$$

since the derivative of $\sin x$ is $\cos x$, and the derivative of $-\cos x$ is $\sin x$.

So far, we can only integrate functions by, so to speak, reading a table of derivatives in the reverse direction. For example, we also know from these tables that

$$\int \sec^2 dx = \tan x + C \quad \text{and} \quad \int \sec x \tan x dx = \sec x + C \ ,$$

but we don't yet know the integral of $\sec x$, or for that matter, $\tan x$ or x^{-1}. Finding integrals in general is a quite complicated process, and as this course proceeds we will study the various techniques of integration.

The first, and most useful of these techniques is that of *substitution*. This is the integration form of the chain rule. It is most conveniently stated in terms of differentials.

Proposition 4.5. Given variables u and x; suppose we know that $u = u(x)$ is a function of x. Suppose their differentials are related by

$$f(x)dx = g(u)du \tag{4.1}$$

for some functions f and g. Then

$$\int f(x)dx = \int g(u)du + C \ .$$

To see this, let $G(u) = \int g(u)du$. Then, treating G as a function of x by the substitution $u = u(x)$, we have

$$\frac{dG}{dx} = \frac{dG}{du}\frac{du}{dx}$$

by the chain rule. But $dG/du = g(u)$, so

$$\frac{dG}{dx} = g(u)\frac{du}{dx} = f(x)$$

by the relation (4.1). Thus $G(u(x))$ is an integral of $f(x)$.

This explains in part the notation for the integral: we should be thinking that it is the differential $f(x)dx$ which we are integrating, rather than the function. For when we change variables by substitution, it is the entire differential which we must consider.

Example 4.4. $\int(5x-3)^5dx = \ ?$

Since we would rather not multiply $5x - 3$ by itself 5 times so we can use Proposition 4.3, we instead introduce the variable $u = 5x - 3$. Then $du = 5dx$, or $dx = (1/5)du$, so

$$(5x-3)^5dx = \frac{1}{5}u^5du \ .$$

We now apply the power rule to the right hand side:

$$\int \frac{1}{5}u^5du = \frac{1}{5}\cdot\frac{1}{6}u^6 + C$$

and then replace u by its expression $5x - 3$ as a function of x:

$$\int(5x-3)^5dx = \frac{1}{30}(5x-3)^6 + C \ .$$

Example 4.5. $\int x(x^2+1)^3 dx = \ ?$

To integrate by substitution, always let u be what is inside the most complicated part. Here we want to let $u = x^2+1$. Then $du = 2xdx$, so $xdx = \frac{1}{2}du$ and we can replace the differential to be integrated as follows:

$$x(x^2+1)^3dx = (x^2+1)^3(xdx) = u^3(\frac{1}{2}du) = \frac{1}{2}u^3du \ .$$

Then

$$\int x(x^2+1)^3dx = \frac{1}{2}\int u^3du = \frac{1}{2}\cdot\frac{1}{4}u^4 + C = \frac{1}{8}(x^2+1)^4 + C \ .$$

Example 4.6. $\int \cos^2(2x+1)\sin(2x+1)dx = \ ?$

Let $u = \cos(2x+1)$. Then $du = -2\sin(2x+1)dx$ or $\sin(2x+1)dx = -\frac{1}{2}du$ so that

$$\cos^2(2x+1)\sin(2x+1)dx = -\frac{1}{2}u^2du \ .$$

This substitution is effective:

$$\int \cos^2(2x+1)\sin(2x+1)dx = -\frac{1}{2}\int u^2du = -\frac{1}{2}\frac{u^3}{3} + C = -\frac{1}{6}\cos^3(2x+1) + C \ .$$

Problem 4.1

1. Find the indefinite integral of $f(x) = (x^2+1)^2x$.

2. Integrate : $\int (x^3+3x+5)^3(x^2+1)dx =$

3. Integrate : $h(x) = x(x^2-1)^{-3}$

4. Integrate : $g(x) = (x^2-1)(x^3-3x)^3$

5. Integrate : $\int (\sin^2 x + 1)\cos x dx =$

6. Find the indefinite integral of $f(x) = x^2 - 3x + x^{-2}$.

7. Find the indefinite integral of

$$g(x) = \sin x + \frac{1}{\cos^2 x} \ .$$

8. Find the indefinite integral of $h(x) = \tan x \sec^2 x$.

9. Find the indefinite integral of $g(x) = \sin^3 x$.

10. Find the indefinite integral of $f(x) = \sin(2x)(\cos(2x))^2$.

11. Find the indefinite integral of

$$\text{a)} \quad \int \sqrt{x}(x+1)dx \qquad \text{b)} \quad \int x\sqrt{x+1}dx \ .$$

12. Find the Indefinite Integrals:

a) $$\int (x^3 - 3x^2 + x^{-2})dx$$

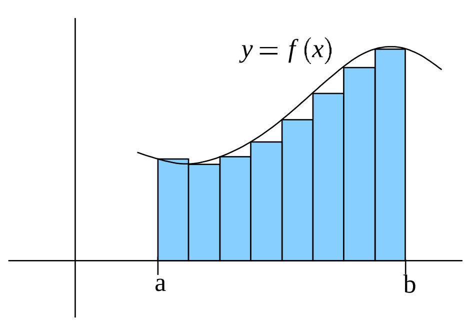
b) $$\int \frac{xdx}{(4x^2+1)^2}$$

4.2 Area

Before the calculus, the idea for the computation of the area of a region was as follows. First of all, the area of a rectangle is the product of the lengths of its sides. Now, fill a region as best as possible with nonoverlapping rectangles, and add up the areas of all the rectangles. This gives an approximate value for the area of the region. If the rectangles are very small, the approximation is good. If we do this cleverly, so that we can distinguish a pattern as we make the rectangles smaller and smaller, perhaps we can go to the limit to find the actual area of the region. This idea was successfully applied by the Greeks to find the area of a circle (and thus to get an approximate value for π), and, most effectively, by Archimedes to approximate the area of a section of a parabola, and more complex figures. This is painstaking work - as we'll see in the next chapter, when we look at 15th and 16th century calculations of this type.

Newton had a better idea: change this to a problem of dynamics. Sweep out the region by a vertical line moving from left to right. For any number x, let $A(x)$ be the area accumulated when the moving line intersects the x-axis at the point x. Now find the rate of change of $A(x)$ with respect to x.

To fix the ideas, start with the graph of a continuous function $y = f(x)$ which lies above the x axis.Let's find the area under the curve and above the x-axis between the lines $x = a$ and $x = b$. Let $A(x)$ be the area accumulated when the line goes through x on the x-axis. So, $A(a) = 0$, and the answer we're looking for is $A(b)$. Fix a point x between a and b, and now move it a small distance Δx, and approximate the area of the region added (see figure 4.1).



Since the curve is almost a straight line over the interval of base length Δx, we can approximate that piece of area by the area of the trapezoid:

$$\Delta A = \frac{1}{2}(f(x) + f(x + \Delta x))\Delta x \quad \text{or} \quad \frac{\Delta A}{\Delta x} = \frac{1}{2}(f(x) + f(x + \Delta x)) \ .$$

Taking the limit as $\Delta x \to 0$, we get

$$dA = f(x)dx \ .$$

so that area under a curve is found by finding the integral $\int f(x)dx$ that has the value 0 at $x = a$.

Example 4.7. Find the area under the curve $y = 4x$ between $x = 1$ and $x = 4$.

This is a trapezoid of side lengths 4,16 and base length 3, so that its area is $(1/2)(4+16)(3) = 30$. But let's use Newton's idea. We first calculate the integral of the defining function:

$$A(x) = \int 4x dx = 2x^2 + C \ .$$

We want the particular function $A(x)$ whose value at $x = 1$ is zero; use that condition to solve for C:

$$0 = 2(1^2) + C$$

so that the precise function for area is $A(x) = 2x^2 - 2$. Finally, the answer is $A(4) = 2(4^2) - 2 = 30$.

Clearly our technique is not so simple as the formula from geometry. The powerful point here is that this works for all continuous functions $y = f(x)$.

Example 4.8. Find the area under the curve $y = 6x^2 + 4x$ between $x = 2$ and $x = 5$.

First, we calculate the indefinite integral:

$$\int (6x^2 + 4x)dx = 2x^3 + 2x^2 + C \ .$$

Now we find the particular integral whose value at $x = 2$ is zero:

$$0 = 2(2)^3 + 2(2)^2 + C \ , \qquad \text{so that} \qquad C = -24 \ ,$$

and the area function is $A(x) = 2x^3 + 2x^2 - 24$. The answer is

$$A(5) = 2(5)^3 + 2(5)^2 - 24 = 276 \ .$$

We now introduce a standard way for codifying these computations, which helps us to make the computations automatic. This starts with the following observation:

Proposition 4.6. Given a positive continuous function $y = f(x)$, the area between the curve and the x-axis from $x = a$ to $x = b$ is given by $F(b) - F(a)$ where F is any indefinite integral of f.

To see this, let F be any indefinite integral of f. If $F(a) = 0$, it is the function we're looking for. If not, then $F(x) - F(a)$ is an indefinite integral whose value at a is 0, so is the function we're looking for, and the area is its value at b: $F(b) - F(a)$.

This evaluation of an indefinite integral is called the *definite integral* of the function f from a to b, and is made explicit by indicating those values at the integral sign:

(4.2) $$\int_a^b f(x)dx = F(b) - F(a) \text{ , where } F \text{ is any antiderivative of } f.$$

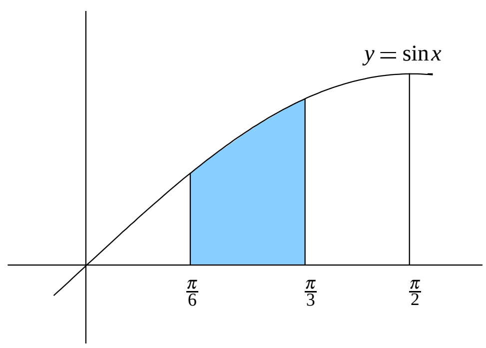
It is customary and convenient to use the notation $F(x)|_a^b$ as an intermediary between setting up the definite integral and its calculation.

Example 4.9. The area under the curve $y = x^2$ between $x = 1$ and $x = 3$ is

$$\int_1^3 x^2 dx = \frac{x^3}{3}\Big|_1^3 = \frac{3^3}{3} - \frac{1^3}{3} = \frac{26}{3}$$

In the first step we found the indefinite integral $x^3/3$, and in the second, evaluated it at 3 and 1, and took the difference.

Example 4.10. Find the area under the curve $y = \sin x$ between $x = \pi/6$ and $x = \pi/3$ (see the figure).

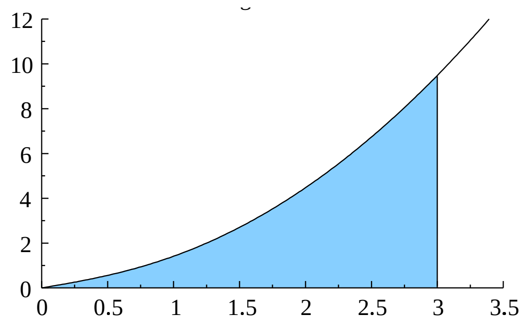


$$\int_{\pi/6}^{\pi/3} \sin x dx = (-\cos x)\Big|_{\pi/6}^{\pi/3} = -\cos(\frac{\pi}{6}) - (-\cos(\frac{\pi}{3})) = \frac{\sqrt{3}-1}{2} \ .$$

Example 4.11. The area under the curve $y = x^n$ between $x = 0$ and $x = 1$ is $1/(n + 1)$:

$$\int_0^1 x^n dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1} \ .$$

Example 4.12. Find the area between the curve $y = x\sqrt{x^2+1}$ and the x-axis, from $x = 0$ to $x = 3$.

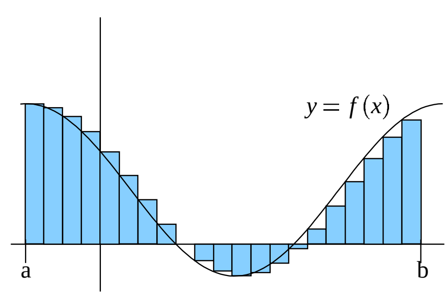


Here $dA = x\sqrt{x^2+1}dx$, so the area is $\int_0^3 x\sqrt{x^2+1}dx$. We integrate by using the substitution $u = x^2+1,\ du = 2xdx$. When $x = 0,\ u = 1$, and when $x = 3,\ u = 10$. The area is

$$\int_0^3 x\sqrt{x^2+1}dx = \frac{1}{2}\int_1^{10} u^{1/2}du = \frac{1}{2}\frac{2}{3}u^{3/2}\Big|_1^{10} = \frac{1}{3}(10\sqrt{10}-1)$$

Notice, that when we make a substitution in a definite integral, we also replace the limits of integration by the values of the new variable at the endpoints. In this way the computation is easier than resubstituting back at the end.

Now, the definite integral $\int_a^b f(x)dx$ has been introduced to find the area under the curve $y = f(x)$ when f is a nonnegative function; however, it can be defined by equation (4.2) for a function f which takes on both positive and negative values. In this case, the definite integral computes the *signed area* between the curve and the x-axis. This is easily seen by noticing that, in the approximation, $\Delta A = f(x)\Delta x$, we get a negative contribution when $f(x) < 0$ (see the figure below for a justification).



Definition 4.1. Let f be a continuous function defined on the interval $[a, b]$. The *signed area* between the curve $y = f(x)$ and the x-axis is the difference of the areas of the regions below the curve and above the x-axis, where $f(x) \geq 0$ and the areas of the regions above the curve and below the x-axis, where $f(x) \leq 0$.

The signed area is calculated by the definite integral:

$$\int_a^b f(x)dx = F(b) - F(a) \text{ , where } F \text{ is any antiderivative of } f. \tag{4.3}$$

Example 4.13. Water is being drawn from a certain reservoir at the rate of 3000 gals/hour. During a certain 4 hour period, water flows into (or leaves) the reservoir at the rate $x^3 - 3x^2 + 2x$ thousand gallons per hour, where x is the time past the beginning of this period, measured in hours. At the end of the four hour period, how much water has the reservoir gained (or lost)?

Let's measure water in thousands of gallons. Thus the rate of exchange of water at time x is

$$f(x) = x^3 - 3x^2 + 2x - 3 \qquad \text{thousand gallons per hour.}$$

Note that, at least at the beginning, there is more water taken out of the reservoir than is put in. The total change in water over the four hour period is

$$\int_0^4 (x^3 - 3x^2 + 2x - 3)dx = \frac{x^4}{4} - x^3 + x^2 - 3x\Big|_0^4 = 64 - 64 + 16 - 12 = 4 \quad \text{thousand gallons .}$$

Problem 4.2

1. $$\int_1^3 (2t + 1)^3 dt =$$

2. $$\int_{-1}^1 (4x^3 - 2x^2 + 1)dx =$$

3. $$\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} dx =$$

4. Integrate : $$\int_1^4 \frac{1}{\sqrt{y}(\sqrt{y} + 1)^2} dy$$

5. Integrate : $$\int_0^{\pi/2} \cos^2 x \sin x dx =$$

6. Find the area of the region in the first quadrant bounded by the curves $y = \sin \frac{\pi}{2} x$ and $y = x$.

7. Find the area of the region under the curve $y = x^2\sqrt{x^3+1}$, above the x-axis and bounded by the lines $x = 0$ and $x = 4$.

8. Find the area under the curve $y = x^2 + x^{-2}$, above the x-axis and between the lines $x = 1$ and $x = 2$.

9. Find the definite Integral: $\int_1^2 (x^2 + x^{-2})dx$.

10. Find the area of the region above the x-axis and below the curve $y = \sec^2 x$ lying between the lines $x = -\pi/4$ and $x = \pi/4$,

11. Find the definite integral: $\int_0^3 (4x+1)^2 dx$.

12. Calculate the definite integrals: $\int_{-4}^4 (x^2 - 3 + \cos x)dx$.

4.3 Separation of Variables

A *first order differential equation* is a relation among the variables y, y', x. For example:

$$y' = x^2 + 1 \ , \quad (y')^2 + \sin^2 x = 1 \ , \quad xy' = y + 2x^3 \ .$$

A *solution* of a differential equation is a function $f(x)$ such that, if we let $y = f(x)$, $y' = f'(x)$ in the equation, we get an identity. So, for the above examples, we check that the following are solutions, respectively:

$$f(x) = \frac{1}{3}x^3 + x + 5 \ , \quad g(x) = \sin x \ , \quad h(x) = x^3 + 2x \ .$$

For the first, we just differentiate: $f'(x) = x^2 + 1$, and for the second we have $g'(x) = \cos x$, and of course $\cos^2 x + \sin^2 x = 1$. The third takes a little calculation. First, $h'(x) = 3x^2 + 2$, so $xh'(x) = 3x^3 + 2x = x^3 + 2x + 2x^3 = h(x) + 2x^3$, and the identity is verified.

In general, it is difficult to find solutions to differential equations, but in one special case, thanks to Proposition 4.5, it is not so hard. That proposition says that if we know that the differentials $f(y)dy$ and $g(x)dx$ are equal, then their integrals differ by constant. In that proposition we assumed the prior knowledge of y as a function of x, but the technique still works without that assumption. In fact, it comes out as a conclusion!

So, suppose that, upon replacing y' by dy/dx, we can rewrite the differential equation as an equation of differentials:

$$f(y)dy = g(x)dx \ ,$$

so that all factors involving y are on one side of the equation, and all those involving x are on the other. In this case we say that "the variables can be separated." Then we can solve by integrating both sides.

Example 4.14. Solve the differential equation $y' = x^2/y$. We rewrite this as $dy/dx = x^2/y$, which can be rewritten in differential form as $ydy = x^2dx$. Now, integrate both sides:

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + C$$

or

$$y^2 = \frac{2}{3}x^3 + C$$

(since C represents a generic constant, so does $2C$, so we can again call it C). Thus the solution is

$$y = \sqrt{\frac{2}{3}x^3 + C}\ ,$$

and this is the general solution.

Example 4.15. Find the particular soluton of the differential equation $y' = x^2/y$ such that $y = 7$ when $x = 3$.

We follow the above argument to the equation

$$y^2 = \frac{2}{3}x^3 + C$$

Now we use the condition $y = 7$ when $x = 3$ to identify C:

$$7^2 = \frac{2}{3}3^3 + C \quad or \quad 49 = 18 + C$$

so that $C = 31$. Then the solution is

$$y = \sqrt{\frac{2}{3}x^3 + 31}\ .$$

Example 4.16. An object moves along the x-axis so that its velocity at time t is $v(t) = t(t^2 - 1)^5$. If the object is at the origin at time $t = 0$, at what point is it at time $t = 1$? At time $t = 2$?

Notice that the velocity is negative until $t = 1$, so the object starts moving to the left, but at $t = 1$ turns around and moves to the right. No matter; its position at time T, $x(T)$, is still given by the definite integral:

$$x(T) = \int_0^T v dt = \int_0^T t(t^2 - 1)^5 dt$$

Make the substitution $u = t^2 - 1,\ du = 2dt$ to obtain the integrand $\frac{1}{2}u^5 du$. When $t = 0,\ u = -1$, and when $t = 1,\ u = 0$, so

$$x(1) = \frac{1}{2}\int_{-1}^0 u^5 du = \frac{1}{12}u^6\Big|_{-1}^0 = -\frac{1}{12}\ .$$

For $t = 2$, we have $u = 3$, so

$$x(2) = \frac{1}{2}\int_{-1}^3 u^5 du = \frac{1}{12}u^6\Big|_{-1}^3 = \frac{1}{12}(3^6 - 1)\ .$$

Example 4.17. Suppose a ball is thrown upward at an initial velocity of 128 ft/sec. How high does it go?

Let $s,\ v,\ a$ represent position (measured upwards, with the surface of the earth at $s = 0$), velocity, acceleration. The acceleration due to gravity is

$$\frac{dv}{dt} = a = -32 \ \ \text{ft/sec}^2 \ .$$

We integrate and conclude that $v = -32t + C$, for some constant C. Now, since $v = 128$ at time $t = 0$, we have

$$\frac{ds}{dt} = v = -32t + 128 \tag{4.4}$$

and integrating again we obtain $s = -16t^2 + 128t + C$. Since $s = 0$ when $t = 0$, we must have $C = 0$, so

$$s = -16t^2 + 128t \tag{4.5}$$

Now, at the maximum height, the velocity is zero. Solving (4.4) for $v = 0$, we have $t = 128/32 = 4$ seconds at the high point. Putting this value of t in (4.5), we obtain the answer

$$s = -16 \cdot 4^2 + 128(4) = 256 \ .$$

Example 4.18. In the above problem, the time it takes to attain maximum height is calculated in order to find that height, but is otherwise irrelevant. In fact, by eliminating the variable t, we can find a more direct relation between position and velocity. Once again, we consider the variables $s,\ v,\ a,\ t$, and relate their differentials by

$$ds = vdt \ , \qquad dv = adt \ .$$

But now, we eliminate dt by multiplying the second equation by v:

$$vdv = avdt = ads$$

from which we conclude, by integrating the differential equation $vdv = ads$:

$$\frac{1}{2}v^2 = \int ads \tag{4.6}$$

which is useful if a is constant, or a function of position alone. In the case of the above problem a is constant, so (4.6) becomes $(1/2)v^2 = as + C$. Setting $s = 0$ as the initial position, and v_0 the initial velocity, we find $C = (1/2)v_0^2$, giving the relation

$$\frac{1}{2}v^2 - \frac{1}{2}v_0^2 = as \tag{4.7}$$

Now, in the preceding problem $a = -32$, $v_0 = 128$ and $v = 0$ at the maximum height, so we get $-(1/2)(128)^2 = -32s$; solving for s gives $s = 256$.

Example 4.19. An automobile travelling at a speed of 60 mph (88 ft/sec) decelerates at a rate of 12 ft/sec^2. How far does it travel before it stops?

Here $a = -12,\ v_0 = 88$ and $v = 0$ when it stops. Putting these data into (4.7):

$$-\frac{1}{2}(88)^2 = -12s$$

giving $s = 322.67$ ft.

Example 4.20. Now, Newton's second law, $F = ma$, says that the acceleration of a body of mass m in motion is proportional to the force F applied to it. If those forces are spacial; that is, functions of s alone, then, by multiplying equation (4.6) by m we get

$$\frac{1}{2}mv^2 = \int F(s)ds \tag{4.8}$$

up to a constant, which is a way of stating the law of conservation of energy: the change in kinetic energy of the moving object (the left hand side) is equal to the work done by the force (the right hand side).

Example 4.21. To illustrate this observation, consider a rocket sitting on the surface of a planet of mass M and radius R. With what initial velocity v_0 should the rocket be propelled so as to escape the gravitational field of the planet?

According to Newton's law of universal gravitation, the force F of gravity is given by

$$F = -G\frac{mM}{s^2}$$

where G is a universal constant, m is the mass of the rocket, and s is the distance of the rocket from the center of the planet. In particular, from $F = ma$, we obtain

$$a = -G\frac{M}{s^2}\ ,$$

and equation (4.6) becomes

$$\frac{1}{2}v^2 = -GM\int s^{-2}ds = GMs^{-1} + C\ .$$

At liftoff, $s = R$ and $v = v_0$, and we find

$$C = \frac{1}{2}v_0^2 - GMR^{-1}\ ,$$

and thus the velocity and distance of the rocket at any future time satisfy the relation

$$v^2 = \frac{2GM}{s} + v_0^2 - \frac{2GM}{R}\ .$$

The rocket will escape the planet if this is never zero. As s increases, the first term gets smaller and smaller, so the only way we can insure this is to take $v_0 \geq \sqrt{2GM/R}$.

In particular, if the planet is earth, then $GM/R^2 = g = 32$ ft/sec^2, and $R = 3900$ miles. First, we convert g to miles per hour per hour:

$$g = 32\frac{\text{ft}}{\text{sec}^2} \cdot \frac{(3600)^2\text{sec}^2}{\text{hr}^2} \cdot \frac{\text{mile}}{5280 \text{ ft}} = \frac{32(3600)))^2}{5280}\ \frac{\text{mi}}{\text{hr}^2} = 78545\frac{\text{mi}}{\text{hr}^2}\ .$$

Then, to escape earth's gravitational pull, we must have

$$v_0 \geq \sqrt{\frac{2GM}{R}} = \sqrt{2gR} = 24,751 \quad \text{miles per hour}\ .$$

Problems 4.3

1. Given

$$\frac{dy}{dx} = x^2\sqrt{y}, \quad y = 4 \quad \text{when} \quad x = 0$$

find y as a function of x.

2. Find y as a function of x, given

$$\frac{dy}{dx} = \frac{\sin x}{y}\ , \qquad y = 5 \quad \text{when} \quad x = 0\ .$$

3. Find the solution to the differential equation $y' = y^2x^2 + y^2$ such that $y(1) = 2$.

4. Find $f(x)$ given that $f(2) = 1,\ f'(1) = -1$ and $f''(x) = x - x^{-3}$.

5. Find the function $y = f(x)$ which satisfies the differential equation $x^2y' + (1 + x^2)y^2 = 0$ such that $f(1) = 2$.

6. Find the function $y = f(x)$ which satisfies the differential equation

$$\frac{dy}{dx} = \frac{1}{yx^2}$$

such that $f(1) = 1$.

7. Solve the differential equation:

$$\frac{dy}{dx} = (1 + x)y^2, \quad y(1) = 2.$$

8. Find the function whose value at 0 is 0 and whose derivative is

$$\frac{x}{(2x^2 + 1)^2}.$$

9. Find the function whose value at 0 is 0 and whose derivative is $\sin x/\cos^4 x$.

10. Find y as a function of x, given that $y = 4$ when $x = 0$ and

$$\frac{dy}{dx} = x + \sin x \ .$$

11. Find the solution to the differential equation

$$\frac{dy}{dx} = \frac{x}{y^2}$$

such that $y(1) = 2$.

12. An automobile is travelling down the road a speed of 100 ft/sec. a) At what constant rate must the automobile decelerate in order to stop in 300 ft.? b) How long does that take?

13. A ball is thrown from ground level so as to just reach the top of a building 150 ft. high. At what initial velocity must the ball be thrown?

14. A 10 g object enters a straight track at a velocity of 12 cm/sec. On the track a force of $-2s$ dynes is applied to this object , where s is distance from the start of the track measured in cm. What is the speed of the object when it is 10 cm down the track?How far does it go before it turns around?

4.4 The Exponential Function

In many dynamical processes we are interested in studying the development of the quantity of a certain material as time progresses where the laws governing the process depend only on the properties of the material. For example: (1) growth of a bacteria culture, (2) decay of a radioactive material, (3) cooling of a hot metal thrust into water, (4) growth of an interest bearing investment fund, (5) spread of an epidemic. Let $x = x(t)$ be the variable in such a process, where t represents time. In the above examples x would be (1) the mass of the bacteria, (2) the remaining mass of the original material, (3) the difference between the temperature of the metal and the ambient temperature, (4) the value of the fund, (5) the number of infected people. These are examples of *growth (or decay) processes*. In studying such processes we consider the *per unit rate of growth*:

$$r = \frac{x'(t)}{x(t)} \ , \tag{4.9}$$

which can be written as the differential equation

$$\frac{dx}{dt} = rx \ . \tag{4.10}$$

In many of these processes, the per unit rate of growth r is constant. For our examples: (1) so long as the source of nourishment is plentiful, we can expect the rate at which the bacteria devleop to remain unchanged, (2) the physical process of fission depends only on the nature of the atom and is the same throughout the material and remains unchanged over time, (3) the rate at which heat is lost depends only on the temperature difference at the interface and the thermal properties of the material, (4) the growth of an investment account is the interest rate, set periodically by the bank. As for (5), the rate of spread of an epidemic depends upon the nature of the disease, and may depend on time, mortality rate as well as the rate of interaction between infected and noninfected people. We will return to this case in section 6.

In case r is constant, equation (4.10) leads to a separable differential equation:

$$\frac{dx}{x} = rdt \ . \tag{4.11}$$

The integral on the right is $rt + C$, and the integral on the left is given by sweeping out the area under the curve $y = x^{-1}$. Although we do not know this function in any explicit sense, we can conclude that the differential equation (4.10) (or (4.11)) has a solution, and we proceed with that knowledge.

Definition 4.2. The *exponential function*, written $x(t) = e^{rt}$, is the solution of the differential equation

$$\frac{dx}{dt} = rx$$

with value $x(0) = 1$. If $r > 0$, r is called the *growth rate*; if $r < 0$, it is the *decay rate*. More precisely, the function e^{rt} is defined by the conditions

$$\frac{d}{dx}e^{rt} = re^{rt} \ , \qquad e^{r0} = 1 \ . \tag{4.12}$$

The exponential function, like the trigonometric functions, is a *transcendental function*. These are functions which cannot be expressed as a quotient of polynomials; in this sense they *transcend* rational functions. In due course, we shall find ways to calculate approximate values of the transcendental functions; for us now it suffices to know that these calculations have been done and are incorporated into our calculators.

Proposition 4.7. The solution of the differential equation $x'(t) = rx(t)$, with initial value $x(0) = x_0$ is

$$x(t) = x_0 e^{rt} \ . \tag{4.12}$$

To verify this, we show that the function defined by (4.12) solves this initial value problem. First, $x(0) = x_0 e^{r0} = x_0$. Differentiating:

$$x'(t) = x_0 \frac{d}{dx} e^{rt} = x_0 r e^{rt} = rx(t) \ .$$

Example 4.22. \$500 is deposited in an account, continuously compounded at an interest rate of 5% per year. What is the value of the account after 5 years?

Let $x(t)$ be the value of the fund at time t. Then $x(0) = 500$. The phrase *continuously compounded* tells us that the fund grows continuously at the given rate, so x satisfies the differential equation (2) with $r = .05$. The solution then is

$$x(t) = 500e^{.05t} \ .$$

At $t = 5$, we calculate: $e^{.05(5)} = e^{.25} = 1.284$, so $x(5) = 500(1.284) = 642.01$ dollars.

Example 4.23. According to the census, the US population in 2000 was 281.4 million. The growth rate over the preceding decade was .1235. Assuming that growth rate continues during the present century, what will be the US population in 2050 ?

Let $x(t)$ be the population of the US in millions, where t is the number of decades after 2000. Then $x(0) = 281.4$, and Proposition 4.6 tells us

$$x(t) = 281.4e^{(.1235)t}$$

At 2050, $t = 5$, so the answer is $x(5) = 281.4e^{.1235(5)} = 521.8$ million.

Example 4.24. The radioactive isotope ^{128}I has a decay rate of .0279 per minute. How many grams of an initial 100g suppy of ^{128}I remain after 20 minutes?

Let $x(t)$ be the amount of ^{128}I present after t minutes. The information we are given is that $x(0) = 100$ g and $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = -.0279x(t) \ .$$

We have the negative sign since the given per unit rate is that of decay. Thus, by Proposition 4.7, $x(t) = 100e^{(-.0279)t}$, and the answer is

$$x(20) = 100e^{-.0279(20)} = 57.23 \text{ g.}$$

We can come upon a way to calculate the exponential function by starting with a comparison of continuously compounded interest with other ways of compounding interest. Suppose I deposit one dollar in a bank account with an interest rate of 10% per year. If the rate is *simple*, that is, it is paid out once only at the end of the year, then at that time the account will have \$1.10. This is considered unfair of the bank, since they have had the use of my dollar throughout the year and have been investing it over and over, but they have transfered my share of the earning only at the end of the year. Suppose instead the bank paid me twice a year, and the amount added to my account after six months were reinvested. Then, after 6 months, I have \$1.05. This is reinvested for another half year, so now, at the end of the year I accrue another 1.05(.05) = .0525, so I will have \$1.1025. We now build on this model by increasing the frequency of recomputation of interest.

So, suppose that the interest rate is r per year, and is paid in n periods per year. Then, in each period I gain r/n of the amount I had at the beginning of the period. Let $P(k)$ represent the amount I have at the end of k periods. I start with $P(0)$ dollars. The law of change here is $P(k) = P(k-1) + (r/n)P(k-1)$: my increment in any period is r/n times the amount at the beginning of the period. Thus

$$P(1) = P(0) + P(0)(r/n) = P(0)(1 + \frac{r}{n}) \ , \quad P(2) = P(1)(1 + \frac{r}{n}) = P(0)(1 + \frac{r}{n})^2 \ ,$$

and after k periods,

$$P(k) = P(0)(1 + \frac{r}{n})^k \ . \tag{4.13}$$

Now, let t be the time (in years) the fund is allowed to grow at this interest rate. The number of periods is $k = nt$, so we can rewrite (4.13) as

$$P(t) = P(0)(1 + \frac{r}{n})^{nt} \ . \tag{4.14}$$

As we let the number of periods get larger and larger, this approaches continuous compounding in the limit; that is, formula (4.14) approaches the formula

$$P(t) = P(0)e^{rt} \ . \tag{4.15}$$

We conclude, taking $P(0) = 1$:

Proposition 4.8. $$e^{rt} = \lim_{n\to\infty}(1 + \frac{r}{n})^{nt} \ .$$

Note that $e^0 = 1$, that is, there is no change in zero time. The number

$$e = e^1 = \lim_{n\to\infty}(1 + \frac{1}{n})^n \ , \tag{4.16}$$

denoted by e, and called *Euler's number*, is approximately $2.71828\ldots$.

Example 4.25. \$500 is invested at 5% per year, compounded quarterly. What is the value of the fund at the end of 5 years? We use formula (4.15) with $P(0) = 500,\ r = .05,\ n = 4, t = 5$:

$$P(5) = 500(1 + \frac{.05}{4})^{20} = 641.01 \ .$$

Notice that this is close to, but less than the answer to example 4.22: 642.01. If the compounding is done only annually, then the answer would be $500(1+.05)^5 = 638.14$. Since the result using simple interest is just 625, clearly any kind of compounding is preferable, and quarterly compounding is already very close to continuous compounding.

Now, we turn to the study of the properties of the exponential function given by a growth rate of $1 = 100\%$:

Definition 4.3. The *exponential function*, $y = e^x$, is the solution of the differential equation

$$\frac{dy}{dx} = y$$

with value $y(0) = 1$. It can be computed by

$$e^x = \lim_{n\to\infty}(1 + \frac{1}{n})^{nx} \ . \tag{4.17}$$

We use exponential notation to denote the exponential function because it obeys the rules of exponents:

Propostion 4.9. For any two numbers A and B:

$$\text{a)}\quad e^{A+B} = e^A \cdot e^B \ ,\ \text{b)}\quad e^{AB} = (e^A)^B \ ,\ \text{c)}\quad e^{-A} = \frac{1}{e^A}$$

Let's start with c). Let $u = 1/e^x$. Then, by the chain rule:

$$\frac{du}{dx} = \frac{-1}{(e^x)^2}\frac{d}{dx}e^x = \frac{-1}{(e^x)^2}e^x = \frac{-1}{e^x} = -u \ .$$

Since $u(0) = 1$, u is the function described in Proposition 4.7 with $r = -1$, so $u = e^{-x}$; that is, $1/e^x = e^{-x}$.

To show a), let B be a number, and consider $v = e^{x+B}e^{-x}$. By the product rule for differentiation:

$$\frac{dv}{dx} = e^{x+B}(-e^{-x}) + e^{x+B}e^{-x} = 0 \ ,$$

so v is constant. But at $x = 0$, $v = e^B$, so

$$e^{x+B}e^{-x} = e^B \ .$$

Now, replacing x by A, and using c), this says $e^{A+B}/e^A = e^B$, which is a).

b) is shown similarly: differentiate the function $w = (e^x)^B$:

$$\frac{dw}{dx} = B(e^x)^{B-1} \cdot e^x = B(e^x)^B = Bw \ ,$$

and $w(0) = 1$, so w is the solution function of Proposition 4.6 with $r = B$. Thus $(e^x)^B = e^{Bx}$, and we get b) by evaluating at $x = A$.

In particular, since $e^x = (e^1)^x$, we can think of the exponential function as raising Euler's number e to the xth power.

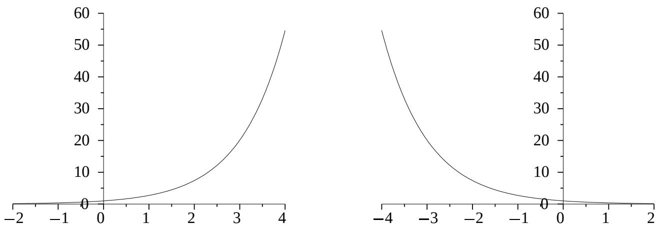
Proposition 4.10. a) The exponential function $y = e^x$ is a strictly increasing function.
b) $\lim_{x\to\infty} e^x = +\infty$; in particular $e^n \geq 2^n$ for all n.
c) $\lim_{x\to-\infty} e^x = 0$.

a) follows from the fact that $e^x > 0$ for all x, so the derivative of e^x is always positive. To see b), we first observe, from the binomial expansion,

$$(1 + \frac{1}{n})^n = 1 + n\frac{1}{n} + \cdots > 2 \ ,$$

so in the limit $e \geq 2$. Thus $e^n \geq 2^n$ for all n. Now, for c):

$$\lim_{x\to-\infty} e^x = \lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \ .$$



$y = e^x$ $\qquad\qquad$ $y = e^{-x}$

Since the second derivative of $y = e^x$ is again e^x which is positive, the curve is always concave up. From this information, we can easily sketch the graph of the exponential function (see figure 4.4). To graph the function $y = e^{-x}$, we just reflect in the y-axis (see figure 4.5).

Since the derivative of e^x is itself, so is its indefinite integral:

Proposition 4.11. $\int e^x dx = e^x + C$.

Example 4.26. Find $\int_0^3 xe^{x^2} dx$.

Let $u = e^{x^2}$, so that (by the chain rule) $du = 2xe^{x^2} dx$. When $x = 0$, $u = 1$ and when $x = 3$, $u = e^9$. Thus

$$\int_0^3 xe^{x^2} dx = \frac{1}{2}\int_1^{e^9} du = \frac{1}{2}u\Big|_1^{e^9} = \frac{1}{2}(e^9 - 1) \ .$$

Problems 4.4

1. Differentiate : $\qquad g(x) = xe^x - e^x + 1$

2. Differentiate : $\qquad f(x) = \dfrac{e^x}{x^2}$

3. Differentiate : $\qquad g(x) = e^{2x^2+3x-1}$

4. Integrate : $\qquad \displaystyle\int_0^3 e^x(e^{2x} + 1)dx$

5. Integrate : $\int e^{\tan x} \sec^2 x dx$

6. Integrate : $\int \frac{xe^{x^2}}{e^{x^2}+1} dx$

7. I invest \$100,000 in a company for five years, with a guaranteed income of 8% per year, compounded semi-annually. How much will I have at the end of 5 years? If the interest were compounded continuously, how much would I have in 5 years?

8. A certain element decays at a rate of .000163/year. Of a piece of this element of 450 kg, how much will remain in ten years?

4.5 The Logarithm

At what rate (compounded continuously) should I invest \$500 so as to have \$800 in the account at the end of 6 years? If r is the unknown rate, the answer is given by solving the equation:

$$800 = 500e^{6r} . \tag{4.18}$$

To answer this, we have to find the number a such that $e^a = 800/500$. That is, we have to *invert* the operation of exponentiation. This is done by the logarithm.

Definition 4.4. Given a positive number x, the *natural logarithm* of x, denoted $y = \ln x$ is that number y such that $e^y = x$.

Any positive number x lies between 0 and ∞, so by the intermediate value theorem appied to the exponential function and Proposition 4.10, there is a number y such that $e^y = x$. Since the exponential function is strictly increasing, there is only one such number. Thus the above definition makes sense. The logarithm is a transcendental function whose values have been calculated and are stored in our calculators.

Example 4.27. To conclude the above discussion, we solve (4.18): $e^{6r} = 8/5$, so $6r = \ln(8/5) = .47$, so $r = .47/6 = .078$: the interest rate must be 7.8%.

Before continuing with further applications, we look into the properties of the logarithmic function.

Proposition 4.12. $$\frac{d}{dx}\ln x = \frac{1}{x} .$$

Let $y = \ln x$, so that $x = e^y$. Taking differentials: $dx = e^y dy$, so

$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x} .$$

The properties (Proposition 4.9) of the exponential function translate into properties of the logarithm:

Proposition 4.13. For any two positive numbers A and B:

$$\text{a)} \quad \ln(AB) = \ln A + \ln B \ , \quad \text{b)} \quad \ln(A^B) = B(\ln A) \ , \quad \text{c)} \quad \ln\frac{1}{A} = -\ln A \ .$$

For c), let $a = \ln A$, so that $A = e^a$. Then, from Proposition 4.9c: $e^{-a} = 1/e^a = 1/A$, so $\ln(1/A) = -a = -\ln A$.

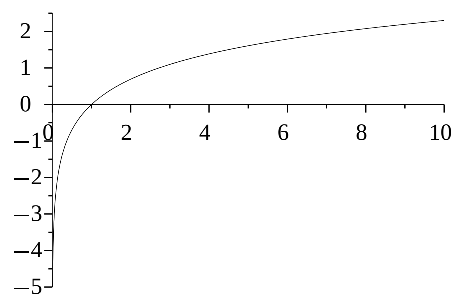
For a), let $b = \ln B$, so that $B = e^b$. Then, again, from Proposition 4.9: $AB = e^a e^b = e^{a+b}$, so $a + b = \ln(AB)$, which is a). b) can be shown in the same way.

Proposition 4.14.
a) $\ln x$ is a strictly increasing function of x for $x > 0$.

b) $$\lim_{x\to\infty} \ln x = \infty$$

c) $$\lim_{x\to 0} \ln x = -\infty$$

These follow directly from the corresponding assertions for the exponential (Prop 4.10). This information suffices to sketch the graph of $y = \ln x$:



We note that for $x < 0$, the function $y = \ln|x| = \ln(-x)$ satisfies the differential equation $dy/dx = 1/x$:

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}(-1) = \frac{1}{x} \ .$$

This then tells us what the indefinite integral of $1/x$ is (for $x \neq 0$):

Proposition 4.15. $$\int \frac{dx}{x} = \ln|x| + C \quad \text{for} \quad x \neq 0 \quad \text{over any interval not containing } 0 \ .$$

Example 4.28. $\int_0^4 (2x-10)^{-1}dx = ?$

Let $u = 2x - 10,\ du = 2dx$. At $x = 0,\ u = -10$, and at $x = 4,\ u = -2$. Thus

$$\int_0^4 (2x+1)^{-1} dx = \frac{1}{2} \int_{-10}^{-2} \frac{du}{u} = \frac{1}{2} \ln |u| \ \Big|_{-10}^{-2} = \frac{1}{2}(\ln 2 - \ln 10) = \ln \frac{1}{5} \ .$$

Example 4.29. $$\int \frac{e^x}{e^x+1} dx = \ ?$$

Let $u = e^x,\ du = e^x dx$. Then

$$\int \frac{e^x}{e^x+1} dx = \int \frac{du}{u} = \ln u = \ln(e^x+1) + C \ .$$

Example 4.30. Find the solution of the differential equation $dy/dx = xy$, $y(0) = 1$.

The equation is separable and becomes: $dy/y = xdx$. Integrating both sides gives $\ln y = x^2/2 + C$. Substituting $x = 1,\ y = 0$ we find $\ln 1 = 0 = C$, so $C = 0$. Thus the solution is given by

$$\ln y = x^2/2 \ , \quad \text{or} \quad y = e^{x^2/2} \ .$$

General Exponentials and Logarithms

We can raise any positive number a to any power p: since $a = e^{\ln a},\ a^p = (e^{\ln a})^p = e^{p(\ln a)}$. This observation allows us to introduce the general exponential and logarithmic functions.

Definition 4.5. For a any positive number, we define the *exponential function with base a* by

$$a^x = e^{(\ln a)x}$$

and the *logarithmic function with base a* as its inverse function:

$$y = \log_a x \quad \text{if and only if} \quad x = a^y \ .$$

To find a formula for $\log_a$, we note that if if $x = a^y$, then $x = e^{(\ln a)y}$, so $\ln x = (\ln a)y$. Replacing y by $\log_a x$, we have

$$\log_a x = \frac{\ln x}{\ln a} \ .$$

From the chain rule, we obtain the following formulas for the derivatives and integrals of these new functions:

Proposition 4.16.

$$\frac{d}{dx} a^x = (\ln a) a^x \ , \quad \frac{d}{dx} \log_a x = \frac{1}{(\ln a)x} \ ,$$

$$\int a^x dx = \frac{a^x}{\ln a} \ .$$

Problems 4.5

1. Solve for x: $2^x = 3(5^x)$.

2. Solve for x: $(e^x)^5 = e^x e^3$.

3. Solve for x: $6^x = 36^{2-x}$.

4. Solve for x: $\ln_3 x = 5$.

5. Solve for x: $\ln_2(x+1) - \ln_2(x-1) = \ln_2 8$.

6. Solve: $\sqrt{\ln x} = \ln(\sqrt{x})$.

7. Differentiate : $f(x) = e^{2\ln x}$

8. Differentiate : $y = \ln(\ln x)$

9. Differentiate : $y = \log_2(x^2+1)$

10. Differentiate : $y = \dfrac{e^{x^2}}{x}$

11. Differentiate : $f(x) = e^x \ln x$

12. Differentiate : $f(x) = \ln(\dfrac{x+1}{x-1})$

13. Differentiate : $g(x) = 5^x \log_5 x$

14. Differentiate : $h(x) = 5^{\log_2(x^2+1)}$

15. Integrate : $\displaystyle\int \frac{(\ln x)^2 + 1}{x} dx \ =$

16. Integrate : $\displaystyle\int e^{\sin x} \cos x dx \ =$

17. Integrate : $\displaystyle\int \frac{x dx}{3x^2+1} \ =$

18. Integrate : $$\int e^{\ln x+1}dx$$

19. Integrate : $$\int_0^3 2^{x^2}xdx$$

20. Two variables are related by the equation $2\ln x + \ln y = x - y$. What is the equation of the tangent line to the graph of this relation at the point (1,1)?

4.6 Growth and Decay

We have already looked at several growth/decay problems in the course of discussing the exponential and logarithmic functions. In this section we look at growth processes in greater depth, and at more general processes.

Example 4.31. How long does it take for a quantity of ^{128}I to be reduced to half its size?

Referring to example 4.23, if we start with an amount $P(0)$ of ^{128}I , the amount we have after t minutes is $P(t) = P(0)e^{-(.0279)t}$. To solve our problem we find t such that

$$\frac{1}{2}P(0) = P(0)e^{-(.0279)t} \; .$$

Factor out $P(0)$ and take the logarithm of both sides:

$$\ln(1/2) = -(.0279)t \quad \text{or} \quad t = \frac{\ln 2}{.0279} = 24.84 \text{ minutes} \; .$$

This time is called the *half-life* of ^{128}I.

The general exponential function allows us to replace the rate of decay of a radioactive element by its half life. For suppose that a certain element has a rate of decay r, and a half-life T. Then $1/2 = e^{-rT}$. Taking logarithms, this gives us $\ln 2 = rT$, giving this relation between the half-life and the growth rate:

$$r = \frac{\ln 2}{T} \; .$$

Now, if an amount A of the element decays for t years, then the amount remaining is

$$A(t) = Ae^{-rt} = Ae^{-\ln 2(t/T)} = 2^{-t/T}A \; . \tag{4.19}$$

Example 4.32. Suppose that the half-life of a certain element is 40 years. How much will remain of a 1 kg sample after 200 years? After 50 years?

In 200 years the sample will have halved 5 times, so what will remain is $1/2^5$ of a kilogram, or 31.25 g. After 50 years, we have $A(t) = 1000(2^{-50/40}) = 435$ grams.

Example 4.33. The rate of decay of the radioactive isotope of carbon (^{14}C) is 1.211×10^{-4} per year. In how many years will it take a certain amount of ^{14}C to be reduced by 10%?

Let $C(t)$ represent the amount of ^{14}C in t years. We can take $C(0) = 1$, and the question is: for what t is $C(t) = .9$? Since $C(t)$ satisfies the differential equation

$$C'(t) = -1.211 \times 10^{-4} C(t) \ .$$

We conclude (using definition 4.2)

$$C(t) = e^{-1.211 \times 10^{-4} t}$$

so we must solve

$$.9 = e^{-1.211 \times 10^{-4} t}$$

Then

$$t = \frac{\ln(.9)}{-1.211 \times 10^{-4}} = \frac{-.1054}{-1.211} \times 10^4 = 870 \text{ years} \ .$$

Example 4.34. At the time an organic material is buried, its ^{14}C content ceases to be replenished by cosmic radiation, so is subject only to the radioactive decay, as described in example 8. Suppose the carbon content of a fossil is discovered to contain 84% of the amount of ^{14}C had it not been buried. How old is it?

Let time $t = 0$ represent the time the fossil was buried, and t the number of years since then. If we take $P(0) = 1$, then $P(t) = .84$, so we must solve the equation

$$.84 = e^{-1.211 \times 10^{-4} t} \ ,$$

or

$$t = \frac{\ln(.84)}{-1.211 \times 10^{-4}} = \frac{-.1744}{-1.211} \times 10^4 = 1440 \text{ years} \ .$$

Example 4.35. To find the half-life T of ^{14}C, we solve

$$\frac{1}{2} = e^{-1.211 \times 10^{-4} T} \ .$$

The answer is $T = \ln 2/(1.211 \times 10^{-4}) = 5724$ years.

Inhibited growth

The growth equation $dx/dt = rx$ does not really work for many biological populations, since they do not appear to continue to grow exponentialy without bound. In fact, one should expect that there are factors present which cause the growth rate to decrease as the population increases. One such is the competition over nutrient: the growth rate of a bacterium in an agar dish diminishes as the population increases and nutrient becomes less accessible. A good model for this is to let the growth rate r decrease linearly as a function of x: $r = a - bx$, where a is the genetic growth factor, and b is the *inhibiting factor*. In this model, the growth equation is replaced by

$$\frac{dx}{dt} = (a - bx)x \ . \tag{4.20}$$

Notice that if $x = a/b$ we have $dx/dt = 0$, so there is no growth. More exactly, the constant function $x(t) = a/b$ is a solution of (4.20). For this reason, a/b is called the *stable* population. Equation (4.20) (called the *logistic equation*) is separable and can be rewritten

$$\frac{dx}{(a-bx)x} = dt \ . \qquad (4.21)$$

To solve this we need a little algebra:

$$\frac{1}{(a-bx)x} = \frac{1}{a}[\frac{1}{x} + \frac{b}{a-bx}]$$

so

$$\int \frac{dx}{(a-bx)x} = \frac{1}{a}\int \frac{dx}{x} + \frac{b}{a}\int \frac{dx}{a-bx} = \frac{1}{a}(\ln x - \ln(a-bx)) + C \ .$$

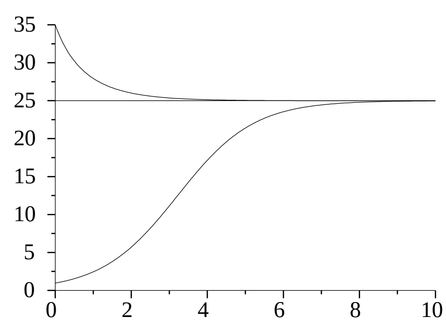
Thus (4.21) becomes

$$t + C = \frac{1}{a}\ln(\frac{x}{a-bx}) \qquad \text{or} \qquad at + C = \ln(\frac{x}{a-bx}) \ .$$

Exponentiating each side gives us $Ke^{at} = x/(a-bx)$, where we have written e^C as K. If we solve for x in terms of t, we obtain

$$x = \frac{aK}{bK + e^{-at}} \ . \qquad (4.22)$$

Since $dx/dt = x(a-bx)$, this function is increasing in the interval $x < a/b$, and is decreasing for $x > a/b$. Finally, from (4.22), we see that $x \to a/b$ as $t \to +\infty$. In figure 4.7, we have drawn three typical solutions of the logistic differential equation.



It is useful to rewrite formula (4.22), replacing the unknown constant K with the value x_0 of x at time $t = 0$. To find that expression, substitute 0 for t and x_0 for x, in equation (4.22) and solve for K. After much tedious algebra we arrive at the formula

$$x(t) = \frac{ax_0}{bx_0 + (a-bx_0)e^{-at}} \ . \qquad (4.23)$$

Example 4.36. A certain bacterium grows at the growth rate of 80% per hour. In a particular agar dish, the inhibiting factor is .002. What is the stable population? If 100 g of the bacterium is put in the dish, what will be the size of the population after 2 hours?

Here $a = .8$, and $b = .002$, and the growth equation is

$$x' = (.8 - .002x)x \ .$$

The stable populations is $.8/.002 = 400$ g. To find the population after 2 hours, we use equation (4.23), with $x_0 = 100$ and $t = 2$:

$$x(2) = \frac{.8(100)}{.002(100) + (.8 - .002(100))e^{-.8(2)}} = \frac{8}{.2 + .6e^{-.16}} = 249.25$$

If there were no inhibition to the growth the population woud be $100e^{1.6} = 495$ g.

Example 4.37. In Example 4.23, we saw that the growth rate of the US population over the decade 1990-2000 was .1235. However, if we looked at the census over the previous few decades, we would find that the growth rate was decreasing. If we attribute that to an inhibiting factor, we would be able to estimate that factor to be about .00032. Given these data, what is the stable population of the US, and what will the population be in 2050 according to this model?

Let $P(t)$ be the US population (in millions) t decades after 2000. We start with the differential equation $P' = (.1235 - .00032P)P$. The stable population is $P = .1235/.00032 = 385.93$ million. Now, to estimate the actual population in 2050, let's use equation (4.21) and evaluate the constant K in the general solution

$$P(t) = \frac{.1235K}{.00032K + e^{-.1235t}} \ .$$

At $t = 0$, $P = 281.4$: this gives $K = 8400$. Then, the population in 2050 is estimated to be

$$P(5) = \frac{.1235(8400)}{.00032(8400) + e^{-.1235(5)}} = 321.45$$

million; a much more realistic estimate than the 521.8 million estimate of example 4.22.

Example 4.38. A certain disease is transmitted by direct contact of an infected person with an infected person. This suggest that the rate of change of the number of infected people is proportional both to the number of infected people and the number of uninfected people. If we let N represent the total population, and $P(t)$ the number of infected people, this assumption can be written as

$$P'(t) = kP(t)(N - P(t))$$

where k represents the probability of direct contact of an infected person with one not infected. This is the logistic equation with $a = kN$ and $b = k$. With these values equation (4.23) becomes

$$P(t) = \frac{NP_0}{P_0 + (N - P_0)e^{-kNt}} \ . \tag{4.24}$$

Notice that the stable population is $a/b = N$; alas, without intervention everyone eventually is infected in this model.

Example 4.39. For a certain transmittable disease, it has been determined that the rate of transmission is .1 percent per day, that is $k = .001$. In a certain community of 500 people it has been found that 5 people have the disease. Assuming that there is no intervention, how many people will have the disease in 4 days? How long before half the population is infected?

Here $N = 500$ and $P_0 = 5$, so $NP_0 = 2500$ and $kn = .5$ and (4.24) becomes

$$P(t) = \frac{2500}{5 + (495)e^{-.5t}} \ .$$

To answer the first question, calculate

$$P(4) = \frac{2500}{5 + (495)e^{-.5(4)}} = 34 \ .$$

For the second question, solve for t:

$$\frac{1}{2}(500) = \frac{2500}{5 + (495)e^{-.5t}} \ .$$

This gives us $5 + 495e^{-.5t} = 10$, which results in the value $t = 9.19$.

Example 4.40. In the study of an epidemic of an airborne disease which is reinforced by interaction among those infected, a first working hypothesis may be that the rate of spread of the disease is proportional to that amount of interaction, which is in turn proportional to the square of the population of infected. If we let $P(t)$ represent the number of people infected at time t, we have $P' = kP^2$ for some constant k. We are interested in knowing how long it will take, unless some action is taken, for the entire population to be infected. To work through an example, let's suppose that $k = .001$ and that at time $t = 0$, 100 people are infected.

The given law of change is

$$\frac{dP}{dt} = .001P^2$$

We can rewrite this in the form

$$P^{-2}dP = .001dt$$

Integrating, we find

$$-P^{-1} = .001t + C \ .$$

Now when $t = 0$, $P = 100$, so $C = -.01$, so we have

$$-P^{-1} = .001t - .01 \ ,$$

or $P(t) = (.01 - .001t)^{-1}$. So, for example, after 5 days the infected population is $(.01 - .005)^{-1} = 200$, and after 8 days it is 500. Worst of all, in ten days, P is infinite: everyone is infected, no matter how large the original healthy population was.

Problems 4.6

1. If I invest \$ 8,000 at 12.5 percent per year (compounded continuously) in how many years will my investment be worth \$ 30,000 ?

2. I want to invest \$5000 in a growth fund so that in 5 years i will have \$8000. What interest rate, compounded continuously will produce that growth?

3. If I invest \$4000 in a fund, with an interest rate of 8%, compounded continuously, how long will it take for the fund to be worth \$10,000? 4. At what rate (continuously compounded) should I invest \$10,000 so as to have \$14,000 in five years?

5. The Zombie National Bank offers accounts which pay 10.5% annually, compounded continuously. How much should I invest today so as to have \$12,000 in 6 years?

6. Carbon^{14} has a half life of 5801 years. How long does it take for a sample to be reduced to 80% its original size?

7. A certain radioactive element decays so that in 100 years it has decreased to 82% its original size. What is its half-life?

8. The half-life of Rossidium$_{312}$ is 4,801 years. How long will it take for a mass of Rossidium$_{312}$ to decay to 98 % of its original size?

9. A certain autocatalytic chemical reaction proceeds according to the differential equation

$$\frac{dx}{dt} = .01\frac{1-x^2}{x}$$

where $x(t)$ is the fraction of the sample which is the resulting compound at time t, in seconds. If we start with $x(0) = .2$, how long does it take for x to reach .9?

10. According to Newton's Law of Cooling, if a hot object is immersed into a cool environment, the rate of decrease of the temperature of the oject is proportional to the difference in the temperature of the object and its environment. If, then, $h(t)$ is the temperature of the object at time t, and T_0 is the temperature of the environment, Newton's law says

$$\frac{dh}{dt} = -k(h(t) - T_0) \ , \tag{1}$$

where k is the coeffiecient of cooling. Suppose that a body at 95° Celsius is immersed in a water bath held at 5° Celsius, and the coefficient of cooling is $k = .08$. What will be the temperature of the body in 10 minutes?

11. Suppose that I wish to make iced tea of tea at the boiling point, to be consumed in three minutes. To get the tea as cold as possible, should I put in an ice cube immediately, or just before the three minutes are up?

12. A lake containing 300,000 acre-feet of water has 20% salinity. Clear water flows in from rivers, and out at a dam, both at the rate of 4,000 acre-feet per day. In how may days will the salinity be reduced to 10%?

13. A pond is in the form of a cylinder of radius 100 ft. and depth 8 ft. Water flows into the pond at the rate of 100 cu. ft./hr and seeps into the ground through the porous bottom at a rate proportional to the volume, where the constant of proportionality is .0005. What is the maximum

height of water in the pond that can be achieved? If the water level in the pond is 2 feet at time t= 0, what is the height after 1000 days?

14. Water flows into an elastic ball at a rate of 4 cu. in/minute. The ball has a puncture out of which water flows at a rate proportional to the volume of water in the ball, where the constant of proportionality is .02. Assuming the ball is empty at the beginning, how much water is in the ball after 20 minutes?

V. Integration: The Accumulation Method

5.1 The Definite Integral and the Fundamental Theorem of the Calculus

In the preceding sections we followed the ideas of Newton on integration: or example we saw how to calculate area of a region by considering the dynamic problem of a line moving across the region, accumulating area behind it. We now turn to Liebniz for his ideas on the subject. In particular, Liebniz took on the problem of defining area for an inrregular region, whereas the issue did not arise in Newtons methods. As we have already observed, before the calculus, there were many area computations: usually by filling up the region with rectangles or other regular shapes in a clever enough way so as to discover a pattern as the dimensions of the shapes get smaller and smaller. We shall illustrate this in the next section. Liebniz, following this approach, considered integration as a method of accumulation; or of approximation and accumulation, to be more precise.

To fix the ideas, we start with the calculation of area under a curve. Suppose that $y = f(x)$ is a non-negative function defined on the interval $[a, b] = \{x; a \leq x \leq b\}$. We want to find the area of the region enclosed by the curve, the x-axis and the lines $x = a,\ x = b$. If we pick some point c between a and b, then we know that the area under the curve $y = f(x)$ and over the interval $[a, b]$ is the sum of the areas over the intervals $[a, c]$ and $[c, b]$. In fact, if we cut the interval $[a, b]$ into any number of little intervals, the area of the whole is the sum of the areas over all the little intervals. Now, if the little intervals are small enough, and if the function is continuous, then the area over that interval is approximately equal to the area of the column over that interval of height $f(c)$, for some c in the interval. See figure 5.1 for a graphic of this process.

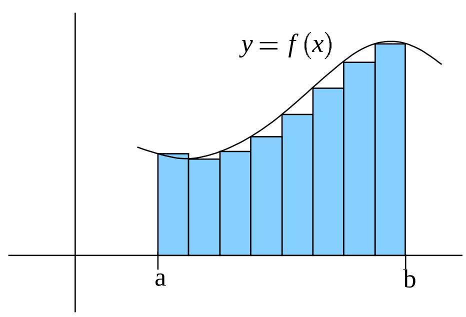


Figure 5.1

Liebniz' notation for this approximation is

$$\sum_a^b f(x)\Delta x$$

where Δx represents the length of the base of a typical column, $f(x)$ its height, and the symbol $\sum$ indicates that we add all these together. Here is where Liebniz takes the great leap: suppose the little intervals are of infinitesimal length; that is Δx is the infinitesimal dx. Then this "approximation" is precise, and we obtain the actual area of the figure, denoted by

$$\int_a^b f(x)dx\ .$$

Now, there are many processes besides that of calculating area (as we shall see later in this chapter) which have this accumulative property: that the whole is the sum of its parts, and we can calculate the value on the whole by adding the values of all of its parts. Thus, Liebniz goes on to discuss this process for any function $y = f(x)$.

Definition 5.1. Let $y = f(x)$ be a function defined on the interval $[a, b]$. The *definite integral* is defined as follows. A *partition* of the interval is any increasing sequence

$$\{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$$

of points in the interval. The corresponding *approximating sum* is

$$\sum_1^n f(x_i')\Delta x_i \tag{5.1}$$

where Δx_i is the length $x_i - x_{i-1}$ of the ith interval, x_i' is any point on that interval, and $\sum$ indicates that we add all these products together (see Figure 5.2).

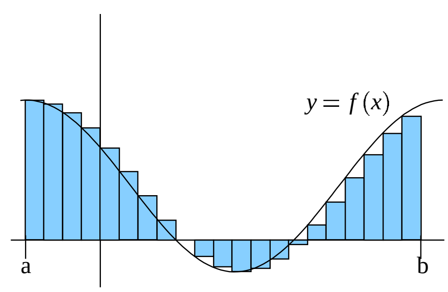


Figure 5.2

If these approximating sums approach a limit as the partition becomes increasingly fine (the lengths of the little intervals go to zero), this limit is the *definite integral* of f over the interval $[a, b]$, denoted

$$\int_a^b f(x)dx\ .$$

This definition raises a serious question. Consider the calculation of area. Surely, as we have observed, if the interval Δx is small enough, then the term $f(x')\Delta x$ is within a very small error of the actual area over the interval. But now we add together a large number of these approximations, and so the errors add. How do we know that the accumulated error is not too large? This issue also took several centuries to be satisfactorily resolved; for us it is enough to know that it does work:

Proposition 5.1. If $y = f(x)$ is a continuous function on the interval $[a, b]$, then the sums (5.1) do converge as the maximum of the Δx_i goes to zero, and thus the integral $\int_a^b f(x)dx$ exists. If f is a nonnegative function, this integral is the area under the curve.

The following properties of the definite integral follow easily from the definition.

Proposition 5.2. Suppose that f and g are continuous functions on the interval $[a, b]$.

a.
$$\int_a^b Cf(x)dx = C\int_a^b f(x)dx \qquad \text{for any constant } C ,$$

b.
$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx .$$

c. If c is any point in $[a, b]$:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx .$$

d. If $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx .$$

In particular, the definite integral of a positive function is positive.

Now the process described in definition 5.1 is almost impossible to carry out in practice. Fortunately, Liebniz' and Newton's methods give the same result:

Proposition 5.3. (The Fundamental Theorem of the Calculus, I). Suppose that $y = f(x)$ is a continuous function on the interval $[a, b]$. If F is any indefinite integral of f, then

$$\int_a^b f(x)dx = F(b) - F(a) . \tag{5.2}$$

We follow Newton's thought: calculate the definite integral by accumulating from left to right. For any x in the interval, let $V(x)$ be the value of the definite integral from a to x. Now, if we go slightly further, say, to $x + \Delta x$, then, by the defining process, $f(x)\Delta x$ is an approximation to the increment in V:

$$\Delta V = V(x + \Delta x) - V(x) = f(x)\Delta x \quad \text{approximately} . \tag{5.3}$$

Now, moving to a differential increment, we obtain the equality

$$dV = f(x)dx \tag{5.4}$$

so V is an indefinite integral of f. Thus $V(x) = F(x) + C$, for some constant C. Since $V(a) = 0$,

$$0 = F(a) + C \ , \quad so \quad C = -F(a)$$

so that $V(x) = F(x) - F(a)$ for all x. In particular, evaluating at $x = b$ gives us (5.2).

In the above argument, we used $V(x)$ to represent $\int_a^x f(t)dt$. Putting this in equation (5.4) gives the second version of the fundamental theorem of the Calculus:

Proposition 5.4. (The Fundamental Theorem of the Calculus, II).

$$\frac{d}{dx}\int_a^x f(t)dt = f(x) \ .$$

From equation (5.2), replacing b by x, we get

$$\int_a^x f(t)dt \ = F(x) - F(a) \ , \tag{5.5}$$

where F is an indefinite integral of f, so $F' = f$. Differentiating both sides of (10) gives us

$$\frac{d}{dx}\int_a^x f(t)dt = F'(x) = f(x) \ .$$

So, to calculate a definite integral $\int_a^b f(x)dx$, follow these steps:

1. Find an indefinite integral F for f;
2. Evaluate $F(b) - F(a)$.

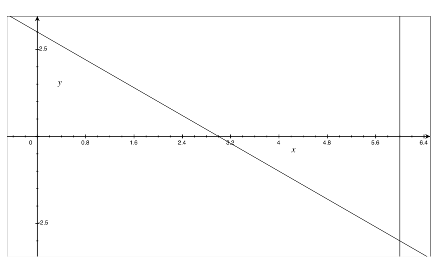
As we have noted in the previous chapter, in actual calculations it is customary and convenient to use the notation $F(x)|_a^b$ as an intermediary between these two steps.

Example 5.1.

$$\int_0^6 (3 - x)dx = 3x - \frac{x^2}{2}\Big|_0^6 = 3(6) - \frac{6^2}{2} = 0 \ .$$

Notice (see figure 5.3) that the contribution to the integral coming from the interval [3.6] is negative, since $f(x) \leq 0$ for x in that interval. The integral of a negative function is the negative of the area between the curve and the x-axis, and so this integral is zero since the two triangles forming the figure are congruent.

Figure 5.3



Example 5.2.

$$\int_{-1}^1 (x^2 - 2x)dx = \frac{x^3}{3} - x^2\Big|_{-1}^1 = \frac{1}{3} - 1 - (-\frac{1}{3} - 1) = \frac{2}{3} \ .$$

In the preceding chapter we discussed area between a the graph of a positive function and the x-axis. For more general regions, follow these guidelines, suggested by the Liebniz definition of the integral.

1. Sketch the region under consideration.
2. Choose a direction in which to accumulate the area.
3. Write down the expression for the differential increment in area:

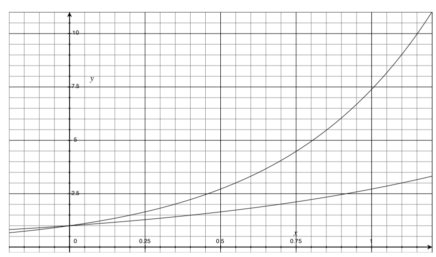
$$dA = L(x)dx \ ,$$

where dx is an infinitesimal increment in the direction of accumulation, and $L(x)$ is the length of the column over that increment.

4. Integrate.

Example 5.3. Find the area of the region bounded by the line $x = 0$, $x = 1$ and the curves $C_1 : y = e^{2x},\ C_2 : y = e^x$.

Figure 5.4



The sketch of this region is given in figure 5.4. Here we will accumulate area in the direction of the x-axis from $x = 0$ to $x = 1$. At a particular value of x, the length of the column is the difference of the values of y on the two curves. Express this in terms of x: $L(x) = e^{2x} - e^x$ so that

$$dA = (e^{2x} - e^x)dx$$

Thus, the area is given by the integral

$$\int_0^1 (e^{2x} - e^x)dx = \frac{e^{2x}}{2} - e^x\Big|_0^1 = \frac{e^2}{2} - e - (1 - 1) = \frac{e}{2}(e - 2) \ .$$

Example 5.4. Find the area of the region bounded by the curve $y^2 = x^3$ and the line $x = 4$.

In drawing the curve, don't forget that y can be negative - in fact, the region is symmetric in the x-axis. The upper curve is $y = x^{3/2}$, and the lower curve is $y = -x^{3/2}$ so that $L(x) = 2x^{3/2}$ and the area is given by

$$\text{Area} = \int_0^4 dA = \int_0^4 2x^{3/2}dx = \frac{4}{5}x^{5/2}\Big|_0^4 = \frac{4}{5}4^{5/2} = \frac{128}{5} \ .$$

Example 5.5. Find the area of the region bounded by the curves $x = -y^2,\ x = y^2,\ y = -2$ and $y = 2$.

See figure 5.5 for the sketch. Here we choose to accumulate area in the y direction. The infinitesimal increment at a particular value of y is $dA = (y^2 - (-y^2))dy = 2y^2 dy$. Thus

$$A = \int_{-2}^{2} 2y^2 dy = \frac{2}{3} y^3 \Big|_{-2}^{2} = \frac{2}{3} 2^3 - (\frac{2}{3}(-2)^3 = \frac{32}{3} \ .$$

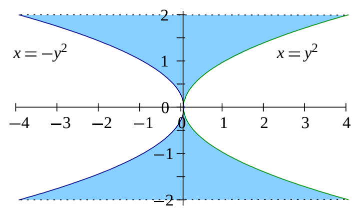


Figure 5.5

Example 5.6 . Let f be a function continuous on the interval $[a, b]$. Suppose that $\int_a^x f(t)dt$ is constant for x between a and b. Then f is identically zero.

Since $\int_a^x f(t)dt$ is constant, its derivative is zero. By the fundamental theorem of the calculus, its derivative is $f(x)$.

Example 5.7. Find $$\frac{d}{dx} \int_1^{2x} \ln(t)dt \ .$$

Let $g(u) = \int_1^u \ln(t)dt$, s o that the function to differentiate is $g(2x)$. By the chain rule

$$\frac{d}{dx} g(2x) = \frac{dg}{du}(2) \ .$$

By the fundamental theorem of the calculus, $dg/du = f(u)$. Thus

$$\frac{d}{dx} \int_1^{2x} \ln(t)dt = 2\ln(2x) \ .$$

Problems 5.1

1. Calculate the definite integral: $\int_{-4}^{4} (x^2 - 3 + \cos x)dx$.

2. Find the area of the region bounded by the line $x = 0$ and the curves $C_1 : y = 9 - x^2,\ C_2 : y = x(3 - x)$.

3. Find the area between the curves $y = x^{-1}$ and $y = x^{-2}$ between $x = 1$ and $x = 10$.

4. Find the area of the region in the first quadrant bounded by the curves $y = \sin \frac{\pi}{2} x$ and $y = x$.

5. Find the area of the region in the right half plane ($x > 0$) bounded by the curves $y = x - x^3$ and $y = x^2 - x$.

6. What is the area of the region bounded by the curves $y = x^3 - 3x$ and $y = 3x$.

7. The *average value* of a function $y = f(x)$ defined over an interval $[a, b]$ is defined to be

$$y_{\text{ave}} = \frac{1}{b-a} \int_a^b f(x) dx \ .$$

Find the average of $y = \sin x$ over the interval $[0, \pi]$.

8. Let $g(x) = x^2 + x^3$ for x in the interval $0 \leq x \leq 10$. Find the average, or mean, value of g on the interval. Find the average slope of the graph of $y = g(x)$ on the interval.

9. Let f be a function continuous on the interval $[a, b]$. Suppose that $\int_a^x f(t)dt$ is a linear function. Show that f is constant.

10. Let f be a function continuous on the interval $[a, b]$. Suppose that $\int_a^x f(t)dt = \int_x^b f(t)dt$ for all x between a and b. Show that $f(x) = 0$ for all x in the interval.

11. $$\frac{d}{dx} \int_0^{2x+1} \cos t dt =$$

12. $$\frac{d}{dx} \int_0^{x^2} t^3 dt =$$

13. $$\frac{d}{dx} \int_0^{2x} \ln t dt =$$

14. $$\frac{d}{dx} \int_0^{x^2+1} e^{t+1} dt =$$

5. 2 Summation and the Definite Integral

In this section we shall make the definition of the definite integral precise, and shall do some computations directly from the definition. The purpose here is to emphasize that the definite integral represents a process of accumulation, as well as to introduce and work with the notation for summation.

Let n be a positive integer, and $a(k)$ a rule that assigns a number to each integer between 1 and n. For example:

$$a(k) = 1 \ for \ all \ k \ ; \quad \text{or} \quad a(k) = 2k + 3 \ ; \quad \text{or} \quad a(k) = k^2 \ ; \quad \text{or} \quad a(k) = \frac{3}{k} \ ,$$

for k running from 1 to $n = 100$. The sum of all these numbers is denoted

$$\sum_{k=1}^{n} a(k) = a(1) + a(2) + \cdots + a(n) \ .$$

This sum can sometimes be easily expressed as a formula in n; more often this is difficult, or impossible. For example, the sums in the first three cases are, respectively

$$n \ ; \qquad n(n+4) \ ; \qquad \frac{n(n+1)(2n+1)}{6} \ ,$$

whereas a formula for the sum in the fourth case is not known. These formulas are not easy to come by, and usually rely on clever tricks. Here are the cases we will need for application to integration.

Proposition 5.5.

a)
$$Z(n) = \sum_{1}^{n} 1 = 1 + 1 + \cdots + 1 = n$$

b)
$$U(n) = \sum_{1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

c)
$$S(n) = \sum_{1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

d)
$$K(n) = \sum_{1}^{n} k^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

a) just says that the sum of n ones is n. To show b) we start with the observation that

$$(k+1)^2 = k^2 + 2k + 1$$

Now add these all together from $k = 1$ to $k = n - 1$. On the left hand side we get the sum of all the squares from 2^2 to n^2, or $S(n) - 1^2$. The sum of the first terms on the right hand side is $S(n-1)$, of the second is $2U(n-1)$ and the last contributes $n - 1$. Thus

$$S(n) - 1 = S(n-1) + 2U(n-1) + n - 1 \quad \textit{or} \quad 2U(n-1) = S(n) - S(n-1) - n \ .$$

Now add $2n$ to both sides, and remember that $S(n) - S(n-1) = n^2$:

$$2U(n) = n^2 + n \ ,$$

which gives b). For c) we use the same idea. Start with

$$(k+1)^3 = k^3 + 3k^2 + 3k + 1$$

Add these all together from $k = 1$ to $k = n - 1$, and calculate each term as above to get:

$$K(n) - 1 = K(n-1) + 3S(n-1) + 3U(n-1) + n - 1 \ .$$

We know $U(n-1)$ from a), and $K(n) - K(n-1) = n^3$, so we get:

$$3S(n-1) = n^3 - 3\frac{(n-1)n}{2} - n$$

Now add $3n^2$ to both sides, to get

$$3S(n) = n^3 - \frac{3}{2}n^2 + \frac{3}{2}n - n + 3n^2$$

from which c) follows. To derive d) we must employ the fourth powers in the same way.

Example 5.6. The sum of the first n odd integers is n^2. We see this using a). The first n odd integers are the numbers $1, 3, 5, \ldots, 2n - 1$. The kth odd integer is $2k - 1$. Thus the sum of the first n odd integers is

$$\sum_1^n (2k-1) = 2(\sum_1^n k) - \sum_1^n 1 = (n+1)n - n = n^2 \ .$$

Sometimes to find the sum of a collection of numbers it helps to write out the first few terms.

Example 5.9. Find $\sum_3^{15}(k^{-1} - (k+1)^{-1})$.

The sum of the first two terms is $(1/3 - 1/4) - (1/4 - 1/5) = 1/3 - 1/5$ Then the sum of the first three terms is $(1/3 - 1/5) + (1/5 - 1/6) = 1/3 - 1/6$. Notice the cancellation. This will happen at each stage, because each time the first term added is the same as the last subtracted. We conclude

$$\sum_3^{15}(k^{-1} - (k+1)^{-1}) = 1/3 - 1/16 \ .$$

Now let us return to the definition of the definite integral. Let $y = f(x)$ be a continuous function on the interval $[a, b]$. Select $n + 1$ points between a and b: $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. This is a *partition* P of the interval $[a, b]$. The *size* of the partition is the maximum difference between consecutive points, denoted $|P|$. Form the sum (called *the Riemann sum* of the function over the partition)

$$\sum_P f(x)\Delta x = \sum_1^n f(x'_k)(x_k - x_{k-1})$$

where x'_k is any point in the interval between x_{k-1} and x_k.

Definition 5.2. The function f is integrable over the interval $[a, b]$ if the Riemann sums converge. That is, there is a number L for which the following condition can be verified. Given any $\epsilon > 0$, there is a $\delta > 0$ such that if the partition satisfies $|P| < \delta$, then

$$|\sum_P f(x)\Delta x - L| < \epsilon \ .$$

L is the definite integral, denoted $\int_a^b f(x)dx$.

Now, we have already observed that all continuous functions are integrable, and we have discovered that the value of the integral is found by evaluating an indefinite integral. However, historically, the above definition (for area) was formulated, in a somewhat vaguer sense, long before the Calculus. And areas were calculated by actually finding this limit. In the sixteenth century Cavalieri succeeded in doing this for the functions $y = x^p$ over the interval $[0, 1]$ for all values of p from 1 to 9. This was a huge effort; completely replaced by one-line calculations using the calculus. Here is how Cavalieri proceeded.

Take the particular partition

$$P: \quad 0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < \frac{n}{n} = 1 \ .$$

Here $x_k = k/n$. Form the Riemann sum with x'_k taken to be x_k:

$$\sum_P x^p \Delta x = \sum_1^n (\frac{k}{n})^p (\frac{k}{n} - \frac{(k-1)}{n}) = \sum_1^n (\frac{k}{n})^p (\frac{1}{n}) = \frac{1}{n^{p+1}} \sum_1^n k^p$$

Now, for $p = 1$, using Propositon 5.5b) we obtain

$$\sum_P x\Delta x = \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n}$$

which converges to 1/2 as $n \to \infty$. Thus the limit exists, and we can conclude that

$$\int_0^1 x dx = \frac{1}{2} \ .$$

For $p = 2$ using Propositon 5.5c) we obtain

$$\sum_P x^2 \Delta x = \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} = \frac{2n^3 + 3n^2 + n}{6n^3}$$

which converges to 1/3. Thus, taking the limit, we have

$$\int_0^1 x^2 dx = \frac{1}{3} \ .$$

Cavalieri calculated (at great pains, in the manner of Proposition 5.5), the sum of the pth powers of the first n integers, and was able to find the limt for all p up to 9. Based on this, he daringly conjectured the correct formula for all integer values of p.

5.3 Volume

In section 5.1 we saw how to calculate areas of planar regions by integration. The relevant property of area is that it is *accumulative*: we can calculate the area of a region by dividing it into pieces, the area of each of which can be well approximated, and then adding up the areas of the pieces. To put it another way, we calculate area by adding piece by piece as we move through the region in a particular direction. Once we have obtained a formula for the differential increment in the area (such as $dA = L(x)dx$), we find the area by integration. This process can be used to calculate values of any accumulative concept, such as volume, arc length and work. The rest of this chapter is devoted to these applications.

Example 5.10. To begin with, let us calculate the volume of a sphere of radius R. We first have to decide on a direction of accumulation; and for that we need a particular coordinate representation of the sphere. Start with the region in the plane bounded by the x-axis and the curve $C : x^2+y^2 = R^2$. If we rotate this region in space about the x-axis, we obtain the sphere of radius R.

We accumulate the volume of the sphere by moving along the axis of rotation, (the x-axis), starting at $x = -R$, and ending at $x = R$. Let $V(x)$ be the volume accumulated when we reach the point x. Then the piece added when we move a distance dx further along the axis is a cylinder of width dx and of radius y, the length of the line from the x-axis to the curve C (see figure 5.6).

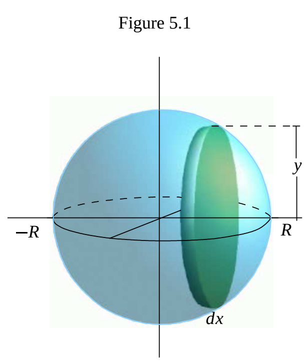


Figure 5.6

The volume of this piece is

$$dV = \pi y^2 dx \ . \tag{5.6}$$

Now, along C, $y = \sqrt{R^2 - x^2}$, from which we obtain $dV = \pi(R^2 - x^2)dx$. The volume thus is the integral of this differential:

$$V = \int dV = \int_{-R}^{R} \pi(R^2 - x^2)dx = \pi\left[R^2x - \frac{x^3}{3}\right]\Big|_{-R}^{R}$$

$$= \pi\left[R^2(R) - \frac{R^3}{3} - (R^2(-R) - \frac{(-R)^3}{3})\right] = \frac{4}{3}\pi R^3 \ .$$

We can find the volume of a spherical segment of depth h the same way (see figure 5.7):

$$V == \int_{R-h}^{R} \pi(R^2 - x^2)dx = \frac{2\pi}{3}h(R^2 - h^2) \ ,$$

after an even longer computation.

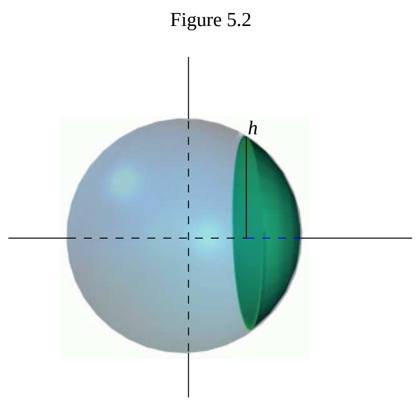


Figure 5.7

In general, we can calculate the volume of a solid by integration if we can see a way of sweeping out the solid by a family of surfaces, and we can calculate, or already know the area of those surfaces. Then we calculate the volume by integrating the area along the direction of sweep. In the above example we swept out the sphere by moving along the x-axis, and associating to each point x the area of the disc which is the perpendicular cross-section of the sphere at x. To summarize:

Volume: General Method.

1. Sketch the region under consideration.
2. Choose a direction in which to accumulate the volume.
3. Write down the expression for the differential increment in volume:

$$dV = A(x)dx \ , \tag{5.7}$$

where dx is an infinitesimal increment in the direction of accumulation, and $A(x)$ is the area of the section of the solid by the plane through point x and perpendicular to the direction of accumulation.

4. Integrate $A(x)dx$.

Of course, if the solid is highly irregular, finding $A(x)$ may still be a problem. In this section we restrict attention to those cases where $A(x)$ is known or is easily found.

Example 5.11. Find the volume of a cone of base radius r and height h.

Figure 5.3

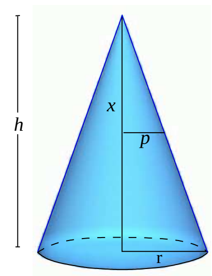


Figure 5.8

We sweep out the cone along its axis starting at the vertex (see figure 5.8), so x ranges from 0 to h. The cross-section of the cone at any x is a disc, let ρ be its radius. Then $dV = \pi\rho^2 dx$. Now, we can find ρ as a function of x by similar triangles:

$$\frac{\rho}{r} = \frac{x}{h} ,$$

so $\rho = rx/h$. Then

$$Volume = \int_0^h \pi \frac{r^2 x^2}{h^2} dx = \pi \frac{r^2}{h^2} \frac{x^3}{3} \Big|_0^h = \frac{1}{3}\pi r^2 h .$$

Example 5.12. Find the volume of a pyramid of height h and square base of side length s.

Figure 5.4

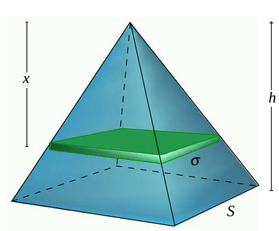


Figure 5.9

Here again, we sweep the pyramid out along its axis, with x representing the distance from the vertex (see figure 5.9). Then x ranges from 0 to h, and at any x, the cross-section is a square of side length σ. The differential increment of volume at x is $dV = \sigma^2 dx$. Again by similar triangles $\sigma = sx/h$. Thus

$$Volume = \int_0^h \frac{s^2}{h^2} x^2 dx = \frac{s^2}{h^2} \frac{x^3}{3} \Big|_0^h = \frac{1}{3} s^2 h .$$

Example 5.13. Let R be the region in the plane bounded by the curves $y = x^2$, $y = -x^2$. Let K be a solid lying over this region whose cross-section is a semicircle of diameter the line segement between the curves (see figure 5.10). Find the volume of the piece of K lying between $x = 1$ and $x = 2$.

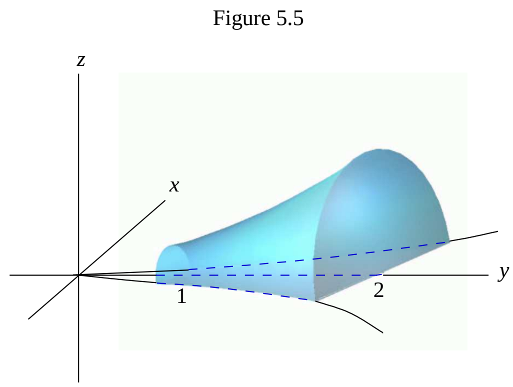


Figure 5.10

We sweep out along the x-axis. Then $dV = (1/2)\pi r^2 dx$, where r is the radius of the semicircle. Now $r = x^2$, so we obtain

$$V = \int_1^2 \frac{1}{2}\pi(x^2)^2 dx = \frac{1}{2}\pi\frac{x^5}{5}\Big|_1^2 = \frac{1}{10}\pi(32-1) = 31\pi \ .$$

Volumes of Revolution.

A solid of revolution is obtained by revolving a region in the plane around an axis in the plane. There are three ways of calculating the volume of such a solid, depending upon how we sweep it out.

Disc method. Suppose, as in figure 5.11, the region lies between a curve $C : y = f(x)$ and the x-axis, which is the axis of rotation. Then a cross section is a disc of radius y, so

$$dV = \pi y^2 dx = \pi(f(x))^2 dx \ .$$

Washer method. Suppose the region to be rotated lies between two curves $C_1 : y = f(x)$ lying above $C_2 : y = g(x)$ in the upper half-plane and the axis is the x-axis (see figure 5.12). Then a cross section is a *washer*: the region between two concentric circles. Its area is the difference of the areas of these circles. The differential increment of volume is then

$$dV = (\pi f(x)^2 - \pi g(x)^2)dx \ .$$

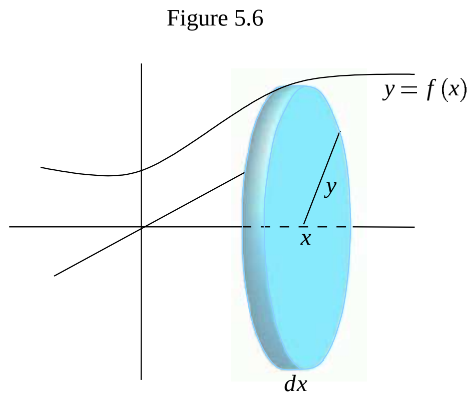


Figure 5.11

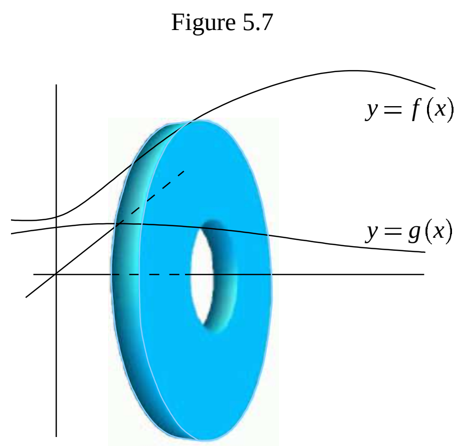
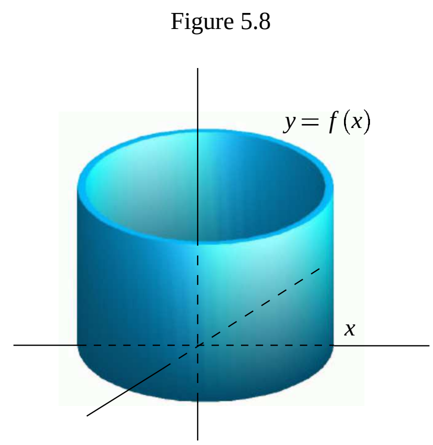


Figure 5.12

Shell method. This method is used when it is convenient to sweep out the volume along an axis perpendicular to the axis of rotation. To be more specific, suppose the region lies in the right half plane, and we rotate it about the y-axis. Then for each x, the incremental surface at x is the cylinder swept out by rotating the line segment perpendicular to the x-axis lying in the region (see figure 5.13) about the axis of rotation. The area of this surface is $2\pi x L$ where L is the length of the line segment. Thus

$$dV = 2\pi x L dx$$



.

Figure 5.13

Example 5.14. Consider the region R between the lines $L_1 : y = x$, $L_2 : y = 2x$ and $x = 3$.

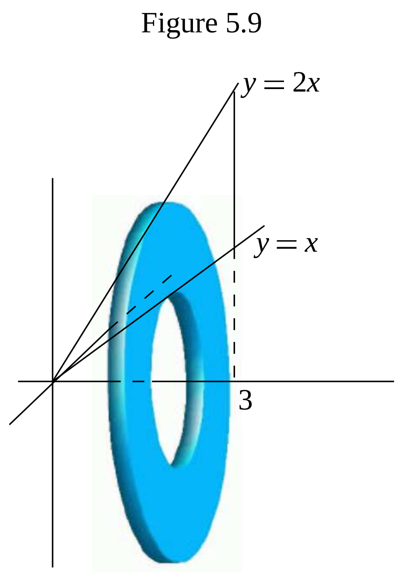


Figure 5.14

Suppose we generate a solid by rotating R about the x-axis (see figure 5.14). We sweep out the solid along the x-axis. At any x the cross-section is the washer generated by the line segment between L_1 and L_2. This is bounded by the circles of radius $2x$, x respectively. Thus $dV = \pi((2x)^2 - x^2))dx = \pi(3x^2)dx$, and so

$$V = \pi \int_0^3 (3x^2)dx = \pi x^3\Big|_0^3 = 27\pi \ .$$

Example 5.15. Suppose instead we rotate the same region R about the y-axis. Then (sweeping along the x-axis), the surface generated at any x is the cylinder of radius x and height the distance between the two curves: $2x - x$. Thus $dV = 2\pi x(x)dx$, and

$$V = 2\pi \int_0^3 x^2 dx = \frac{2}{3}\pi(3^3 - 0^3) = 18\pi \ .$$

Figure 5.10

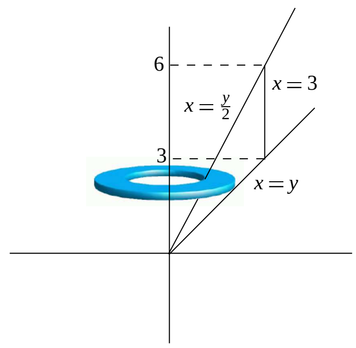


Figure 5.15

Example 5.16. We could also do example 5.15 by accumulating volume along the axis of rotation (the y-axis), but this is more complicated. As we see from figure 5.15, y ranges from 0 to 6, and the curve describing the outer boundary changes at the point $y = 3$.

So we have to split this up into two computations. For the first, y ranges from 0 to 3, and the region is bounded by the curves $x = y/2,\ x = y$, and for the second, y ranges from 3 to 6 and the region is bounded by the curves $x = y/2,\ x = 3$. In both cases, a slice at a fixed y is a washer, so

$$dV = \pi(R^2 - r^2)dy \ ,$$

where R is the outer radius, and r is the inner radius. For the first computation then

$$V_1 = \int_0^3 \pi(y^2 - (\frac{y}{2})^2)dy = \frac{\pi}{4}\int_0^3 3y^2 dy = \frac{27}{4}\pi \ .$$

For the second:

$$V_2 = \int_3^6 \pi(3^2 - (\frac{y}{2})^2)dy = \pi\int_3^6 (9 - \frac{y^2}{4})dy = \pi(9y - \frac{y^3}{12}\Big|_3^6 = 9\pi + \frac{9}{4}\pi.$$

The total volume is $V_1 + V_2 = 18\pi$.

Example 5.17 . Let R be the region above the x-axis, below the curve $y = x^{-1}$ and between the lines $x = 1$ and $x = 8$. FInd the volume of the solid obtained by rotating R about the x-axis.

We accumulate along the x-axis, using the disc method. The radius of the disc at x is x^{-1}, so $dV = \pi x^{-2}dx$, and the volume is

$$\pi\int_1^8 x^{-2}dx = -\pi x^{-1}\Big|_1^8 = \pi(1 - \frac{1}{8}) = \frac{7}{8}\pi \ .$$

Now to find the volume of the figure obtained by rotating this region about the y-axis, we use the shell method, we have $dV = 2\pi xgdx = 2\pi xx^{-1}dx = 2\pi dx$, so that the volume is $2\pi(8-1) = 14\pi$.

Figure 5.11

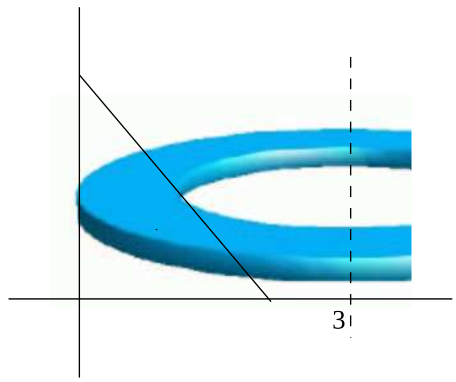


Figure 5.16

Example 5.18. Rotate the triangle bounded by the coordinate axes and the line $x + y = 1$ about the line $x = 3$, and find the volume (see figure 5.16).

We'll accumulate the volume in the direction of the axis of rotation, so the variable is y, ranging from 0 to 1. For a fixed y, the surface generated by the rotation is a washer of outer radius 3, and inner radius $3 - x = 3 - (1 - y) = 2 + y$. Thus $dV = \pi(3^2 - (2 + y)^2) = 5 - 4y - y^2$, and the volume is

$$\int_0^1 (5 - 4y - y^2)dy = (5y - 2y^2 - \frac{y^3}{3}\Big|_0^1 = \frac{8}{3} .$$

Problems 5.3

1. A solid is formed over the region in the first quadrant bounded by the curve $y = \sqrt{10 - x}$ so that the section by any plane perpendicular to the x-axis is a semicircle. What is the volume of this solid?

2. A solid is formed over the region in the first quadrant bounded by the curve $y = \sqrt{4 - x}$ so that the section by any plane perpendicular to the x-axis is a square. What is the volume of this solid?

3. A solid is formed over the region in the first quadrant bounded by the curve $y = 2x - x^2$ so that the section by any plane perpendicular to the x-axis is a semicircle. What is the volume of this solid?

4. The region in the first quadrant bounded by $y = \sqrt{x^2 - 1},\ y = 0,\ x = 1,\ x = 4$ is revolved around the x-axis. Find the volume of the resulting solid.

5. Find the volume of the solid obtained by rotating about the y-axis the region bounded by $y = x^2,\ x = 2$ and the x-axis.

6. The region in the first quadrant under the curve $y^2 = 2x - x^2$ is rotated about the y-axis. Find the volume of the resulting solid.

7. The region in the first quadrant bounded by $y = x^4$ and $x = 1$ is revolved around the y-axis. Find the volume of the resulting solid.

8. The region in the first quadrant bounded by $y = x - x^2$ and $y = x - x^3$ is revolved around the x-axis. Find the volume of the resulting solid.

9. The region bounded by the curves $y = e^x$ and $y = e^{2x}$ and the lines $x = 0$ and $x = 1$ is rotated around the x-axis for form a solid D. What is the volume of D?

10. The area between the curves $y = x$, $y = x^{-1}$ and the lines $x = 1$ and $x = 2$ is rotated around the x-axis to form a solid D, and around the y-axis to form a solid E. What are the volumes of D and E?

11. An ellipsoid is formed by rotating the curve $4x^2 + y^2 = 1$ around the x-axis. What is its volume? What is the volume of the ellipsoid obtained by rotating this curve about the y-axis?

12. A pencil sharpener is made by drilling a cone out of a sphere; the cone has as its axis a diameter of the sphere, and its vertex is on the surface of the sphere. If the ratio of the height to base radius in the cone is 4 to 1, and the sphere has a 1 inch radius, what is the volume of the pencil sharpener?

13. Consider the region in the first quadrant bounded by $y = \sin x,\ x = 0,\ x = \pi$. Find the volume of the solid obtained by rotating this region about the x-axis. Hint: If you use the disc method you'll need to know: $\cos(2x) = 1 - 2\sin^2 x$.

14. Now use the shell method to solve problem 13. Hint: The derivative of $\sin x - x\cos x$ is $x \sin x$.

15. A solid is formed over the region in the first quadrant bounded by the curve $y = 2x - x^2$ so that the section by any plane perpendicular to the x-axis is a semicircle. What is the volume of this solid?

5.4 Arc Length

Parametric Representation of a Curve

We have seen that a curve in the plane can be described *explicitly* as the graph of a function $y = f(x)$ or *implicitly*, as a relation $F(x, y) = C$ between the variable x and y. A third way a curve can be described is *parametrically* in the form

$$x = x(t) \qquad y = y(t) \tag{5.8}$$

as the *parameter* t ranges over an interval (a, b). For example, a particle may be moving on the plane, and its position at time t is $(x(t), y(t))$.

Example 5.19. The circle of radius R can be described parametrically by taking as the parameter the angle the ray through the point makes with the x-axis, so that $(R\cos t, R\sin t)$ are the coordinates of the point for a given angle t (see the figure). Since $\cos^2 t + \sin^2 t = 1$, these points all satisfy the implicit relation $x^2 + y^2 = R^2$.

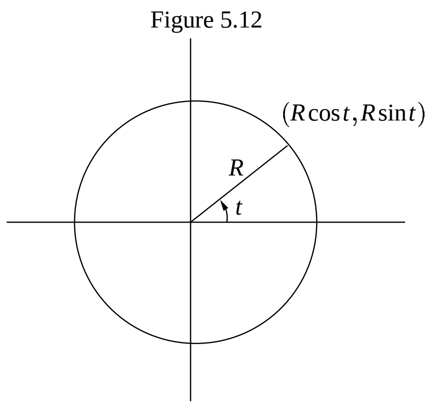


Figure 5.17

Suppose an object is moving in a plane perpendicular to the earth's surface so that the only force acting on it is gravity. Let its position at time t be $(x(t), y(t))$. Then dx/dt is the horizontal velocity of the object, and dy/dt the vertical velocity, in the sense that these are the rates of change of the motion in those directions. We say that the pair (also called *vector*) $(dx/dt, dy/dt)$ is the *velocity* of the object. Similarly, its *acceleration* is the pair $(d^2x/dt^2, d^2y/dt^2)$, where d^2x/dt^2 is the *horizontal acceleration* and d^2y/dt^2 is the *vertical acceleration*. In our case, since the only force is vertical, that of gravity, we have

$$\frac{d^2x}{dt^2} = 0 \quad \text{and} \quad \frac{d^2y}{dt^2} = -32 \ \text{ ft/sec}^2 \ .$$

We can integrate these equations to get the equations of motion of the object:

$$(5.9) \qquad \frac{dx}{dt} = v_x \qquad \frac{dy}{dt} = -32t + v_y$$

where v_x is the initial (and constant) horizontal velocity, and v_y is the initial vertical velocity. The position is then given by integrating again:

$$(5.10) \qquad x(t) = v_x t + x(0) \ , \qquad y(t) = -16t^2 + v_y t + y(0)$$

where at time $t = 0$ the particle is at $(x(0), y(0))$

Example 5.20. A rifle is fired from a prone position at an angle of 6 degrees from the horizontal. The bullet leaves the muzzle with an initial velocity of 900 ft/sec. Assuming the ground is level, how far does the bullet travel before it hits the ground again? How long does this take?

Let $(x(t), y(t))$ represent the position of the bullet at time t, assuming it starts at the origin, so $x(0) = 0$ and $y(0) = 0$. We are told that the tangent to the trajectory of the bullet at $t = 0$ makes an angle of 6 degrees with the horizontal, and that it starts out in this direction at 900 ft/sec. In an increment of time dt it travels a distance $900dt$ feet along this line, so the corresponding horizontal and vertical displacements (see figure 5.18) are $dx = 900\cos 6dt = 895dt$ and $dy = 900\sin 6dt = 94dt$. Thus $v_x = 895$ ft/sec and $v_y = 94$ ft/sec. Thus, according to (10), the position of the bullet at time t is

$$x(t) = 895t \qquad y(t) = -16t^2 + 94t \ .$$

Now, the bullet is at ground level when $y(t) = 0$. Solving that equation, $t = 0$ or $t = 94/16 = 5.88$ sec, and $x = 895(5.88) = 5262$ feet.

Figure 5.13

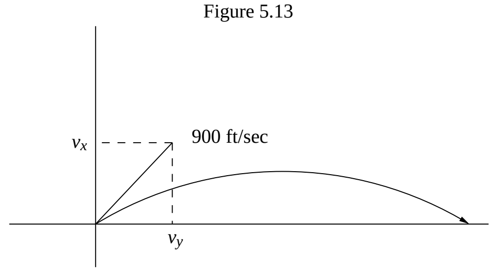


Figure 5.18

Example 5.21. Suppose a ball is hit horizontally at a height of 5 ft. at 120 mph (176 ft/sec). How far from the hitter will it hit the ground?

Putting the origin of the coordinates at the batter's feet, we have $x(0) = 0,\ y(0) = 5$. Since the ball is struck horizontally, we have $v_x = 176,\ v_y = 0$. Thus, from (2), the equations of motion are

$$x(t) = 176t \qquad y(t) = -16t^2 + 5\ .$$

Ground level is at $y(t) = 0$, so we can solve the second equation for $t:\ 0 = -16t^2 + 5$, from which we find $t = \sqrt{5/16} = .559$. Thus the answer is $x(.559) = 176(.559) = 98.39$ ft.

When a curve C is given parametrically as $x = x(t),\ y = y(t)$, it may not be easy to describe the curve. One way to do that is to try to eliminate the variable t, so as to get a relation between x and y.

Example 5.22. Suppose C is given parametrically by

$$x(t) = 15t + 10 \qquad y(t) = -16t^2 + 32t\ .$$

To eliminate t we solve the first equation for t in terms of $x:\ t = (x - 10)/15$, and then put that in the equation for y:

$$y = -16(\frac{x-10}{15})^2 + 32(\frac{x-10}{15})\ ,$$

from which we see that y is a quadratic function of x along C, so the curve is a parabola.

Example 5.23. Sometimes a little algebraic ingenuity is required. Given

$$x(t) = t^3 + t + 1,\ y(t) = t^2\ ,$$

we eliminate t in this way:

$$x = t(t^2+1)+1 = \sqrt{y}(y+1)+1 \quad \text{or} \quad (x-1)^2 = y(y+1)^2 \ ,$$

so the relation between x and y is quadratic in x and cubic in y.

Now let us see how to calculate the length of a curve. Suppose that C is a curve in the plane running from point P to Q. First of all, observe that arc length is accumulative: if we break the curve up into many pieces, the length of the curve is the sum of the lengths of the pieces. We thus anticipate that arc length can be found by integration. In order to discover what to integrate, we seek a differential relationship between arc length and the coordinates. For R any point on the curve, we consider the length of the arc from P to R. It is customary to use the letter s as the function representing arc length along the curve: thus $s(R)$ is the length of the curve from P to R. If we now move along the curve C a small distance ds from R, then the variables are displaced by small amounts dx and dy (see the figure).

Figure 5.14

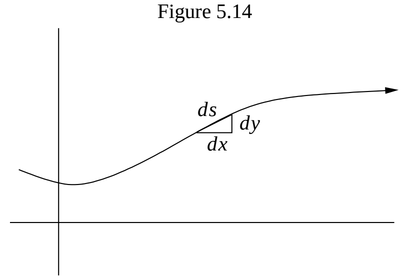


Figure 5.19

Near R the curve is almost its tangent line, so we use the tangent line approximation to the length. By the pythagorean theorem we have

$$ds^2 = dx^2 + dy^2 \tag{5.11}$$

and thus the length of arc from P to Q is

$$L(C) = \int_P^Q ds = \int_P^Q \sqrt{dx^2+dy^2} \ . \tag{5.12}$$

To do this integration, we use the equations describing a curve, explicit or parametric. We warn the reader that usually this computation leads to an expression we do not yet know how to integrate; in this case we shall leave it as a definite integral.

Example 5.24. What is the length of the line $y = 3x+2$ between the points $P(1,5)$ and $Q(3,11)$?

Here we can express the differential (5.11) in terms of dx. We have $dy = 3dx$, so

$$ds^2 = dx^2 + dy^2 = dx^2 + 3^2dx^2 = 10dx^2$$

so $ds = \sqrt{10}dx$. Then

$$L = \int_P^Q ds = \int_1^3 \sqrt{10}dx = \sqrt{10}(3-1) = 2\sqrt{10}$$

which is, of course, the same answer we would have gotten from the distance formula of chapter I.

Example 5.25. Find the length of the arc of a circle of radius R subtended by an angle α.

Here we parametrize the circle as in example 5.18 (see figure 5.18): $x = R\cos t,\ y = R\sin t$. Then $dx = -R\sin t dt,\ dy = R\cos t dt$ and

$$ds^2 = dx^2 + dy^2 = (-R\sin t)^2 dt^2 + (R\cos t)^2 dt^2 = R^2(\sin^2 t + \cos^2 t)dt^2 = R^2 dt^2\ .$$

Our length is thus

$$L = \int ds = \int_0^\alpha R dt = R\alpha\ ,$$

another standard formula.

Example 5.26. Find the length of the curve $x^2 = y^3$ from (0,0) to (27,9).

First we have to find a suitable parametrization of the curve. Since the variables are positive in this range, we can write $y = t^2$, and then $x^2 = y^3 = (t^2)^3 = (t^3)^2$, so we can take $x = t^3$, as t ranges from 0 to 3. Now $dy = 2tdt$ and $dx = 3t^2dt$, so

$$ds^2 = dx^2 + dy^2 = 9t^4dt^2 + 4t^2dt^2 = t^2(9t^2+4)dt^2\ ;$$

that is, $ds = t\sqrt{9t^2+4}dt$. Then the length is

$$L = \int ds = \int_0^3 t\sqrt{9t^2+4}dt == \frac{1}{18}\int_4^{85} u^{1/2}du$$

using the substitution $u = 9t^2+4, du = 18tdt$. The limits of integration in u are found by evaluating u at $t = 0$ and $t = 3$. The length is

$$L = \frac{1}{18}\frac{2}{3}u^{3/2}\Big|_4^{85} = \frac{1}{27}(85^{3/2} - 8)\ .$$

Example 5.27. Find the length of the parabolic segment $y = 5x - x^2$ from $x = 0$ to $x = 5$.

We have $dy = (5-2x)dx$, so $ds^2 = (1+(5-2x)^2)dx^2$, and the length is

$$L = \int_0^5 \sqrt{1+(5-2x)^2}dx = \frac{1}{2}\int_{-5}^5 \sqrt{1+u^2}du\ ,$$

making the substitution $u = 2x-5,\ du = 2dx$. But that didn't help; we still end up with an integral we have yet to learn how to integrate.

Problems 5.4

1. Find the length of the curve $y = t^3$, $x = t^2$, $0 \le t \le 1$.

2. The equations $x = e^{at}\cos t,\ y = e^{at}\sin t$ define the logarithmic spiral. Find the length of this curve for $0 \le t \le 2\pi$.

3. Find the integral giving the length of the curve $y = \sqrt{1+x^2}$ for $0 \le x \le 1$.

4. Find the area of the surface obtained by rotating the line segment $y = 3x - 3,\ 3 \le x \le 5$ about the x-axis.

5. Find the area of the bowl obtained by rotating the parabola $y = x^2,\ 0 \le x \le a$ about the x-axis.

5.5 Work

Work is the product of force and distance. The unit of force in the British system is the foot-pound; in the metric system it is the joule, or newton-meter. For example, to move a box a distance of 12 feet against a force of friction of 40 lbs takes $40 \cdot 12 = 480$ foot-pounds. To lift a weight of 25 pounds a distance of 8 feet takes $25 \cdot 8 = 200$ foot-pounds. Remember that the weight (at the surface of the earth) of an object is a measure of its gravitational force, and is equal to mg, where m is the mass of the object and g is the acceleration of gravity. In the metric system, the gram and the kilogram are measures of mass, so to find the weight (in joules) we must multiply the mass (in kilograms) by g = 9.8 m/sec^2.

Example 5.28. How much work is done in lifting a mass of 1.2 kg a distance of 12 meters? The weight of the mass is (1.2)(9.8) newtons, so the work is (1.2)(9.8)(12) = 141.12 joules.

Now, suppose that the force is not constant, but varies over the distance traversed. For example, to put a rocket in orbit, we must calculate the energy (work) required. But the force on the rocket (its effective weight) decreases as it leaves the surface of the earth.

So, suppose we have a force acting along a line; denote the value of the force at the point x as $F(x)$. If $F(x)$ is positive, it works in the direction of increasing x, and if negative, it works opposite to that direction. That is, if the force is positive, it does work as we progress, and if negative, it causes work as we progress. The question we ask is this: what work is done by the force through a given distance on the line. Work is accumulative: If we break the interval up into pieces, the total work done is the sum of the work done over each piece. We set up a work calculation by integration, once we know the differential relation. Let $W(x)$ be the work done to get from a to x. If we move a small distance dx further, we may assume the force to be constant, and equal to $F(x)$ over this interval. Then the increment in work is $dW = F(x)dx$. We now find the total work by integrating:

$$W = \int_a^b F(x)dx \ .$$

In particular, if this is positive, the force is doing work over the interval, and if negative, we must supply the work to overcome it.

Definition 5.3. Let $F(x)$ be a force acting along the real axis, such that $F(x) > 0$ when F is acting in the direction of increasing x. Let $a < b$. The *work* done by the force over the interval $[a, b]$ is $W = \int_a^b F(x)dx$.

Example 5.29. Let $F(x) = x(1 - x^2)$ be a force acting on the x axis. How much work is done by the force a) between the points $0, 1$? b) between the points $0, 2$?

a)
$$W = \int_0^1 (x - x^3)dx = \frac{x^2}{2} - \frac{x^4}{4}\Big|_0^1 = 1/2$$

b)
$$W = \int_0^2 (x - x^3)dx = \frac{x^2}{2} - \frac{x^4}{4}\Big|_0^2 = -2$$

Thus the force does work for us between 0 and 1, but against us between 1 and 2.

Springs

Consider a spring hanging vertically. If the spring is extended or compressed, a force will be exerted, called the *restoring force*, directed toward the inital rest position. If a weight is carefully attached to the spring, it will extend to a new rest position, at which the weight exactly balances the restoring force. This is the *equilibrium position* for that weight. According to Hooke's Law, the magnitude of the restoring force and the displacement from rest are proportional. That is, if we let x represent the displacement from the equilibrium position, and $F(x)$ the force in the spring, then we have

$$F(x) = -kx \tag{5.13}$$

for some constant k, called the *spring constant*. The negative sign indicates that the force acts in the direction opposite to the displacement.

Example 5.30. Suppose a 5 lb weight extends a spring by 1.5 inches. a) What is the spring constant? b) How far will a 12 lb. weight extend the spring?

The force in the spring at 1.5 inches, or .125 feet is -5 lbs, so by (13), $-5 = -k(.125)$, so $k = 40$ lb/ft, and Hooke's law for this spring is $F(x) = -40x$. When the 12 lb weight is attached, the restoring force is $F(x) = -12$, so we find x by solving $-12 = -40x$, or $x = .3$ ft or 3.6 inches.

Example 5.31. Suppose a 3 pound weight displaces a spring 2 inches. How much work is required to displace the spring two feet?

First we use Hooke's law with the given information to find the spring constant. In our situation, when $x = 1/6$ foot, $F = -3$ pounds. From Hooke's law, $-3 = -k(1/6)$, so the spring constant is $k = 18$, and Hooke's law for this spring is $F(x) = -18x$. Now, for work, we have $dW = F(x)dx$, so to extend the spring 2 feet, we calculate the work done by the spring over this distance:

$$W = \int_0^2 -18x dx = -9x^2\Big|_0^2 = -36$$

ft-lbs. Thus the work needed to extend the spring 2 feet must counterbalance this, so is 36 foot-pounds.

Now if a spring with an object at its end is extended a certain distance and then released, it will vibrate. To understand this motion, we recall the discussion of Example 4.18. There we saw that for a body in motion $vdv = adx$, where x is the displacement, v the velocity and a the acceleration. If we multiply by the mass m and use Newton's law, $F = ma$, we get

$$mvdv = Fdx \ . \tag{5.14}$$

The expression on the right is dW, the differential of work, and the expression on the left is the differential of $(1/2)mv^2$, the kinetic energy: the change in kinetic energy is equal to the work done. This is then the differential expression of the law of conservation of energy. Now, for a spring, $F(x) = -kx$, so (5.14) becomes

$$mvdv + kxdx = 0$$

which integrates to

$$mv^2 + kx^2 = \text{constant} \tag{5.15}$$

during the motion of a vibrating spring of spring constant k and a mass m at its end.

Example 5.32. A spring of spring constant $k = 4$ ft/lb with an object of weight 3 lbs attached to its end is at rest. The object is extended a distance of 2 feet and then released. At what velocity does the object pass its equilibrium position?

The mass of the object is $m = w/g = 3/(32) = .0938$. Thus, in this situation, (15) tells us that $.0938v^2 + 4x^2$ is constant during the motion. At the moment of release, $v = 0$ and $x = 2$, so this constant is $.0938(0)^2 + 4(2)^2 = 16$. At the moment the object passes the equilibrium point, $x = 0$, so we can solve:$.0938v^2 = 16$, for $v = 18.48$ ft/sec.

Example 5.33. A 1 kg mass extends a spring 8 cm. Suppose that a mass of 3 kg is attached to the spring, then extended 20 cm. beyond equilibrium and released. What will be the velocity of the mass as it passes the 10 cm. point?

First we must find the spring constant. The weight of a 1 kg mass is $w = mg = (1)(9.8) = 9.8$ newtons. From Hooke's law we have $-9.8 = -k(.08)$, so $k = 9.8/.08 = 122.5$ newtons/meter. For our 3 kg mass in motion, equation (7) is $3v^2 + 122.5x^2 =$ constant. At the moment of release, $v = 0$ and $x = .2$, so the constant is $122.5(.2)^2 = 4.9$. At the 10 cm point, we solve $3v^2 + 122.5(.1)^2 = 4.9$, giving $v = 1.107$ meters/second.

Example 5.34. What is the energy required to lift a payload of 1000 lbs a distance of 250 miles from the surface of the earth?

We have to calculate the work required to move the payload 250 miles vertically from the earth's surface. Let s represent distance from the center of the earth, and let R be the radius of the earth. According to Newton's law of gravitation, the force of gravity is

$$F(s) = -\frac{k}{s^2}$$

for some constant k. Now, when $s = R$, for our payload, $F(R) = -10^3$ lbs, so $k = 10^3R^2$, and $F(s) = -10^3R^2/s^2$. Thus the work "done by" gravity is

$$W = -\int_R^{R+250} \frac{10^3R^2}{s^2} ds = -10^3R^2(-\frac{1}{s})\Big|_R^{R+250} = -10^3R^2(\frac{250}{R(R+250)}) \ .$$

Taking $R = 3900$ miles, we see that the energy required to lift this payload to 250 miles is

$$10^3(3900)\frac{250}{4150} = 2.34(10^5) \quad \text{mi} \cdot \text{lbs} = 12.4(10^9) \quad \text{ft} \cdot \text{lbs}$$

Example 5.35. A cistern is a hole in the ground (usually lined with steel or cement) used to collect water. Suppose that we have a cylindrical cistern of radius 8 feet and of depth 20 feet. If it is full of water, how much work is required to empty the cistern?

In this problem, the force is the weight of the water (one cubic foot of water weighs 62.5 lbs), but the distance the water has to be moved depends upon how deep into the cistern we have gotten. So, we again approach the problem dynamically, where the dynamic variable is the distance y between the top of the cistern and the surface of the water. At this depth, the work required to lift the next slab of water (of thickness dy) is $y\times$ (weight of that slab). The weight of that slab is $62.5\times$(volume). The slab is a cylinder of radius 8 feet and thickness dx. Thus

$$dW = (62.5)\pi(8)^2 ydy \ .$$

Since the cistern is 20 feet deep, the total work then is the integral

$$\int_0^{20} 62.5\pi(8)^2 ydy = 4000\pi\frac{y^2}{2}\Big|_0^{20} = 8 \times 10^5\pi \text{ ft} \cdot \text{lbs} \ .$$

Example 5.36. Suppose instead that the cistern, of the same depth, has a parabolic profile, following the curve $y = (5/16)x^2$. Now, to lift the slab of thickness dy at a depth y takes the amount of work

$$dW = (62.5)\pi x^2 ydy \ ,$$

because the radius at this level is now x. Since $x^2 = (16/5)y$, the total work is given by

$$\int_0^{20} 62.5\pi\frac{16}{5}y^2 dy = 200\pi\frac{y^3}{3}\Big|_0^{20} = 5.33 \times 10^5\pi$$

Problems 5.5

1. Find the work done in pumping all the oil (whose density is 50 lbs. per cubic foot) over the edge of a cylindrical tank which stands on end. Assume that the radius of the base is 4 feet, the height is 10 feet and the tank is full of oil.

2. John Brown has a parabolic cistern in the ground with a depth of 12 feet and a diameter at the top of 4 feet. This can be viewed as formed by revolving the curve $y = 3x^2$ around the y-axis, where the line $y = 12$ represents ground level. How much work does it take to pump out the cistern when it is full of water (the density of water is 62.5 lb/ft^3)?

3. A cylindrical reservoir of base radius 50 feet and height 15 feet is built 300 feet above the surface level of a lake. How much work is required to fill the reservoir with lake water (assuming the lake is large enough that its surface level does not change during this process)? Recall that the density of water is 62.5 lb/ft^3.

4. A 2 lb. weight will extend a certain spring 5 inches. How much work is done in extending the spring 14 inches?

5. A 10 kg mass extends a spring 45 cm, to a new equilibrium position. The spring is then extended another meter and released. With what velocity does it pass the equilibrium position?

5.6 Mass and Moments

Mass is another concept which is accumulative, and so can be calculated by integration. An object is said to be *homogeneous* if its composition is everywhere the same. In this case, mass is proportional to volume, where the constant of proportionality is denoted δ, the density. Thus, for example, the density of water is 1 g/cc, or 62.5 lbs/cu.ft. If the object is inhomogeneous, then its density will vary as we move around the object. For x a point on the object, we let $\delta(x)$ be its density at that point - in the sense that $dM = \delta(x)dV$ at the point: the ratio of the mass to volume of small cube centered at x is approximately $\delta(x)$.

Example 5.37. A cistern is formed by rotating the curve $y = x^4$ from $x = 0$ to $x = 1$ yd. around the y-axis. The cistern is filled with muddy water which has settled, so that the density of the water at a height y from the bottom is $\delta(y) = (1.02 - .02y^2)62.5$ lbs/cu.ft (see figure 5.21). What is the total weight of the fluid contained in the cistern?

We calculate the weight by accumulating fluid along the y-axis, from $y = 0$ to 1. At a height y, the weight of the disc of height dy is $dW = \delta(y)dV$, and $dV = \pi r^2 dy$, where r is the radius of that disc. Since the profile is the curve $x = y^{1/4}$, we have $dW = \pi\delta(y)y^{1/2}dy$. Finally, since our spatial measurements are given in yards, we convert the density to lbs/yd: $\delta(y) = (1.02 - .02y^2)(62.5)(27)$ lbs/cu.yd. Thus the weight is

$$W = \int_0^1 (62.5)(27)\pi(1.02 - .02y^2)y^{1/2}dy = 5301.3\int_0^1 (1.02y^{1/2} - .02y^{5/2})dy$$

$$= 5301.3((1.02)\frac{2}{3}y^{3/2} - (.02)\frac{2}{7}y^{7/2})\Big|_0^1 = 3574.6 \text{ lbs}$$

Figure 5.15

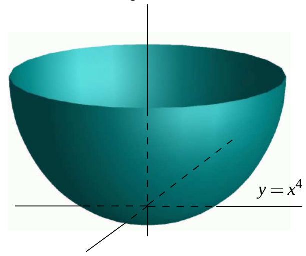


Figure 5.20

Example 5.38. A baseball bat can be considered as a cone of height 28 in, and of base radius 1.5 in. If the bat is made of hickory, of density .0347 lb/in^3, what is the weight of the bat?

We accumulate the weight of the bat from its vertex to its base, At a distance x in from the vertex, a slice of thickness dx has volume $dV = \pi r^2 dx$, where r is the radius at that point. By similar triangles, $r/1.5 = x/28$, so $r = 1.5x/28 = .0535x$. Then the weight of the bat is

$$W = \int \delta dV = \int_0^{28} (.0347)\pi(.0535x)^2 dx = (3.12 \times 10^{-4}) \int_0^2 8x^2 dx = 2.283 \text{ lbs}$$

Moments

Suppose that we have two small objects situated on a plane, and L is a line which runs between the objects (see the figure on the left).

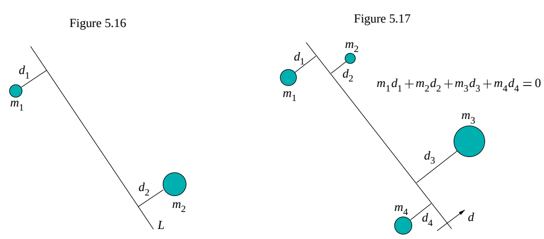


Figure 5.21

Archimedes observed that the objects are balanced around the line L if the product of the mass and the distance to the line is the same for the two objects. By "balance" we mean this: if the line L were a rod in space which is free to rotate, and the objects were attached to this rod by arms perpendicular to the rod, then the rod will not rotate if this condition is met. This product we call the *moment* of the object about the line L : $Mom_L = (mass)(distance)$. If we take distance to be directed: negative on one side of the line, and positive on the other, Archimedes' Law is that a system is in balance about a line L if the sum of the moments about L is zero. Phrased this way, the law applies to a system of many masses; for example the system on the right in figure 5.21 is in balance.

Thus, moment is an accumulative concept, and we can discover the moment for any object by integration. We shall consider only planar homogeneous objects of density 1, so that mass and area are the same. The first method for finding moment about a given line is to sweep out the region in the direction perpendicular to the line; at each stage adding the moment of a differential rectangle at a fixed distance from the line.

Suppose, to start with, we wish to find the moment about the y-axis $\{x = 0\}$ of a region is bounded by the curves $y = f(x),\ y = g(x)$, from $x = a$ to $x = b$ (see figure 5.22). We calculate the moment by adding the moments of infinitesimal strips, starting at $x = a$, and going to $x = b$. At a point x, the next strip has height $f(x) - g(x)$ and width dx, so its mass is $(f(x) - g(x))dx$, and the moment about the y-axis is $dMom_{x=0}$ = distance $\cdot$ mass $= x(f(x) - g(x))dx$. The moment of the entire region is the integral of this differential from a to b:

$$Mom_{\{x=0\}} = \int_a^b x(f(x) - g(x))dx$$

for the region of figure 5.22.

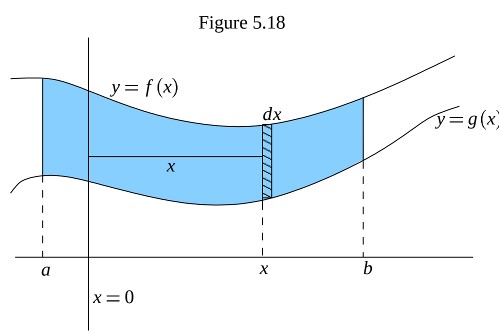


Figure 5.22

Example 5.39. Consider the region bounded by the coordinate axes and the line $4x + 3y = 12$ (see figure 5.23). Find the moments of this region about the coordinate axes.

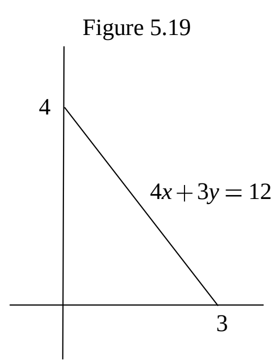


Figure 5.23

To find $Mom_{\{x=0\}}$, we sweep out along the x axis from $x = 0$ to $x = 3$. The height of the strip at a point x is $y = (12 - 4x)/3$, and the distance from the line $x = 0$ is x. Thus $dMom_{x=0} = (1/3)x(12 - 4x)dx$, and the moment is

$$Mom_{\{x=0\}} = \int_0^3 \frac{1}{3}(12x - 4x^2)dx = \frac{1}{3}(6x - \frac{4x^3}{3})\Big|_0^3 = 6\ .$$

To find the moment about the x-axis, we sweep the region out in the vertical direction from $y = 0$ to $y = 4$. At a point y, the width of the strip is $x = (12 - 3y)/4$, so $dMom_{y=0} = (y/4)(12 - 3y)dy$, so

$$Mom_{\{y=0\}} = \int_0^4 \frac{1}{4}(12y - 3y^2)dy = 8 \ .$$

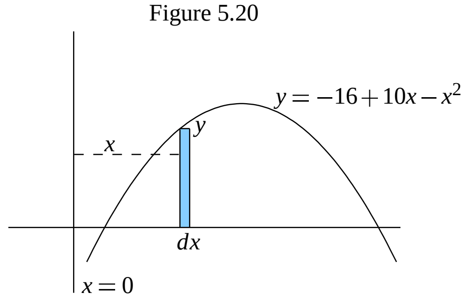


Figure 5.24

Example 5.40. Find the moment about the y-axis of the region bounded by the x-axis and the curve $y = -16 + 10x - x^2$ (see figure 5.24).

The region will be swept out along the x-axis from the points where the curve intersects the x-axis. We solve $0 = -16 + 10x - x^2 = -(8 - x)(2 - x)$. The region thus lies between the lines $x = 2$ and $x = 8$. At an intermediate point x, the differential strip is at distance x from the y-axis, and the mass is $(-16 + 10x - x^2)dx$, so

$$dMom_{\{x=0\}} = x(-16 + 10x - x^2)dx$$

and

$$Mom_{\{x=0\}} = \int_2^8 x(-16 + 10x - x^2)dx = \int_2^8 (-16x + 10x^2 - x^3)dx$$
$$= (-8x^2 + \frac{10}{3}x^3 - \frac{x^4}{4})\Big|_2^8 = 180 \ .$$

Example 5.41. Find the moment about the line $x = 4$ of the same region.

Here we have the same analysis, but now the distance from the strip to the balance axis is $x - 4$, so

$$dMom_{\{x=4\}} = (x - 4)(-16 + 10x - x^2)dx \ .$$

Note that if $x < 4$ this will be negative, and if $x > 4$ it is positive - this is just what we want because we want the contributions of the pieces on either side of the axis to be opposite. To find the moment we integrate;

$$Mom_{\{x=4\}} = \int_2^8 (x - 4)(-16 + 10x - x^2)dx = \int_2^8 (64 - 56x + 14x^2 - x^3)dx = 36 \ .$$

Since the answer is positive, this region is overbalanced to the right. Note that the region is symmetric about the line $x = 5$, so that it is perfectly balanced about the line $x = 5$. If we calculate $Mom_{[\{x=5\}}$, we'll get 0.

Example 5.42. Find the moment of a rectangular strip of width w and height h about the line through one end.

This is $Mom_{\{y=0\}}$ of the region sketched in the accompanying figure.

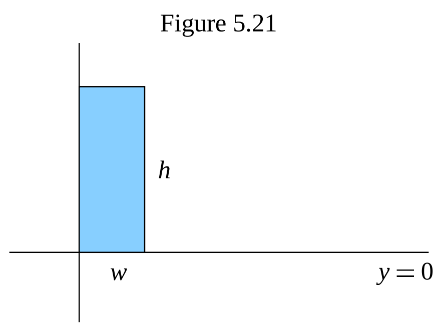


Figure 5.25

Sweeping out from $y = 0$ to $y = h$, we have $dMom_{\{y=0\}} = ywdy$, so

$$Mom_{\{y=0\}} = \int_0^h ywdy = \frac{1}{2}wh^2 \ .$$

Writing the moment as $wh(h/2)$, we see that this is the moment of a point object of the same mass (wh situated at the midpoint of the strip. This observation leads to an alternative method for calculating the moment about $y = 0$ without having to change the variable of integration.

Consider the region bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, as in figure 5.22. We calculate the moment about the x-axis by sweeping out in the x direction. At any point x the differential piece to be added is a vertical rectangular strip of mass $(f(x)-g(x))dx$. The midpoint of this rectangle is $(1/2)(f(x) + g(x))$. Thus

$$dMom_{\{y=0\}} = \frac{1}{2}(f(x) + g(x))(f(x) - g(x))dx \ .$$

Example 5.43. Find the moment about the x-axis of the region in the first quadrant bounded by the lines $y = 3x, \ y = 2x, \ x = 5$ (figure 5.26).

Figure 5.22

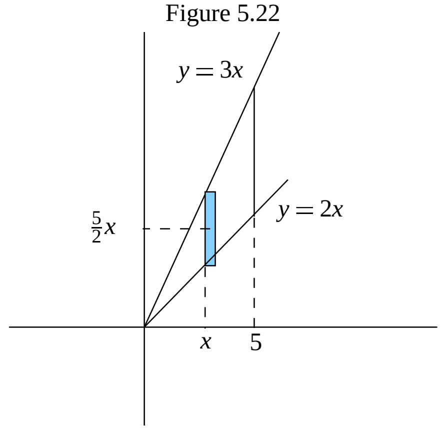


Figure 5.26

We sweep out from $x = 0$ to $x = 5$. The differential strip at the point x has the masss $(3x-2x)dx = xdx$, and its midpoint is $(1/2)(3x+2x) = (5/2)x$. Thus

$$dMom_{\{y=0\}} = \frac{5}{2}x^2dx\ ,$$

so

$$Mom_{\{y=0\}} = \int_0^5 \frac{5}{2}x^2dx = \frac{625}{6}\ .$$

To find the moment about the y-axis, we have

$$dMom_{\{x=0\}} = x(3x-2x)dx = x^2dx$$

so $Mom_{\{x=o\}} = \int_0^5 x^2dx = 125/3$.

Example 5.44. Find the moment about the x-axis of the region of example 5.33, bounded by the x-axis and the curve $y = -16 + 10x - x^2$.

At a point x between 2 and 8, the differential mass is ydx, and the midpoint of that strip is $y/2$, so $dMom_{\{y=0\}} = (y^2/2)dx$. Thus

$$Mom_{\{y=0\}} = \frac{1}{2}\int_2^8 (-16+10x-x^2)^2dx = \frac{648}{5}\ .$$

Centroids

Let R be a region in the plane and L a line in the plane. If R lies to the right of the line, then the moment of R about L, $Mom_L R$, is positive, and if R is to the left of L, then $Mom_L R$ is negative. So, if we look at all lines with a given slope, as we move from one side of R to the other, the moment about that line changes sign. Thus, there is a particular line with a given slope for which the moment of R is zero. The region R is *balanced* about this line: if R were to walk a tightrope in this direction, it would have the tightrope directly below this line. If we change the slope, we get another line of balance in the new direction. These two lines intersect in a point. It turns out that the line of balance of any slope goes through this point, called the *centroid*, or *center of mass* of the region R.

Definition 5.4. The *centroid* of a region R in the plane is that point in the plane such that for any line L through that point, $Mom_L R = 0$.

To calculuate the centroid, it is enough to look at lines of two different slope, in particular, horizontal and vertical lines. Suppose $(\bar{x}, \bar{y})$ is the centroid of R, so that the moment about the line $x = \bar{x}$ is zero. The distance from any point (x, y) to this line is $x - \bar{x}$, so we have

$$Mom_{\{x=\bar{x}\}}(R) = \int (x - \bar{x})dA = 0$$

or $\int xdA = \bar{x} \int dA$, which says that $Mom_{\{x=0\}}(R) = \bar{x}Mass(R)$. Similarly, $Mom_{\{y=0\}}(R) = \bar{y}Mass(R)$. Thus the coordinates for the centroid are

$$\bar{x} = \frac{Mom_{\{x=0\}}(R)}{Mass(R)}\ , \qquad \bar{y} = \frac{Mom_{\{y=0\}}(R)}{Mass(R)}\ .$$

Example 5.45. Find the centroid of the triangle in example 5.39.

This is a right triangle with sides of length 4, 3, so the mass is $(1/2(3)(4) = 6$. We found $Mom_{\{x=0\}} = 6$, $Mom\{y = 0\} = 8$, so the centroid is at $(\bar{x}, \bar{y}) = (6/6, 8/6) = (1, 1.33)$ (see figure 5.27).

Figure 5.23

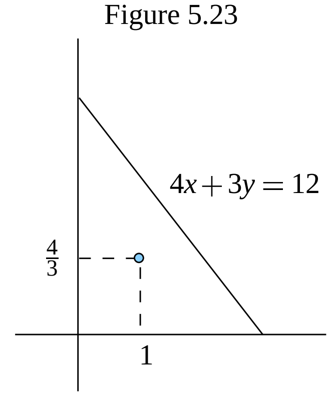


Figure 5.27

Example 5.46. Find the centroid of the region in the first quadrant bounded by the lines $y = 3x,\ y = 2x,\ x = 5$ (see figure 5.26).

In example 5.43, we found $Mom_{\{x=0\}} = 125/3,\ Mom_{\{y=0\}} = 625/6$. The mass is

$$Mass = \int_0^5 (3x - 2x)dx = \frac{x^2}{2}\Big|_0^5 = \frac{25}{2}\ .$$

Thus $\bar{x} = (125/3)/(25/2) = 10/3$ and $\bar{y} = (625/6)/(25/2) = 25/3$.

Example 5.47. Find the centroid of the region bounded by the y axis and the curve $y = -16 + 10x - x^2$ (see figure 5.24).

First of all, since the region is symmetric about the line $x = 5$, we have $\bar{x} = 5$. In example 5.40, we found $Mom_{\{y=0\}} = 648/5$. It remains to calculate the mass of the region which is

$$\int_2^8 (-16 + 10x - x^2)dx = (-16x + 5x^2 + \frac{x^3}{3})\Big|_2^8 = 36 \ .$$

Thus $\bar{y} = (648/5)/36 = 3.6$. The centroid is at $(5, 3.6)$.

As a final application of moments, we derive

Pappus' Theorem. Let R be a region in the right half plane, and consider the solid obtained by rotating R about the y-axis. The volume of this solid is the product of the area of R and the distance traveled by the centroid of R.

This is easy to see using the shell method for finding the volume. By that method,

$$Volume = 2\pi \int x dA$$

where the integration is taken in the x direction between the bounding lines. But $\int x dA = Mom_{\{x=0\}}$ and this is $\bar{x}$ $(Area)$, where the centroid has coordinates $(\bar{x}, \bar{y})$. So

$$Volume = 2\pi\bar{x}(Area) \ ,$$

which is what is asserted by by Pappus' theorem, since, in the rotation, the centroid travels around the circle of radius $\bar{x}$. Although at Pappus' time the calculus didn't even exist (in fact, neither did algebra), he demonstrated this result in essentially the same way, using Archimedes' theory of moments.

Example 5.48. Find the volume of the solid ring obtained by rotating the disc $(x-5)^2 + y^2 = 16$ about the y-axis.

The region being rotated is the circle centered at $(5, 0)$ and of radius 4. Clearly the centroid of a disc is its center, so $\bar{x} = 5$. Thus the volume is $2\pi(5)\pi(4)^2 = 160\pi^2$.

Problems 5.6

1. Kansas can be modelled as a rectangle of length 500 miles and of height 300 miles. The population density decreases as one moves west through the state; in fact the density x miles west of the eastern border is about

$$\delta(x) = 40 - 35\left(\frac{x}{500}\right)^2$$

people per square mile. About what is the population of Kansas? Where is the geographical center of Kansas?

2. Find the center of mass of the region bounded by the curves $y = x - x^3$ and $y = x - x^2$.

3. Find the center of mass of the homogeneous region in the first quadrant bounded by the curve $x^4 + y = 1$.

4. Find the centroid of the region bounded by the parabola $y^2 = 16x$ and the line $x = 9$.

5. Let P be the region obtained by rotating the curve $y = x^3 : \ 0 \leq x \leq 1$ about th y-axis. This bowl is filled with a fluid whose density along the disc $y = h$ is $1 - h$ g/cm^3. What is the total mass of this fluid? 6. Find the centroid of the region above the x-axis, under the curve $y = x^{-2}$ and between the lines $x = 1$ and $x = 8$.