Guiding Stress with Discrete Networks

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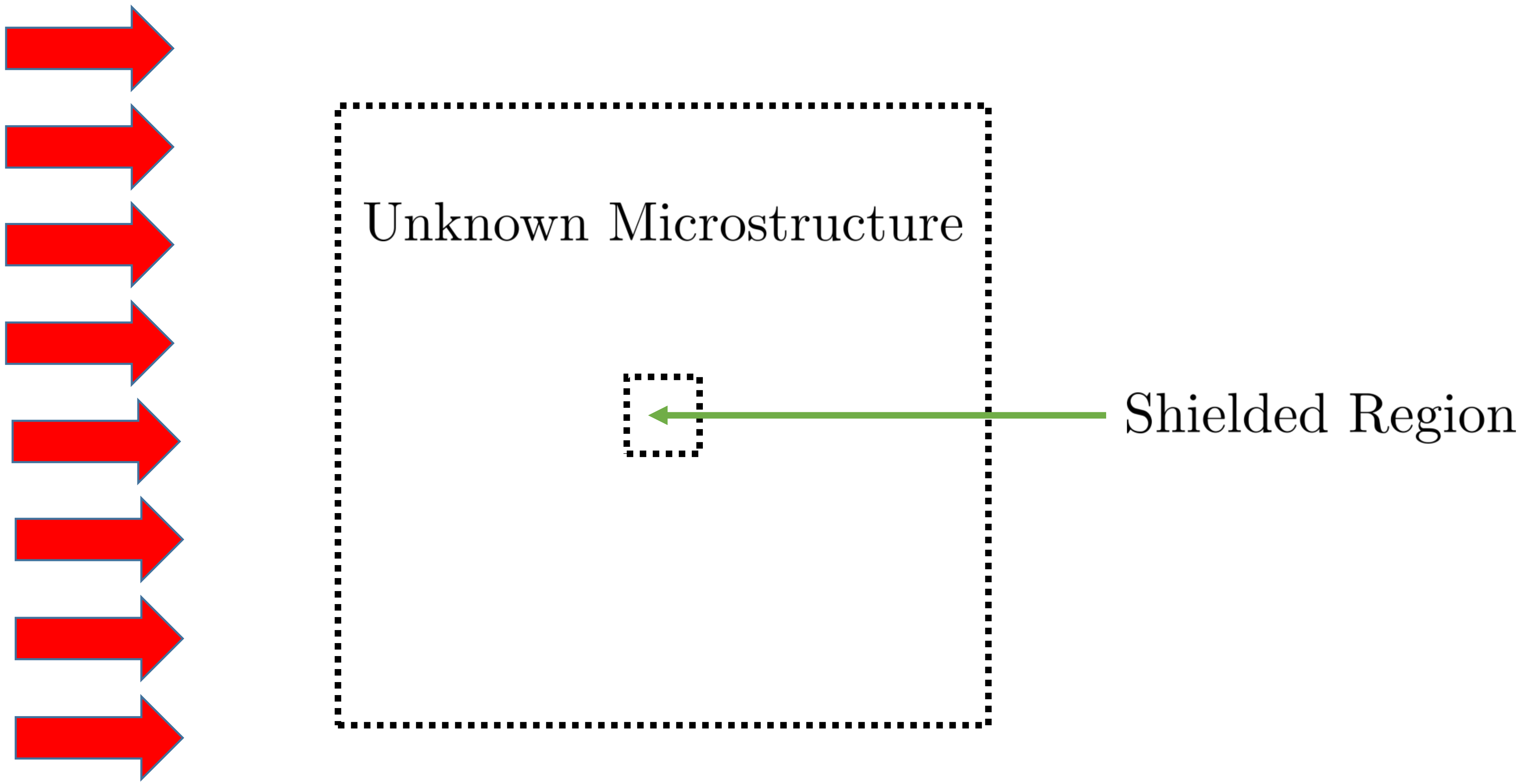
Two Problems:

(1) Concentrating a field into a region.

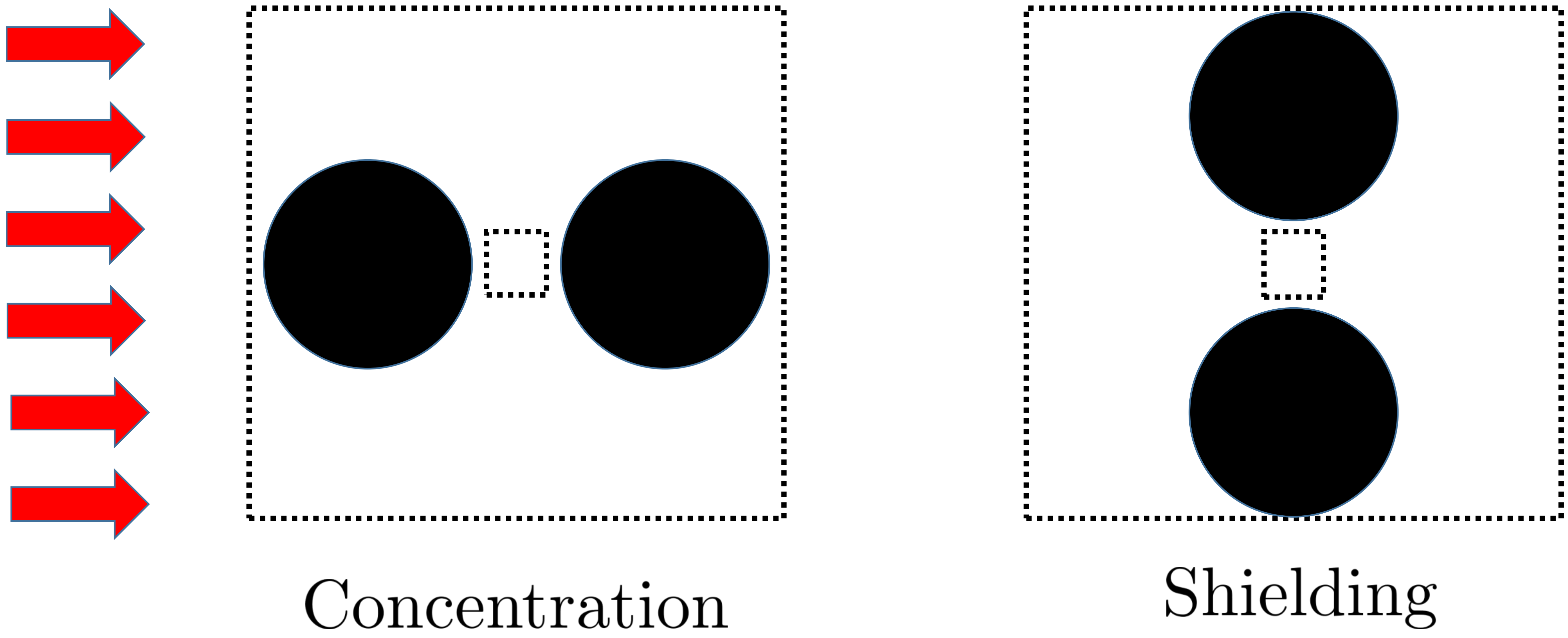
(2) Shielding a region from fields.

Sharp corners concentrate fields

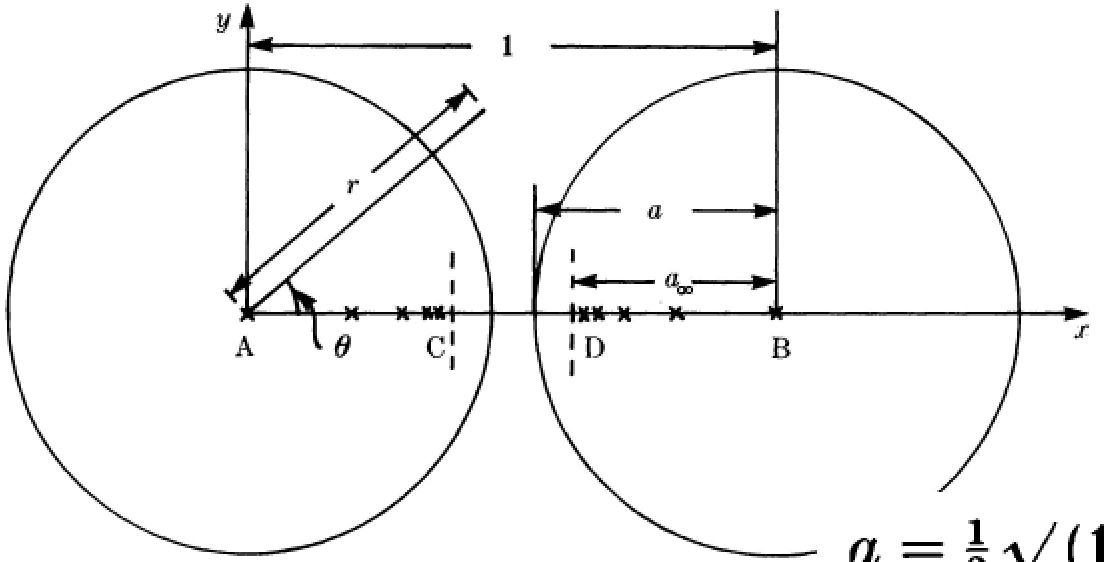
For concentration or shielding problems it seems reasonable to require that there is no microstructure in the concentration region or shielded region and that the microstructure is localized in a box.



Using Disks:



Field between two highly conducting disks close to touching



$$a = \tfrac{1}{2}\sqrt{(1-1/c^2)}.$$

McPhedran, Poladian, GWM (1988)

$$B_1 = \frac{-(c/2)\,(1-1/c)}{2s\ln(c)+1-2s[\gamma+\psi(1+s)]}.$$

$$a = \tfrac{1}{2}\sqrt{(1-1/c^2)}. \quad a_\infty = \tfrac{1}{2}(1-1/c).$$

ψ: Psi or Digamma function

$$\rho_-(a^2/x) = -\eta\rho_+(x)$$

$$\eta = (\sigma-1)/(\sigma+1).$$

$$\rho_+(1-x) = -\rho_-(x),$$

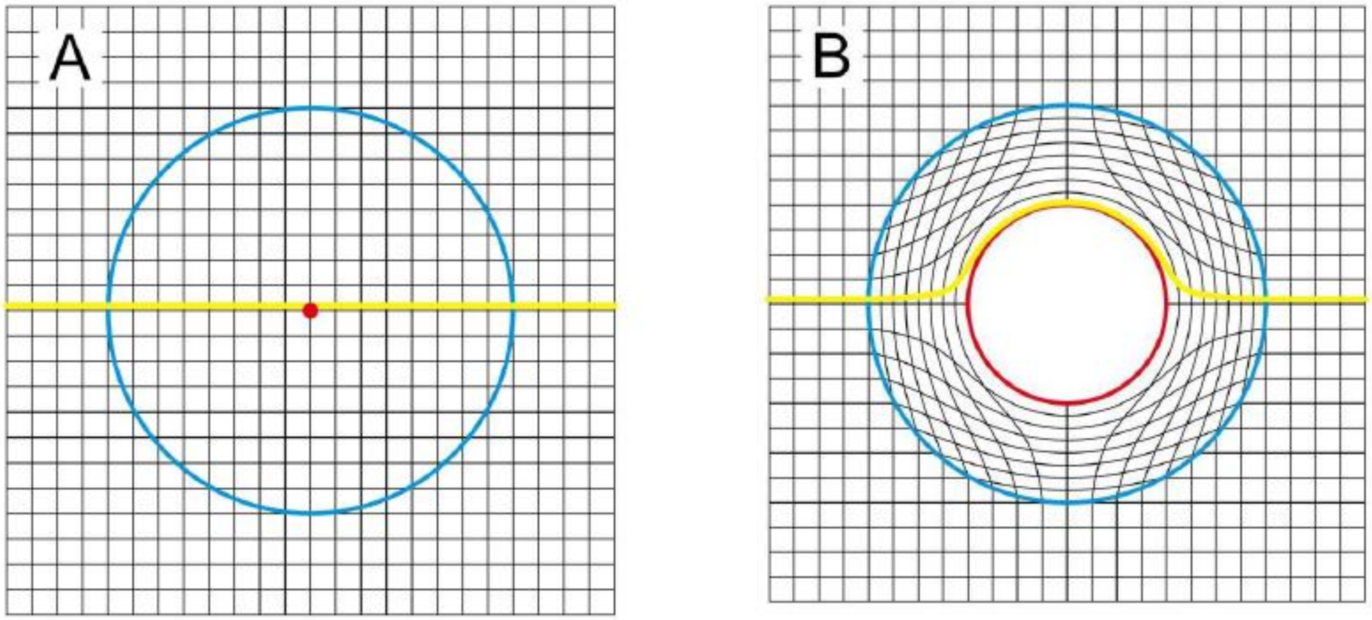
$$\rho_-[a^2/(1-x)] = \eta\rho_-(x)$$

$$\rho_-(x) = A[(a_\infty-x)/(1-a_\infty-x)]^s$$

$$s = \ln(\eta)/\ln[a_\infty/(1-a_\infty)]$$

Rigorous Analysis: Lim and Yu (2015)

Could use the transformation based approach of Greenleaf, Lassas, and Uhlmann

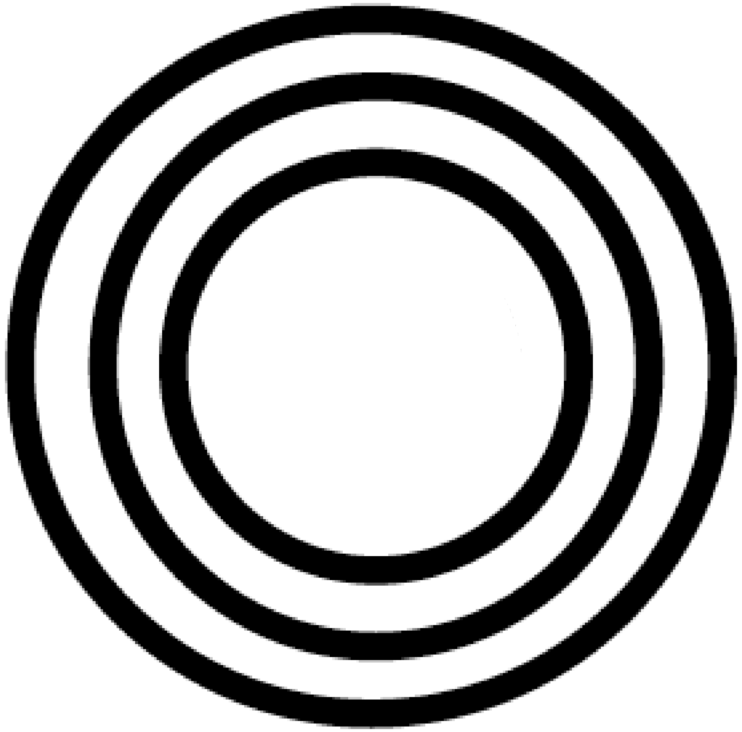


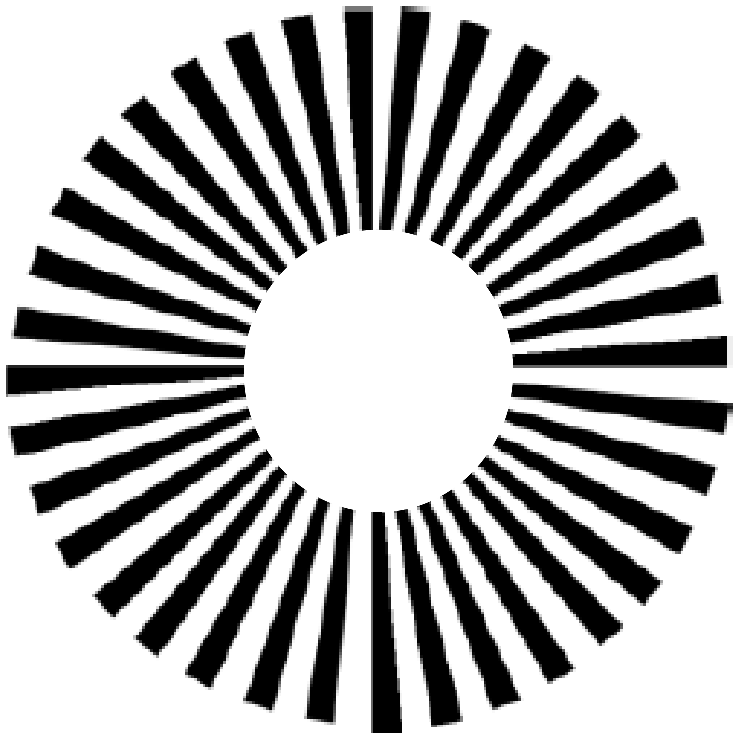
Stretching space (From Ulf Leonhardt)

Advantages: Works for any external field and creates no disturbance

Disadvantages: Requires extreme conductivities, and if one truncates the solution there is no reason to expect it is optimal.

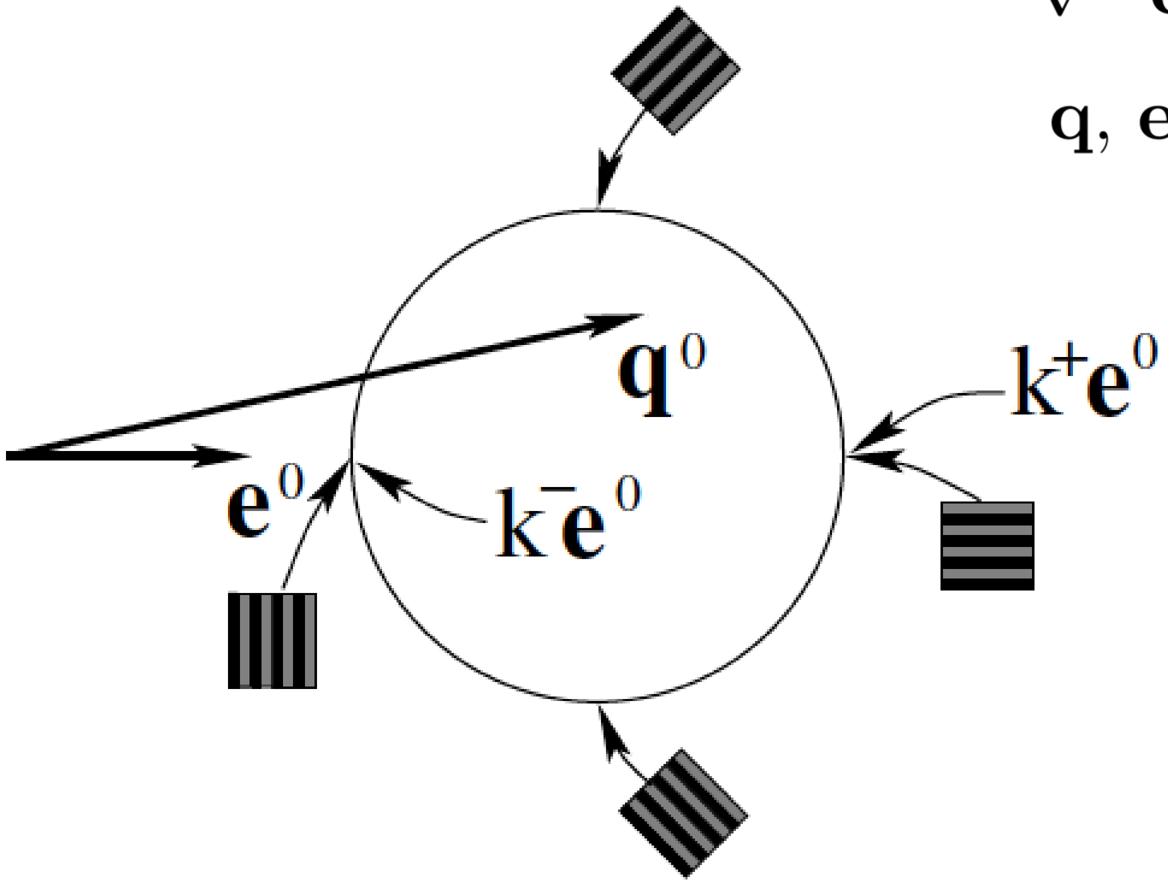
Or Maybe?





Seems like we are just guessing. Is there a more systematic approach, at least in the case where we use just 2 conducting materials, and we are seeking shielding or concentration for just one applied field?

Possible (average heat current, $\mathbf{q}^0$, average temperature gradient, $\mathbf{e}^0$) pairs in a two phase conducting composite (Raitum, 1978).



$$\nabla \cdot \mathbf{q} = 0, \quad \mathbf{q}(\mathbf{x}) = k(\mathbf{x})\mathbf{e}(\mathbf{x}), \quad \mathbf{e} = -\nabla T$$

$\mathbf{q}$, $\mathbf{e}$ periodic, $\langle \mathbf{q} \rangle = \mathbf{q}^0, \quad \langle \mathbf{e} \rangle = \mathbf{e}^0$,

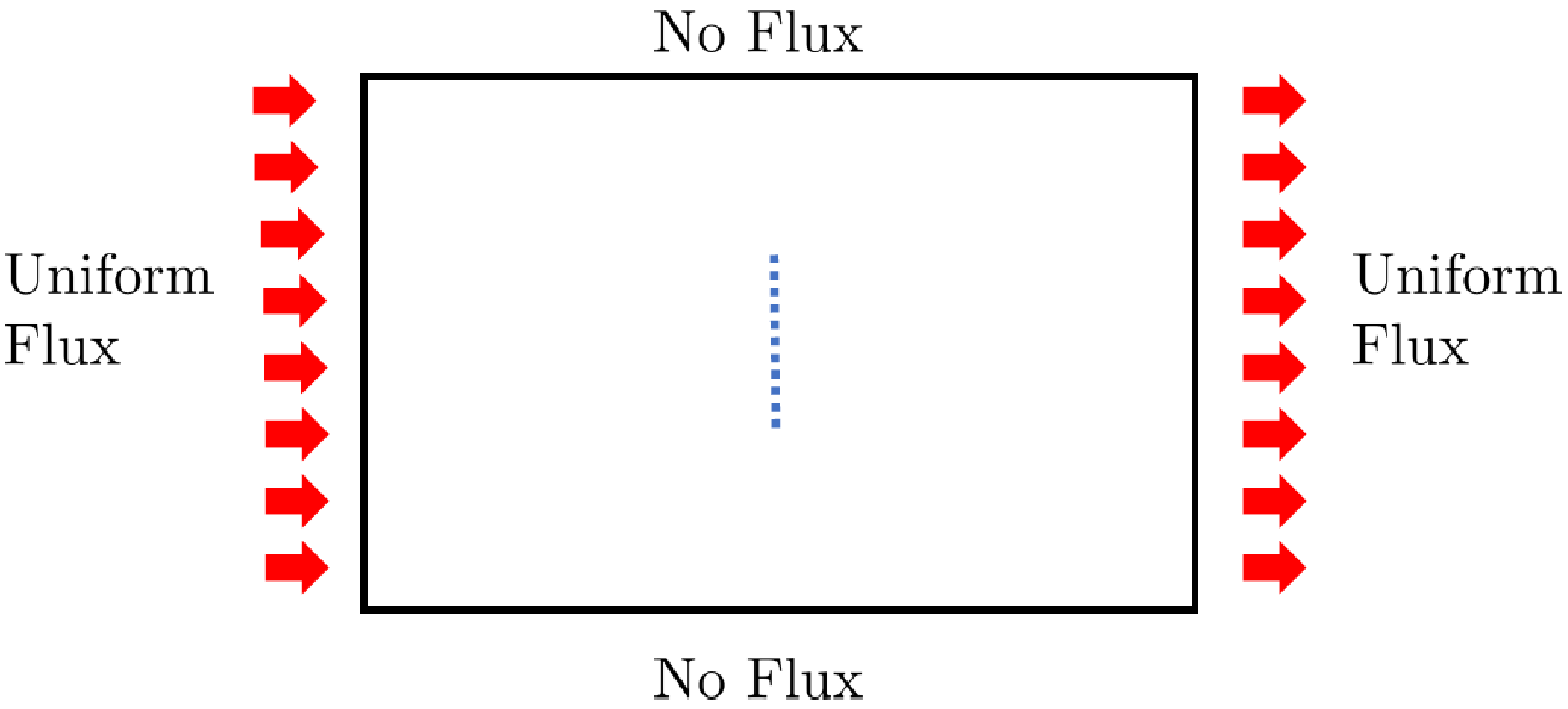
Follows from the Wiener bounds:

$$\begin{aligned} k^- \mathbf{I} &\leq \mathbf{k}^* \leq k^+ \mathbf{I} \\ k^+ &= f k_1 + (1-f) k_2 \\ k^- &= (f/k_1 + (1-f)/k_2)^{-1} \end{aligned}$$

Solution of the "weak G-closure" problem for conductivity

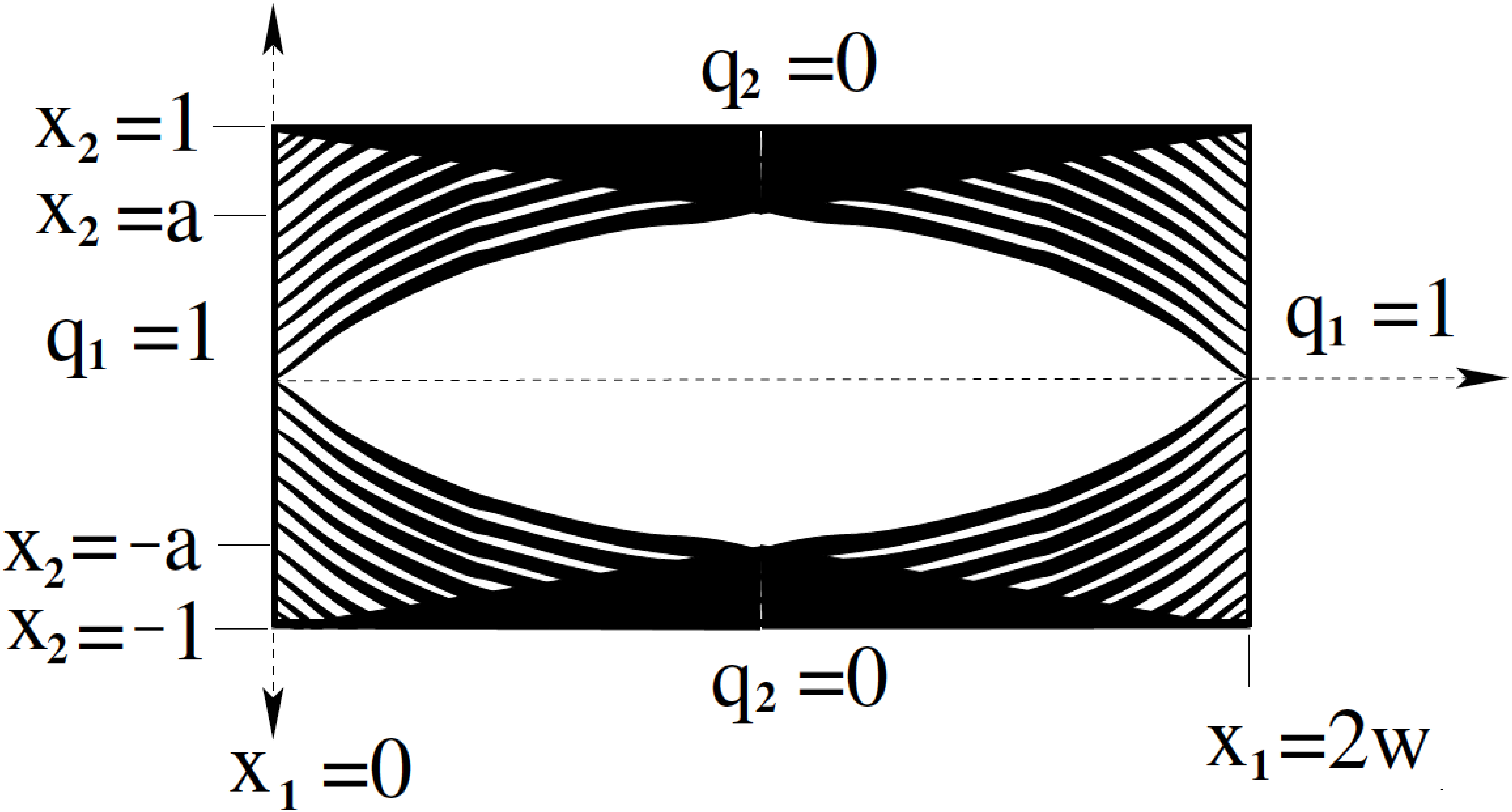
The heat lens problem: Gibiansky, Lurie and Cherkaev (1988)

Aim: Shield or concentrate flux in the blue dashed interval

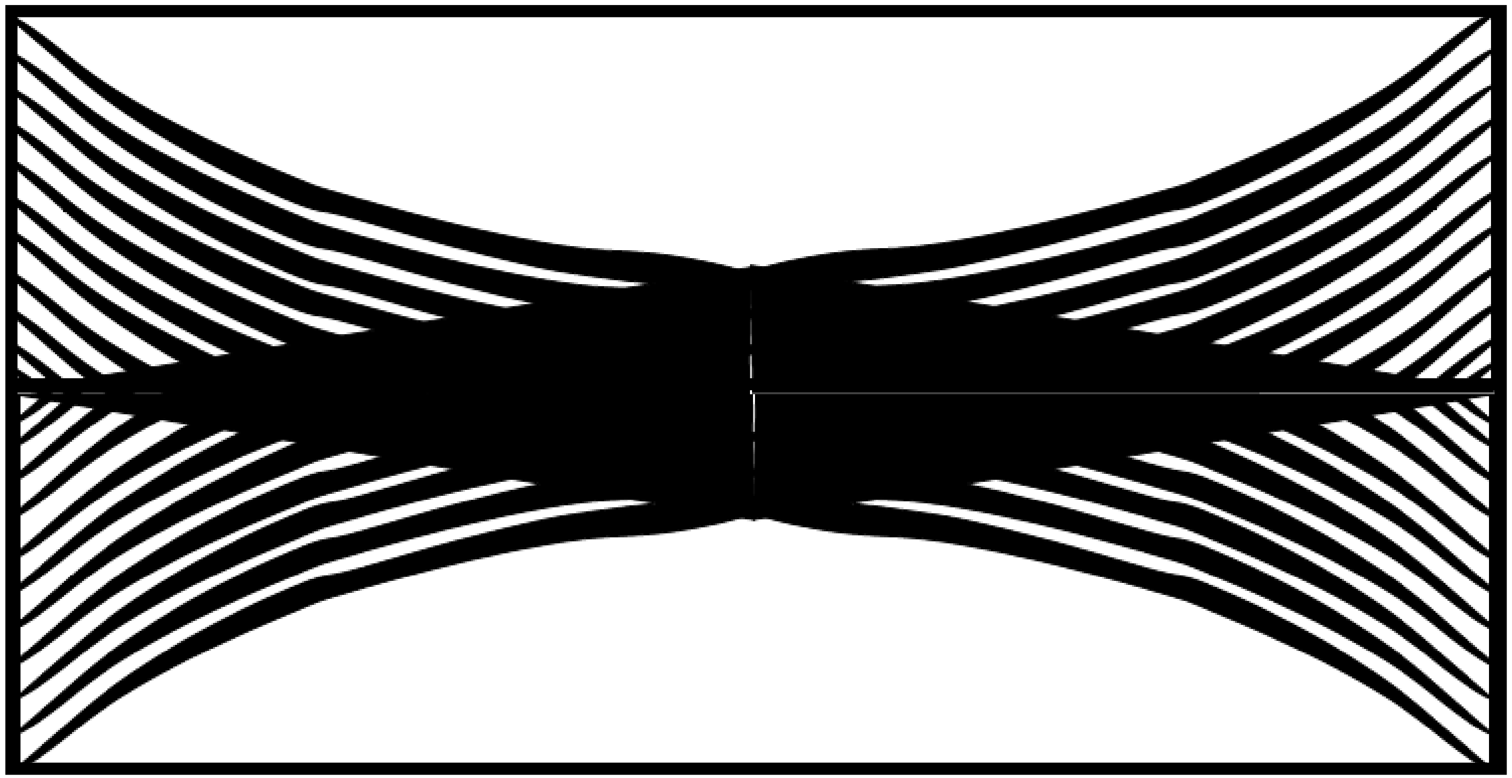


How does one optimally distribute a poor and good conductor to do this?

Field Shield: (Black, good conductor)



Field Concentrator:



To solve similar optimization problems for elasticity, can we find the “weak G-Closure” for 3d-elasticity?

At least in the case for 3d printed materials when one phase is void and the other elastically isotropic?

A difficult problem: need to characterize possible (average strain $\boldsymbol{\epsilon}^0$, average stress $\boldsymbol{\sigma}^0$) pairs,

Can assume $\boldsymbol{\sigma}^0$ is diagonal and normalized : 2 parameters
Then $\boldsymbol{\epsilon}^0$ has 6 parameters.

So the “weak G-Closure” is described by a set in an 8-dimensional space, 11 if one includes the volume fraction, and bulk and shear moduli of the initial elastic material.

One constraint implied by sharp bounds on the minimum compliance energy:

$$W_f(\boldsymbol{\sigma}^0) \leq \boldsymbol{\sigma}^0 : \epsilon^0, \quad (*)$$

Explicit expression for $W_f(\boldsymbol{\sigma}_{\mathbf{0}})$ given by Gibiansky and Cherkaev (1987) and Allaire (1994). Note $W_f(c\mathbf{A}) = c^2 W_f(\mathbf{A})$

Our result is that these optimal bounds on the compliance energy also provide optimal bounds on $(\boldsymbol{\epsilon}^0, \boldsymbol{\sigma}^0)$-pairs. Given $\boldsymbol{\sigma}^0$ they constrain $\boldsymbol{\epsilon}^0$ to lie on one-side of a hyperplane.

Explicit Formula for Bound: (can skip)

$$W_f(\boldsymbol{\sigma}^0) = \boldsymbol{\sigma}^0 : \mathbf{C}_1^{-1}\boldsymbol{\sigma}^0 + \frac{f}{2\mu} g(\mathbf{C}_1, \boldsymbol{\sigma}^0), \quad \text{(Using Allaire's notation.)}$$

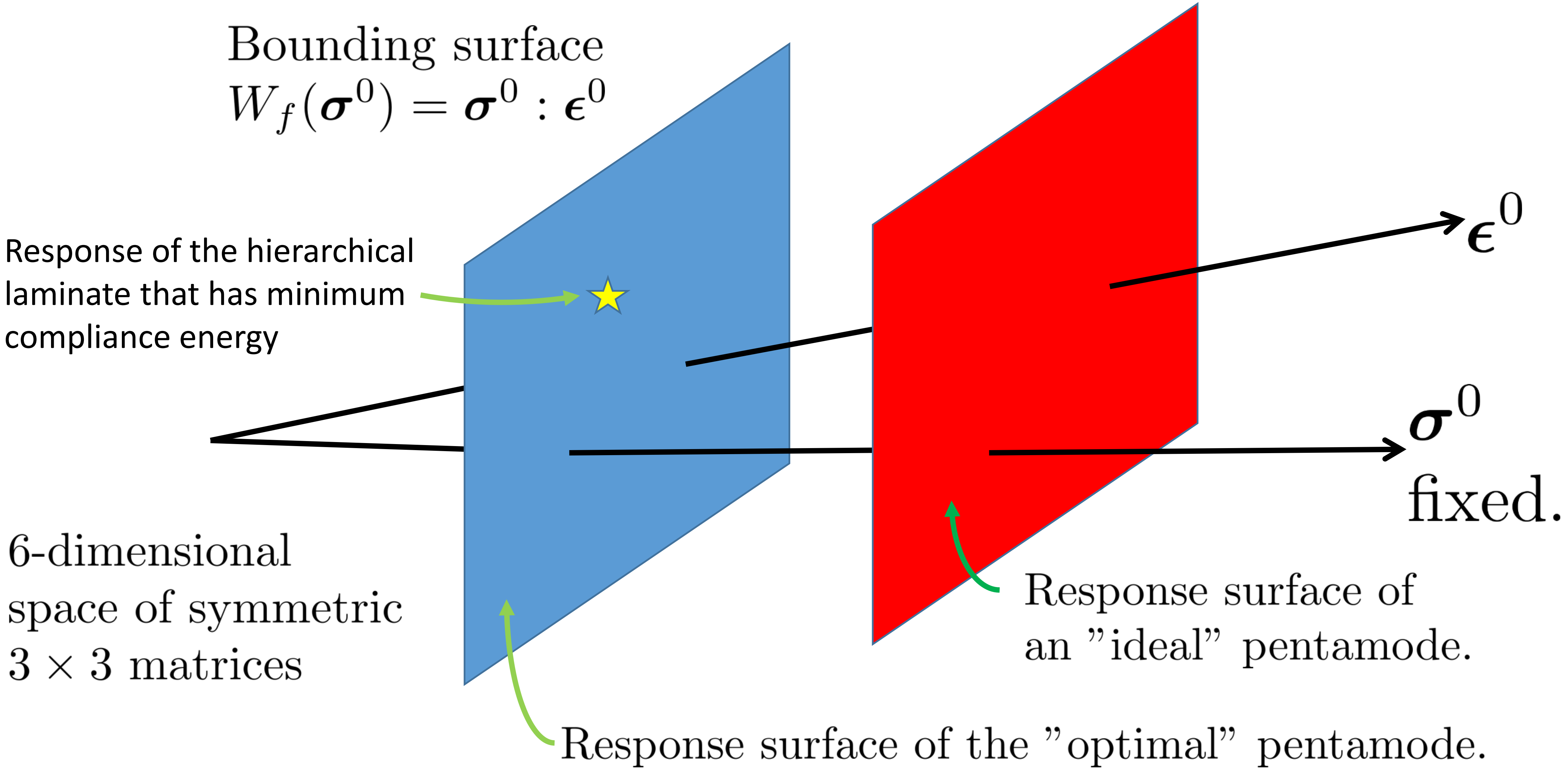
Suppose the stress has eigenvalues σ_1, σ_2 and σ_3. Can assume at most one eigenvalue is negative, and $\sigma_1 \leq \sigma_2 \leq \sigma_3$. When all are non-negative, and $\lambda > 0$:

$$\begin{aligned} g(\mathbf{C}, \boldsymbol{\sigma}) &= \frac{2\mu + \lambda}{2(2\mu + 3\lambda)}(\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 \leq \sigma_1 + \sigma_2, \\ &= (\sigma_1 + \sigma_2)^2 + \sigma_3^2 - \frac{\lambda}{2\mu + 3\lambda}(\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 \geq \sigma_1 + \sigma_2, \end{aligned}$$

while when one eigenvalue, namely σ_1, is negative,

$$\begin{aligned} g(\mathbf{C}, \boldsymbol{\sigma}) &= \frac{2\mu + \lambda}{2(2\mu + 3\lambda)}\left(\sigma_3 + \sigma_2 - \frac{\mu + 2\lambda}{\mu + \lambda}\sigma_1\right)^2 \\ &\quad \text{if } \sigma_3 + \sigma_2 \geq \frac{-\mu}{\mu + \lambda}\sigma_1 \text{ and } \sigma_3 - \sigma_2 \leq \frac{-\mu}{\mu + \lambda}\sigma_1, \\ &= (\sigma_3 + \sigma_2)^2 + \sigma_1^2 - \frac{\lambda}{2\mu + 3\lambda}(\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 + \sigma_2 \leq \frac{-\mu}{\mu + \lambda}\sigma_1, \\ &= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{2\mu}{\mu + \lambda}\sigma_1\sigma_2 - \frac{\lambda}{2\mu + 3\lambda}(\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 - \sigma_2 \geq \frac{-\mu}{\mu + \lambda}\sigma_1. \end{aligned}$$

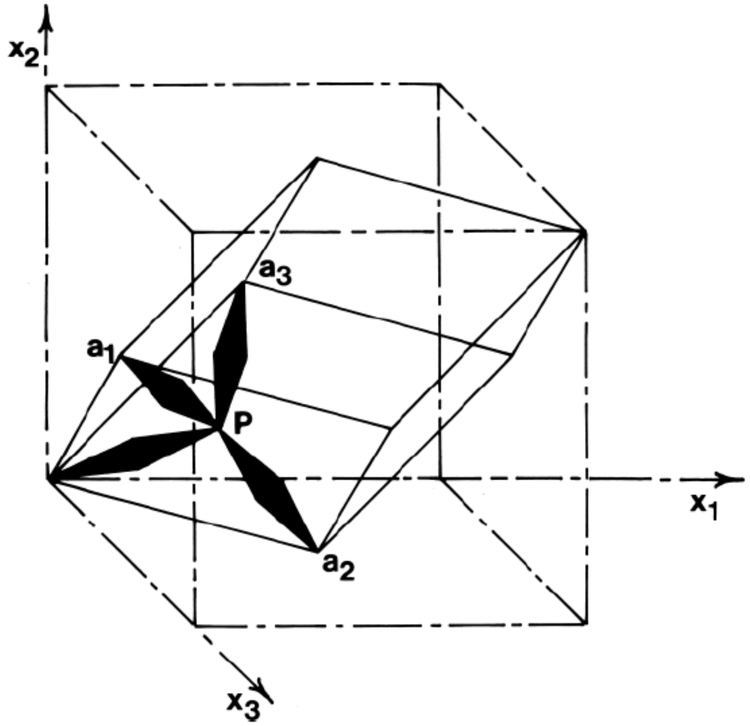
The bound: very similar to the conductivity case when $k_2 = 0$.



What are pentamodes?

New classes of elastic materials (with Cherkaev, 1995)

A three dimensional pentamode material which can support any prescribed loading



Many other important papers on pentamodes.

Like a fluid it only supports one loading, unlike a fluid that loading may be anisotropic. Desired support of a given anisotropic loading is achieved by moving P to another position in the unit cell.

KEY POINT is the coordination number of 4 at each vertex: the tension in one double cone connector, by balance of forces, determines uniquely the tension in the other 3 connecting double cones, and by induction the entire average stress field in the material.

Pentamode structures are a sort of anisotropic inhomogeneous fluid

$$\mathbf{C}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) \otimes \mathbf{A}(\mathbf{x}), \quad \nabla \cdot \mathbf{A} = 0,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x}), \quad \nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\epsilon} = [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]/2$$

have the solution

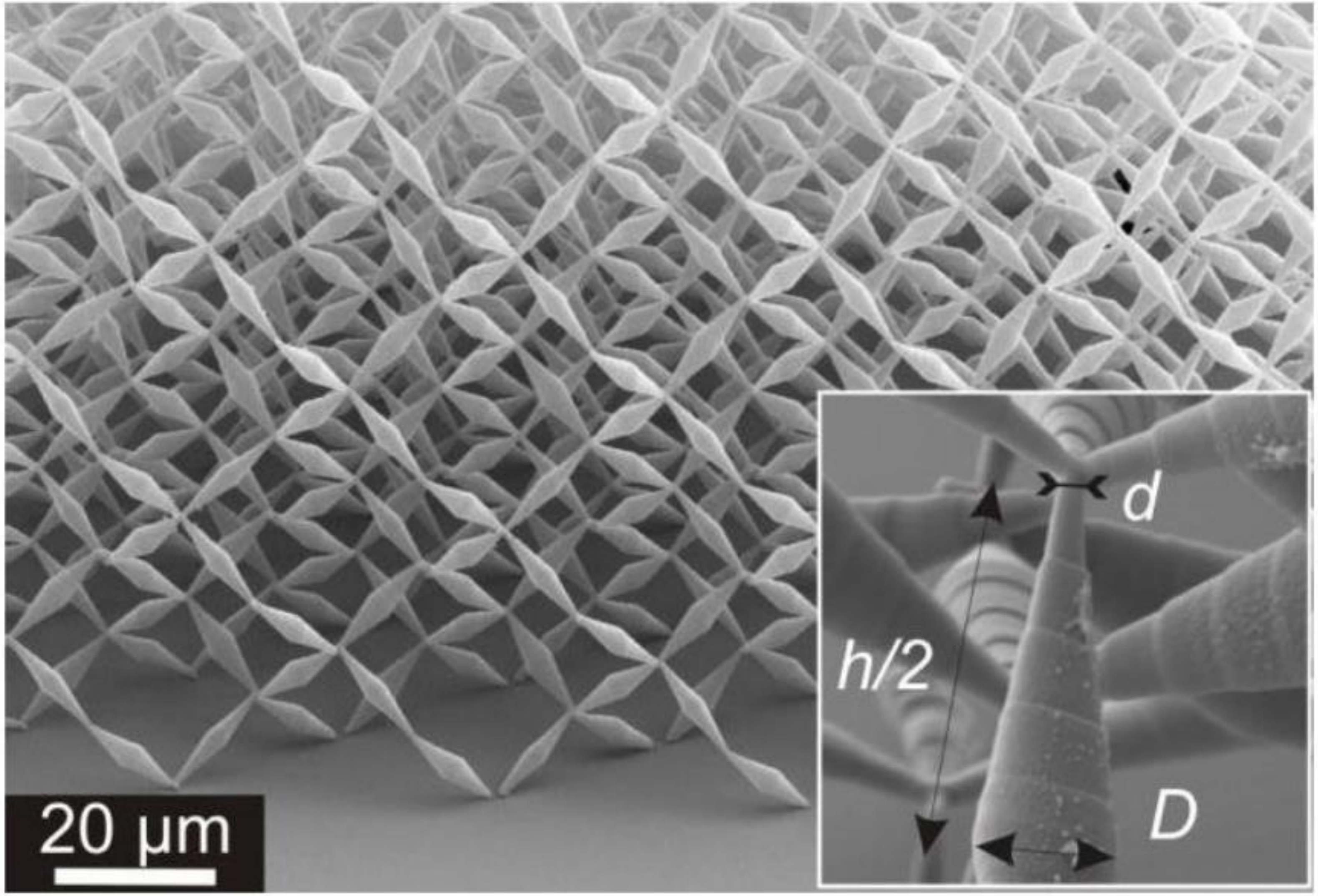
$$\boldsymbol{\sigma}(\mathbf{x}) = \alpha\mathbf{A}(\mathbf{x})$$

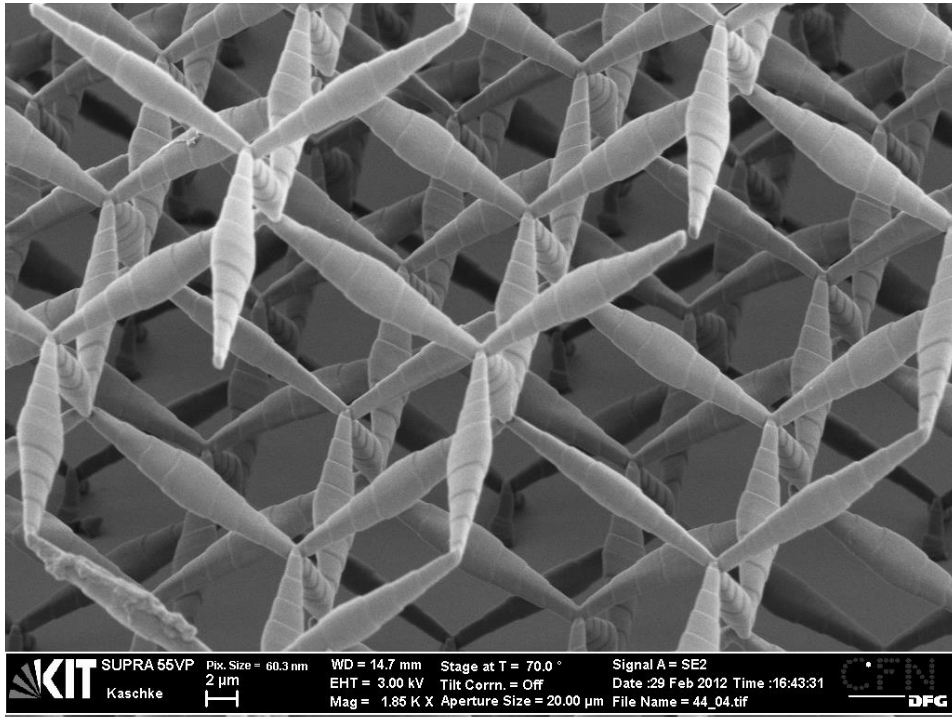
where $\alpha =$ "a constant" is the analog of pressure, and

$$\alpha = \mathrm{Tr}[\mathbf{A}(\mathbf{x})\nabla\mathbf{u}],$$

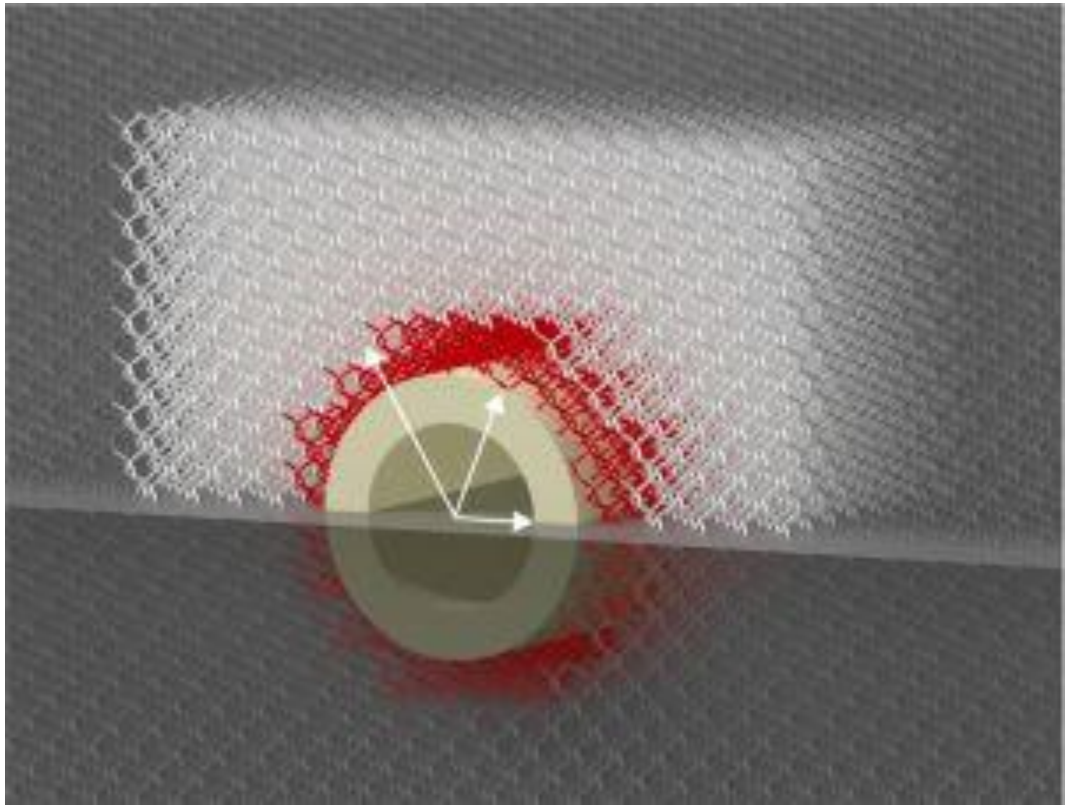
constrains $\nabla\mathbf{u}$. Thus $\mathbf{A}(\mathbf{x})$ is a sort of anisotropic "compressibility"

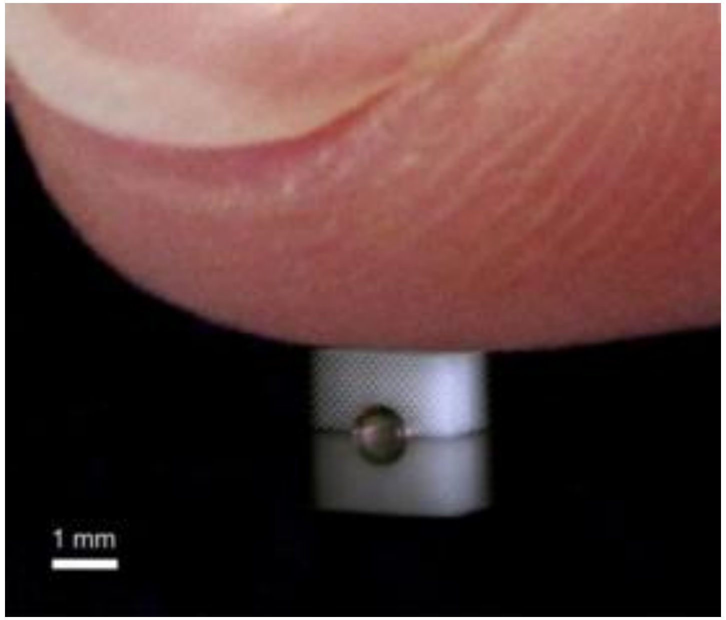
Realization of Kadic et.al. 2012



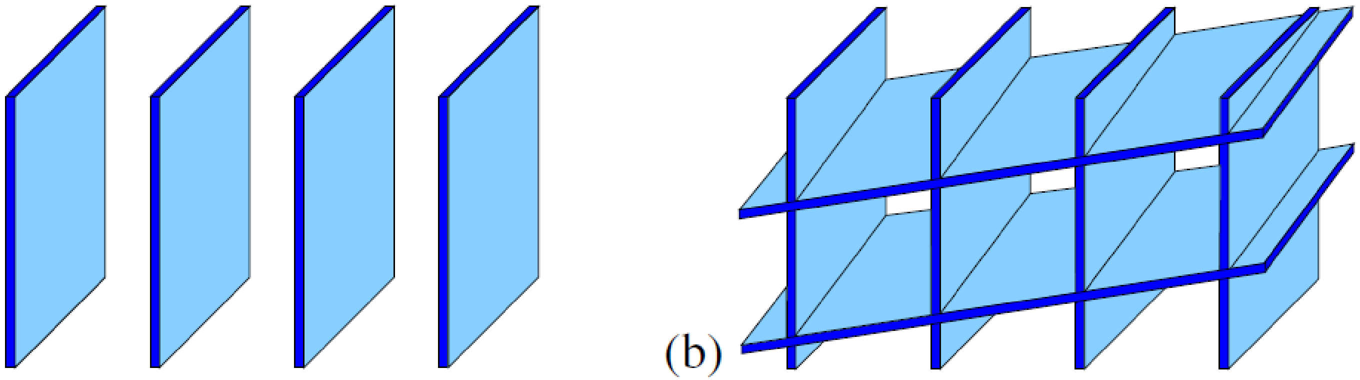


Cloak making an object “unfeelable”: Buckmann et. al. (2014)





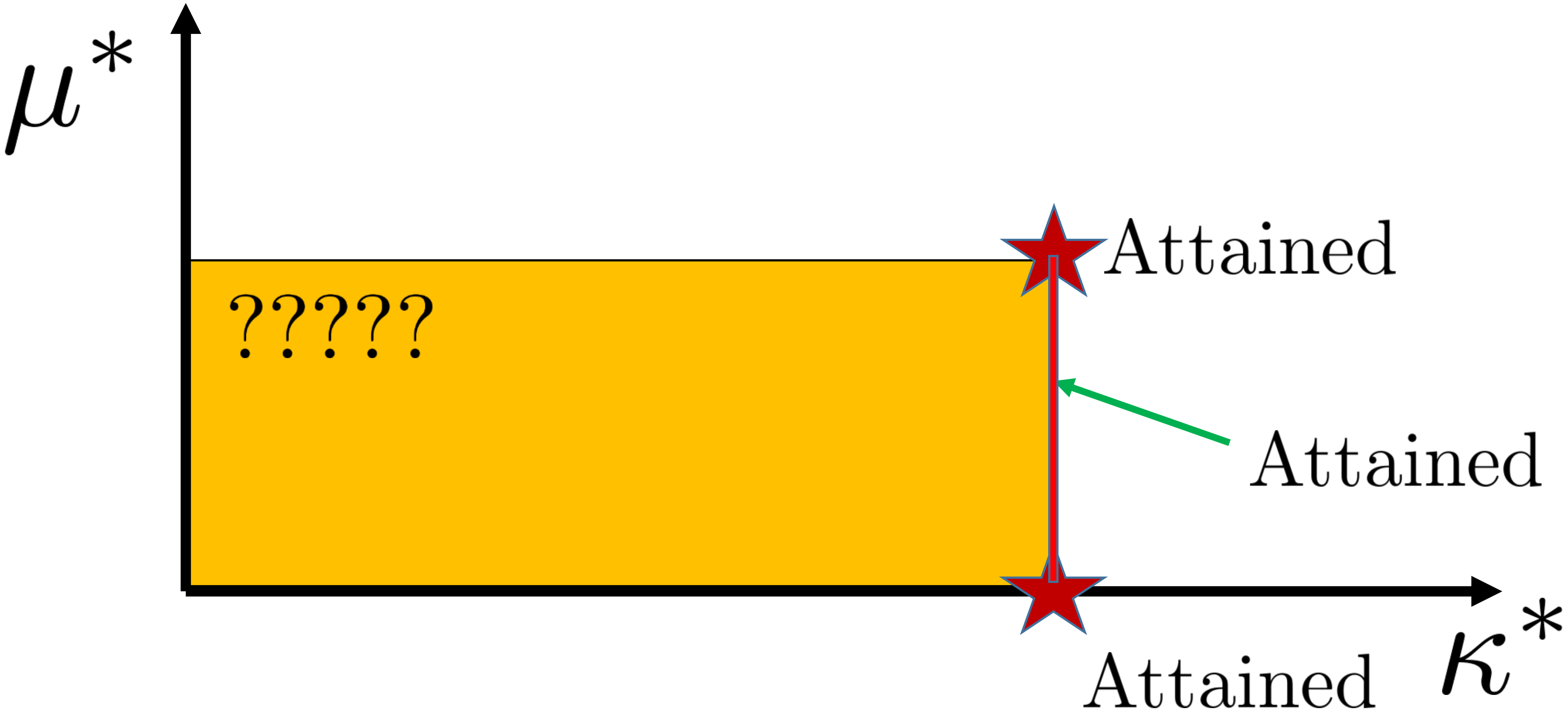
Idea of proof: Insert into the material attaining the energy bounds a thin walled structure with sets of parallel walls:



(a) (b)

Inside the walls put the appropriate modified ideal pentamode material. Thus we obtain an optimal pentamode attaining (arbitrarily closely) the energy bounds.

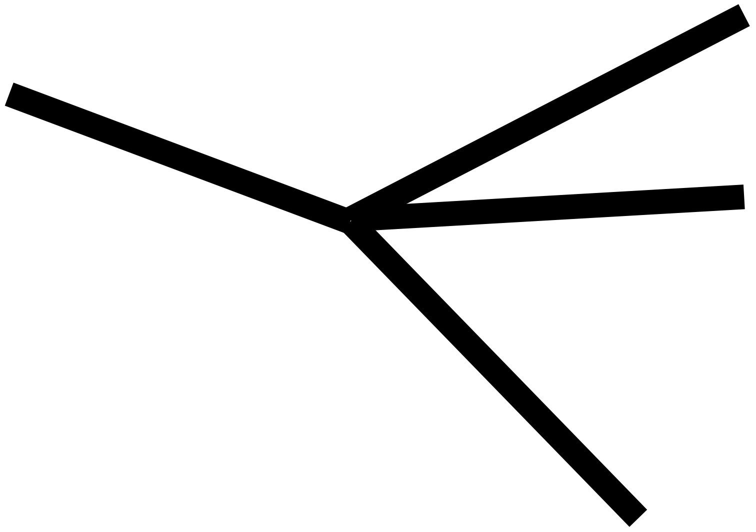
Hashin-Shtrikman bounding box when one phase is void, and the volume fraction is prescribed



See also Ostanin, Ovchinnikov, Tozoni, and Zorin (results in 2d)
https://doi.org/10.1016/j.jmps.2018.05.018

What about discrete networks? Guiding of stress usually achieved by adjusting tension in wires.

However the flow of stress is quite different to the flow of electrical current, e.g. consider a junction of four conducting rods:



Current in one does not determine the current in the others, only their sum.

But the tension in one does uniquely determine the tension in the three rods that meet it, by balance of forces, provided they are not co-planar.

Suggests that the flow of stress can be controlled by geometry alone by carefully making appropriate junctions.

The “spider-web” problem

Given a set of forces $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ at prescribed points $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$, when does there exist a web under tension (possibly with internal nodes) that supports these forces?

Mathematically: given a loading

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{t}_i \delta(\mathbf{x} - \mathbf{x}_i)$$

when does there exist a positive semidefinite stress field $\boldsymbol{\sigma}(\mathbf{x})$ vanishing at infinity such that

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{f}$$

Theorem *A set of points* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ *at the vertices of a convex polygon, numbered clockwise, can support balanced forces* $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n$ *at these vertices, with a truss with all its elements under tension, if and only if for all* i *and* j,

$$\sum_{k=j}^{i-1} (\mathbf{x}_k - \mathbf{x}_j) \cdot [\mathbf{R}_\perp \mathbf{t}_k] \geq 0,$$

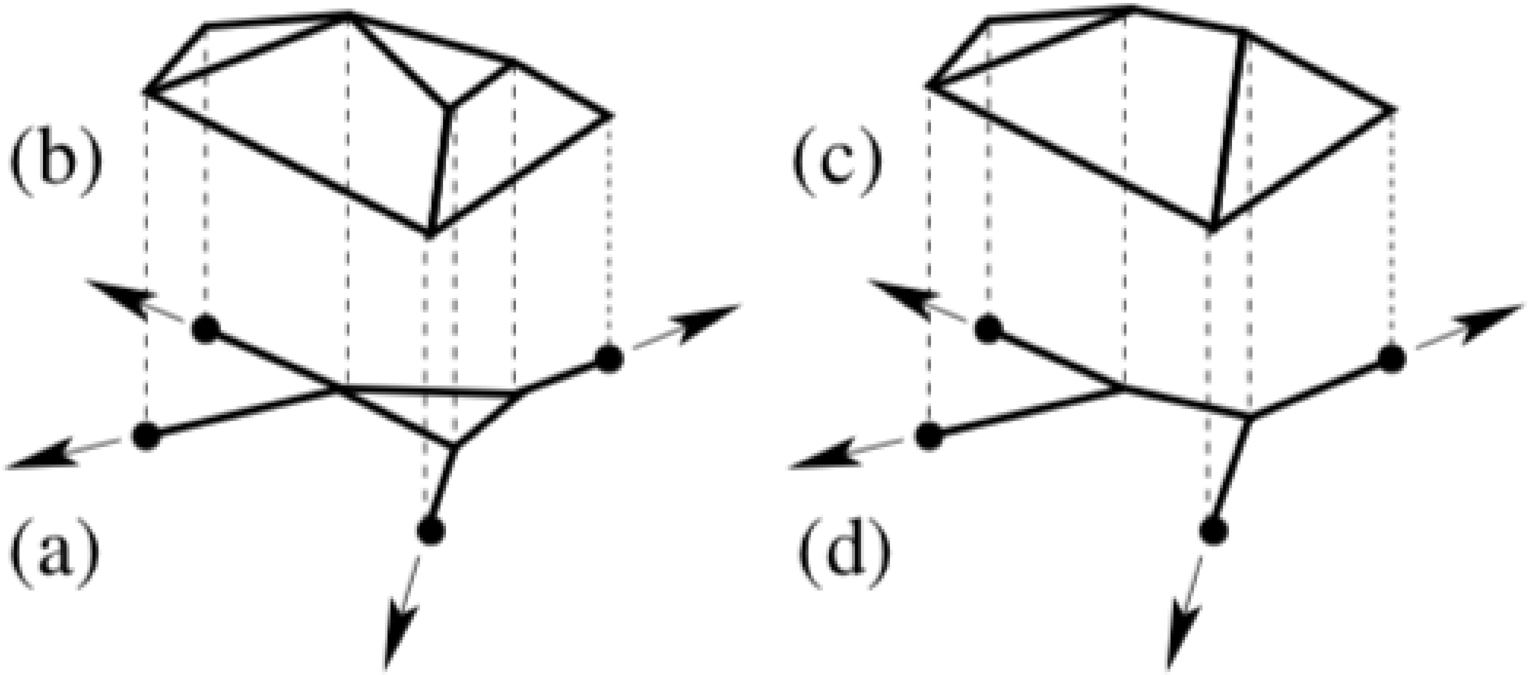
and we have assumed $i > j$, *if necessary by replacing* i *by* $i+n$ *and identifying where necessary* $\mathbf{x}_k$ *and* $\mathbf{t}_k$ *with* $\mathbf{x}_{k-n}$ *and* $\mathbf{t}_{k-n}$.

That is: A web exists if and only if the net torque around any segment of the boundary is non-negative

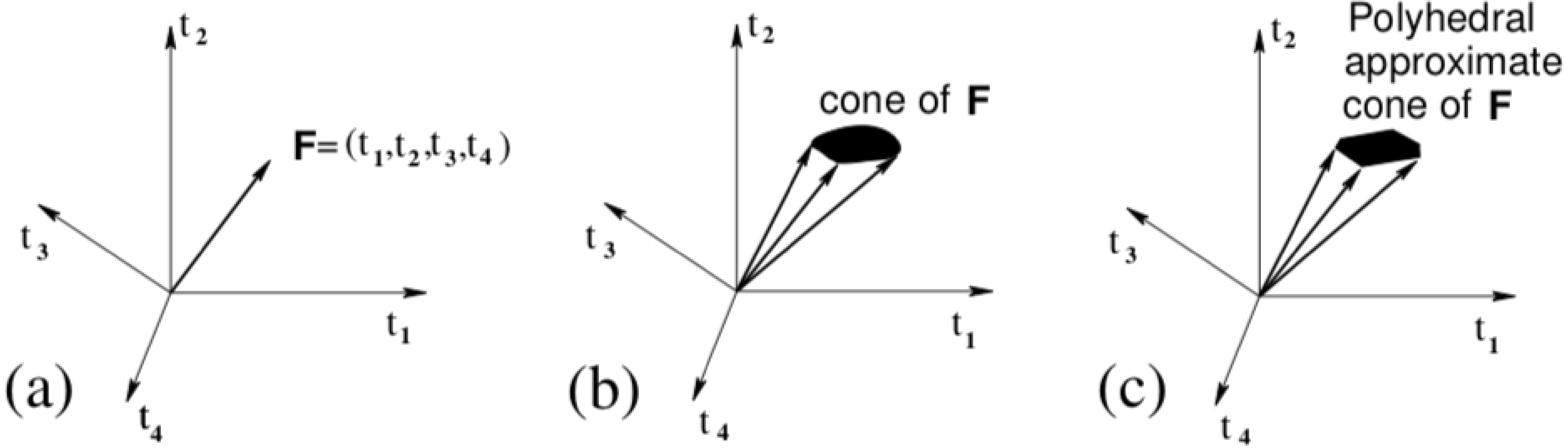
Idea of Proof:

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{R}_{\perp}^{T} \nabla\nabla\phi(\mathbf{x})\mathbf{R}_{\perp},$$

$\boldsymbol{\sigma}(\boldsymbol{x}) \geq 0$ if and only if $\phi(\boldsymbol{x})$ is concave.

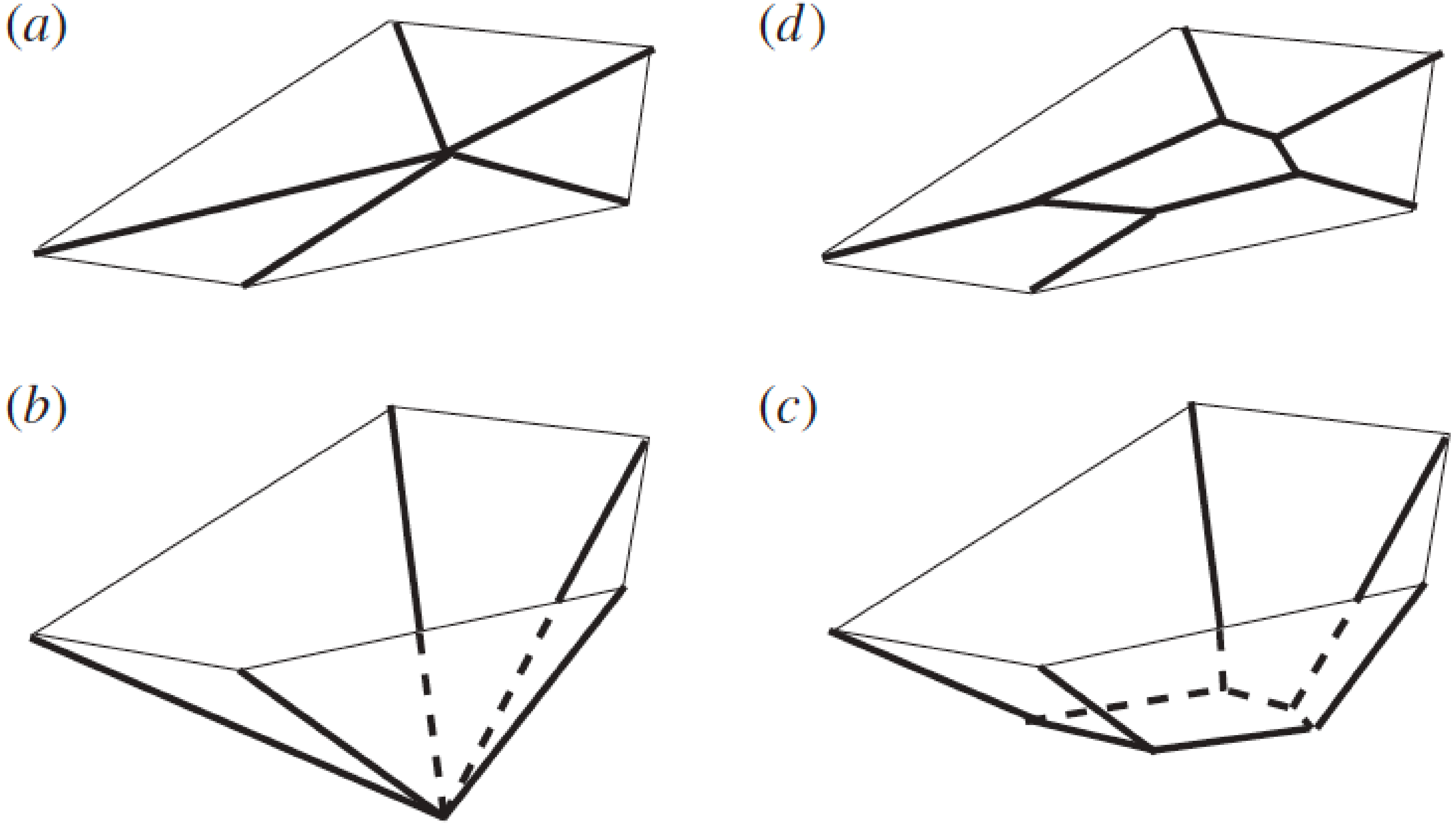


Force Cones::

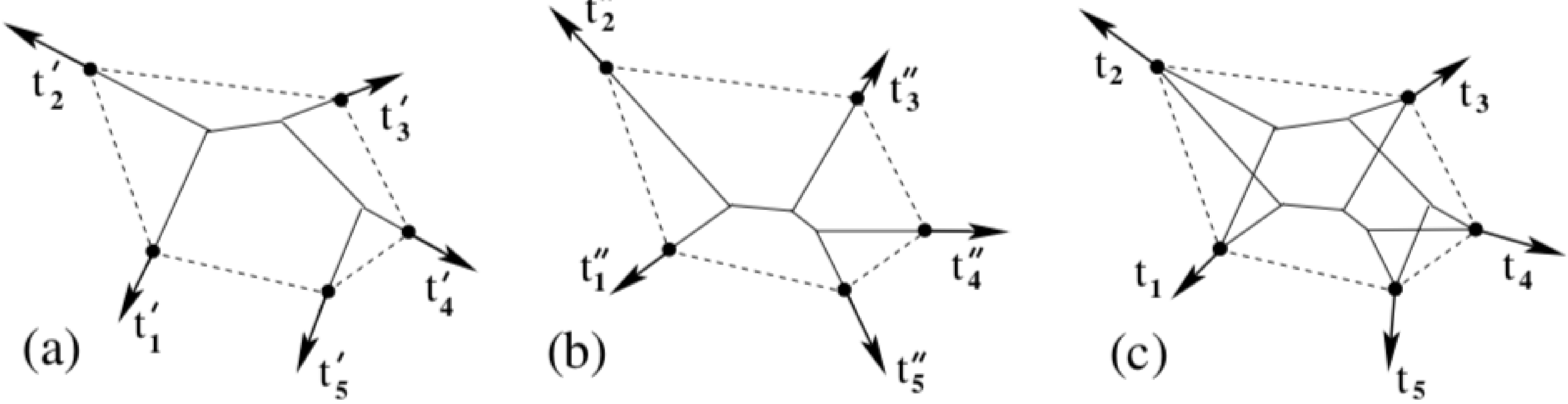


t 2
F= (t 1,t 2,t 3,t 4)
t 3
t 1
(a)
t 4
t 2
cone of F
t 3
t 1
(b)
t 4
t 2
Polyhedral
approximate
cone of F
t 3
t 1
(c)
t 4

Making a 2d-web uniloadable:

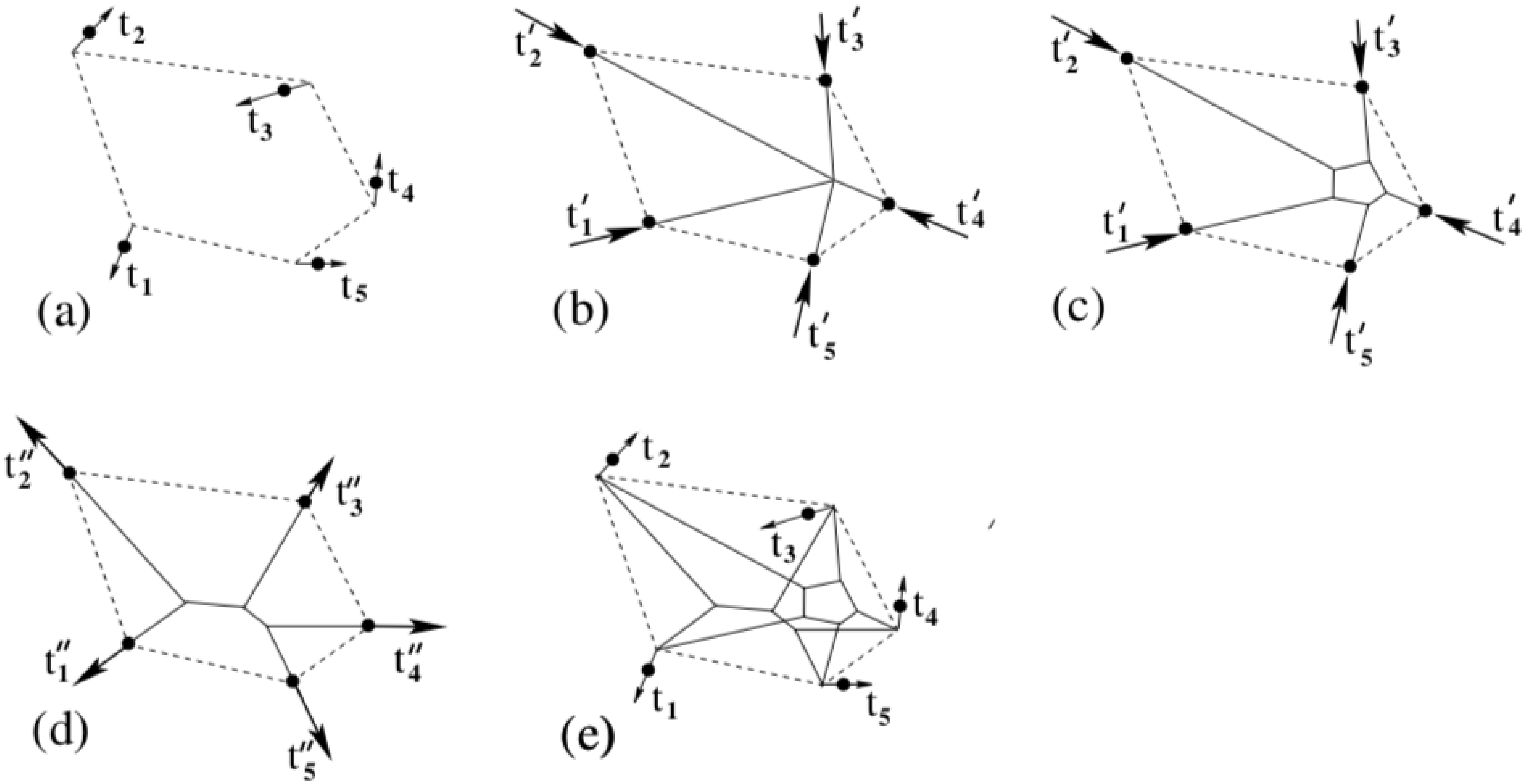


Superimposing uniloadable webs



By superposition one can realize any polyhedral force cone that satisfies the torque condition

Combining a web under tension with a web under compression to support any desired loading at the vertices of a convex polygon



What about if the points $\mathbf{x}_1, \mathbf{x}_2, \ldots \mathbf{x}_n$ are not the vertices of a convex polygon?

What happens in 3d?

Let the forces now be $\mathbf{f}_1, \mathbf{f}_2, \ldots \mathbf{f}_n$

Now if $\boldsymbol{\epsilon}_0(\mathbf{x}) = [\nabla \mathbf{u}_0(\mathbf{x}) + (\nabla \mathbf{u}_0(\mathbf{x}))^T]/2, \boldsymbol{\epsilon}_0(\mathbf{x}) \geq 0$

Then $0 \leq \int \mathrm{Tr}(\boldsymbol{\sigma}\boldsymbol{\epsilon}_0) = \sum_i \mathbf{f}_i \cdot \mathbf{u}_0(\mathbf{x}_i)$

Physically the constraint $\boldsymbol{\epsilon}_0(\mathbf{x}) \geq 0$ implies $\mathbf{u}_0(\mathbf{x})$ corresponds to a displacement field where everything is stretched and hence

$$[\mathbf{u}_0(\mathbf{x}_i) - \mathbf{u}_0(\mathbf{x}_j)] \cdot (\mathbf{x}_i - \mathbf{x}_j) \geq 0$$

Conversely, if this is satisfied for all i and j then there exists an interpolating field $\mathbf{u}_0(\mathbf{x})$ such that $\boldsymbol{\epsilon}_0(\mathbf{x}) \geq 0$

In a space of any dimension d we have proved:

Theorem (existence of a web under tension). *Let $\mathcal{A}_X$ be the cone of displacements* $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_N)$ *at points* $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ *defined by*

$$\mathcal{A}_X := \{\mathbf{U} \in (\mathbb{R}^d)^N : \forall 1 \leq i < j \leq N, (\mathbf{u}_i - \mathbf{u}_j) \cdot (\mathbf{x}_i - \mathbf{x}_j) \geq 0\}.$$

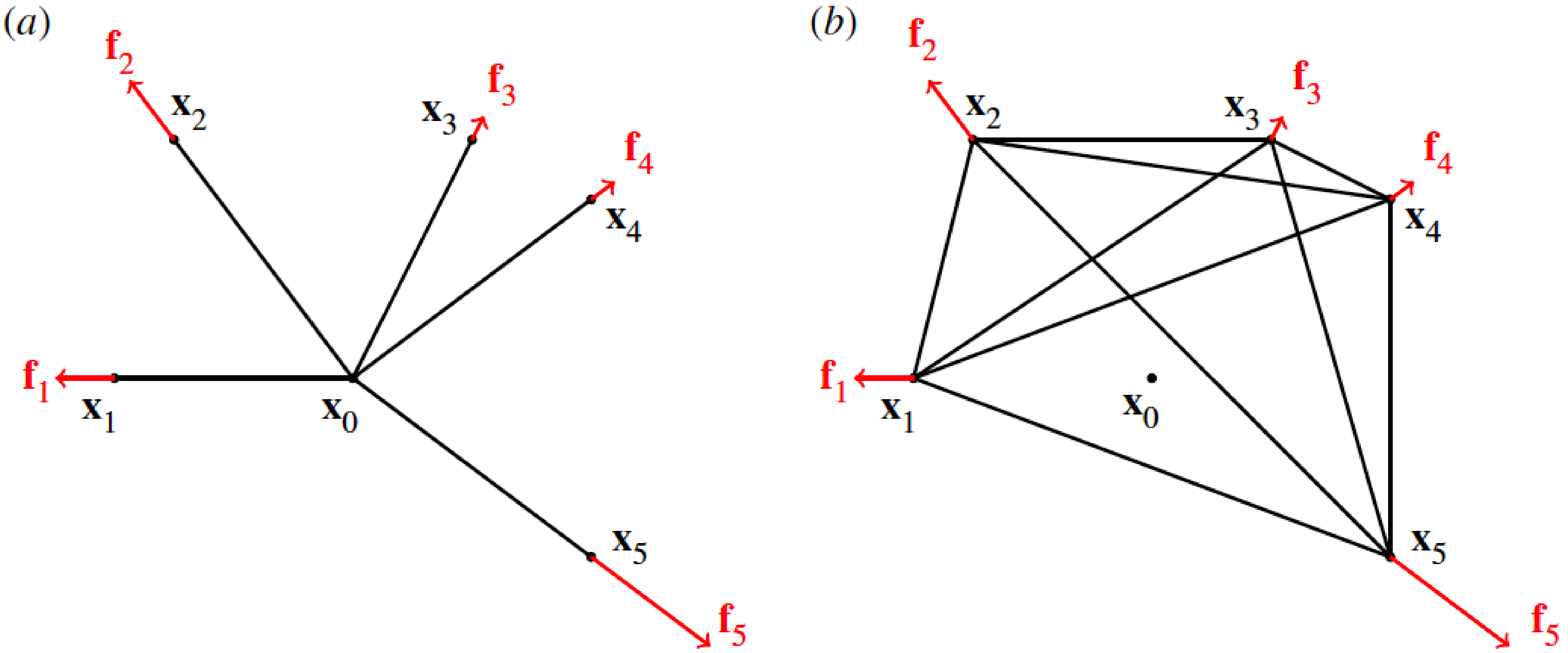
Then, the following condition: $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_N)$ *must satisfy*

$$\inf_{\mathbf{U} \in \mathcal{A}_X} \mathbf{F} \cdot \mathbf{U} \geq 0$$

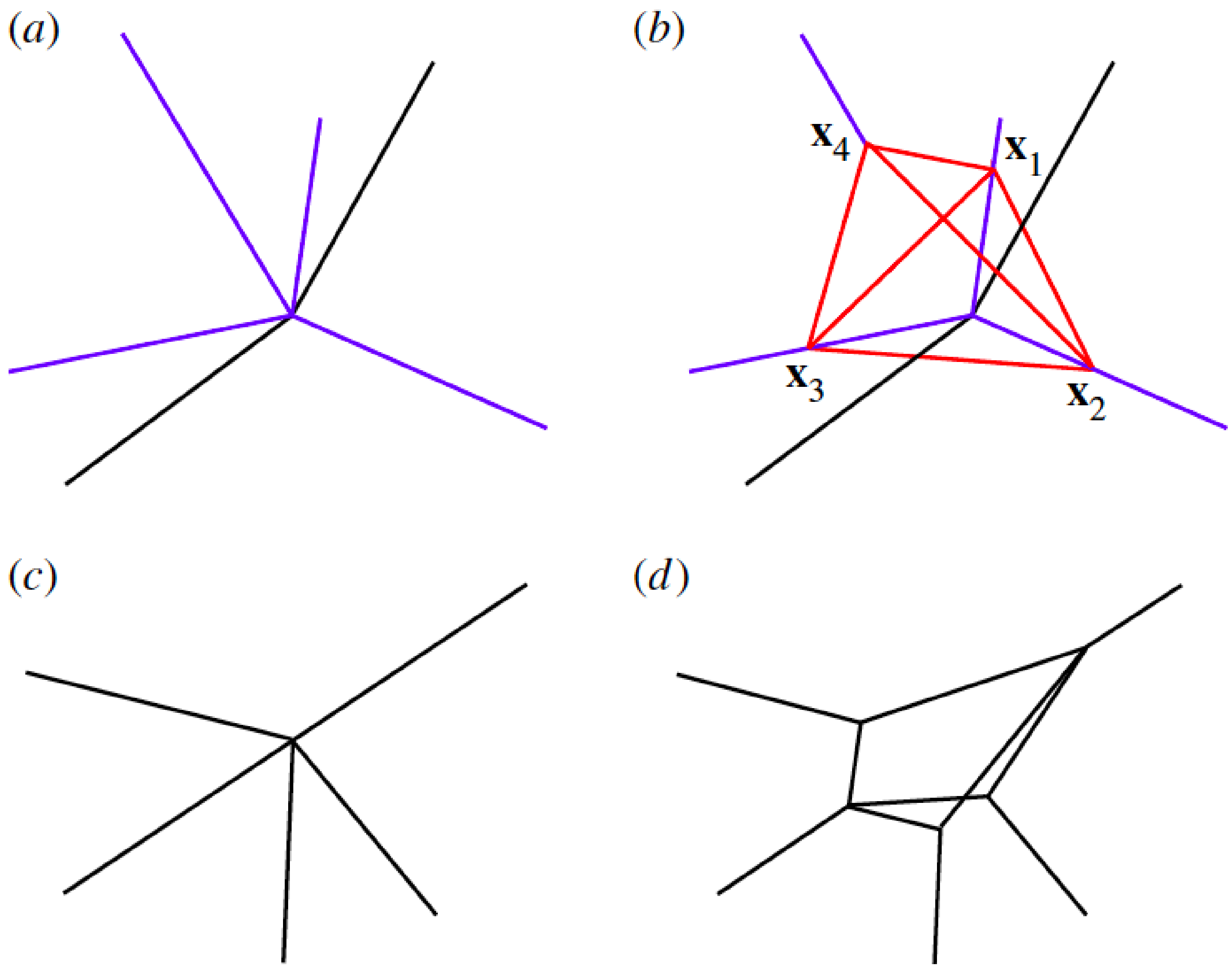
is necessary and sufficient to ensure the existence of a finite web under tension that supports the loading $\mathbf{F}$ *at points* $\mathbf{X}$*. In such a case, the web connecting the terminal points* $\mathbf{X}$ *pairwise supports the loading* $\mathbf{F}$.

So the existence, or not, of a web under tension that supports the desired loading, reduces to a finite dimensional linear programming problem, that is easily solved numerically.

Example:



Making a 3d-web uniloadable:



Thank You!

References:

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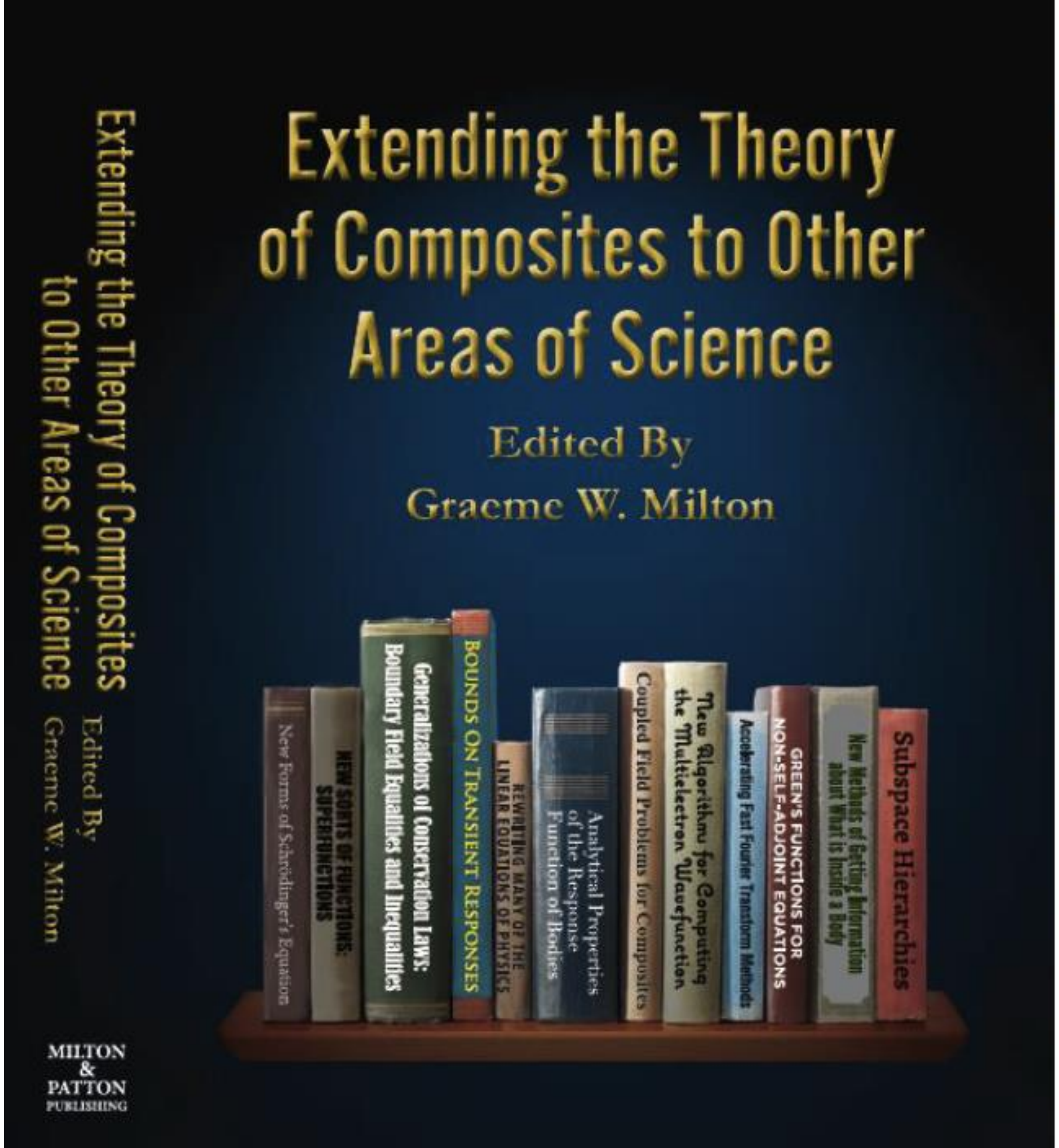

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