14.4 Green's Theorem

Theorem. Let
- R be a region in the xy-plane.
- C be a simple, closed curve enclosing R
- $\vec{F}(x,y) = M\hat{\imath} + N\hat{\jmath}$ be continuously differentiable over R & C

Flux Version

Line Integral		Area (Iterated) Integral
$\oint_C M\,dy - N\,dx$	$=$	$\iint_R (M_x + N_y)\,dx\,dy$

vector form → $\oint \vec{F}\cdot\vec{n}\,ds = \iint_R \nabla\cdot\vec{F}\,dA$

↑ unit normal vector to C

Circulation Version

$$\oint_C M\,dx + N\,dy = \iint_R (N_x - M_y)\,dx\,dy$$

vector form → $\oint_C \vec{F}\cdot d\vec{r}$

$$= \oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R (\nabla\times\vec{F})\cdot\hat{k}\,dA$$

↑ unit tangent vector to C

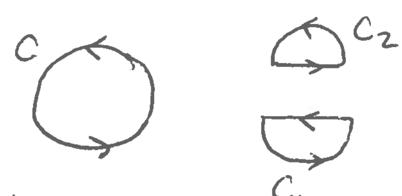
Questions / Thoughts

What is the idea behind Green's Theorem?

Idea behind proof of Green's Theorem

Observation $\int_{(a,b)}^{(c,d)} f\,ds = -\int_{(c,d)}^{(a,b)} f\,ds$

Question 1: Subdivide regions



$\oint_C f\,ds \;\square\; \int_{C_1} f\,ds + \int_{C_2} f\,ds$

↑ How are these integrals related and why?

Question 2: What happens if we divide a region into infinitely many sub regions?

14.4 (Cont)

Ex 1 Given $\oint_C \sqrt{y}\,dx + \sqrt{x}\,dy$ where C is the closed curve formed by $y=0$, $x=2$, $y=\frac{x^2}{2}$.

a.) Does this integral model flux/circulation.

b.) Draw the closed curve C.

c.) Calculate the integral using Green's theorem.

Ex 2

Ex 2 Given the vector field $\vec{F}(x,y) = x\hat{i} + 2y\hat{j}$ and curve C, given by $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$.

a.) Draw the vector field, the curve C & make predictions about the flux / the circulation.

b.) Calculate $\oint_C F \cdot n \, ds$

c.) Calculate $\oint_C F \cdot T \, ds$

Ex3

Find the area between $y=\sqrt{x}$ and $y=\frac{1}{4}x$ using the formula to the right.

Check your answer using a different method.

Let C be a closed, simple, region enclosing a region R.

Then the area of R is given by

$$A = \oint_C -\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy$$

14.5 Surface Integrals

Theorem:

Let

- R be a closed, bounded region in the x-y plane
- f be a function with first-order partial derivatives on R
- G be a surface over R given by $z = f(x,y)$.
- $g(x,y,z) = g(x,y,f(x,y))$ be continuous on R.

Then

$$\iint_{G} g(x,y,z)\, dS = \iint_{R} g(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\; dy\,dx$$

Ex 1 Evaluate $\iint_G g(x,y,z)\, dS$ given by $g(x,y,z) = y$,

$G: z = 4 - y^2, \quad 0 \le x \le 3, \quad 0 \le y \le 2.$

14.5 (Cont)

Ex 2. Evaluate $\iint_S 3z \, dS$ where G is the top of the tetrahedron bounded by the coordinate planes and the plane $2x + 6y + 3z = 6$.

14.5 Cont

Ex 3 Evaluate the flux across G where $F(x, y, z) = 2\hat{i} + 5\hat{j} + 3\hat{k}$ & G is the part of the cone $z = \sqrt{x^2+y^2}$ outside the cylinder $x^2+y^2=1$ and inside the cylinder $x^2+y^2=4$.

14.6 Gauss's Divergence Theorem

Green's Theorem
(or Gauss's theorem in the plane)

Let

- R be

- $\vec{F}(x,y)$ be

- C be

- $\hat{n}$ be

Then

$$\oint_C \vec{F}(x,y)\cdot\hat{n}\,ds = \iint \nabla\cdot\vec{F}\,dA$$

_____ integral   _____ integral

Gauss's Theorem

Let

- S be

- $\vec{F}(x,y,z)$ be

- δS be

- $\hat{n}$ be

Then

$$\iint_{\delta S} \vec{F}(x,y,z)\cdot\hat{n}\,dS = \iiint_S \nabla\cdot\vec{F}\,dV$$

_____ integral   _____ integral

Question: Why is it called the divergence theorem?

Ex1 Let $\vec{F}(x,y,z) = 4z$ and S be the upper hemisphere with radius 3 and center $(0,0)$

a) Calculate $\iint_{\delta S} \vec{F} \cdot \hat{n} \, dS$ as a surface area integral

b.) Calculate a.) using Gauss's Theorem.

14.6 (Cont.)

Ex2 Calculate $\iint_{\partial S} \vec{F} \cdot \hat{n} \, dS$, where $\vec{F} = xy\hat{\imath} + e^x\hat{\jmath} + z^3\hat{k}$

over the box $0 \le x \le 3$, $1 \le y \le 2$, $0 \le z \le 1$

Ex 3 Calculate $\iint_{\partial S} \vec{F}\cdot\vec{n}\, dS$ where $\vec{F} = 3x\hat{\imath} + 2\hat{\jmath} + 2z^2\vec{k}$

and S is the solid between the paraboloid $z = x^2+y^2$, the cylinder $x^2+y^2=1$ & the xy-plane.

14.7 Stoke's Theorem

Green's Theorem (Circulation Version)

Picture

Let

- R be a region in the xy-plane
- C be a simple, closed curve bounding R
- $\vec{F}(x,y)$ be a vector field continuously differentiable over R & C
- $\vec{T}$ be the unit tangent vector to C

Then

$$\oint_C \vec{F}\cdot\vec{T}\,ds = \iint_R (\nabla\times F)\cdot\hat{k}\,dA$$

Stoke's Theorem

Picture

Let

- S be a 3-D surface,
- δS be the boundary of S,
- If you "stand" on the boundary and look in the direction it is oriented, the "top" of S is to the left,
- $\vec{T}$ be unit tangent vector to δS,
- $\vec{n}$ be the unit normal vector to S, pointing away from the "top"

Then

$$\oint_{\delta S} \vec{F}\cdot\vec{T}\,ds = \iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS$$

______ integral　　______ integral

14.7 (Cont)

Ex 1 Use Stoke's theorem to calculate $\iint_S (\nabla \times F) \cdot n \, dS$ where $F = \langle xy, yz, xz \rangle$, S is the triangular surface with vertices $(0,0,0)$, $(1,0,0)$ & $(0,2,1)$ and $\vec{n}$ is the upper normal.

14.7 Cont

Ex 2 Use Stoke's theorem to calculate $\iint_S (\nabla \times F) \cdot n \, dS$ where $F = (z-y)\hat{\imath} + (z+x)\hat{\jmath} - (x+k)\hat{k}$, S is the part of the paraboloid $z = 2 - x^2 - y^2$ above the $z = 1$ plane, & $\hat{n}$ is the upward normal.

14.7 (Cont)

Ex3 Use Stoke's Theorem to calculate $\oint_C F \cdot T\,ds$ where $F = \langle x^2+y^2,\ -x(x^2+y^2),\ 0\rangle$, C is the rectangular path from $(0,0,0)$ to $(1,0,0)$ to $(1,1,1)$ to $(0,1,1)$ to $(0,0,0)$.