Fontaine Rings and Local Cohomology

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Background

The main aim of the research outlined in this talk is to study properties of R^+ where R is a local domain of mixed characteristic.

We will describe recent results in a program to do this by studying the Fontaine rings of various rings associated to R and using the Frobenius map on these rings, which are rings of positive characteristic.

We recall that if R is an integral domain, then R^+ is the absolute integral closure of R; that is, the integral closure of R in the algebraic closure of its quotient field.

The basic setup.

Let R_0 be a Noetherian local ring of mixed characteristic with maximal ideal $\mathfrak{m}_0$ and perfect residue field k. We assume that R_0 is a complete integral domain, and let p be the prime number with $p \in \mathfrak{m}_0$ (and $p \neq 0$).

Let S_0 be a power series ring of the form $V[[y_2, \ldots, y_t]]$ that maps onto R_0, where V is a complete discrete valuation ring with maximal ideal generated by p. Let x_i be the image of y_i for each i; we can assume that $p(= x_1), x_2, \ldots, x_d$ form a system of parameters for R_0. We will sometimes refer to $\{x_1, \ldots, x_t\}$ as a set of generators of R_0.

Let U be a ring of mixed characteristic. The *Fontaine Ring* of U, denoted $E(U)$, is the inverse limit of

$$\cdots U/pU \xrightarrow{F} U/pU \xrightarrow{F} U/pU \xrightarrow{F} U/pU,$$

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The Fontaine ring has the following properties:

1. $E(U)$ is a perfect ring of characteristic p.
2. If U satisfies certain conditions, U can be reconstructed from $E(U)$ up to p-adic completion.

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Remark: An element of $E(U)$ can be represented by a sequence

$$(u_0, u_1, u_2, \ldots)$$

with $u_i \in U$ and $u_i^p \equiv u_{i-1}$ modulo p.

Recovering U from $E(U)$

The relation between $E(U)$ and U is carried out through the use of the ring of Witt vectors. If E is a perfect ring of characteristic p, the ring of Witt vectors, denoted $W(E)$, is a ring of mixed characteristic p such that $W(E)/pW(E) \cong E$, p is a non-zero-divisor, and $W(E)$ is complete in the p-adic topology.

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We have a map

$$\phi_U : W(E(U)) \to \hat{U},$$

where $\hat{U}$ is the p=adic completion of U. It is defined on $E(U)$ by

$$\phi_U((u_0, u_1, u_2, \ldots)) = \lim_{n\to\infty} u_n^{p^n}.$$

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We will discuss conditions for the map ϕ_U to be useful below.

What we are looking for: Almost Cohen-Macaulay Algebras

Let A be a ring between R_0 and R_0^+.

We recall that A is Cohen-Macaulay if the local cohomology $H^i_{\mathfrak{m}_0}(A)$ is zero for $i = 0, \ldots, d-1$, where d is the dimension of R_0 (and $A/\mathfrak{m}_0 A \neq 0$).

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Definition

The algebra A is almost Cohen-Macaulay *if*
for every i with $0 \leq i \leq d-1$
for every $x \in H^i_{\mathfrak{m}_0}(A)$
there is a $c \neq 0 \in A$ such that c^{1/p^n} annihilates x for all n (and c^{1/p^n} does not annihilate $A/\mathfrak{m}_0 A$ for some n).

In his proof of the Direct Summand Conjecture in dimension 3, Heitmann showed that R^+ is almost Cohen-Macaulay in dimension 3 (mixed characteristic) and showed that this is enough to imply the Direct Summand Conjecture.

Review of almost Cohen-Macaulay algebras in positive characteristic

Let T_0 be a Noetherian local domain of positive characteristic. Then the perfect closure of T_0 is almost Cohen-Macaulay, where the perfect closure is the direct limit of

$$T_0 \xrightarrow{F} T_0 \xrightarrow{F} T_0 \xrightarrow{F} \cdots$$

where F is the Frobenius map.

The starting point is that there is a nonzero c that annihilates the local cohomology of T_0 in degrees less than d.

We would like to do something similar in mixed characteristic.

Outline of a plan to construct almost Cohen-Macaulay algebras using Fontaine rings

1. Start with a Noetherian ring R_0 as above.
2. Adjoin some p^nth roots to get a ring R.
3. Take the Fontaine ring $E(R)$.
4. Take the ring of Witt vectors $W(E(R))$ and divide by a non-zero-divisor $P - p$ to get a quotient $W(E(R))/(P - p)$.
5. Show that $W(E(R))$ is almost Cohen-Macaulay and that there is a map from R_0 to $W(E(R))/(P - p)$.

Recall: For a ring of mixed characteristic U we have a map $\phi_U : W(E(U)) \to \hat{U}$ which we want to use to relate properties of $E(U)$ to properties of U.

An element of $E(U)$ can be represented by a sequence

$$(u_0, u_1, u_2, \dots)$$

with $u_i \in U$ and $u_i^p \equiv u_{i-1}$ modulo p.

Let $p = x_1, \dots, x_t$ be a set of generators for R_0. Let R be the ring obtained by adjoining p^nth roots of the x_i. Then

$$\phi_R : W(E(R)) \to \hat{R}$$

is surjective.

Note that the element $X_i = (x_i, x_i^{1/p}, x_i^{1/p^2}, \dots)$ satisfies $\phi_R(X_i) = x_i$.

The map ϕ_U is never injective in the situation we are considering. However..

We assume that we have adjoined p^nth roots of p and of the x_i. We now let

$$P(= X_1) = (p, p^{1/p}, p^{1/p^2}, \ldots,).$$

We then have that $\phi_U(P) = p$. Hence $\phi_U(P - p) = 0$ and $P - p$ is in the kernel of ϕ_U.

The ideal situation: ϕ_U induces an isomorphism from $W(E(U))/(P-p)$ to $\hat{U}$.

Why not just use $W(E(R))/(P-p)$ as our almost Cohen-Macaulay algebra?

The reason–we would need to have a map from R_0 to $W(E(R))/(P-p)$, which means that every element w with $\phi_R(w) = 0$ is a multiple of $P-p$. More precisely, let E_0 be the subring of $W(E(R))$ "generated" by V and the X_i. Let I_0 be the kernel of the map induced by ϕ from E_0 to R_0. We want

$$I_0 \subseteq (P-p)W(E(R)).$$

First Method:

Take a set of generators for I_0. Let $(a_0, a_1, \ldots)$ be such a generator. Then one can solve recursively for x_i to get

$$(a_0, a_1, \ldots) = (P - p)(x_0, x_1, \ldots).$$

One can derive formulas (very complicated ones) for x_i, which are elements of $E_0[1/P]$.

Second method: Extend R.

Theorem

The following are equivalent.

1. *The kernel of ϕ_R is generated by $P - p$.*
2. *If $r \in R_p$ and $r^{p^n} \in R$ for some n, then $r \in R$.*
3. *The kernel of $E(R) \to R/pR$ is generated by P.*

The map in (3) sends $(r_0, r_1, \ldots)$ to r_0.

We say that R is *root closed* if it satisfies these properties. Usually R will not be root closed, so we define the *root closure* of R to be

$$C = \{r \in R_p | r^{p^n} \in R \text{ for some } n\}.$$

C is a subring of R_p.

Theorem

The map

$$\phi_C : W(E(C))/(P - p)W(E(C)) \to \hat{C}$$

is an isomorphism.

All the elements we got from the first method will be in $E(C)$.

The ring C can be considered as an analogue of the perfect closure in positive characteristic.

Example: Let $R_0 = V[[x, y, z]]/(x^3 + y^3 + z^3)$, and let R be the ring obtained from R_0 by adjoining the p^nth roots of p, x, y, and z. If C is the root closure of R, then

$$\frac{x^{3/p} + y^{3/p} + z^{3/p}}{p^{1/p}}$$

is in C but not in R.

Properties of C/pC

Let the letter α denote a rational number of the form k/p^m . We let

$$J = \cup_{\alpha>0} p^{\alpha} C.$$

Let

$$T = \cup V[[P^{1/p^n}, X_2^{1/p^n}, \ldots, X_t^{1/p^n}]] \subseteq W(E(C)).$$

We have a surjective homomorphism from T to R, and this induces a homomorphism ϕ from T/pT to C/pC, where k is the residue field of V.

Proposition

If $c^p \in p^\alpha C$, then $c \in p^{\alpha/p} C$.

In fact, this is really the main property of the root closure. Suppose that $x^p \in p^\alpha C$. Then $(x/p^{\alpha/p})^p \in C$, so $(x/p^{\alpha/p})^{p^n} \in R$ for some n. Hence $x \in p^{\alpha/p} C$.

Proposition

The map induced by ϕ from T/pT to C/J is surjective.

Proof Let c be an element of C; we may assume that c is not in J. We wish to represent c as the sum of an element in the image of ϕ and an element of J.

By the definition of C, we have that $c = r/p^k$ for some k and $r \in R$, and $c^{p^n} \in R$ for some n. Since the map is clearly surjective to $R/J_R R$, where J_R is the ideal of R generated by positive fractional powers of p, we can write

$$c^{p^n} = \phi(t_0) - p^b s$$

for some $t_0 \in T$, $b > 0$ and $s \in R$. Let $t = t_0^{1/p^n}$.

Then

$$(\phi(t) - c)^{p^n} \cong \phi(t)^{p^n} - c^{p^n} \cong \phi(t^{p^n}) - c^{p^n} \cong \phi(t_0) - c^{p^n} \cong p^b s$$

modulo pC. Thus if we let a be the minimum of b and 1, we can write

$$(\phi(t) - c)^{p^n} = p^a u$$

for some $u \in C$, and we have

$$\left[\frac{(\phi(t) - c)^{p^n}}{p^{a/p^n}}\right]^{p^n} = u$$

Since C is root closed, this implies that $\phi(t) \cong c$ modulo p^{a/p^n}.

Proposition

C/J is the perfect closure of R_0/pR_0.

Proof We know that the Frobenius map on C/pC, and hence also on C/J, is surjective. On the other hand, since C is root closed, if x^p is in J, then $x^p \in p^\alpha C$ for some $\alpha > 0$, so $x \in p^{\alpha/p} C$, and thus $x \in J$. Thus C/J is perfect.

The fact that it is the perfect closure of R_0/pR_0 follows essentially from the fact that we have a map from R_0/pR_0 to C/J which and every element of C/J has a power that is in the image.

The above properties imply that the "associated graded" ring $\oplus_{\alpha \geq 0} p^\alpha C / p^\alpha J$ is almost Cohen-Macaulay.

However, it is not clear whether this fact implies very much about C itself.

The final property of C/pC is that it is a limit of Noetherian rings for which $x_2, \ldots, x_d$ form a system of parameters. This follows from the fact that C is an integral extension of R_0.

Since we have an isomorphism

$$E(C)/PE(C) \cong C/pC,$$

all of the above properties hold for $E(C)/PE(C)$. We would like to know in addition that $E(C)$ is a limit of Noetherian rings for which $P, X_2, \ldots, X_d$ form a system of parameters. This would imply that $W(E(C))/(P-p)W(E(C))$ is an almost Cohen-Macaulay algebra for R_0.

An Example

Let $R_0 = V[[x, y, u, v, w]]/I$, where I is the ideal generated by

1. The 2 by 2 minors of $\begin{pmatrix} p & x & y \\ u & v & w \end{pmatrix}$
2. $p^3 + x^3 + y^3, p^2u + x^2v + y^2w, pu^2 + xv^2 + yw^2, u^3 + v^3 + w^3$.

R_0 is a normal non-Cohen-Macaulay domain. One can show using the first method that the image of $H^2_{\mathfrak{m}_0}(R_0)$ in $H^2(W(E(C))/(P - p)$ is almost zero.

K. Shimomoto has used these methods combined with Hochster's method of modifications to construct an algebra A with the following properties:

1. $A/(p, x_2, \ldots, x_d) \neq 0$.
2. $x_2, \ldots, x_d$ form a regular sequence on A/pA.
3. p is not nilpotent on A and $(0 :_A p)$ is annihilated by p^ϵ for all positive rational epsilon.

It is not known whether $A/(p, x_2, \ldots, x_d)$ is almost zero; if not, then A is an almost Cohen-Macaulay algebra.