Optimal design for shielding or field enhancement in electrostatics and linear elasticity

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Two Problems:

(1) Concentrating a field into a region.

(2) Shielding a region from fields.

Sharp corners concentrate fields

Large Fields also very important for Raman Spectroscopy:

Effect goes as the 4^{th} power of the field intensity.

Well known that rough surfaces enhance Raman Spectroscopy,
by orders of magnitude (SERS)

Shielding: Think of Faraday cage to shield Electromagnetic Field,
Shielding from Magnetic Fields, Thermal Currents
Shielding from Vibrations, Sonar

How to measure this?
Threshold exponents on L^γ integrability:

$$\gamma^- \equiv \inf_\gamma : \int_B |\mathbf{E}(\mathbf{x})|^\gamma \, d\mathbf{x} < \infty$$

$$\gamma^+ \equiv \sup_\gamma : \int_B |\mathbf{E}(\mathbf{x})|^\gamma \, d\mathbf{x} < \infty$$

B is any Ball containing Ω.

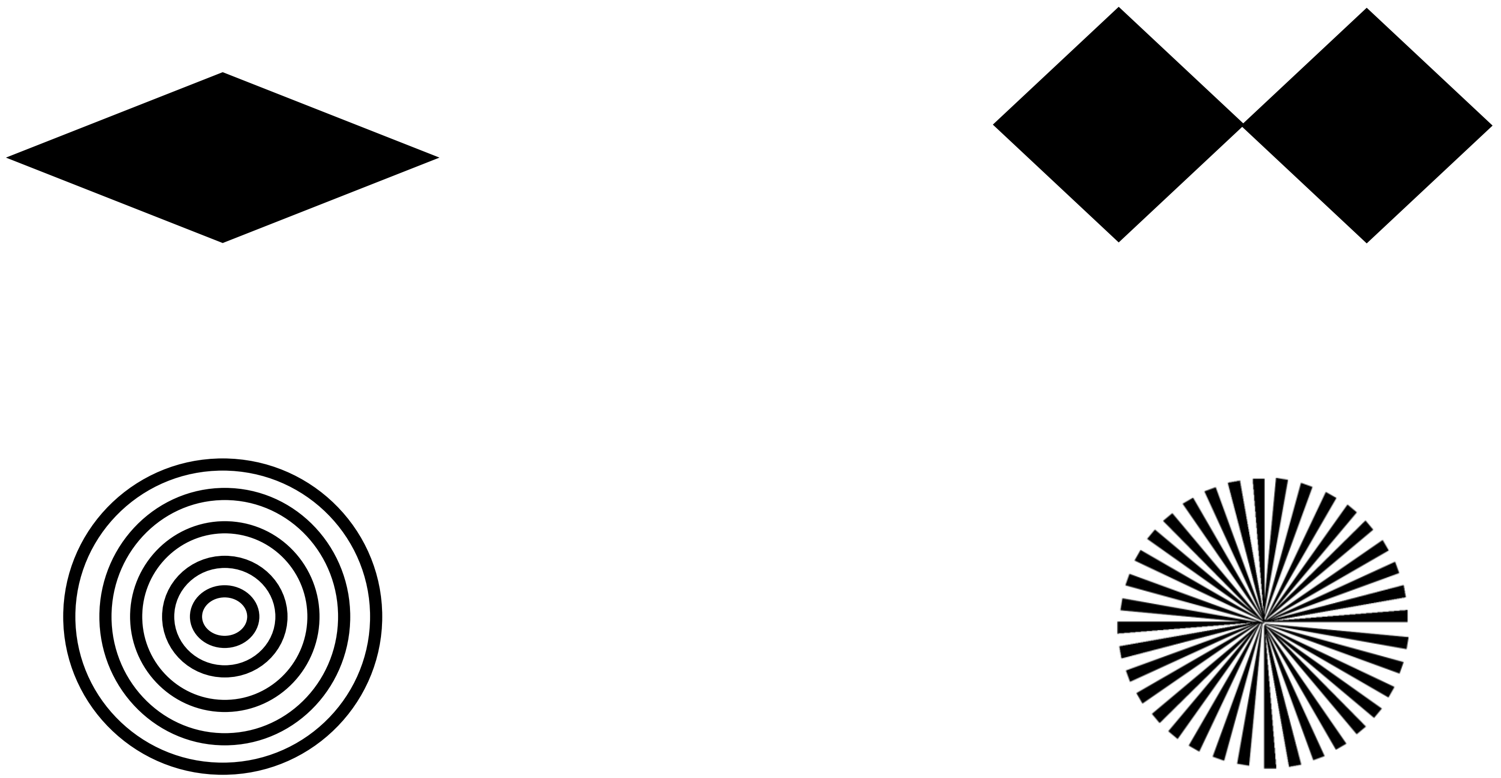
Equivalently, given a (possibly disconnected) subregion $Q \subset \Omega$ of small subvolume $|Q|$ one can maximize or minimize

$$\int_Q |\mathbf{E}(\mathbf{x})|^2 \, d\mathbf{x}$$

and ask how this depends on $|Q|$ asymptotically as $|Q| \to 0$

Two isotropic conductors, conductivities σ_1, σ_2.
Uniform field at infinity

Some Candidates:



Allow for multiscale inclusions:

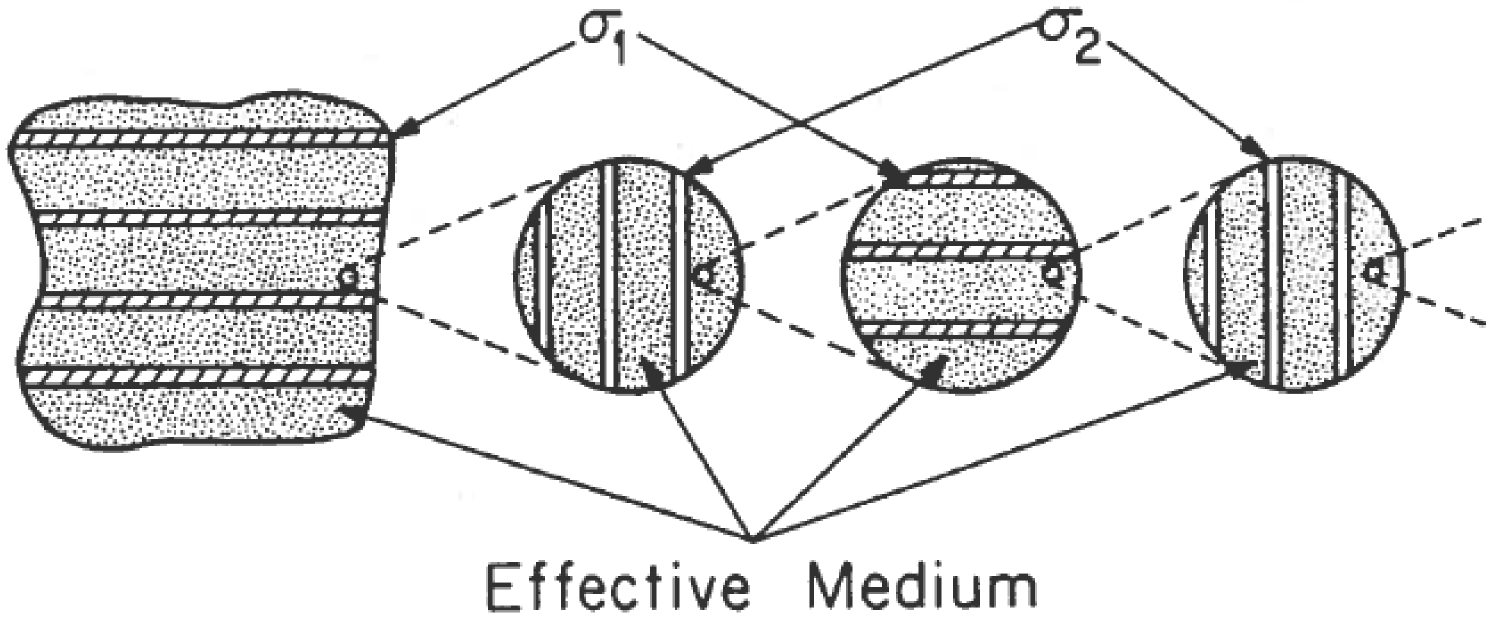
$$\gamma^- \equiv \inf_{\gamma} : \int_B d\mathbf{x} \int_{Y^n} d\mathbf{y}_1, \ldots, d\mathbf{y}_n \; |\mathbf{E}(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n)|^\gamma < \infty$$

$$\gamma^+ \equiv \sup_{\gamma} : \int_B d\mathbf{x} \int_{Y^n} d\mathbf{y}_1, \ldots, d\mathbf{y}_n \; |\mathbf{E}(\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_n)|^\gamma < \infty$$

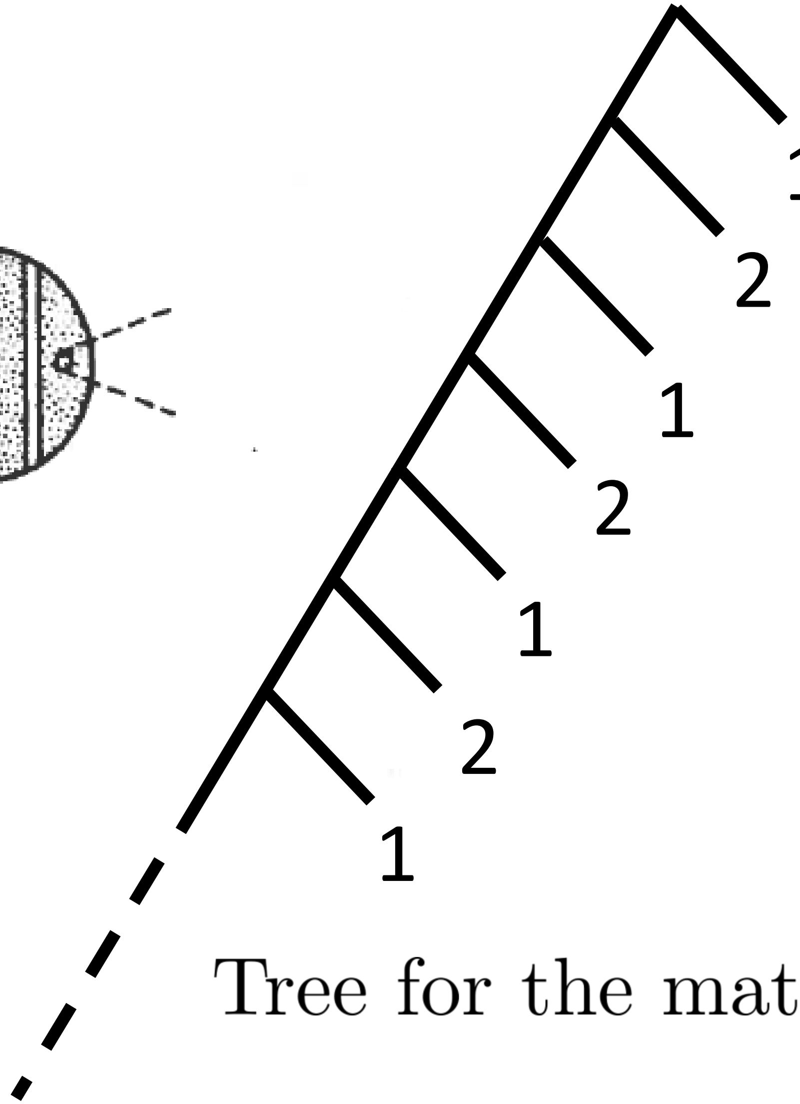
B is any Ball containing Ω.

$\mathbf{y}_1, \ldots, \mathbf{y}_n$ represent finer and finer length scales and $\mathbf{E}$ is periodic in them with period cell Y^n.

Best:

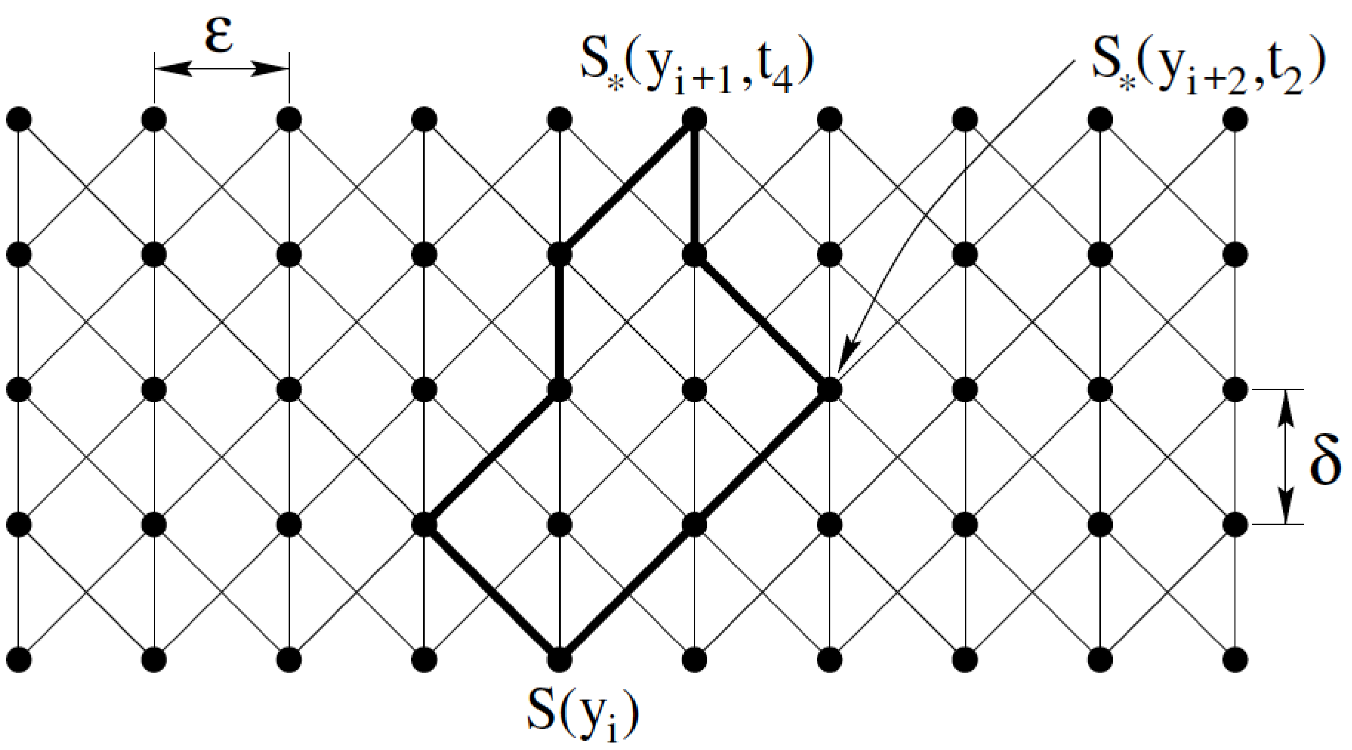


GWM (1986)



Tree for the material

Totally crazy microstructures: partial differential materials



Replace tree by lattice of materials

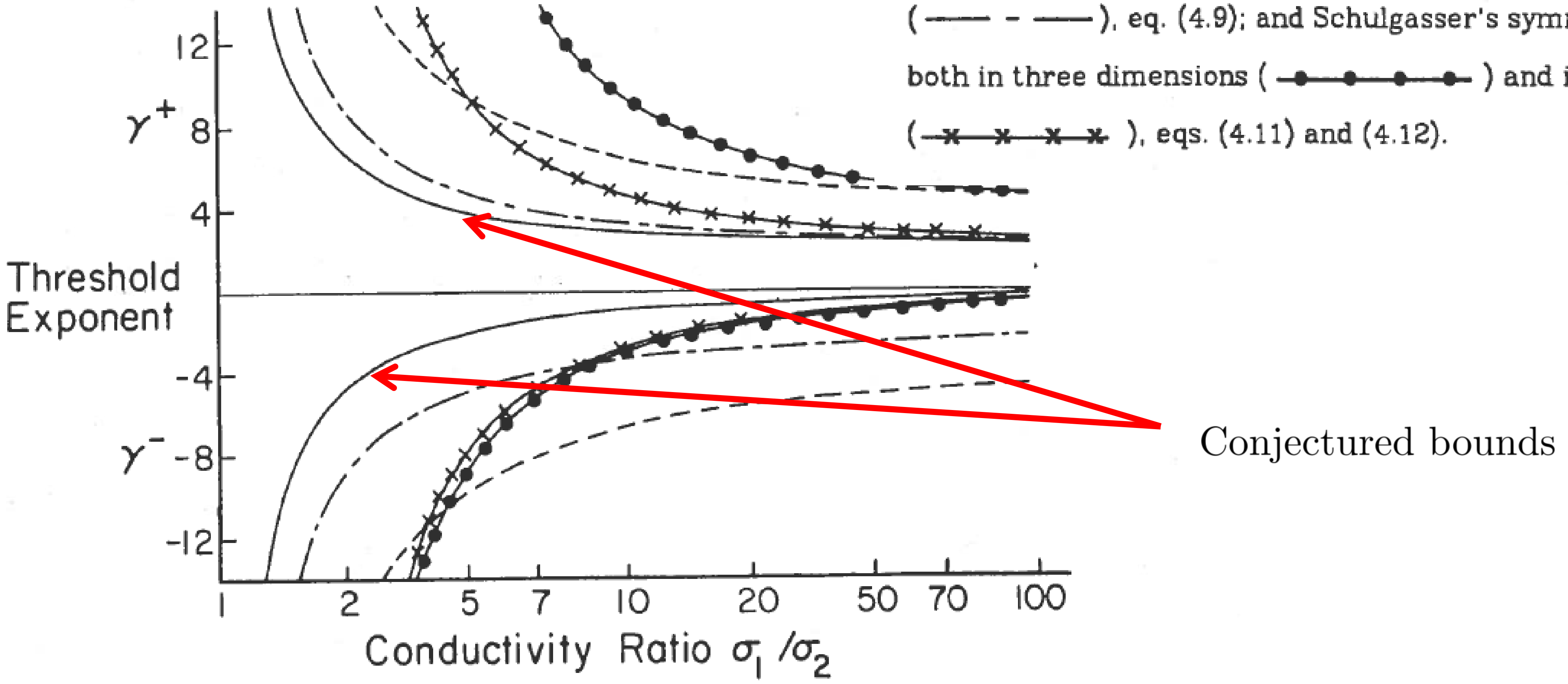
$$\frac{\partial \boldsymbol{S}_*}{\partial t} = q\left\{\boldsymbol{S}_* - \boldsymbol{\Gamma}_1(\boldsymbol{n}) - [\boldsymbol{S}_* - \boldsymbol{\Gamma}_1(\boldsymbol{n})][\boldsymbol{S}_0 - \boldsymbol{\Gamma}_1(\boldsymbol{n})]^{-1}[\boldsymbol{S}_* - \boldsymbol{\Gamma}_1(\boldsymbol{n})]\right\}$$

$$+f\left\{\frac{\partial^2 \boldsymbol{S}_*}{\partial y^2} - 2\frac{\partial \boldsymbol{S}_*}{\partial y}[\boldsymbol{S}_* - \boldsymbol{\Gamma}_1(\boldsymbol{n})]^{-1}\frac{\partial \boldsymbol{S}_*}{\partial y}\right\}.$$

Riccati type PDE

Beauty Contest (GWM, 1986):

Fig. 9. Comparison of threshold exponents for the laminate of Fig. 8. (———), eqs. (4.13) and (4.18); an array of diamond shaped grains (– – – – –), eq. (4.8); a checkerboard of the two components (—— - ——), eq. (4.9); and Schulgasser's symmetric material, both in three dimensions (—●—●—●—●—) and in two dimensions (—×—×—×—×—), eqs. (4.11) and (4.12).



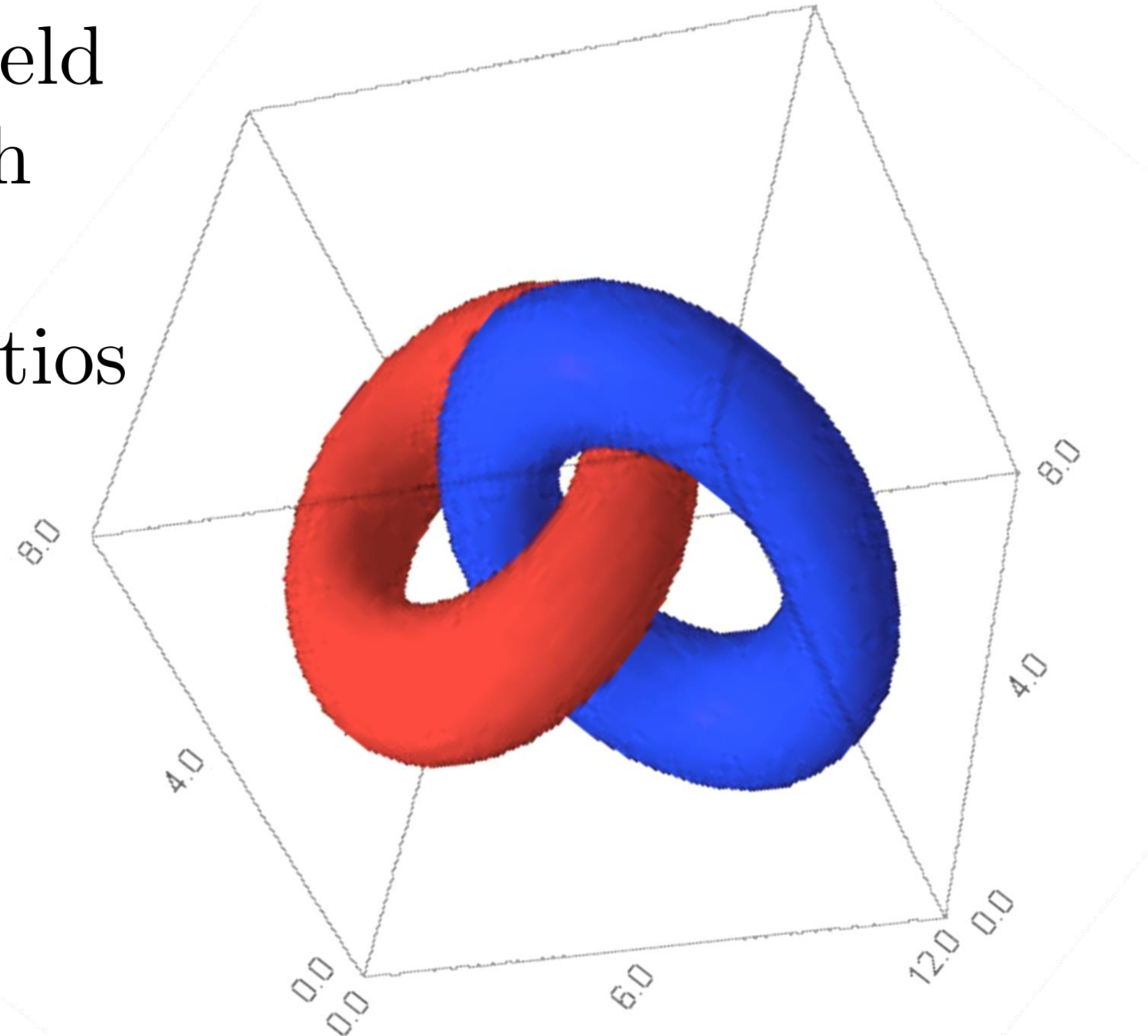
Proof of this micostructure independent Lower Bound on γ^+: Morrey (1938); Boyarski (1957)

Proof of this microstructure Upper Bound on γ^-: Leonetti and Nesi (1997)

See Also Faraco (2003)

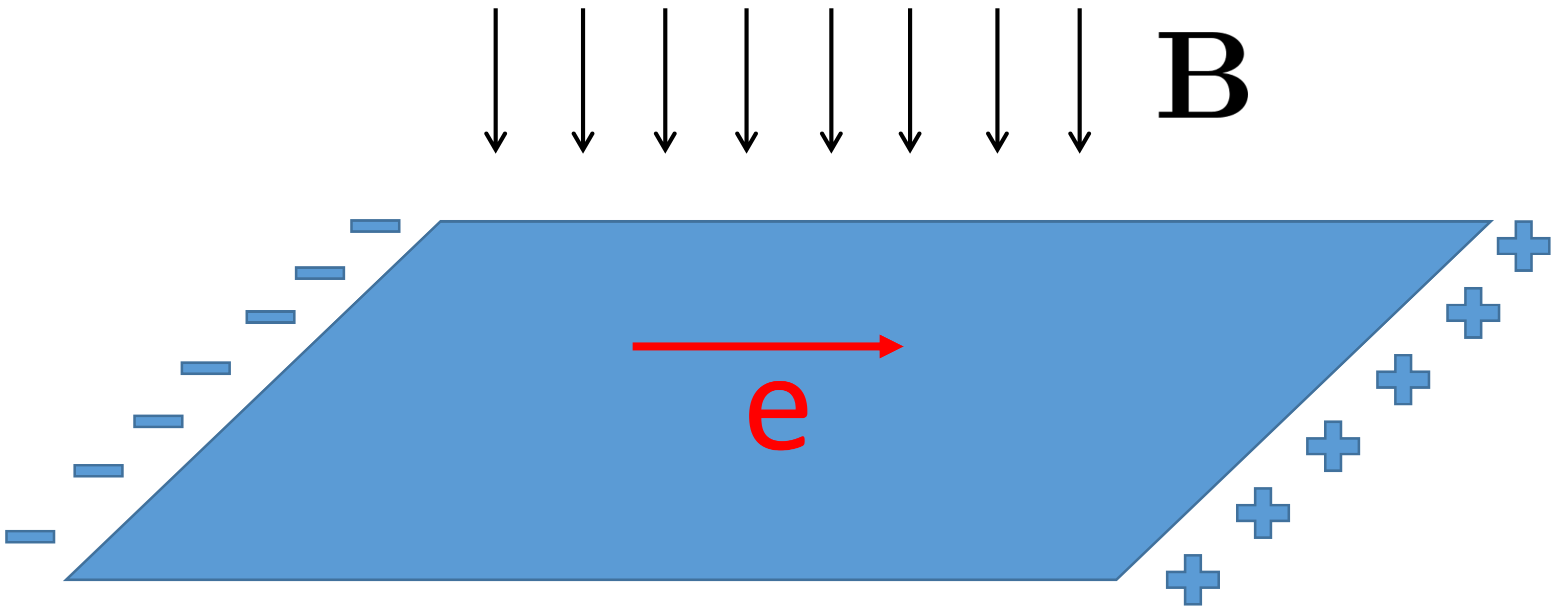
What about 3d?

For a uniform applied field the local field can vanish between the torii, even at finite conductivity ratios



It's constantly a surprise to find what properties a composite can exhibit.

One interesting example:



Hall Voltage

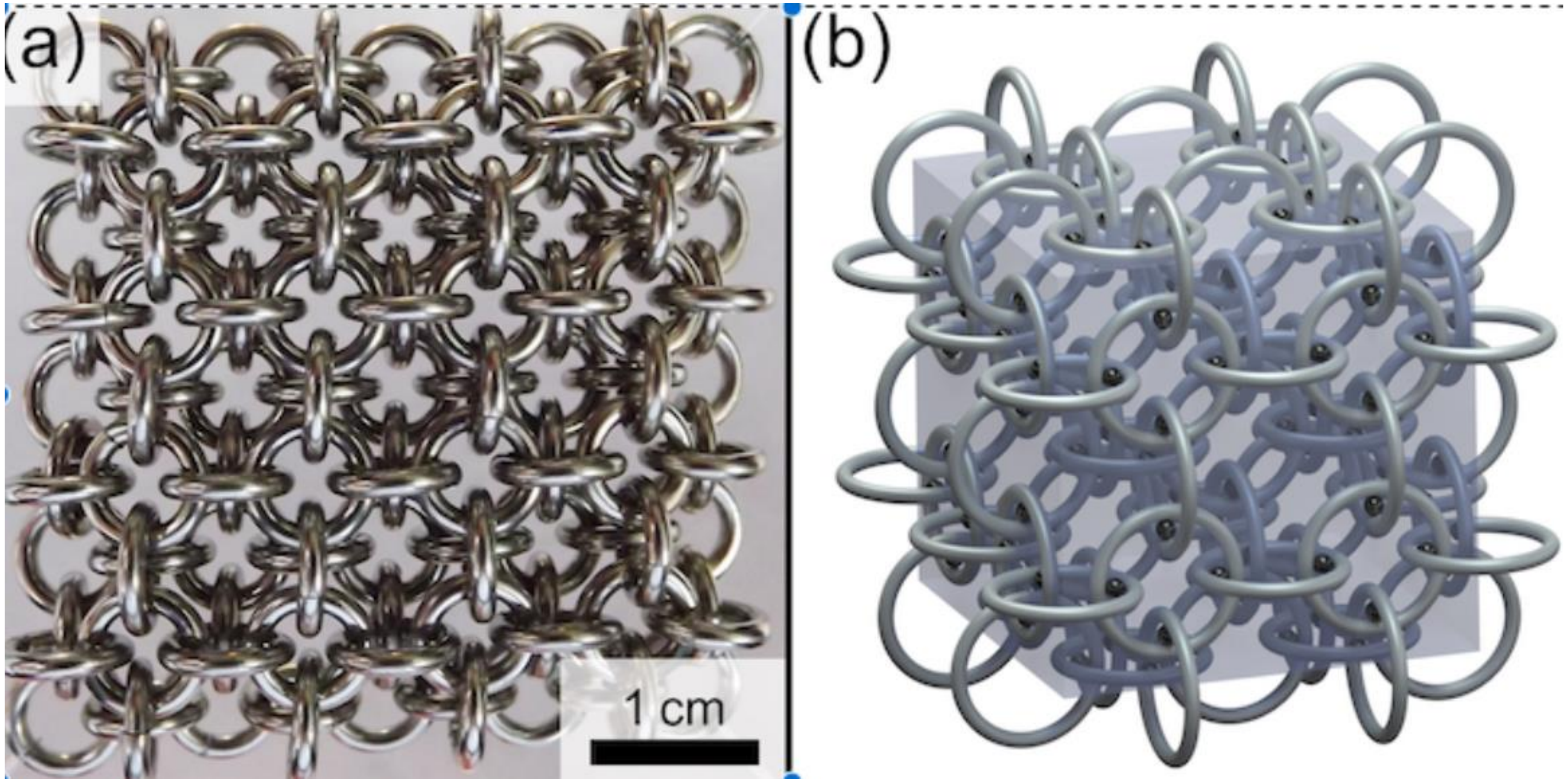
$$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

Non-symmetric conductivity matrix with the antisymmetric part proportional to $\mathbf{B}$

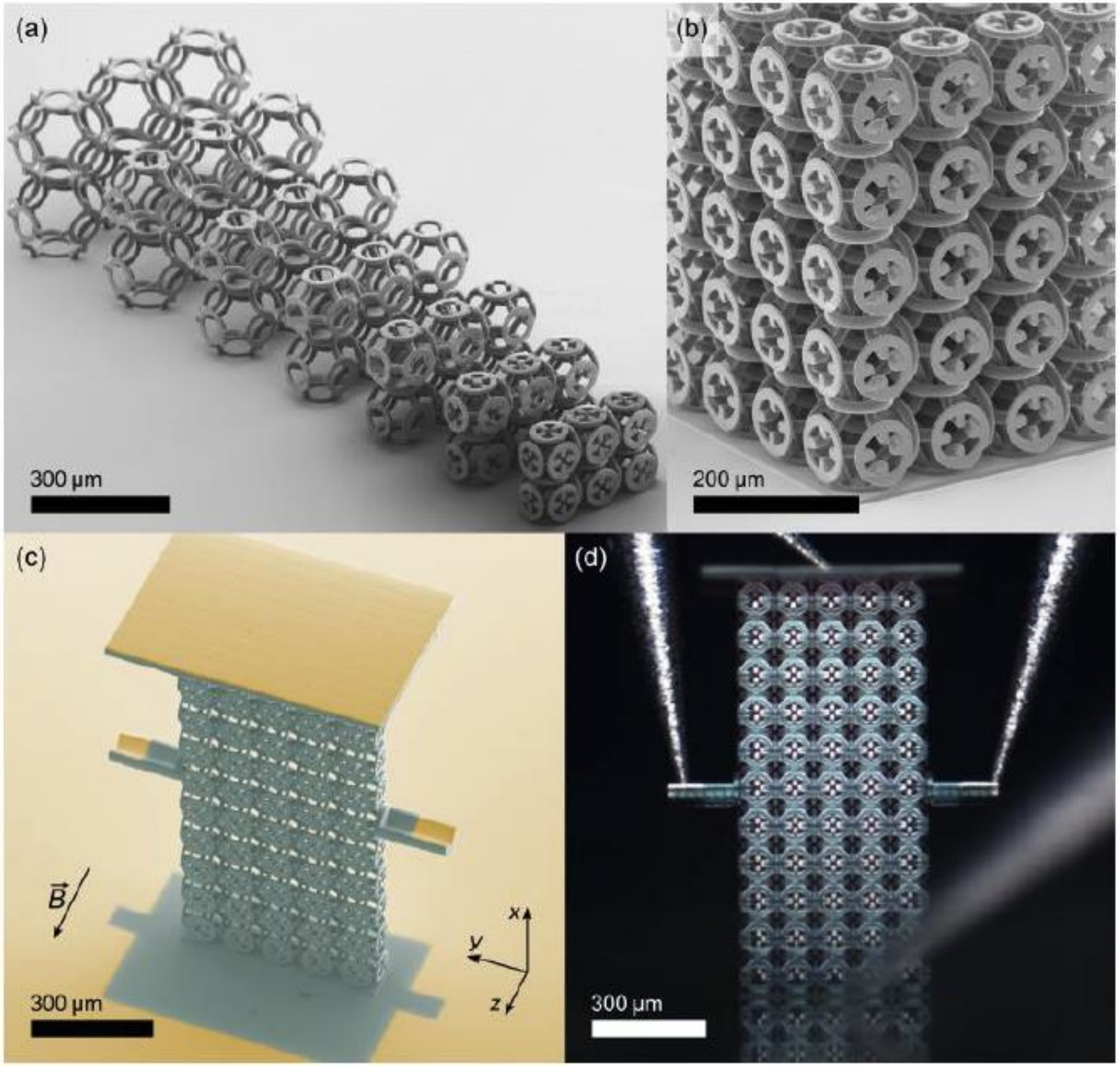
In elementary physics textbooks one is told that in classical physics the sign of the Hall coefficient tells one the sign of the charge carrier.

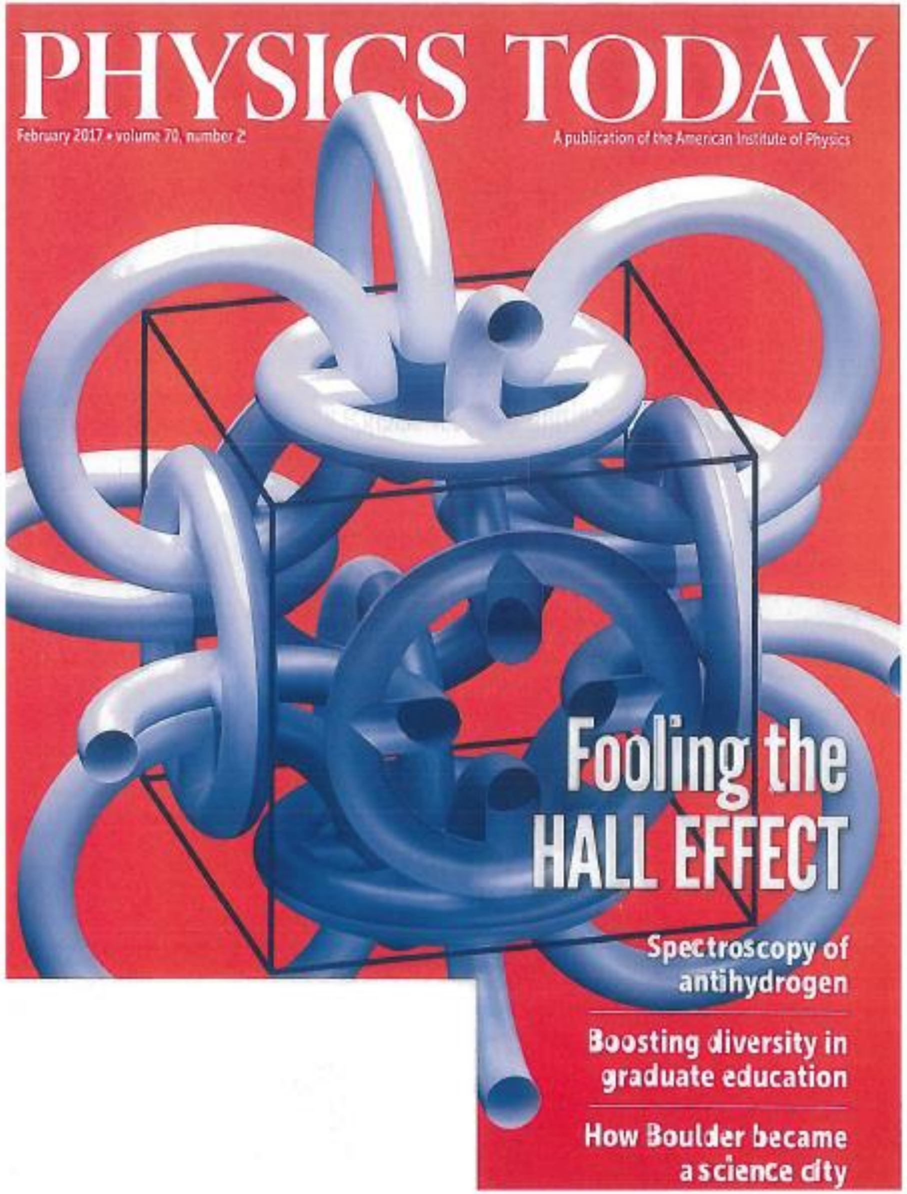
However there is a counterexample!

Geometry suggested by artist Dylon Whyte



A material with cubic symmetry having a Hall Coefficient opposite to that of the constituents (with Marc Briane)

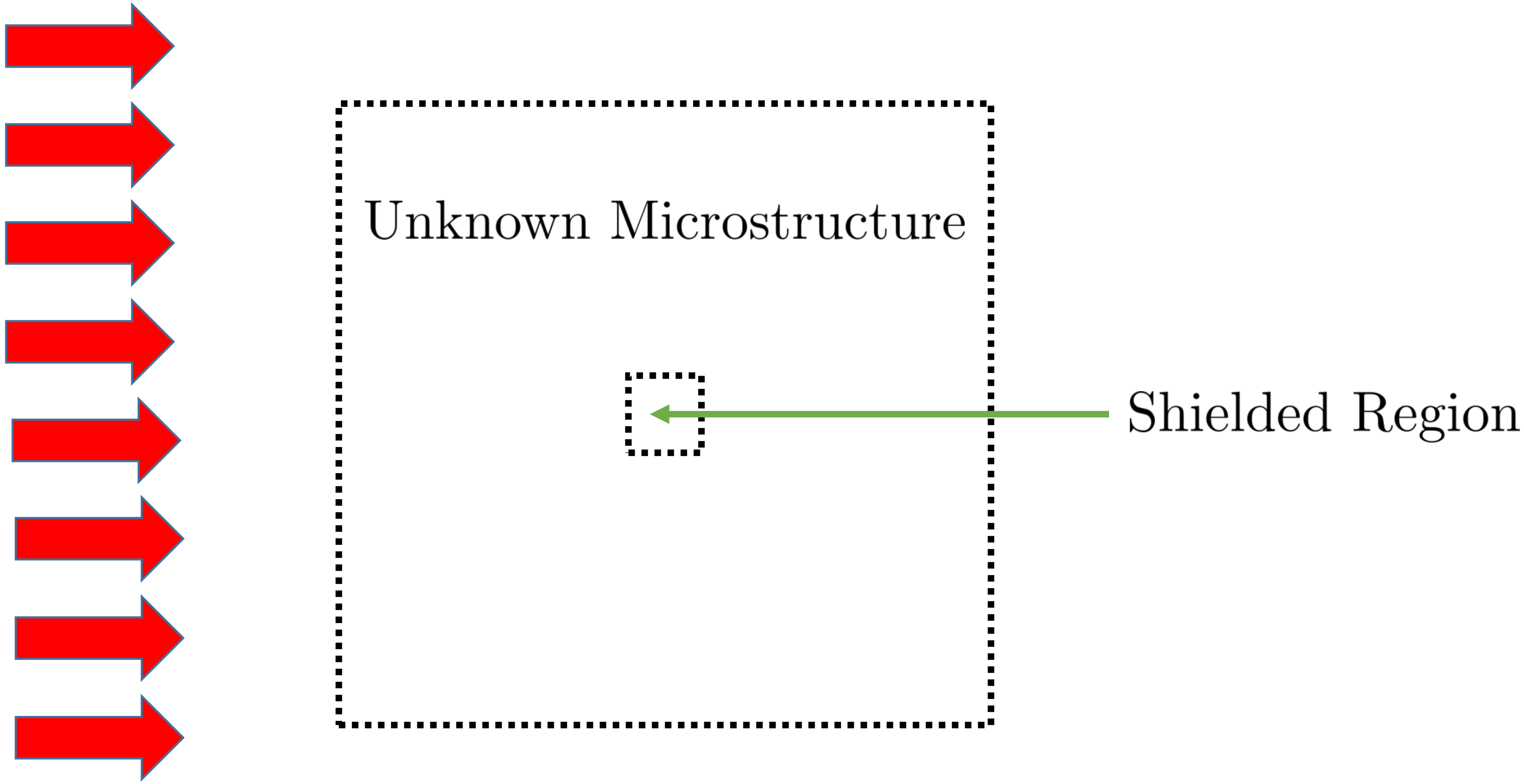




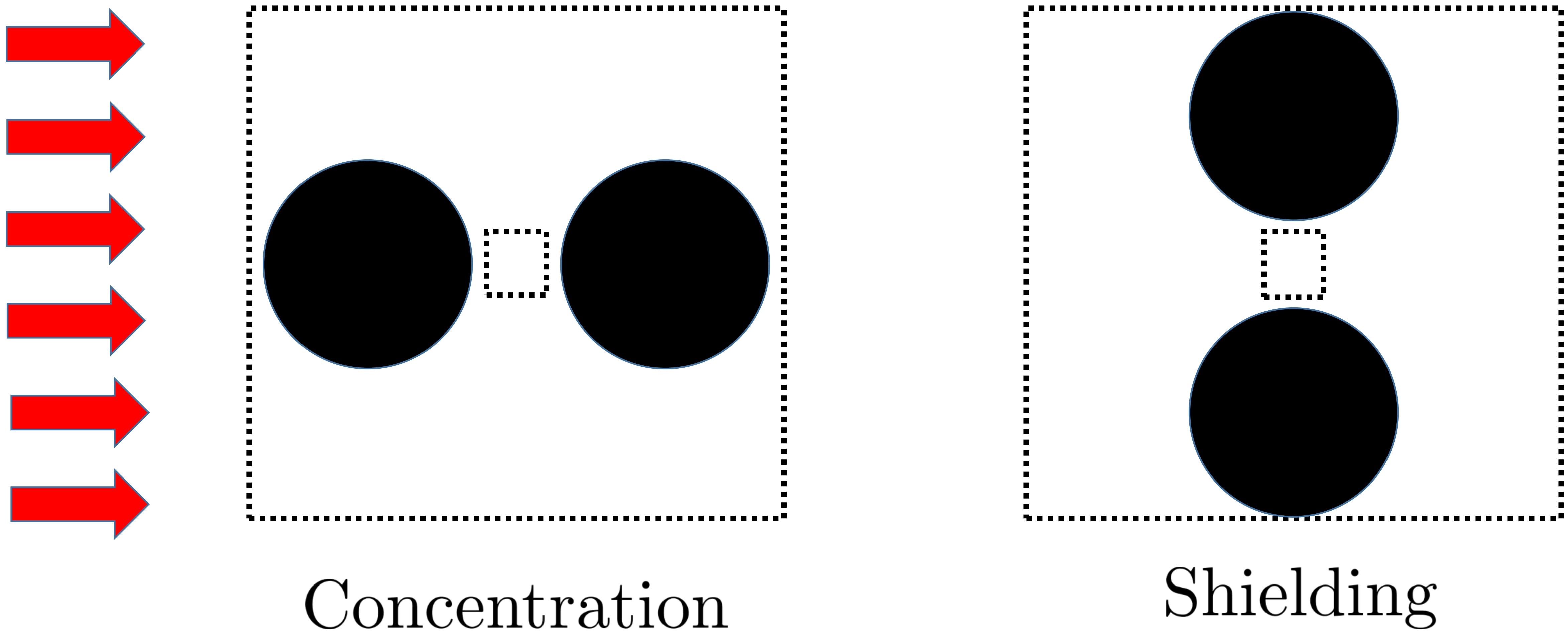
Experimental Realization of Kern, Kadic, Wegener

Back to the shielding problem:

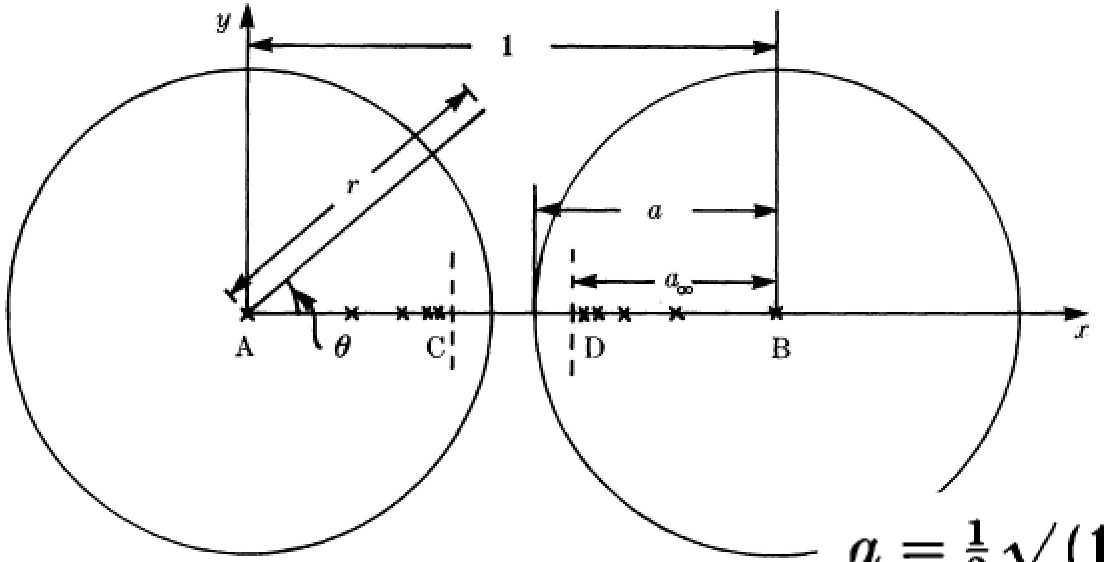
It seems more reasonable to require that there is no microstructure in the shielded region and that the microstructure is localized in a box.



Using Disks:



Field between two highly conducting disks close to touching



$$a = \tfrac{1}{2}\sqrt{(1-1/c^2)}.$$

McPhedran, Poladian, GWM (1988)

$$B_1 = \frac{-(c/2)\,(1-1/c)}{2s\ln(c)+1-2s[\gamma+\psi(1+s)]}.$$

$$a = \tfrac{1}{2}\sqrt{(1-1/c^2)}. \quad a_\infty = \tfrac{1}{2}(1-1/c).$$

ψ: Psi or Digamma function

$$\rho_-(a^2/x) = -\eta\rho_+(x)$$

$$\eta = (\sigma-1)/(\sigma+1).$$

$$\rho_+(1-x) = -\rho_-(x),$$

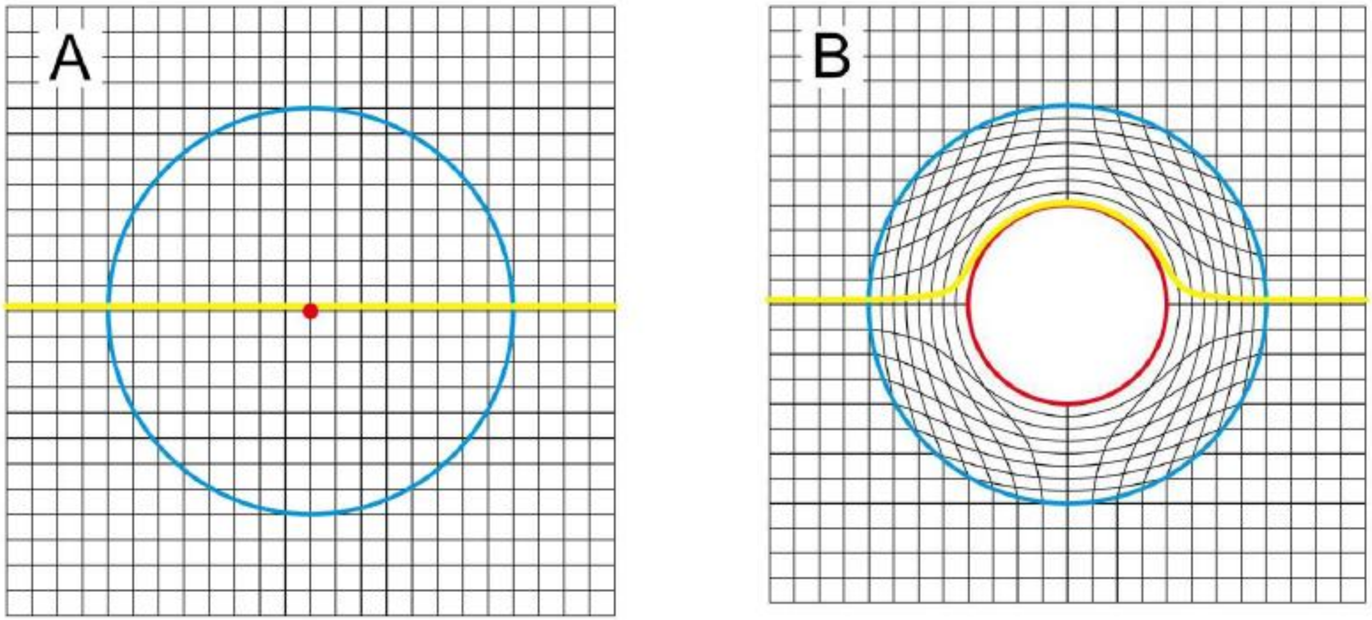
$$\rho_-[a^2/(1-x)] = \eta\rho_-(x)$$

$$\rho_-(x) = A[(a_\infty - x)/(1-a_\infty-x)]^s$$

$$s = \ln(\eta)/\ln[a_\infty/(1-a_\infty)]$$

Rigorous Analysis: Lim and Yu (2015)

Could use the transformation based approach of Greenleaf, Lassas, and Uhlmann

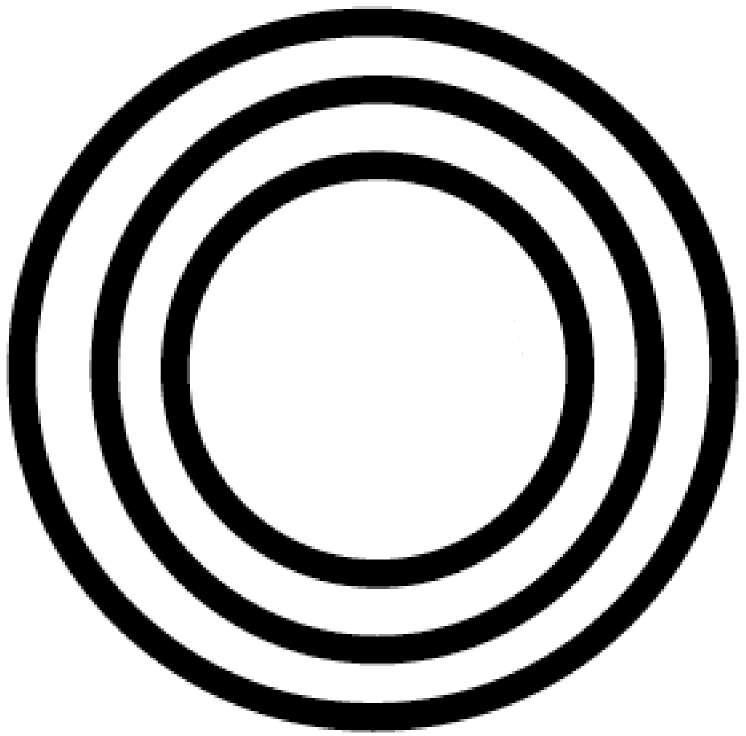


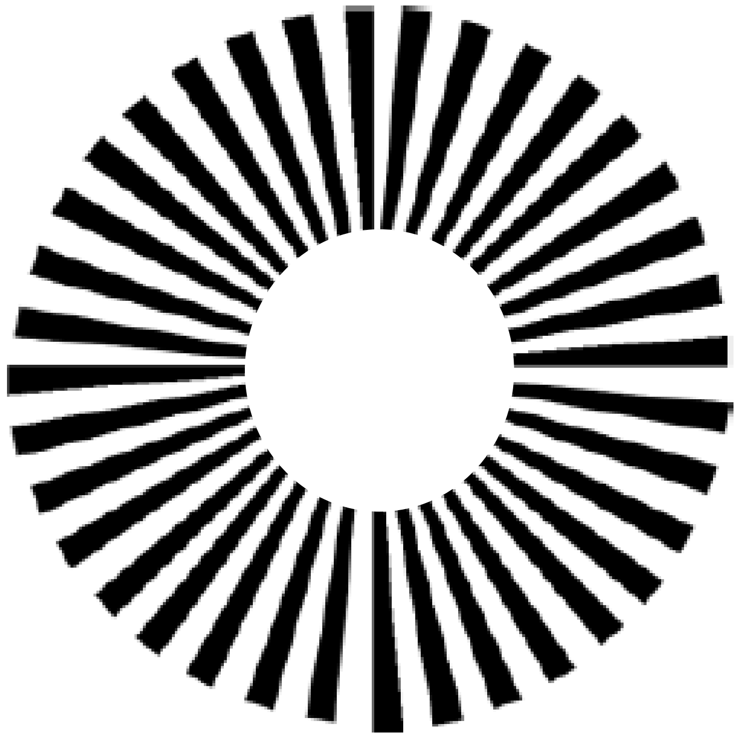
Stretching space (From Ulf Leonhardt)

Advantages: Works for any external field and creates no disturbance

Disadvantages: Requires extreme conductivities, and if one truncates the solution there is no reason to expect it is optimal.

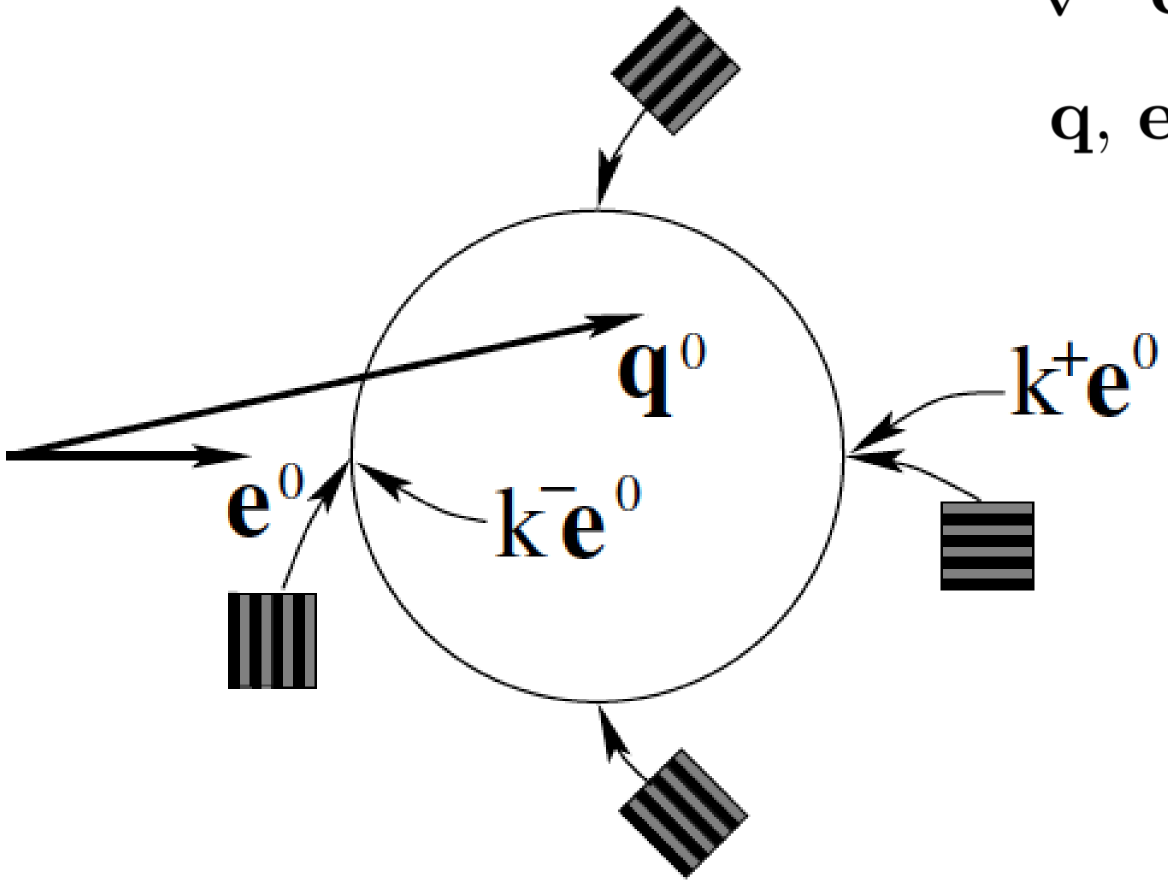
Or Maybe?





Seems like we are just guessing. Is there a more systematic approach, at least in the case where we use just 2 conducting materials, and we are seeking shielding or concentration for just one applied field?

Possible (average heat current, $\mathbf{q}^0$, average temperature gradient, $\mathbf{e}^0$) pairs in a two phase conducting composite (Raitum, 1978).



$$\nabla \cdot \mathbf{q} = 0, \quad \mathbf{q}(\mathbf{x}) = k(\mathbf{x})\mathbf{e}(\mathbf{x}), \quad \mathbf{e} = -\nabla T$$

$\mathbf{q}$, $\mathbf{e}$ periodic, $\langle \mathbf{q} \rangle = \mathbf{q}^0, \quad \langle \mathbf{e} \rangle = \mathbf{e}^0,$

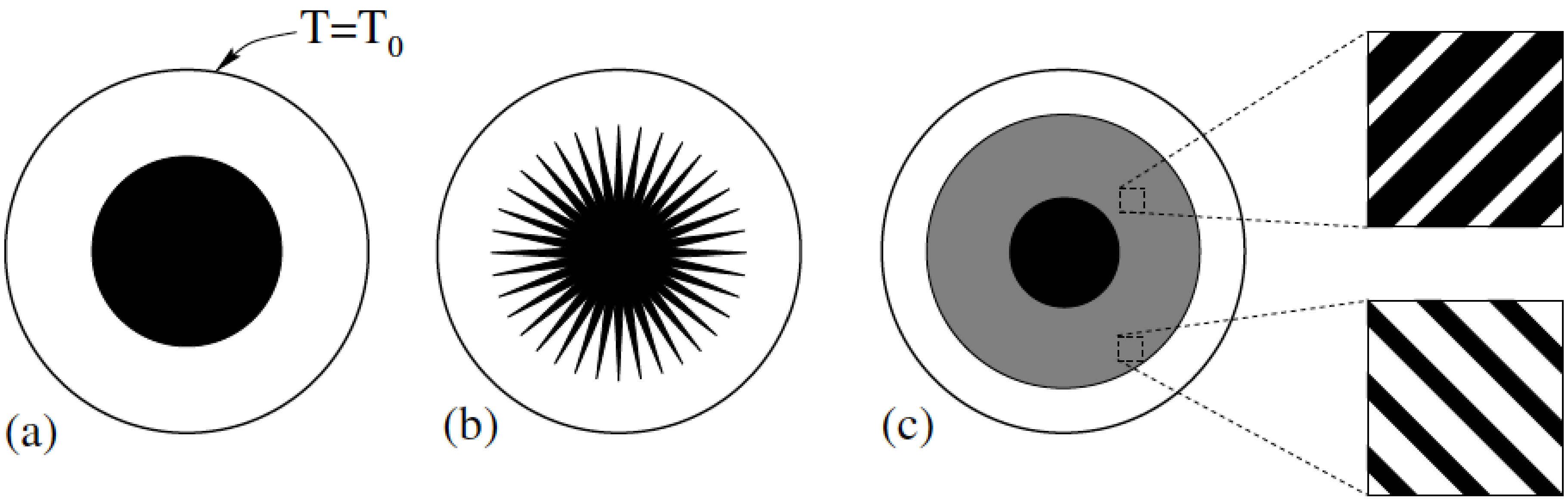
Follows from the Wiener bounds:

$$\begin{aligned} k^- \mathbf{I} &\leq \mathbf{k}^* \leq k^+ \mathbf{I} \\ k^+ &= f k_1 + (1-f) k_2 \\ k^- &= (f/k_1 + (1-f)/k_2)^{-1} \end{aligned}$$

Solution of the "weak G-closure" problem for conductivity

A model optimization problem:

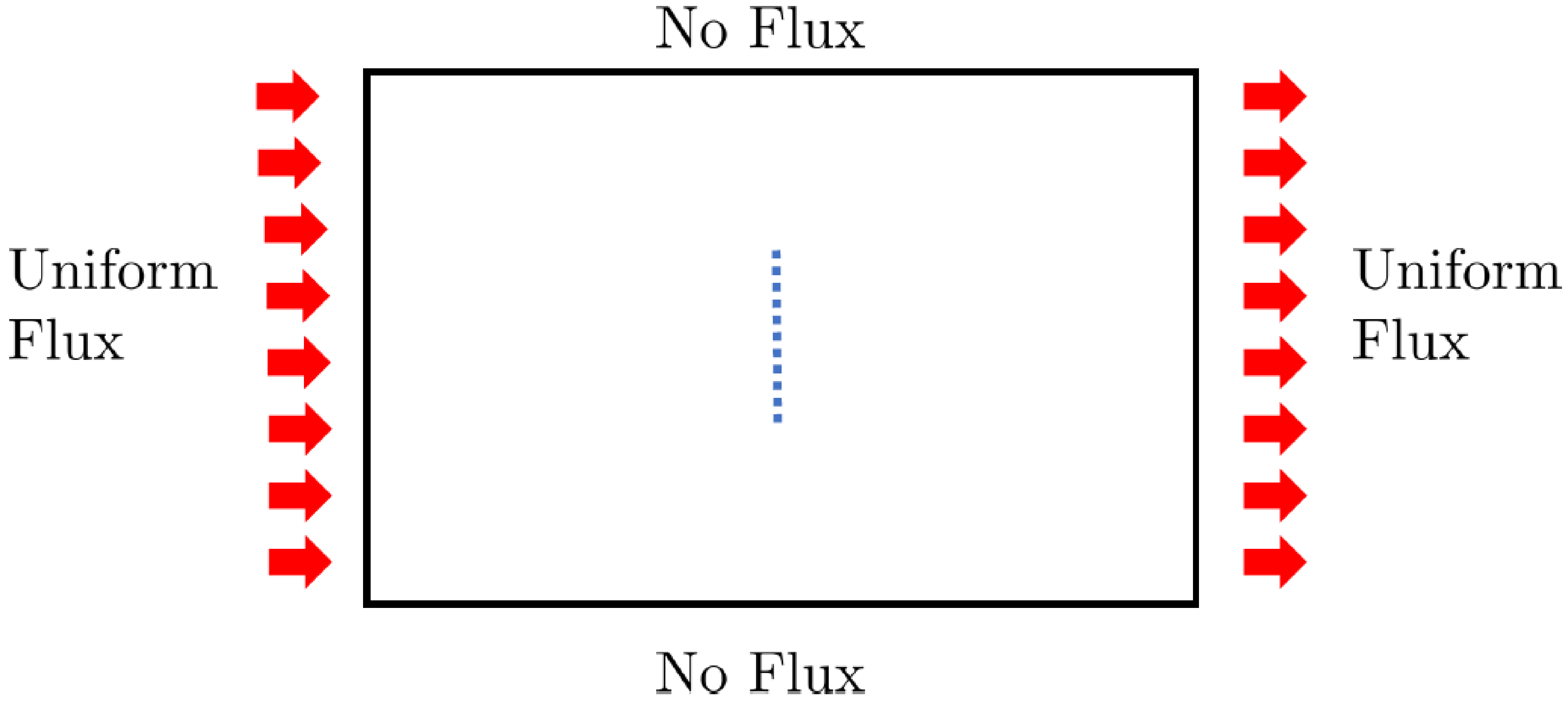
$$-\nabla \cdot \sigma(\mathbf{x}) \nabla T = 1, \quad \sigma = \sigma_1 \text{ or } \sigma_2$$



Solution minimizes $\int_\Omega T(\mathbf{x})\, d\mathbf{x}$, given fixed amounts of the two materials.

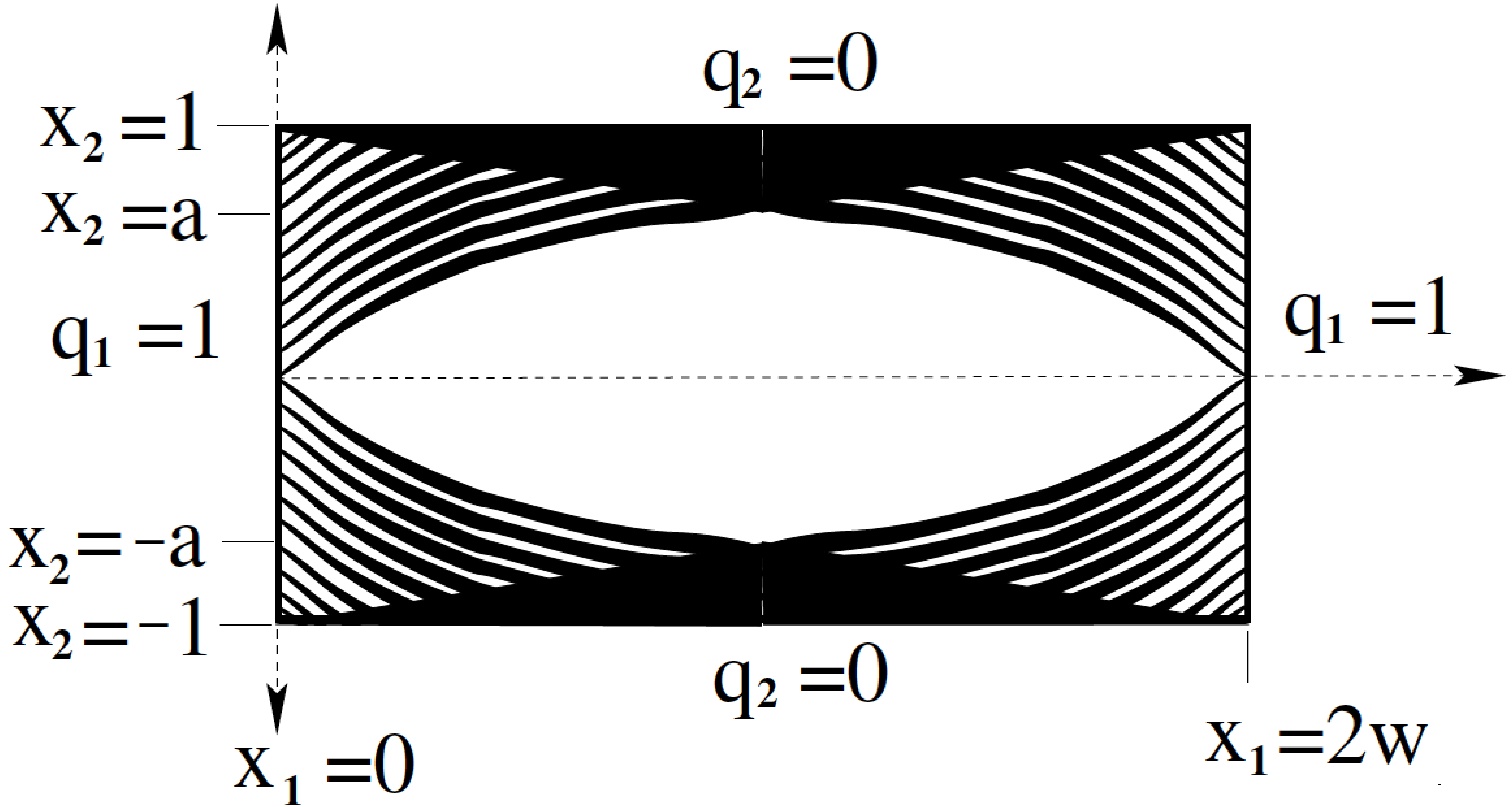
The heat lens problem: Gibiansky, Lurie and Cherkaev (1988)

Aim: Shield or concentrate flux in the blue dashed interval

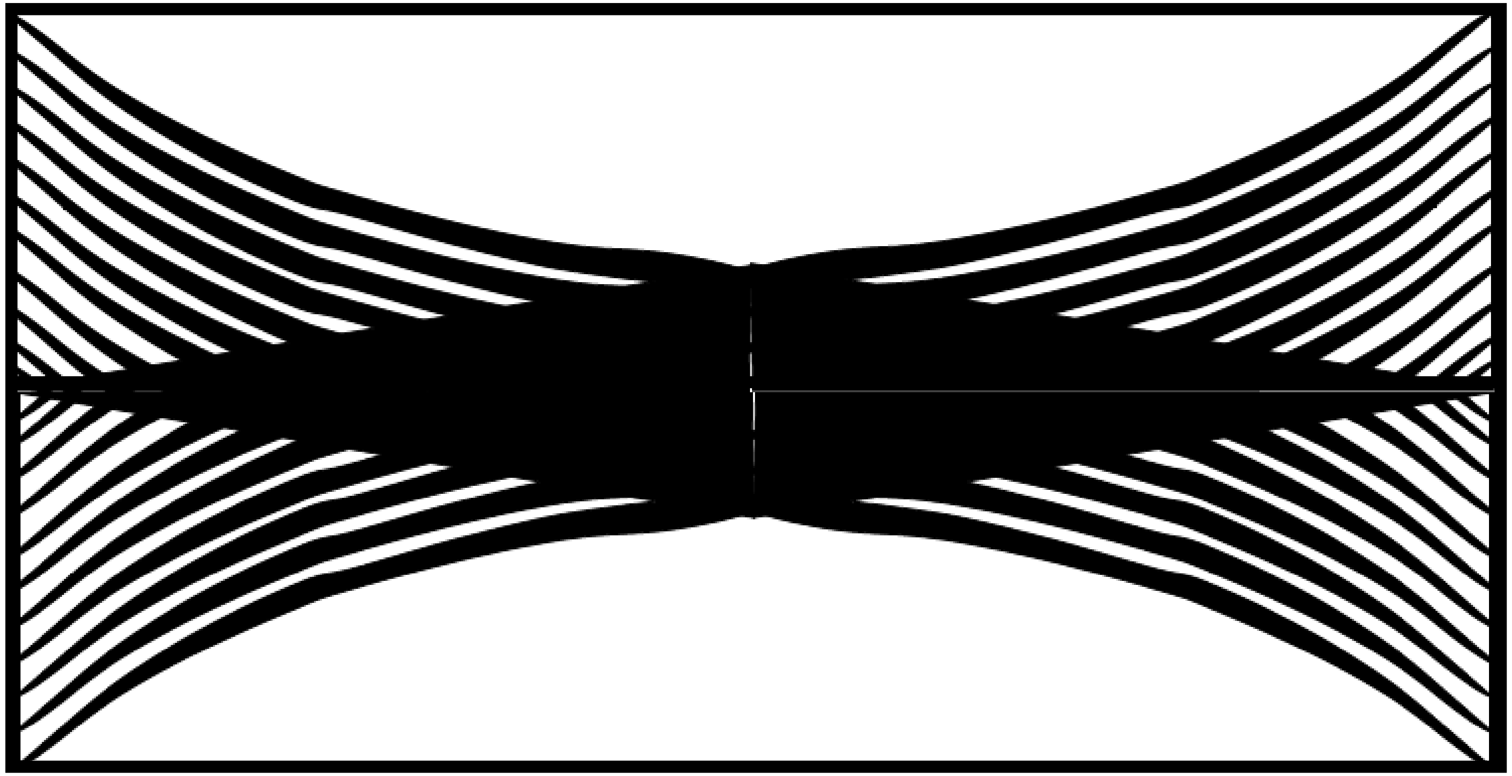


How does one optimally distribute a poor and good conductor to do this?

Field Shield: (Black, good conductor)



Field Concentrator:



What if $k_2 = 0$?

Given $\mathbf{q}^0$ the weak G-closure provides a linear constraint on $\mathbf{e}^0$:

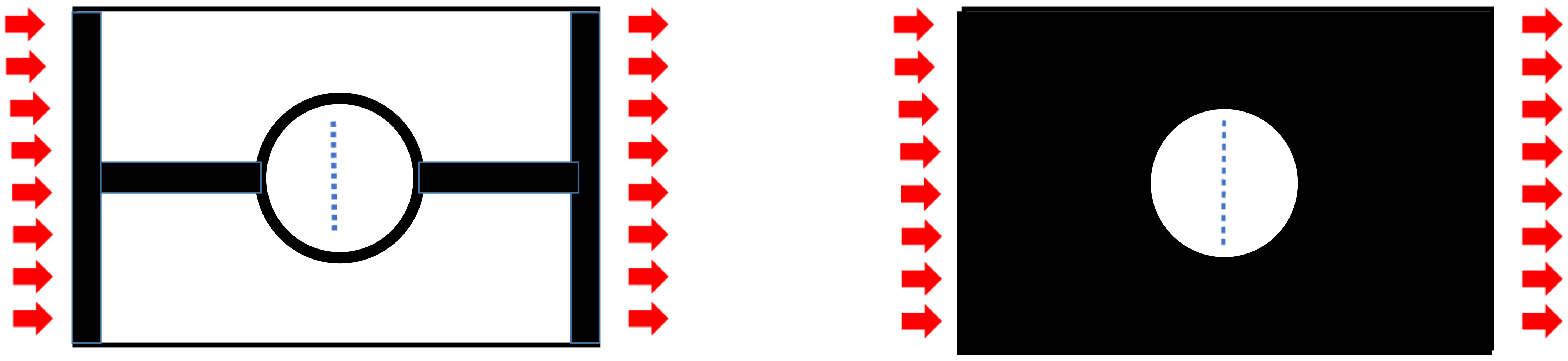
$$\mathbf{q}^0 \cdot \mathbf{q}^0/(f_1 k_1) \leq \mathbf{q}^0 \cdot \mathbf{e}^0$$

It is attained for laminate geometries but also wire geometries where the effective tensor takes the form:

$$\mathbf{k}^* = f_1 k_1 \mathbf{a} \otimes \mathbf{a}, \quad \mathbf{a} \cdot \mathbf{a} = 1$$

Makes sense: wires are best for conducting current

Many Solutions to the shielding problem:



The weak G-closure is still needed if we:

(1) Want to minimize the thermal resistance.

(2) Not use too much of the highly conducting phase (may, e.g., be expensive or heavy.

To solve similar optimization problems for elasticity, can we find
the "weak G-Closure" for 3d-elasticity?

At least in the case for 3d printed materials when one phase is void and the other elastically isotropic?

A difficult problem: need to characterize possible
(average strain $\boldsymbol{\epsilon}^0$, average stress $\boldsymbol{\sigma}^0$) pairs,

Can assume $\boldsymbol{\sigma}^0$ is diagonal and normalized : 2 parameters
Then $\boldsymbol{\epsilon}^0$ has 6 parameters.

So the "weak G-Closure" is described by a set in an 8-dimensional space,
11 if one includes the volume fraction, and bulk and shear moduli of the initial elastic material.

Problem:

$\boldsymbol{\sigma}(\mathbf{x})$, $\boldsymbol{\epsilon}(\mathbf{x})$ periodic,

$$\nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x}), \quad \boldsymbol{\epsilon} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2.$$

$$\mathbf{C}(\mathbf{x}) = \mathbf{C_1}\chi(\mathbf{x}) + \mathbf{C}_2(1 - \chi(\mathbf{x})), \quad \boldsymbol{\sigma}^0 = \langle \boldsymbol{\sigma} \rangle, \quad \boldsymbol{\epsilon}^0 = \langle \boldsymbol{\epsilon} \rangle, \quad f = \langle \chi \rangle$$

Given f what is the range of values the pairs $(\boldsymbol{\sigma}^0, \boldsymbol{\epsilon}^0)$ take in the limit $\mathbf{C}_2 \to 0$ as the microgeometry varies $\chi(\mathbf{x})$ varies over all possible configurations?

One constraint is immediately implied by sharp bounds on the compliance energy:

$$W_f(\boldsymbol{\sigma}^0) \leq \boldsymbol{\sigma}^0 : \epsilon^0, \quad (*)$$

Explicit expression for $W_f(\boldsymbol{\sigma}_{\mathbf{0}})$ given by Gibiansky and Cherkaev (1987) and Allaire (1994). Note $W_f(c\mathbf{A}) = c^2 W_f(\mathbf{A})$

Our main result is that these optimal bounds on the compliance tensor also provide optimal bounds on $(\boldsymbol{\epsilon}^0, \boldsymbol{\sigma}^0)$-pairs. Given $\boldsymbol{\sigma}^0$ they constrain $\boldsymbol{\epsilon}^0$ to lie on one-side of a hyperplane.

$$W_f(\boldsymbol{\sigma}^0) = \boldsymbol{\sigma}^0 : \mathbf{C}_1^{-1}\boldsymbol{\sigma}^0 + \frac{f}{2\mu} g(\mathbf{C}_1, \boldsymbol{\sigma}^0), \quad \text{(Using Allaire's notation.)}$$

Suppose the stress has eigenvalues σ_1, σ_2 and σ_3. Can assume at most one eigenvalue is negative, and $\sigma_1 \leq \sigma_2 \leq \sigma_3$. When all are non-negative, and $\lambda > 0$:

$$\begin{aligned} g(\mathbf{C}, \boldsymbol{\sigma}) &= \frac{2\mu + \lambda}{2(2\mu + 3\lambda)}(\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 \leq \sigma_1 + \sigma_2, \\ &= (\sigma_1 + \sigma_2)^2 + \sigma_3^2 - \frac{\lambda}{2\mu + 3\lambda}(\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 \geq \sigma_1 + \sigma_2, \end{aligned}$$

while when one eigenvalue, namely σ_1, is negative,

$$\begin{aligned} g(\mathbf{C}, \boldsymbol{\sigma}) &= \frac{2\mu + \lambda}{2(2\mu + 3\lambda)}\left(\sigma_3 + \sigma_2 - \frac{\mu + 2\lambda}{\mu + \lambda}\sigma_1\right)^2 \\ &\quad \text{if } \sigma_3 + \sigma_2 \geq \frac{-\mu}{\mu + \lambda}\sigma_1 \text{ and } \sigma_3 - \sigma_2 \leq \frac{-\mu}{\mu + \lambda}\sigma_1, \\ &= (\sigma_3 + \sigma_2)^2 + \sigma_1^2 - \frac{\lambda}{2\mu + 3\lambda}(\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 + \sigma_2 \leq \frac{-\mu}{\mu + \lambda}\sigma_1, \\ &= \sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \frac{2\mu}{\mu + \lambda}\sigma_1\sigma_2 - \frac{\lambda}{2\mu + 3\lambda}(\sigma_1 + \sigma_2 + \sigma_3)^2 \text{ if } \sigma_3 - \sigma_2 \geq \frac{-\mu}{\mu + \lambda}\sigma_1. \end{aligned}$$

The required geometries are pentmodes, materials with elastic tensor

$$\mathbf{C}^* = \alpha \mathbf{A} \otimes \mathbf{A}, \quad \mathbf{A} : \mathbf{A} = 1$$

that are optimal in the sense that

$$\alpha = 1/W_f(\mathbf{A})$$

Given any $\boldsymbol{\sigma}_0$ and $\boldsymbol{\epsilon}_0$ so that $(*)$ holds as an equality, we choose

$$\mathbf{A} = \boldsymbol{\sigma}_0 / \sqrt{\boldsymbol{\sigma}_0 : \boldsymbol{\sigma}_0}$$

and then

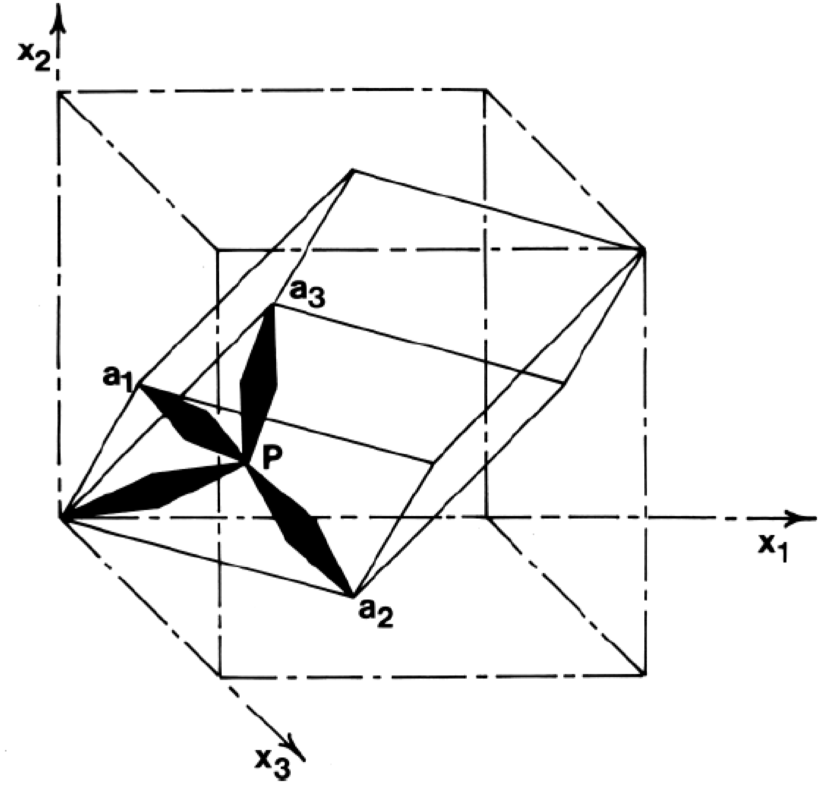
$$\mathbf{C}^* \boldsymbol{\epsilon}_0 = \alpha \boldsymbol{\sigma}_0 W_f(\boldsymbol{\sigma}_0)/(\boldsymbol{\sigma}_0 : \boldsymbol{\sigma}_0) = \alpha \boldsymbol{\sigma}_0 W_f(\mathbf{A}) = \boldsymbol{\sigma}_0$$

as desired.

What are pentamodes?

New classes of elastic materials (with Cherkaev, 1995)

A three dimensional pentamode material which can support any prescribed loading



Like a fluid it only supports one loading, unlike a fluid that loading may be anisotropic

Pentamode structures are a sort of anisotropic inhomegeneous fluid

$$\mathbf{C}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) \otimes \mathbf{A}(\mathbf{x}), \quad \nabla \cdot \mathbf{A} = 0,$$

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x}), \quad \nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\epsilon} = [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]/2$$

have the solution

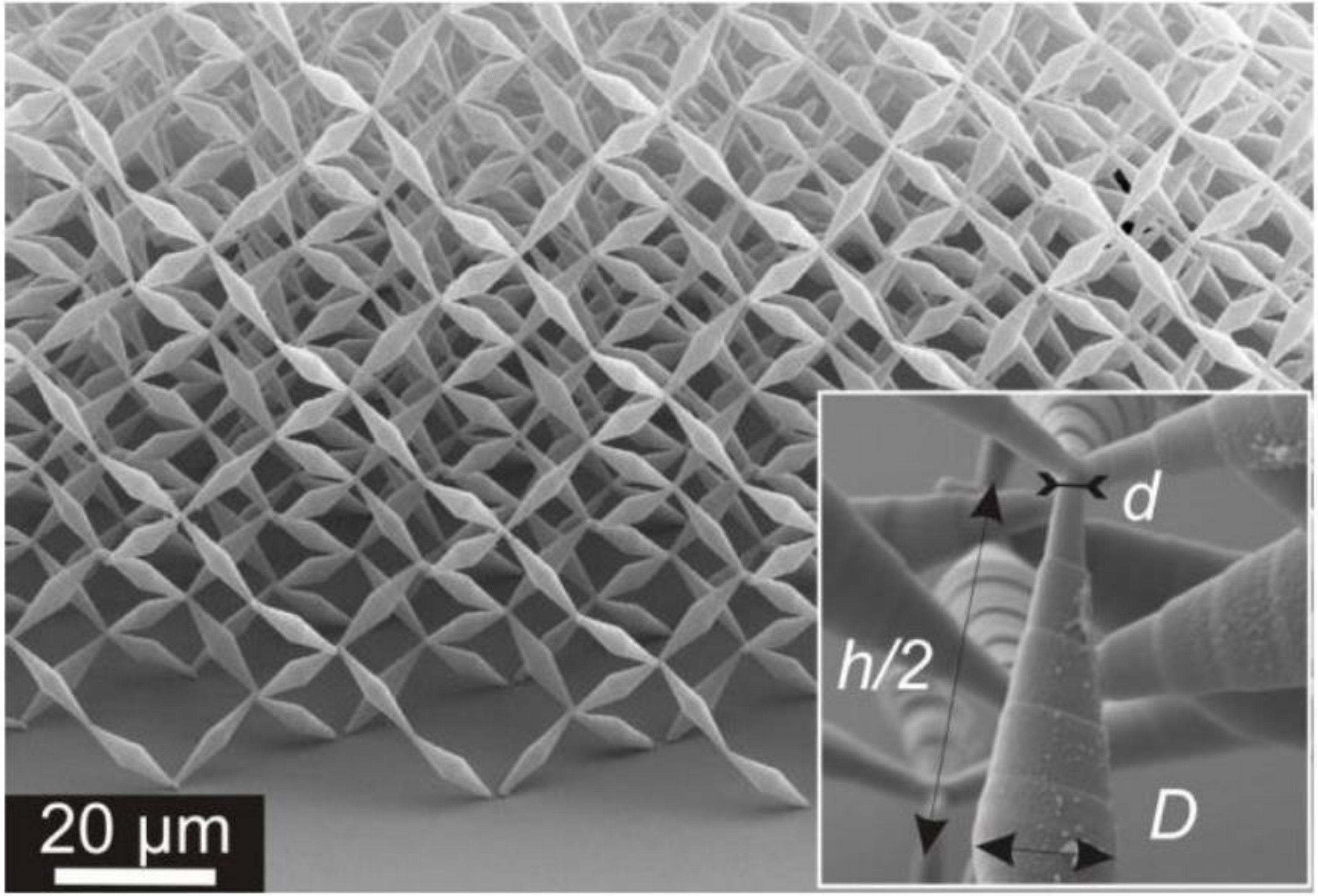
$$\boldsymbol{\sigma}(\mathbf{x}) = \alpha\mathbf{A}(\mathbf{x})$$

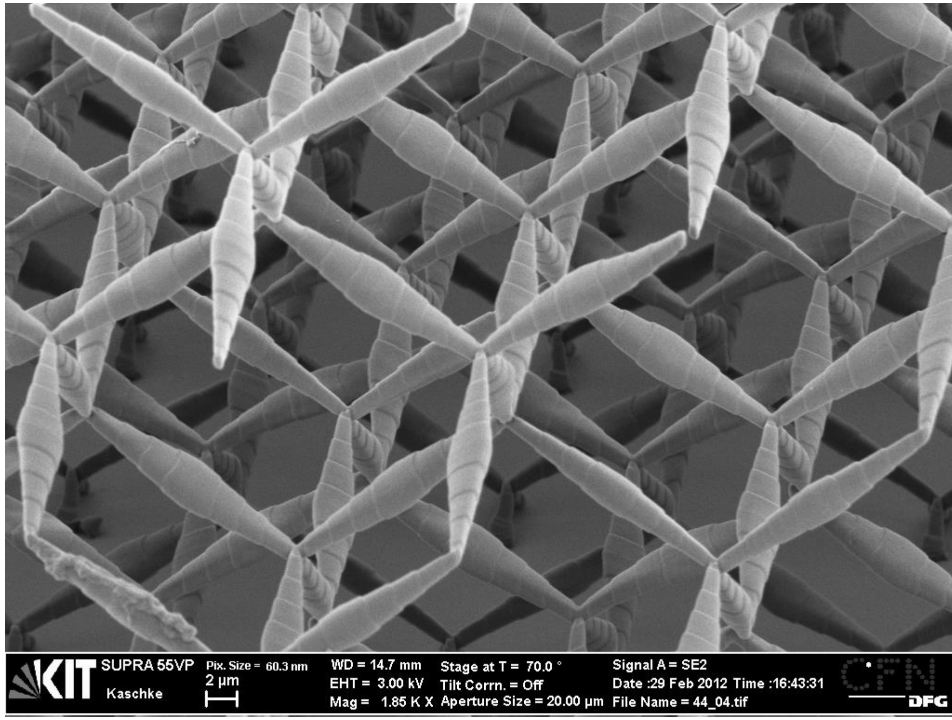
where $\alpha =$ "a constant" is the analog of pressure, and

$$\alpha = \mathrm{Tr}[\mathbf{A}(\mathbf{x})\nabla\mathbf{u}],$$

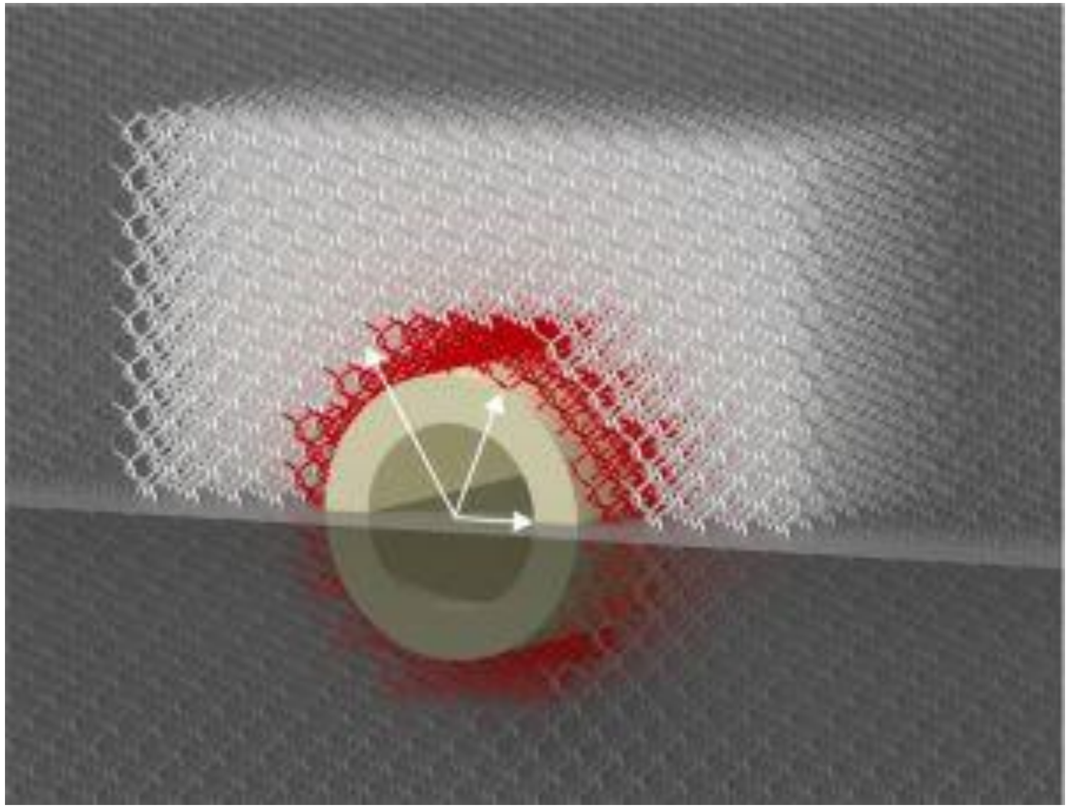
constrains $\nabla\mathbf{u}$. Thus $\mathbf{A}(\mathbf{x})$ is a sort of anisotropic "compressibility"

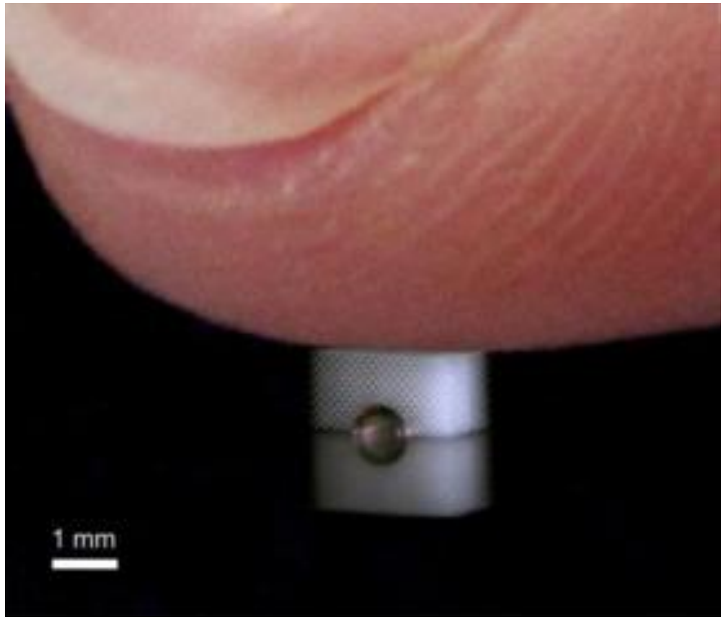
Realization of Kadic et.al. 2012

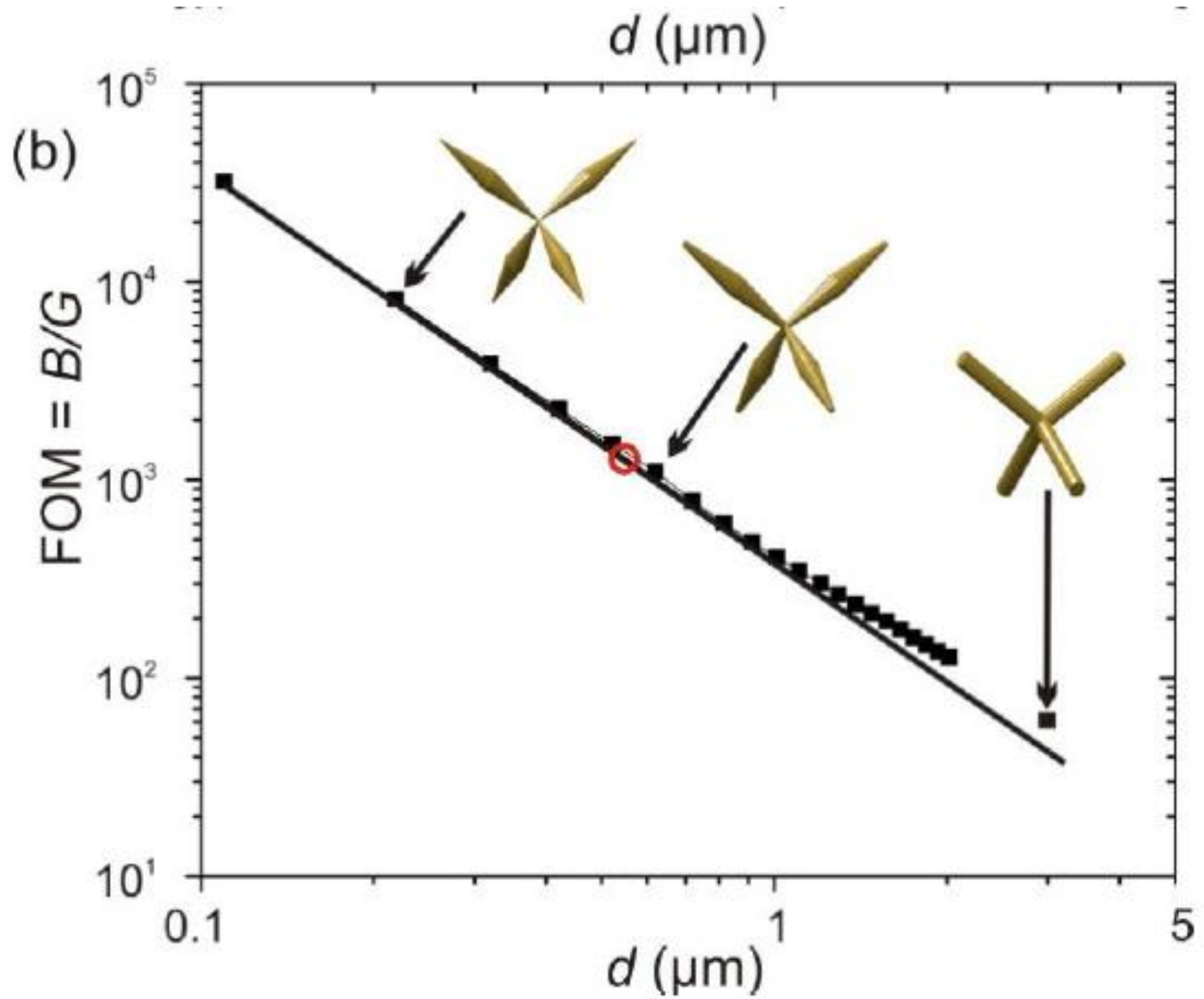




Cloak making an object "unfeelable": Buckmann et. al. (2014)



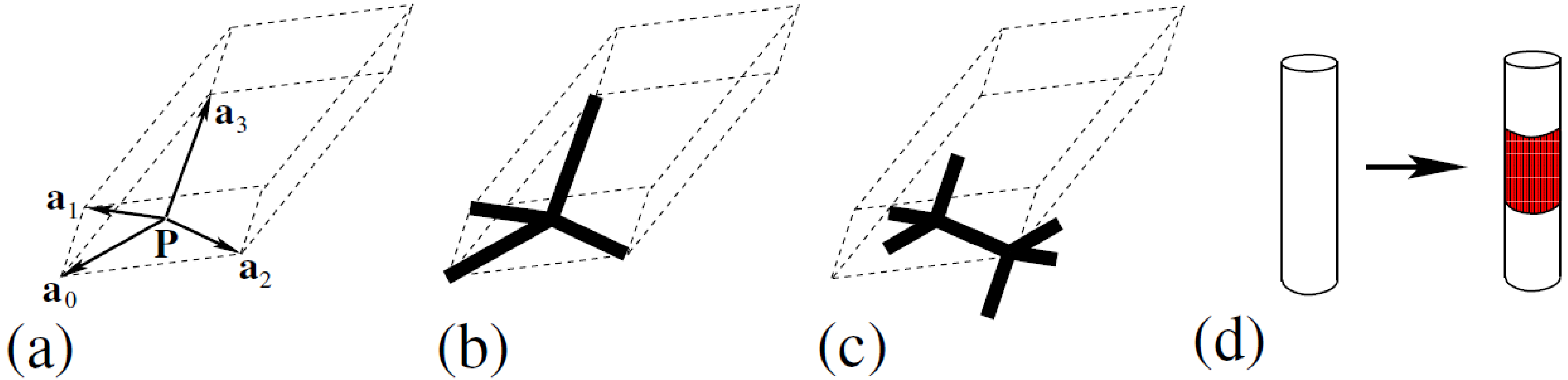


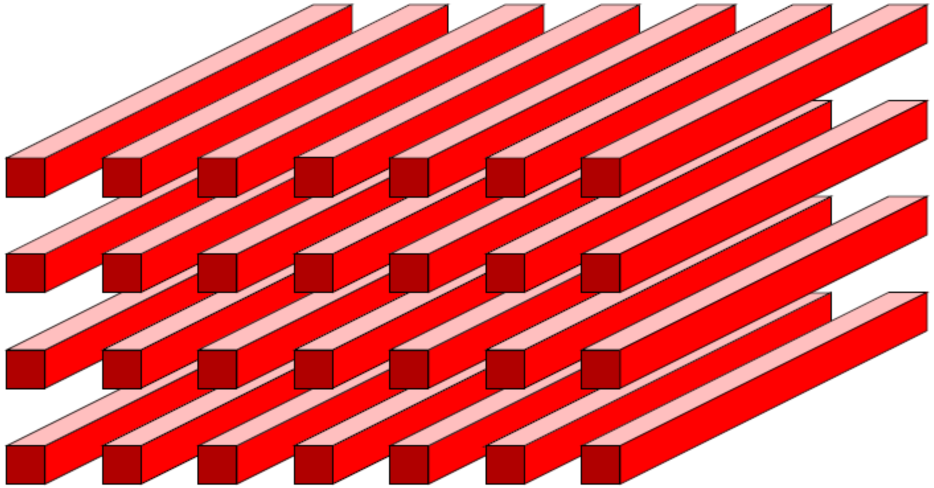


Kadic. et.al 2012

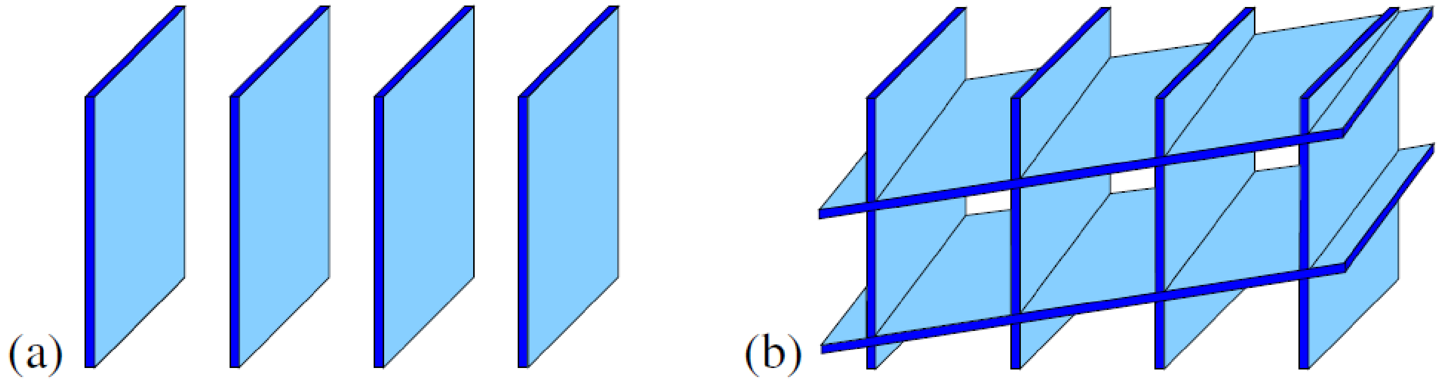
Disadvantage: not only does the shear modulus go to zero as they are made more ideal, but also the bulk modulus goes to zero

Modifying the pentamodes:





Idea of proof: Insert into the material attaining the energy bounds a thin walled structure with sets of parallel walls:



Inside the walls put the appropriate modified pentamode material. Thus we obtain an optimal pentamode attaining the energy bounds.

For elastically isotropic materials one has the Hashin-Shtrikman Bounds

$$\kappa_* \geq f_1\kappa_1 + f_2\kappa_2 - \frac{f_1 f_2(\kappa_1-\kappa_2)^2}{f_2\kappa_1 + f_1\kappa_2 + 4\mu_2/3},$$

$$\mu_* \geq f_1\mu_1 + f_2\mu_2 - \frac{f_1 f_2(\mu_1-\mu_2)^2}{f_2\mu_1 + f_1\mu_2 + \mu_2(9\kappa_2+8\mu_2)/[6(\kappa_2+2\mu_2)]}$$

$$\kappa_* \leq f_1\kappa_1 + f_2\kappa_2 - \frac{f_1 f_2(\kappa_1-\kappa_2)^2}{f_2\kappa_1 + f_1\kappa_2 + 4\mu_1/3},$$

$$\mu_* \leq f_1\mu_1 + f_2\mu_2 - \frac{f_1 f_2(\mu_1-\mu_2)^2}{f_2\mu_1 + f_1\mu_2 + \mu_1(9\kappa_1+8\mu_1)/[6(\kappa_1+2\mu_1)]}$$

The optimal pentamode supporting hydrostatic stress $\boldsymbol{\sigma}^0 = \mathbf{I}$, is a material that for fixed $f_1 = 1 - f_2$ in the limit $\kappa_2, \mu_2 \to 0$ attains the bulk modulus upper bound, yet has zero shear modulus, $\mu_* = 0$.

We can go much further and go a long way to completely characterizing the G-closure of 3d (and 2d) printed materials.

Joint work with Marc Briane and Davit Harutyunyan

Problem:

$\boldsymbol{\sigma}(\mathbf{x})$, $\boldsymbol{\epsilon}(\mathbf{x})$ periodic,

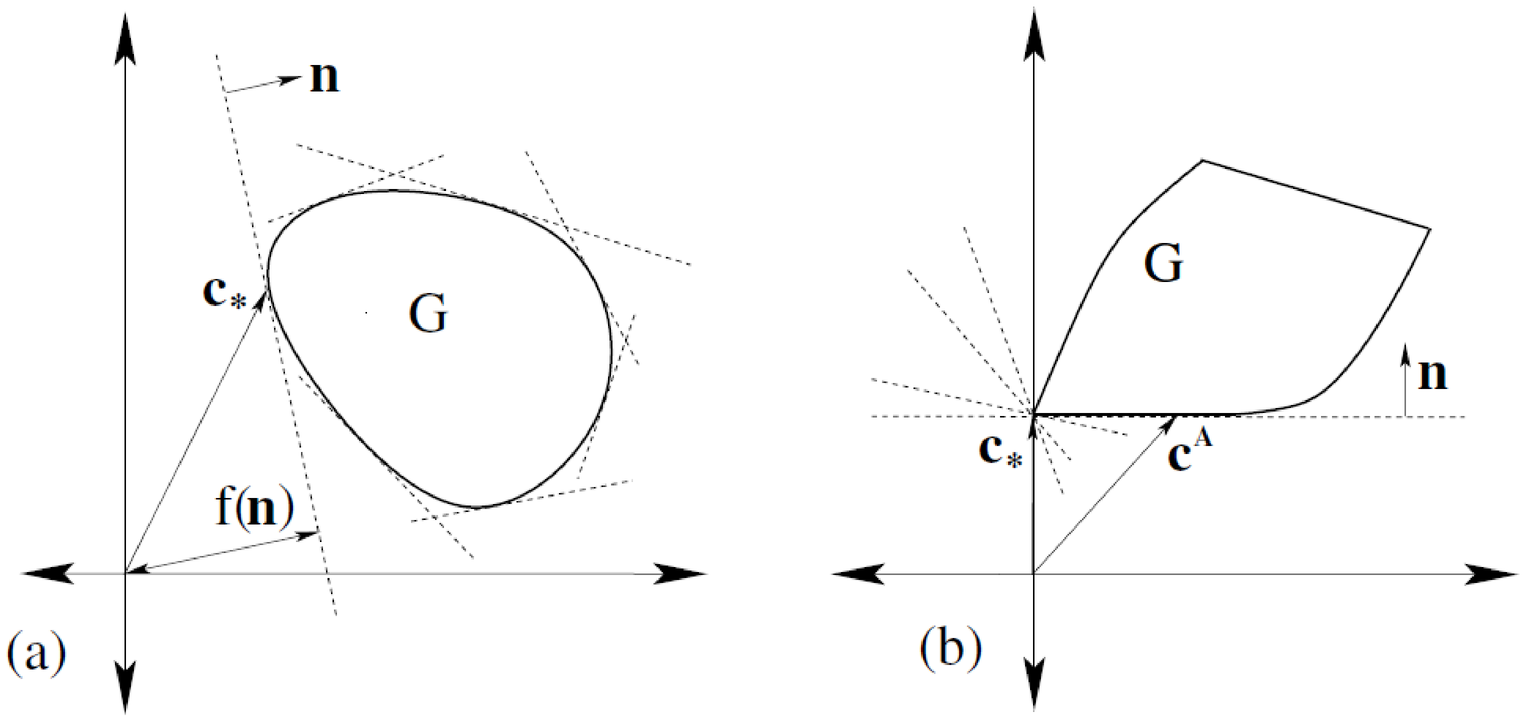
$$\nabla \cdot \boldsymbol{\sigma} = 0, \quad \boldsymbol{\sigma}(\mathbf{x}) = \mathbf{C}(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x}), \quad \boldsymbol{\epsilon} = [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]/2.$$

$$\mathbf{C}(\mathbf{x}) = \mathbf{C_1}\chi(\mathbf{x}) + \mathbf{C}_2(1 - \chi(\mathbf{x})), \quad \boldsymbol{\sigma}^0 = \langle \boldsymbol{\sigma} \rangle, \quad \boldsymbol{\epsilon}^0 = \langle \boldsymbol{\epsilon} \rangle, \quad f = \langle \chi \rangle$$

By linearity $\boldsymbol{\sigma}^0 = \mathbf{C}^* \boldsymbol{\epsilon}^0$. Given f what is the range of values the effective tensor $\mathbf{C}^*$ takes in the limit $\mathbf{C}_2 \to 0$ as the microgeometry varies $\chi(\mathbf{x})$ varies over all possible configurations?

Recall: A convex set G can be characterized by its Legendre transform:

$$f(\mathbf{n}) = \min_{\mathbf{c} \in G} \mathbf{n} \cdot \mathbf{c}.$$



G-closures are not convex sets but can be characterized by their W-transform

$$W_f(\mathbf{N}, \mathbf{N}') = \min_{\mathbf{C}_* \in GU_f} (\mathbf{C}_*, \mathbf{N}) + (\mathbf{C}_*^{-1}, \mathbf{N}'),$$

$$(\mathbf{N}, \mathbf{C}) = N_{ijk\ell} C_{ijk\ell}$$

$$\bigcap_{\substack{\mathbf{N},\mathbf{N}' \geq 0 \\ \mathbf{N}\mathbf{N}' = 0}} \{\mathbf{C} : (\mathbf{C}, \mathbf{N}) + (\mathbf{C}^{-1}, \mathbf{N}') \geq W_f(\mathbf{N}, \mathbf{N}')\} = GU_f.$$

W-transforms generalize the idea of Legendre transforms

$$\mathbf{N} = \sum_{i=1}^{2} \boldsymbol{\epsilon}_i^0 \otimes \boldsymbol{\epsilon}_i^0, \quad \mathbf{N}' = \sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 \otimes \boldsymbol{\sigma}_j^0,$$

Need to know the 7 energy functions

$$W_f^0(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\sigma}_6^0) = \min_{\mathrm{C}_* \in GU_f} \sum_{j=1}^{6} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1} \boldsymbol{\sigma}_j^0,$$

$$W_f^1(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \epsilon_1^0) = \min_{\mathrm{C}_* \in GU_f} \left[\epsilon_1^0 : \mathrm{C}_* \epsilon_1^0 + \sum_{j=1}^{5} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1} \boldsymbol{\sigma}_j^0 \right],$$

$$W_f^2(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \epsilon_1^0, \epsilon_2^0) = \min_{\mathrm{C}_* \in GU_f} \left[\sum_{i=1}^{2} \epsilon_i^0 : \mathrm{C}_* \epsilon_i^0 + \sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1} \boldsymbol{\sigma}_j^0 \right],$$

$$W_f^3(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0) = \min_{\mathrm{C}_* \in GU_f} \left[\sum_{i=1}^{3} \epsilon_i^0 : \mathrm{C}_* \epsilon_i^0 + \sum_{j=1}^{3} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1} \boldsymbol{\sigma}_j^0 \right],$$

$$W_f^4(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0) = \min_{\mathrm{C}_* \in GU_f} \left[\sum_{i=1}^{4} \epsilon_i^0 : \mathrm{C}_* \epsilon_i^0 + \sum_{j=1}^{2} \boldsymbol{\sigma}_j^0 : \mathrm{C}_*^{-1} \boldsymbol{\sigma}_j^0 \right],$$

$$W_f^5(\boldsymbol{\sigma}_1^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0) = \min_{\mathrm{C}_* \in GU_f} \left[\left(\sum_{i=1}^{5} \epsilon_i^0 : \mathrm{C}_* \epsilon_i^0 \right) + \boldsymbol{\sigma}_1^0 : \mathrm{C}_*^{-1} \boldsymbol{\sigma}_1^0 \right],$$

$$W_f^6(\epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0, \epsilon_6^0) = \min_{\mathrm{C}_* \in GU_f} \sum_{i=1}^{6} \epsilon_i^0 : \mathrm{C}_* \epsilon_i^0.$$

Orthogonality conditions

$$(\boldsymbol{\epsilon}_i^0, \boldsymbol{\sigma}_j^0) = 0, \qquad (\boldsymbol{\epsilon}_i^0, \boldsymbol{\epsilon}_k^0) = 0, \qquad (\boldsymbol{\sigma}_j^0, \boldsymbol{\sigma}_\ell^0) = 0$$

for all i, j, k, ℓ with $i \neq j,\ i \neq k,\ j \neq \ell$.

Result of Avellaneda (1987): If $\mathbf{C}_1 \geq \mathbf{C}_2$ then

$$W_f^0(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\sigma}_6^0) = \min_{\mathbf{C}_* \in GU_f} \sum_{j=1}^{6} \boldsymbol{\sigma}_j^0 : \mathbf{C}_*^{-1} \boldsymbol{\sigma}_j^0,$$

$$W_f^6(\boldsymbol{\epsilon}_1^0, \boldsymbol{\epsilon}_2^0, \boldsymbol{\epsilon}_3^0, \boldsymbol{\epsilon}_4^0, \boldsymbol{\epsilon}_5^0, \boldsymbol{\epsilon}_6^0) = \min_{\mathbf{C}_* \in GU_f} \sum_{i=1}^{6} \boldsymbol{\epsilon}_i^0 : \mathbf{C}_* \boldsymbol{\epsilon}_i^0.$$

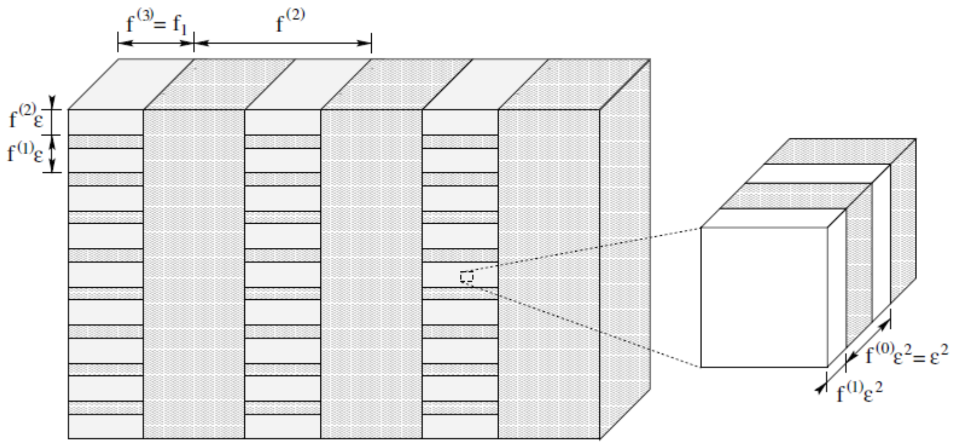
can be computed

They are attained by sequentially layered laminates, and we call the material which attains the minimum in

$$W_f^0(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\sigma}_6^0) = \min_{\mathbf{C}_* \in GU_f} \sum_{j=1}^{6} \boldsymbol{\sigma}_j^0 : \mathbf{C}_*^{-1} \boldsymbol{\sigma}_j^0,$$

the Avellaneda material, with elasticity tensor

$$\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \boldsymbol{\sigma}_6^0)$$



Maxwell (1873)

Obvious bounds:

$$
\begin{aligned}
\sum_{j=1}^{5} \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^1(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \epsilon_1^0), \\
\sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^2(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \epsilon_1^0, \epsilon_2^0), \\
\sum_{j=1}^{3} \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^3(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0), \\
\sum_{j=1}^{2} \boldsymbol{\sigma}_j^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0 &\leq W_f^4(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0), \\
\boldsymbol{\sigma}_1^0 : [\mathbf{C}_f^A(\boldsymbol{\sigma}_1^0, 0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_1^0 &\leq W_f^5(\boldsymbol{\sigma}_1^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0), \\
0 &\leq W_f^6(\epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0, \epsilon_6^0).
\end{aligned}
$$

Main result: in many cases these bounds are sharp

Theorem (GWM, Briane, Harutyunyan):

$$\lim_{\delta\to 0} W_f^3(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0) = \sum_{j=1}^{3} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0,$$

$$\lim_{\delta\to 0} W_f^4(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0) = \sum_{j=1}^{2} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0,$$

$$\lim_{\delta\to 0} W_f^5(\boldsymbol{\sigma}_1^0, \epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0) = \boldsymbol{\sigma}_1^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, 0, 0, 0, 0, 0)]^{-1} \boldsymbol{\sigma}_1^0,$$

$$\lim_{\delta\to 0} W_f^6(\epsilon_1^0, \epsilon_2^0, \epsilon_3^0, \epsilon_4^0, \epsilon_5^0, \epsilon_6^0) = 0.$$

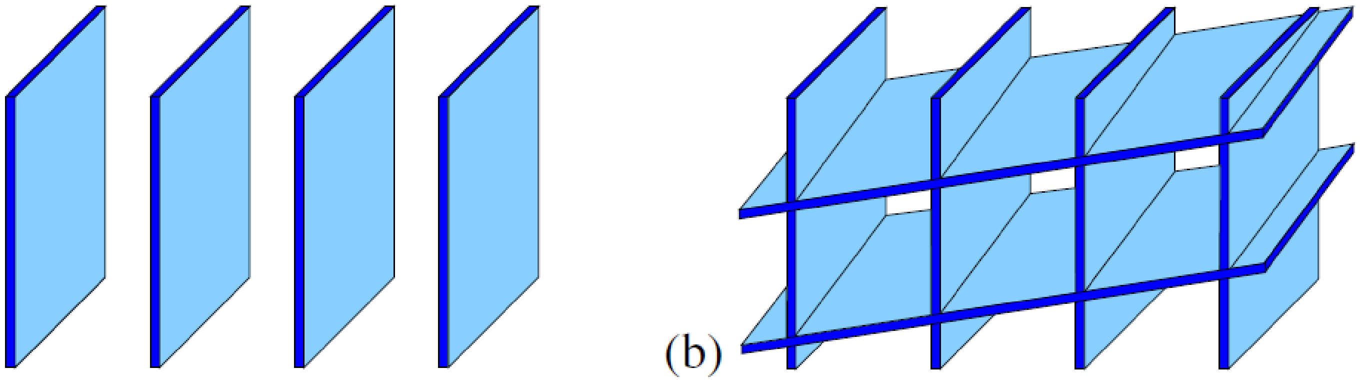
When ϵ_1^0 has one zero eigenvalue, and the other eigenvalues of opposite signs,

$$W_f^1(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, \epsilon_1^0) = \sum_{j=1}^{5} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \boldsymbol{\sigma}_5^0, 0)]^{-1} \boldsymbol{\sigma}_j^0$$

When $\det(\epsilon_1^0 + t\epsilon_2^0) = 0$ has at least two roots and $\epsilon(t) = \epsilon_1^0 + t\epsilon_2^0$ is never positive or negative definite

$$W_f^2(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, \epsilon_1^0, \epsilon_2^0) = \sum_{j=1}^{4} \boldsymbol{\sigma}_j^0 : [\mathrm{C}_f^A(\boldsymbol{\sigma}_1^0, \boldsymbol{\sigma}_2^0, \boldsymbol{\sigma}_3^0, \boldsymbol{\sigma}_4^0, 0, 0)]^{-1} \boldsymbol{\sigma}_j^0$$

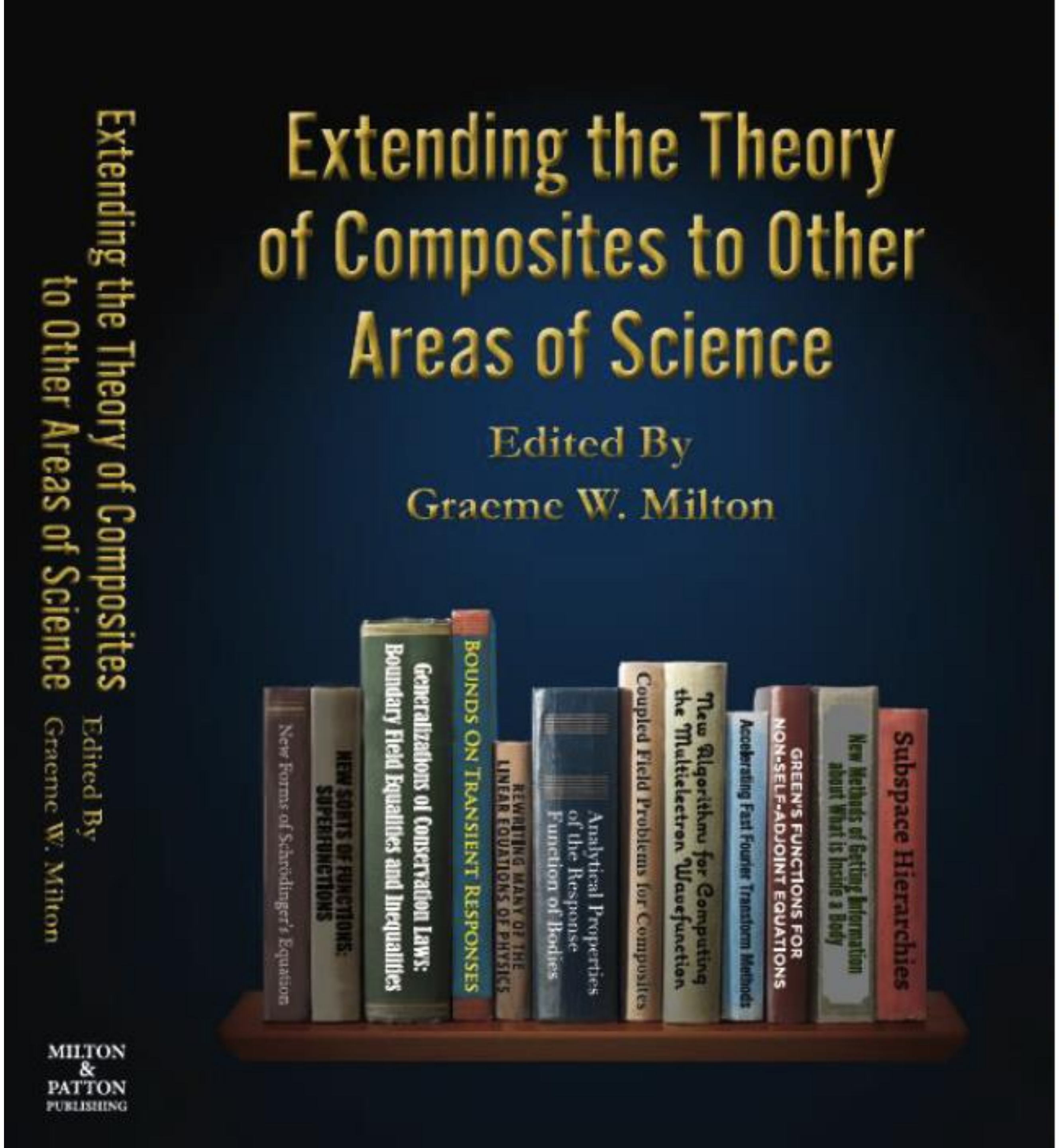
Idea of proof: Insert into the Avellaneda material a thin walled structure with sets of parallel walls:



(a) (b)

Inside the walls put the appropriate multimode material

Thank You!



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