A panorama of problems in material science and networks where Stieltjes and Herglotz functions emerge.

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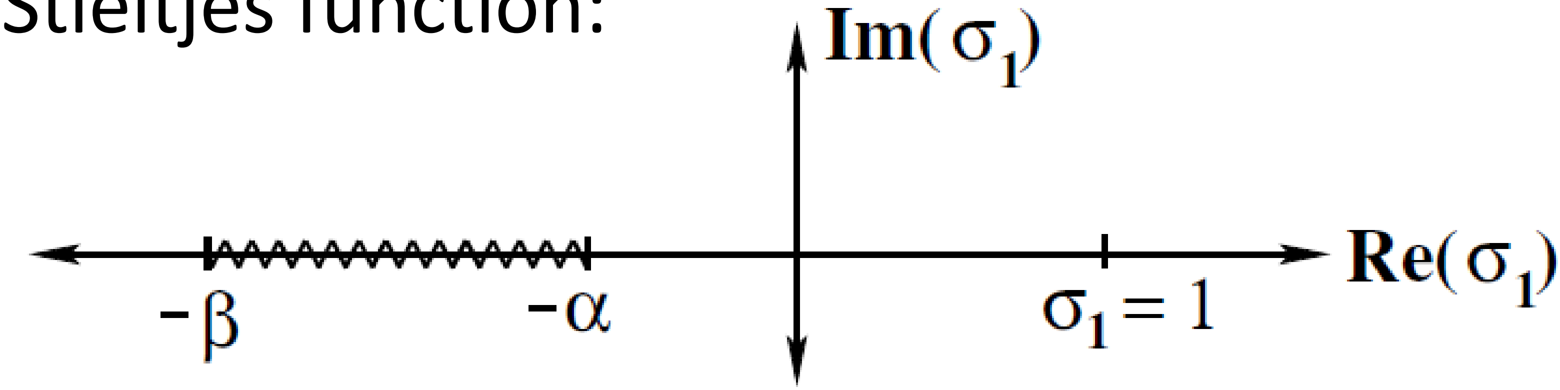
Properties of the effective conductivity

The effective conductivity σ_* is an analytic function of the component conductivities σ_1 and σ_2

With $\sigma_2 = 1$, $\sigma_*(\sigma_1)$ has the properties of a Stieltjes function:



Bergman 1978 (pioneer, but faulty arguments)

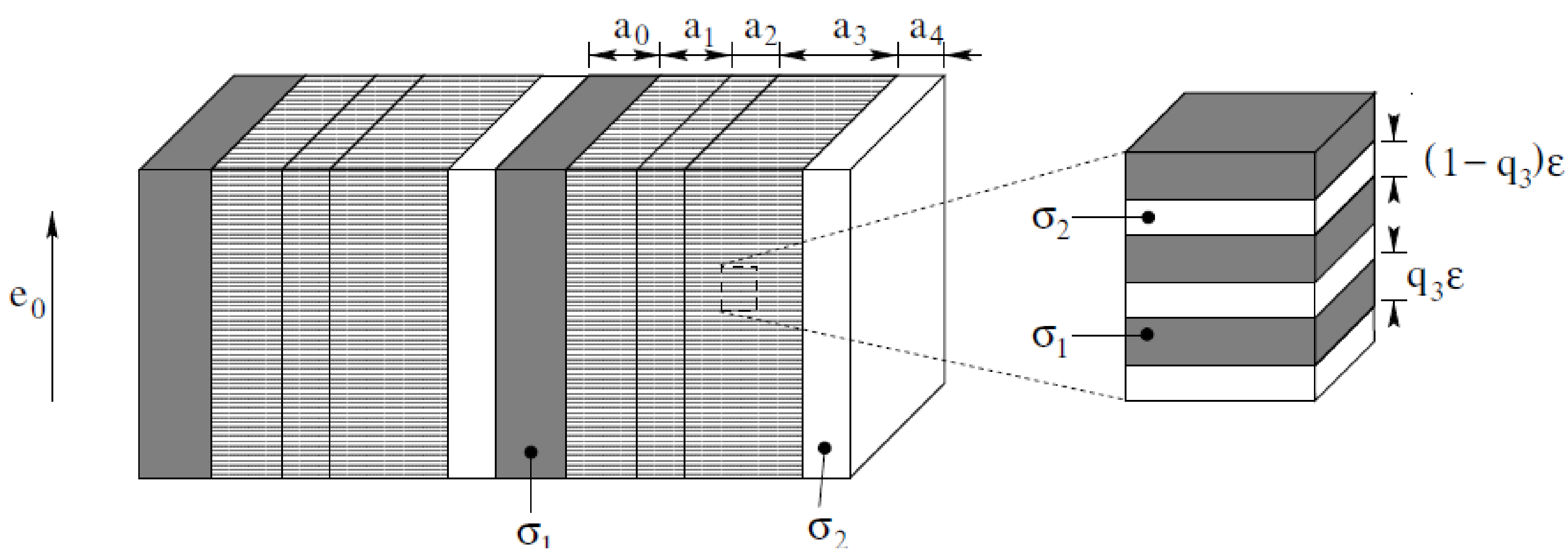
Milton 1981 (limit of resistor networks)

Golden and Papanicolaou 1983 (rigorous proof)

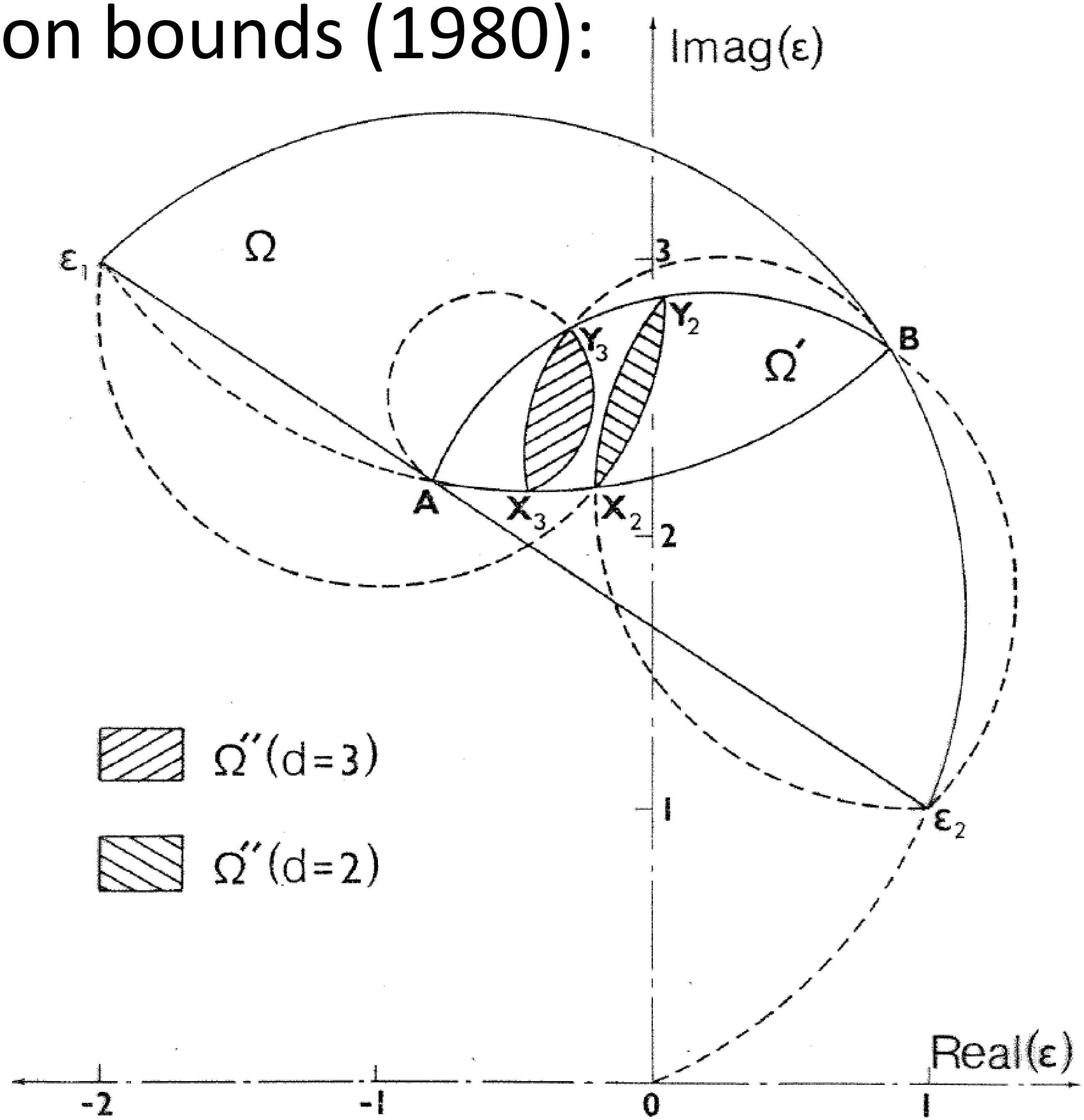
Representation:

$$\sigma_e(\sigma_1, \sigma_2) = \sum_{\alpha=0}^{m+1} a_\alpha \sigma_*^{(\alpha)} = \sum_{\alpha=0}^{m+1} \frac{a_\alpha}{q_\alpha/\sigma_1 + (1 - q_\alpha)/\sigma_2}.$$

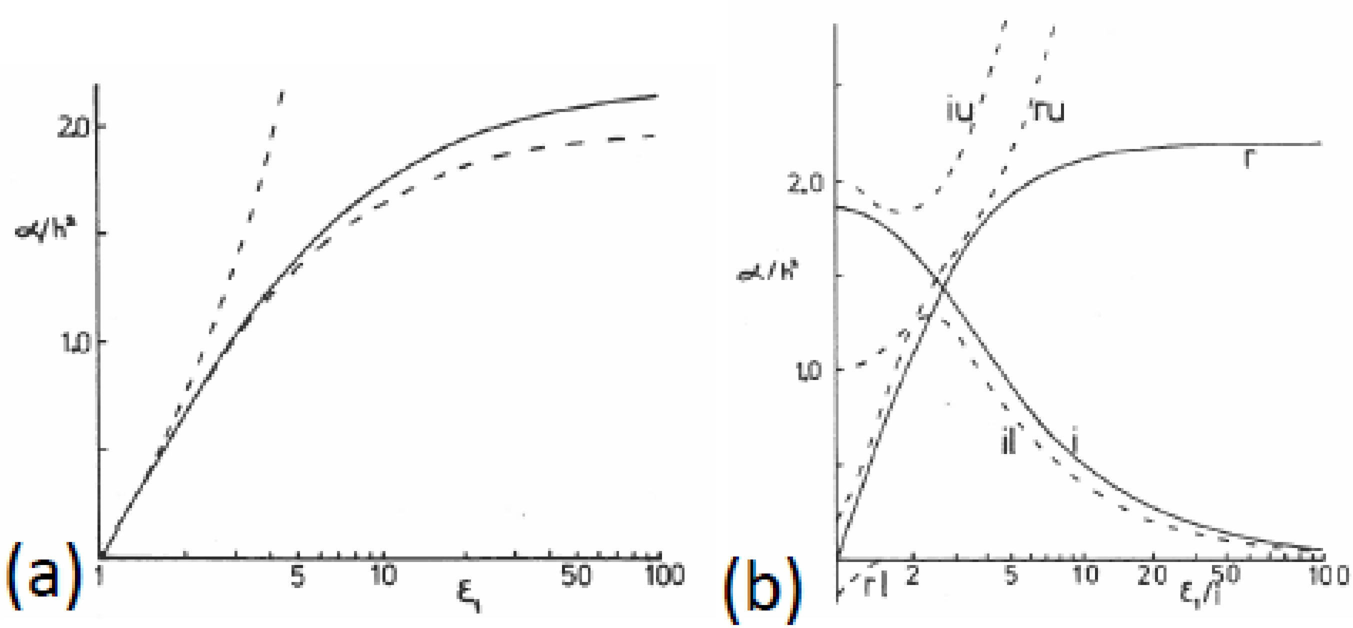
Realization:



Bergman-Milton bounds (1980):



Associated Bounds on Real & Complex Polarizabilities of an inclusion (Milton, McKenzie, McPhedran, 1981)



Solid Curves are Polarizability of a Square

Rediscovered by Miller et. al. 2014

For two-dimensional composites it is convenient to introduce

$$\mathbf{S}_*(s) = [\mathbf{I} - \boldsymbol{\sigma}_*(1-1/s, 1)]^{-1} \qquad \mathbf{R}_\perp = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Then

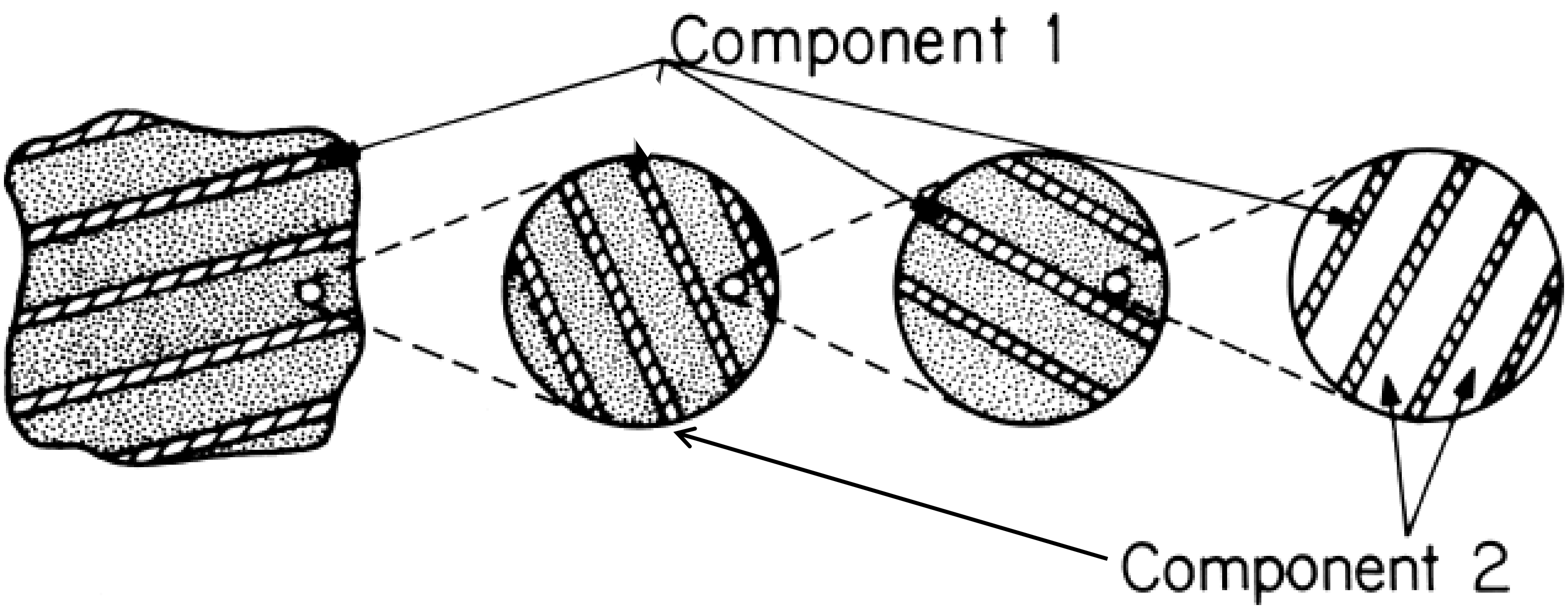
$$\mathbf{S}_*(s) + \mathbf{R}_\perp \mathbf{S}(1-s)\mathbf{R}_\perp^T = \mathbf{I}, \qquad \mathbf{S}_*(\overline{s}) = \overline{\mathbf{S}_*(s)},$$
$$\mathbf{S}_*(s) \geq \mathbf{I} \quad \text{for all real } s \geq 1, \qquad \mathbf{S}_*(s) \leq 0 \quad \text{for all real } s \leq 0,$$
$$\mathrm{Im}[\mathbf{S}_*(s)] \geq 0 \quad \text{whenever} \quad \mathrm{Im}(s) \geq 0.$$

Representation formula:

$$\mathbf{S}_*(s) = s(1 + \mathrm{Tr}\,\mathbf{A})\mathbf{I} - \mathbf{A} + \sum_{i=1}^{m} \frac{\mathbf{S}_i}{s_i - s} + \frac{\mathbf{R}_\perp^T \mathbf{S}_i \mathbf{R}_\perp}{(1-s_i) - s},$$

$$1 > s_i > 0, \quad \mathbf{S}_i \geq 0, \quad \mathbf{A} \geq \sum_{i=1}^{m} \frac{\mathbf{S}_i}{s_i} + \frac{\mathbf{R}_\perp^T \mathbf{S}_i \mathbf{R}_\perp}{1 - s_i}.$$

Gives a complete characterization of the analytic properties:
Sequentially layered laminates are a representative class of materials



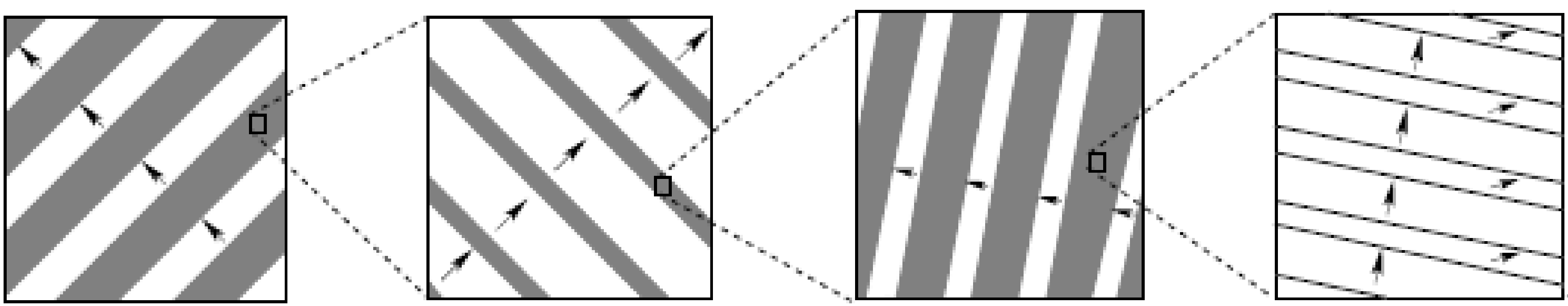
Alternate layering the two components

Also a realization for the matrix valued function

$$\sigma^*(\sigma_{\mathbf{0}})$$

as a function of the matrix $\sigma_{\mathbf{0}}$ for two-dimensional polycrystals.

Representative structures:



with Karen Clark (1994),

Abstract Theory of Composites

Hilbert Space $\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J}$

Operator $\mathbf{L} : \mathcal{H} \to \mathcal{H}$

Given $\mathbf{E}_0 \in \mathcal{U}$

Solve $\mathbf{J}_0 + \mathbf{J} = \mathbf{L}(\mathbf{E}_0 + \mathbf{E})$

With $\mathbf{J}_0 \in \mathcal{U}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{E} \in \mathcal{E},$

Then $\mathbf{J}_0 = \mathbf{L}_* \mathbf{E}_0$ defines $\mathbf{L}_* : \mathcal{U} \to \mathcal{U}$

Example: Conducting Composites

$\mathcal{H}$ - Periodic fields that are square integrable over the unit cell

$\mathcal{U}$ - Constant vector fields

$\mathcal{E}$ - Gradients of periodic potentials

$\mathcal{J}$ - Fields with zero divergence and zero average value

$\mathbf{E}_0 + \mathbf{E}(\mathbf{x})$ - Total electric field

$\mathbf{J}_0 + \mathbf{J}(\mathbf{x})$ - Total current field

$\mathbf{L} = \boldsymbol{\sigma}(\mathbf{x})$ - Local conductivity

$\mathbf{L}_* = \boldsymbol{\sigma}_*$ - Effective conductivity

For multicomponent composites Golden and Papanicolaou used the integral representation with the Szego kernel but it wasn't useful for getting bounds

Now $\mathcal{H} = \mathcal{U} \oplus \mathcal{E} \oplus \mathcal{J} = \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,$

χ_i projects onto the space of fields $\mathcal{P}_i$ that are non-zero only inside phase i.

Γ_1 projects onto the space $\mathcal{E}$.

Γ_0 projects onto the space $\mathcal{U}$.

$\sigma = \sum_{i=1}^{n} \lambda_i \chi_i$

The field equation recursion method (1987): obtain a canonical representation of the operators

$$\Gamma_1 = \begin{bmatrix} 0 & & & & & \mathbf{0} \\ & \mathbf{U}'^{(1)} & \hat{\mathbf{U}}'^{(1)}\hat{\mathbf{V}}'^{(1)} & & & \\ & \hat{\mathbf{U}}'^{(1)}\hat{\mathbf{V}}'^{(1)} & \mathbf{V}'^{(1)} & & & \\ & & & \mathbf{U}'^{(2)} & \hat{\mathbf{U}}'^{(2)}\hat{\mathbf{V}}'^{(2)} & \\ & & & \hat{\mathbf{U}}'^{(2)}\hat{\mathbf{V}}'^{(2)} & \mathbf{V}'^{(2)} & \\ & \mathbf{0} & & & & \ddots \end{bmatrix} \quad \chi_a = \begin{bmatrix} \mathbf{W}_a^{(0)} & \hat{\mathbf{Y}}_a^{(1)} & & & \mathbf{0} \\ \hat{\mathbf{Y}}_a^{(1)T} & \hat{\mathbf{Y}}_a^{(1)T}\tilde{\mathbf{W}}_a^{(0)}\hat{\mathbf{Y}}_a^{(1)} & & & \\ & & \mathbf{W}_a^{(1)} & \hat{\mathbf{Y}}_a^{(2)} & \\ & \mathbf{0} & \hat{\mathbf{Y}}_a^{(2)T} & \hat{\mathbf{Y}}_a^{(2)T}\tilde{\mathbf{W}}_a^{(1)}\hat{\mathbf{Y}}_a^{(2)} & \\ & & & & \ddots \end{bmatrix}$$

$$\mathbf{U}'^{(j)} + \mathbf{V}'^{(j)} = \mathbf{I}^{(j)}, \quad \sum_{a=1}^{n} \mathbf{W}_a^{(j)} = \mathbf{I}^{(j)},$$

$$\tilde{\mathbf{W}}_a^{(j)} = [\mathbf{W}_a^{(j)}]^{-1}, \qquad \hat{\mathbf{Y}}^{(j)}_{a\lambda,b\eta} = \delta_{ab}\mathbf{W}^{(j)}_{a,\lambda,\eta} - \mathbf{W}^{(j)}_{a,\lambda,\tau}\mathbf{W}^{(j)}_{b,\tau,\eta}$$

And a continued fraction representation for the effective tensor:

$$L_* = \sum_{i=1}^{n} W_i \lambda_i - \sum_{a,b \neq h} (\lambda_a - \lambda_h) I_a [\Upsilon + V^{-1} Y_* V^{-1}]^{-1} I_b^T (\lambda_b - \lambda_h),$$

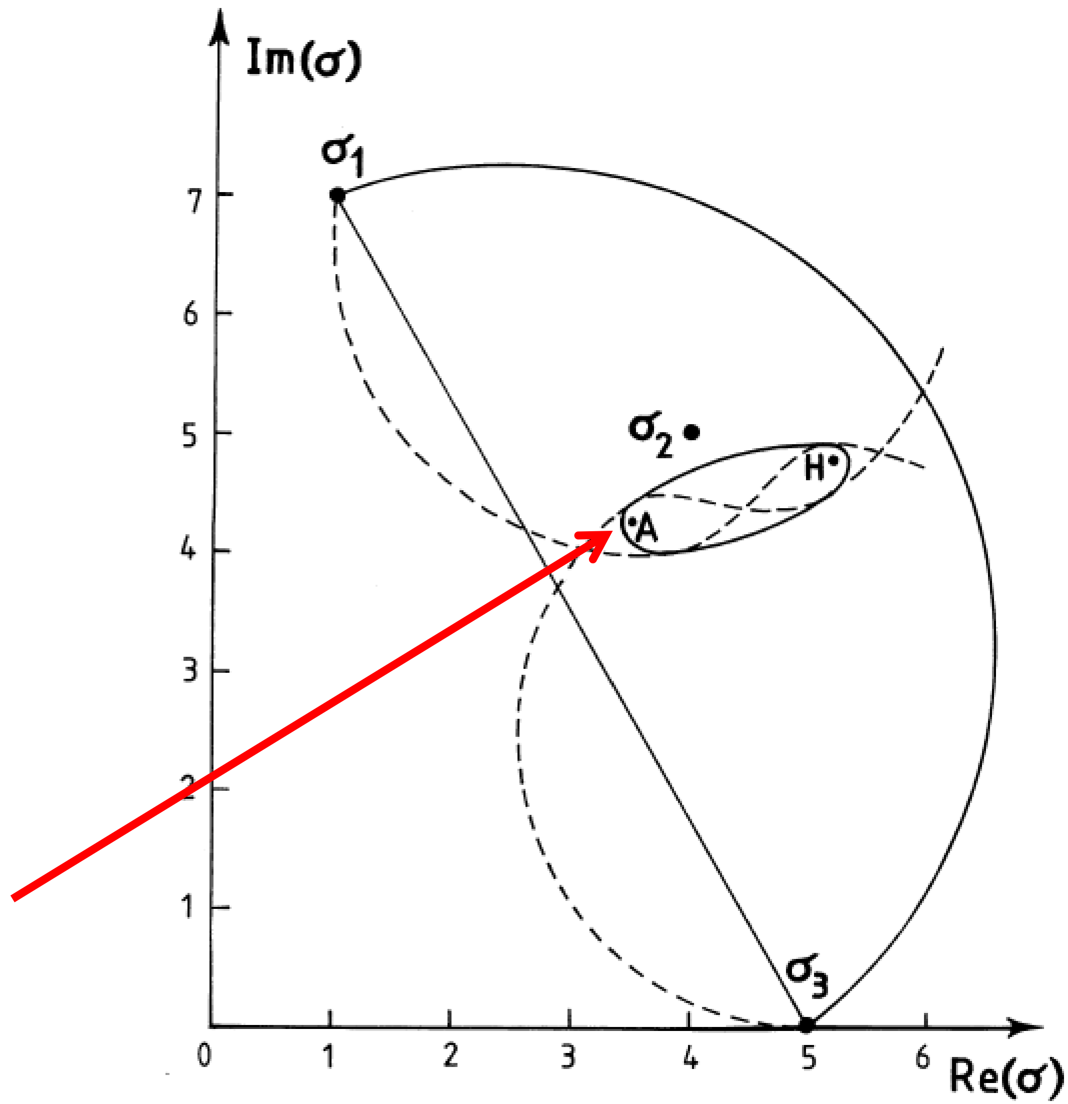
$$\{\Upsilon\}_{aj,bk} = \{\lambda_h W_h^{-1} + \delta_{ab} \lambda_b W_b^{-1}\}_{jk}.$$

$$Y_* = V G_1^{-1} L_*^{(1)} G_1^{-1} V.$$

$$U^{(1)} = G_1 - G_1 V^{-1} G_1$$

$$L_* = \sum_{i=1}^{n} W_i \lambda_i - \sum_{a,b \neq h} (\lambda_a - \lambda_h) I_a [\Upsilon + G_1^{-1} L_*^{(1)} G_1^{-1}]^{-1} I_b^T (\lambda_b - \lambda_h),$$

The optimal bounds involve parts of Figure 8's not just circles!



Setting of a “Y”-Problem

$\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{H}$, $\quad$ $\mathbf{L}$ maps $\mathcal{H}$ to $\mathcal{H}$,

Given an element $\mathbf{E}_1$ of $\mathcal{V}$ find

$$\mathbf{E}_2, \mathbf{J}_2 \in \mathcal{H}, \quad \mathbf{J}_1 \in \mathcal{V}, \quad \text{such that}$$

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \in \mathcal{E}, \quad \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \in \mathcal{J}, \quad \mathbf{J}_2 = \mathbf{L}\mathbf{E}_2.$$

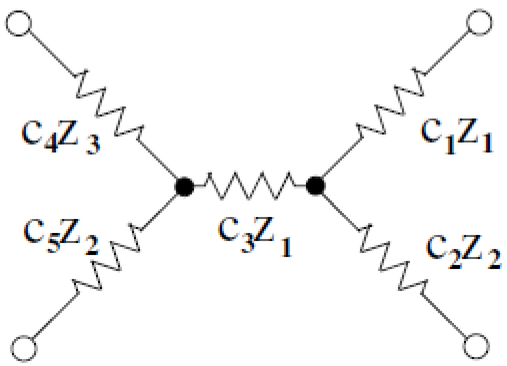
Then

$$\mathbf{J}_1 = -\mathbf{Y}_*\mathbf{E}_1$$

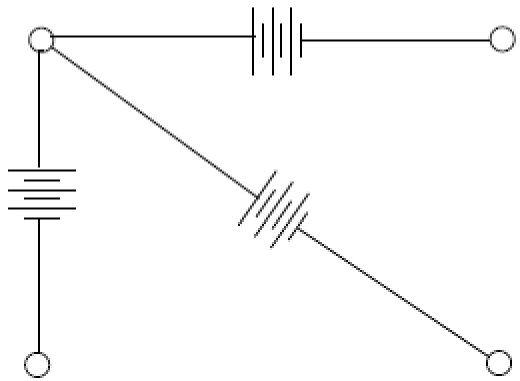
defines $\mathbf{Y}_*$, which maps $\mathcal{V}$ to $\mathcal{V}$, or to a subspace of $\mathcal{V}$.

A canonical example of a "Y-problem"

We should consider a resistor network in conjunction with its batteries



(a)

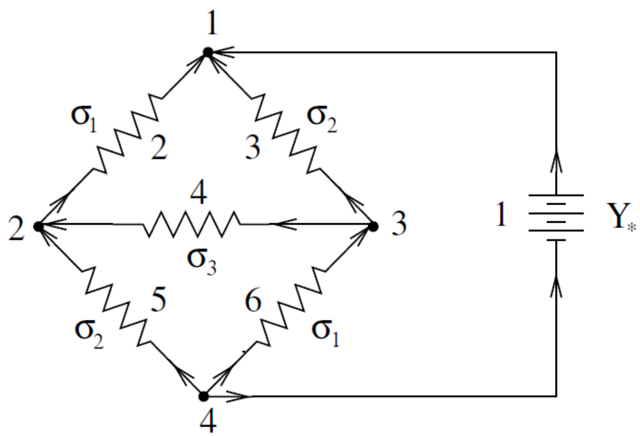


(b)

Space $\mathcal{H}$ Space $\mathcal{V}$

Combined Space $\mathcal{K} = \mathcal{H} \oplus \mathcal{V}$

Incidence Matrices:



$$M = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

$$\begin{aligned} M_{ij} &= +1 \text{ if the arrow of bond } i \text{ points towards node } j, \\ &= -1 \text{ if the arrow of bond } i \text{ points away from node } j, \\ &= 0 \text{ if bond } i \text{ and node } j \text{ are not connected.} \end{aligned}$$

Two natural subspaces:

$\mathcal{J}$ the null space of M^T (current vectors)

$\mathcal{E}$ the range of M (potential drops)

These are orthogonal spaces and $\mathcal{K} = \mathcal{E} \oplus \mathcal{J}$

Other spaces:

Divide the bonds in $\mathcal{H}$ into n groups (representing the different impedances).

Define $\mathcal{P}_i$ as the space of vectors P with elements P_j that are zero if bond j is not in group i.

The projection Λ_i onto the space $\mathcal{P}_i$ is diagonal and has elements

$$\begin{aligned}\{\Lambda_i\}_{jk} &= 1 \text{ if } j = k \text{ and bond } j \text{ is in group } i, \\ &= 0 \text{ otherwise.}\end{aligned}$$

Thus $\mathcal{K} = \mathcal{E} \oplus \mathcal{J} = \mathcal{V} \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \oplus \cdots \oplus \mathcal{P}_n,$

This is an orthogonal subspace collection Y(n)

The next big breakthrough: minimization variational principles for quasistatics in lossy media-made by Cherkaev and Gibiansky (1994)

Beauty of the method: easily applied to multiphase or polycrystalline composites recovers existing bounds on the complex dielectric constant + new ones

Consider the electrostatic equation

$$\nabla \cdot \varepsilon \nabla V = 0,$$

or equivalently

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{E} = -\nabla V, \quad \nabla \cdot \mathbf{D} = 0,$$

in a body Ω with dielectric constant $\varepsilon(\mathbf{x})$

Assume complex dielectric constant

$$\varepsilon''(\mathbf{x}) > 0$$

Rewrite constitutive law: $\mathbf{D} = \varepsilon \mathbf{E}$ as

$$\begin{pmatrix} \mathbf{D}'' \\ \mathbf{D}' \end{pmatrix} = \begin{pmatrix} \varepsilon'' & \varepsilon' \\ \varepsilon' & -\varepsilon'' \end{pmatrix} \begin{pmatrix} \mathbf{E}' \\ \mathbf{E}'' \end{pmatrix}$$

Partial Legendre transforms convert saddle-shaped quadratic functions into convex quadratic functions.

Equivalent to rewriting constitutive law:

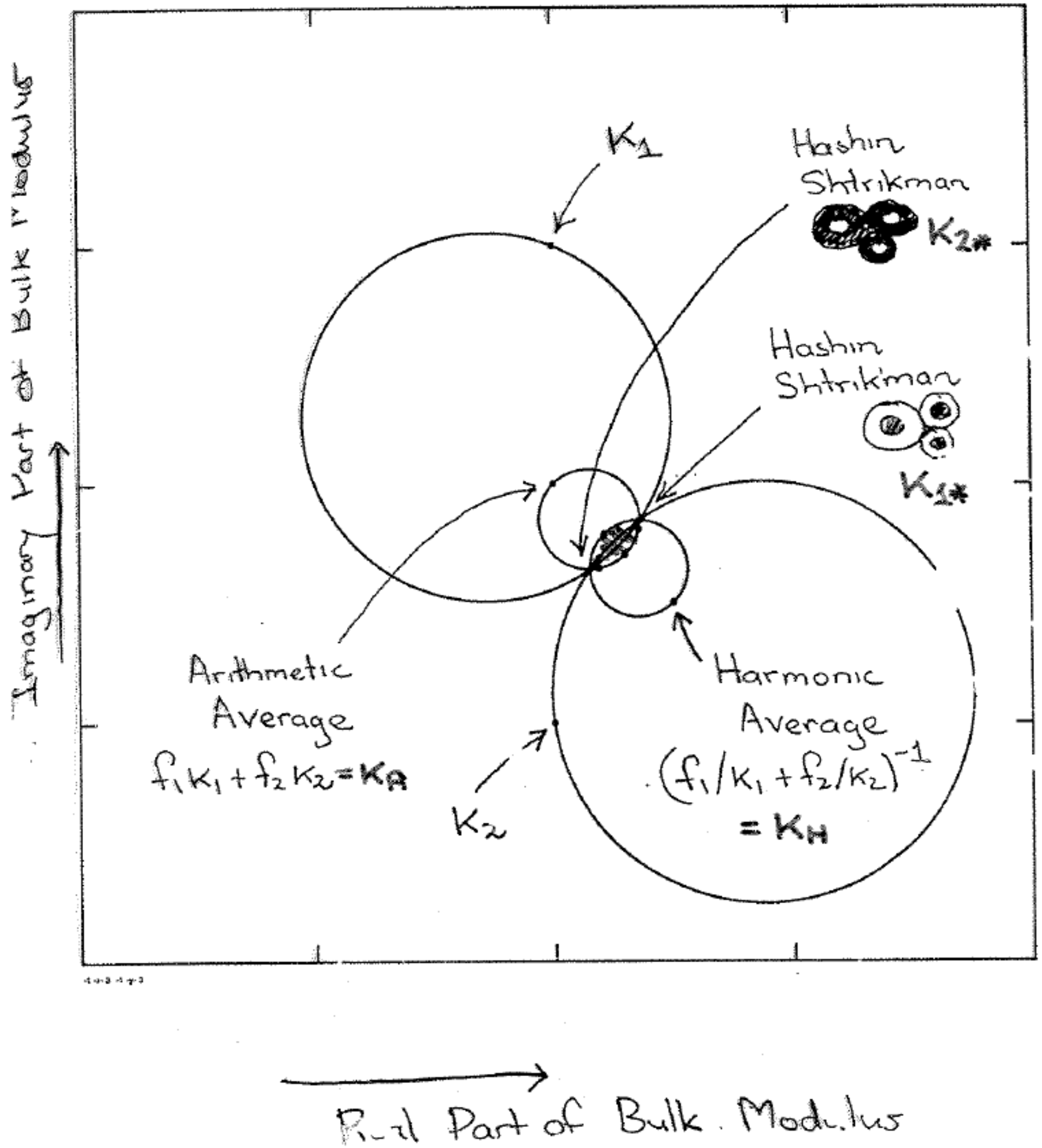
$$\begin{pmatrix} \mathbf{D}'' \\ \mathbf{E}'' \end{pmatrix} = \boldsymbol{\mathcal{E}} \begin{pmatrix} \mathbf{E}' \\ -\mathbf{D}' \end{pmatrix}$$

$$\boldsymbol{\mathcal{E}} = \begin{pmatrix} \varepsilon'' + (\varepsilon')^2/(\varepsilon'') & \varepsilon'/\varepsilon'' \\ \varepsilon'/\varepsilon'' & 1/\varepsilon'' \end{pmatrix}$$

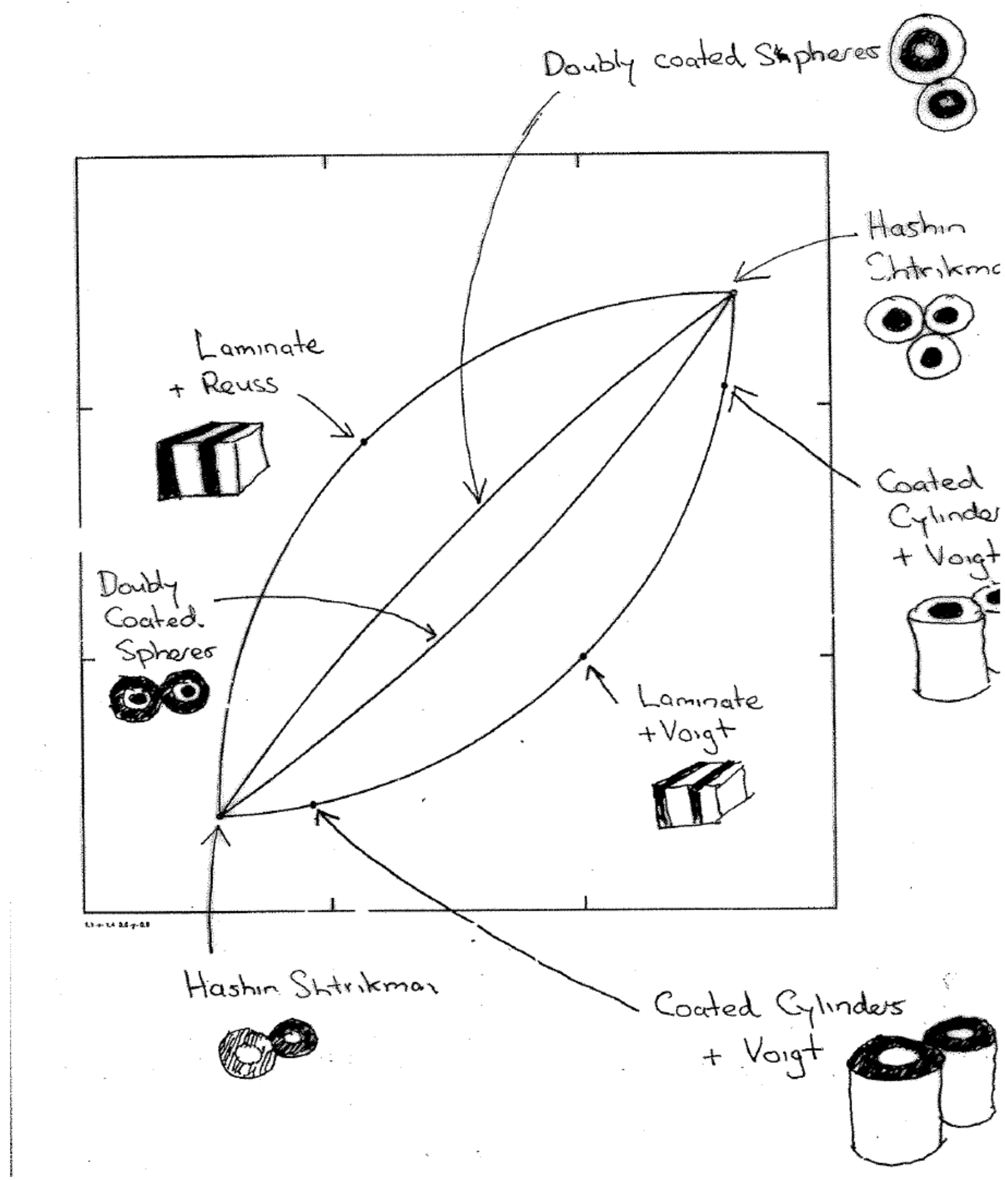
Positive Definite!

Gibiansky-Milton bounds

Bounds using Variational Principles go hand in hand with bounds using Analytic Properties



$$K_{1*} = K_1 + \frac{f_2}{\frac{1}{(K_2-K_1)} + \frac{3f_1}{3K_1+4\mu_1}}, \quad K_{2*} = K_2 + \frac{f_1}{\frac{1}{K_1-K_2} + \frac{3f_2}{3K_2+4\mu_2}}$$



Complex shear modulus bounds also obtained with us and Berryman

New Methods for Imaging

Key idea: making a direct link between Dirichlet-to-Neumann maps for bodies and effective tensors for composites.

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Operator $\mathbf{L} : \mathcal{H} \rightarrow \mathcal{H}$

Given $\mathbf{E}_0 \in \mathcal{U}$

Solve $\mathbf{J}_0 + \mathbf{J} = \mathbf{L}(\mathbf{E}_0 + \mathbf{E})$

With $\mathbf{J}_0 \in \mathcal{U}, \quad \mathbf{J} \in \mathcal{J}, \quad \mathbf{E} \in \mathcal{E},$

Then $\mathbf{J}_0 = \mathbf{L}_* \mathbf{E}_0$ defines $\mathbf{L}_* : \mathcal{U} \rightarrow \mathcal{U}$

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$\mathbf{J}_0 + \mathbf{J}(\mathbf{x})$ - Total current field

$\mathbf{L} = \boldsymbol{\sigma}(\mathbf{x})$ - Local conductivity

$\mathbf{L}_* = \boldsymbol{\sigma}_*$ - Effective conductivity

Variational principles if $\mathbf{L}$ is self-adjoint and positive definite:

$$(\mathbf{J}_0, \mathbf{L}_*^{-1}\mathbf{J}_0) = \inf_{\underline{\mathbf{J}} \in \mathcal{J}} (\mathbf{J}_0 + \underline{\mathbf{J}}, \mathbf{L}^{-1}(\mathbf{J}_0 + \underline{\mathbf{J}}))$$

$$(\mathbf{E}_0, \mathbf{L}_*\mathbf{E}_0) = \inf_{\underline{\mathbf{E}} \in \mathcal{E}} (\mathbf{E}_0 + \underline{\mathbf{E}}, \mathbf{L}(\mathbf{E}_0 + \underline{\mathbf{E}}))$$

Leading to the elementary bounds:

$$\mathbf{L}_* \geq 0, \quad \mathbf{L}_* \leq \mathbf{\Gamma}_0\mathbf{L}\mathbf{\Gamma}_0, \quad \mathbf{L}_*^{-1} \leq \mathbf{\Gamma}_0\mathbf{L}^{-1}\mathbf{\Gamma}_0,$$

$\mathbf{\Gamma}_0$ is the projection onto $\mathcal{U}$

Formula for the effective operator

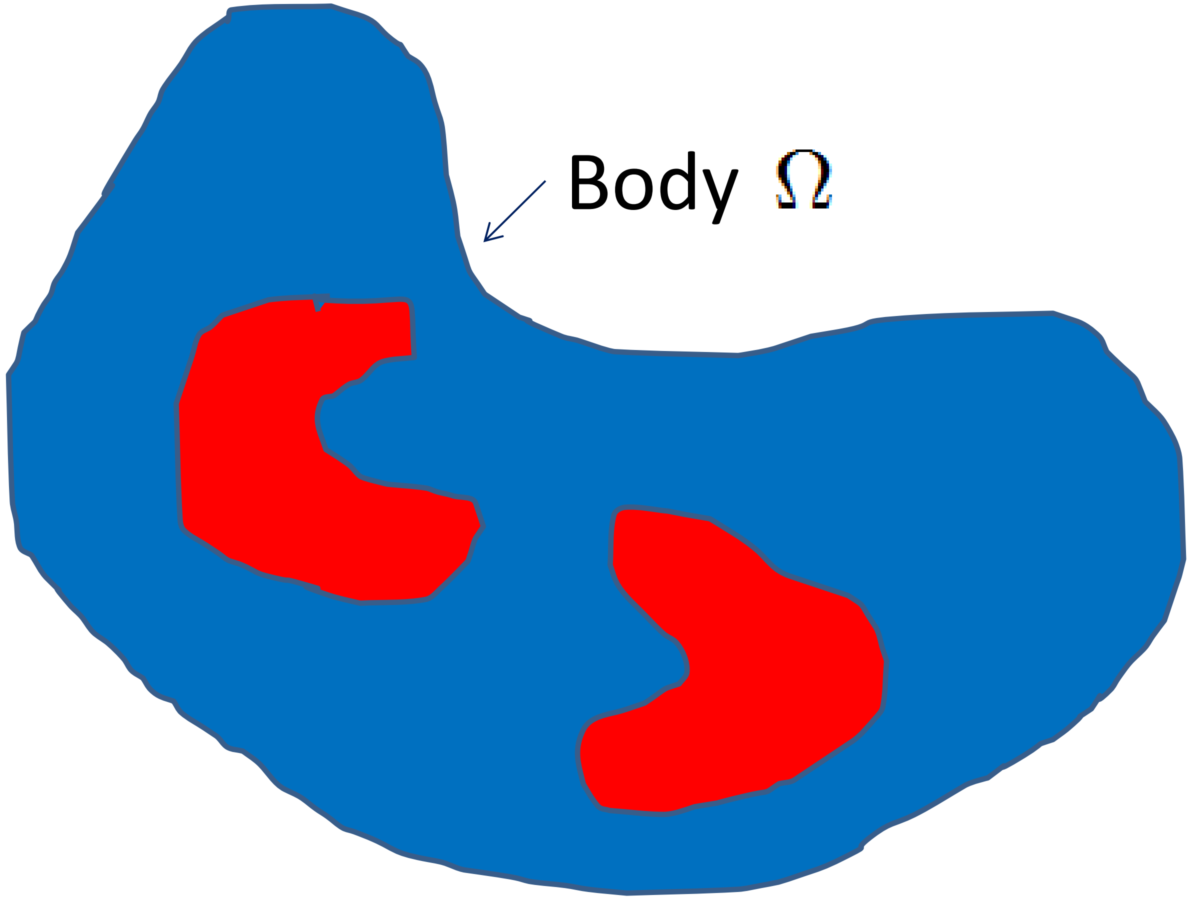
$$\mathbf{L}_* = \mathbf{\Gamma}_0\mathbf{L}[\mathbf{I} + \mathbf{\Gamma}_1(\mathbf{L}/\sigma_0 - \mathbf{I})]^{-1}\mathbf{\Gamma}_0.$$

where $\mathbf{\Gamma}_1$ is the projection onto $\mathcal{E}$.

Leads to series expansions:

$$\begin{aligned}
\mathbf{L}_* &= \sum_{j=0}^{\infty}\mathbf{\Gamma}_0\mathbf{L}[\mathbf{\Gamma}_1(\mathbf{I}-\mathbf{L}/\sigma_0)]^j\mathbf{\Gamma}_0,\\
\mathbf{J}_0 &= \sum_{j=0}^{\infty}\mathbf{\Gamma}_0\mathbf{L}[\mathbf{I}-\mathbf{\Gamma}_1(\mathbf{L}/\sigma_0)]^j\mathbf{E}_0,\\
\mathbf{E} &= \sum_{j=0}^{\infty}[\mathbf{\Gamma}_1(\mathbf{I}-\mathbf{L}/\sigma_0)]^j\mathbf{E}_0,\\
\mathbf{J} &= \sum_{j=0}^{\infty}\mathbf{\Gamma}_2\mathbf{L}[\mathbf{\Gamma}_1(\mathbf{I}-\mathbf{L}/\sigma_0)]^j\mathbf{E}_0.
\end{aligned}$$

Dirichlet-to-Neumann Map

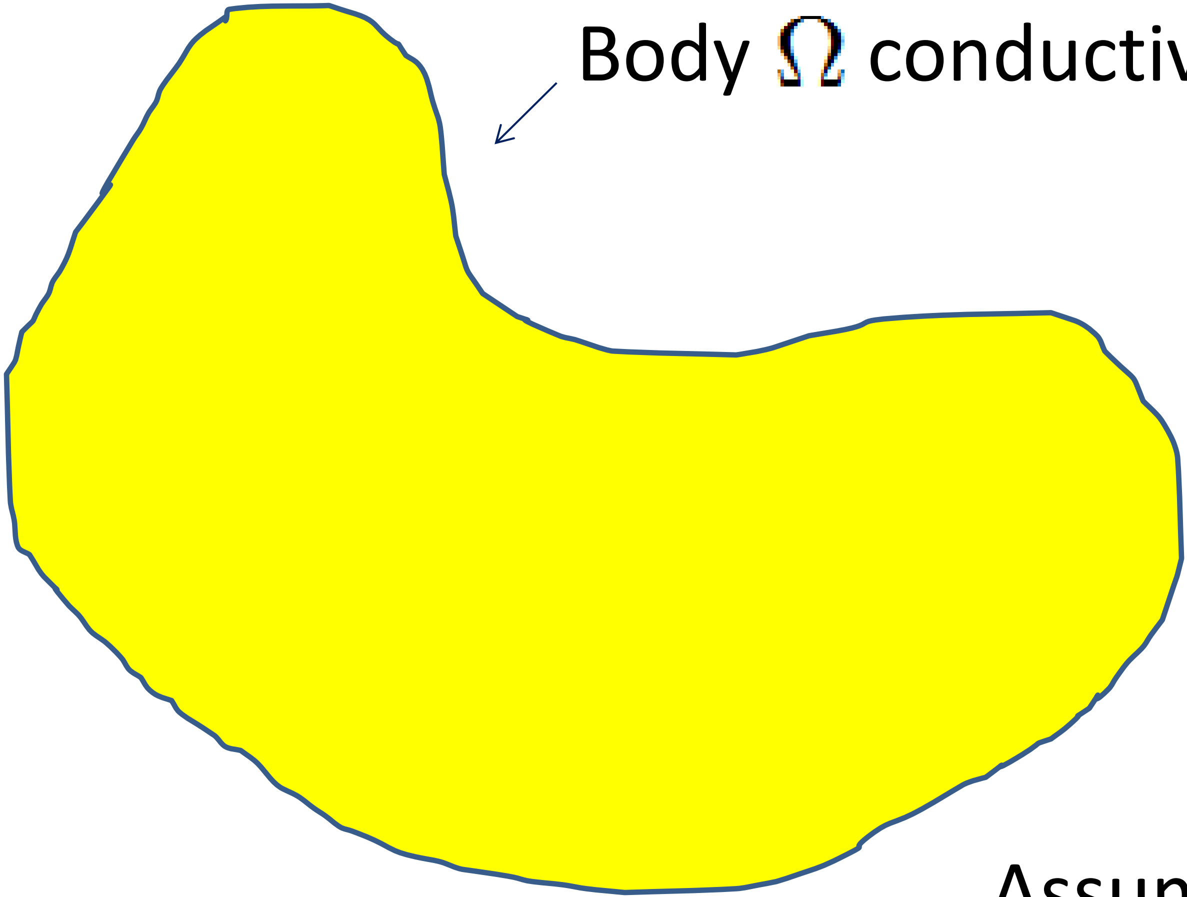


Specify boundary potential $V_0(\mathbf{x})$

Measure current flux $\mathbf{n} \cdot \mathbf{j}(\mathbf{x})$

We want to reformulate it as a problem in the abstract theory of composites, so we can apply the machinery of the theory of composites.

Remove boundary conditions, by expressing the problem in terms of the fields that solve the problem when Ω is filled with a homogeneous material



Body Ω conductivity $\boldsymbol{\sigma}(\mathbf{x}) = 1$

$$\mathbf{j}(\mathbf{x}) = \mathbf{e}(\mathbf{x})$$

$$\nabla \cdot \mathbf{j} = 0$$

$$\mathbf{e} = -\nabla V$$

$$V = V_0 \text{ on } \partial\Omega$$

Assume solved

Now let

$\mathcal{U}$ consist of those fields $\mathbf{j}(\mathbf{x}) = \mathbf{e}(\mathbf{x})$ that solve the equations as the boundary potential $V_0(\mathbf{x})$ varies.

$\mathcal{E}$ consist of fields $\mathbf{E} = -\nabla V$ with $V(\mathbf{x}) = 0$

$\mathcal{J}$ consist of fields $\mathbf{J}$ with $\nabla \cdot \mathbf{J} = 0$ and $\mathbf{n} \cdot \mathbf{J} = 0 \text{ on } \partial\Omega$

Three spaces are orthogonal

Note that fields $\mathbf{j}(\mathbf{x}) = \mathbf{e}(\mathbf{x})$ in $\mathcal{U}$
can be parameterized either by the boundary values of $V = V_0 \text{ on } \partial\Omega$ or by the boundary values of $\mathbf{n} \cdot \mathbf{j}(\mathbf{x})$.

The abstract problem in composites consists in finding for a given field $\mathbf{e}(\mathbf{x})$ in $\mathcal{U}$
(with associated boundary potential $V_0(\mathbf{x})$) the fields which solve:

$$\mathbf{j}'(\mathbf{x})+\mathbf{J}(\mathbf{x})=\sigma(\mathbf{x})[\mathbf{e}(\mathbf{x})+\mathbf{E}(\mathbf{x})]$$

with

$$\mathbf{j}'(\mathbf{x})\in\mathcal{U},\quad \mathbf{J}(\mathbf{x})\in\mathcal{J},\quad \mathbf{E}(\mathbf{x})\in\mathcal{E}$$

which is exactly the conductivity problem we would solve for the Dirichlet problem.

Furthermore if we knew the effective operator

$$\mathbf{L}_*: \mathcal{U} \to \mathcal{U}$$

Then we have

$$\mathbf{j}' = \mathbf{L}_* \mathbf{e}$$

and the boundary values of $\mathbf{n} \cdot \mathbf{j}'(\mathbf{x})$ allow us to determine the Dirichlet-to-Neumann map assuming the fields in $\mathcal{U}$ have been numerically calculated

Analyticity properties of effective tensors as functions of the moduli of the component materials (Bergman, Milton, Golden and Papanicolaou) extend to the Dirichlet-to-Neumann map

$$\sigma(\mathbf{x}) = \mathbf{R}(\mathbf{x})^T [\textstyle\sum_{i=1}^n \chi_i(\mathbf{x})\sigma_i]\mathbf{R}(\mathbf{x})$$

The Dirichlet-to-Neumann map is a Herglotz function of the matrices $\sigma_1, \sigma_2, \ldots \sigma_n$ in the domain where these have positive definite imaginary parts, modulo a rotation in the complex plane.

Easiest to prove using an approach of Bruno:
The truncated series expansion,

$$\mathbf{L}_* \approx \sum_{j=0}^{m} \boldsymbol{\Gamma}_0 \mathbf{L}[\boldsymbol{\Gamma}_1(\mathbf{I}-\mathbf{L}/\sigma_0)]^j \boldsymbol{\Gamma}_0$$

with $\mathbf{L}=\boldsymbol{\sigma}(\mathbf{x})$ is a polynomial in the matrix elements of $\sigma_1, \sigma_2, \ldots \sigma_n$ and hence $\mathbf{L}_*$ will be an analytic function of them in the domain of convergence of the series

One obtains integral representation formulas for $\mathbf{L}_*$

Time Harmonic Equations:

Acoustics:

$$\underbrace{\begin{pmatrix} -i\mathbf{v} \\ -i\nabla\cdot\mathbf{v} \end{pmatrix}}_{\mathcal{G}(\mathbf{x})} = \underbrace{\begin{pmatrix} -(\omega\rho)^{-1} & 0 \\ 0 & \omega/\kappa \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla P \\ P \end{pmatrix}}_{\mathcal{F}(\mathbf{x})},$$

Elastodynamics:

$$\underbrace{\begin{pmatrix} -\boldsymbol{\sigma}(\mathbf{x})/\omega \\ i\mathbf{p}(\mathbf{x}) \end{pmatrix}}_{\mathcal{G}(\mathbf{x})} = \underbrace{\begin{pmatrix} -\mathcal{C}/\omega & 0 \\ 0 & \omega\rho \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \underbrace{\begin{pmatrix} \nabla\mathbf{u} \\ \mathbf{u} \end{pmatrix}}_{\mathcal{F}(\mathbf{x})},$$

Maxwell:

$$\underbrace{\begin{pmatrix} -i\mathbf{h} \\ i\nabla\times\mathbf{h} \end{pmatrix}}_{\mathcal{G}} = \underbrace{\begin{pmatrix} -[\omega\mu(\mathbf{x})]^{-1} & 0 \\ 0 & \omega\varepsilon(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}} \underbrace{\begin{pmatrix} \nabla\times\mathbf{e} \\ \mathbf{e} \end{pmatrix}}_{\mathcal{F}},$$

Schrödinger equation,

$$\begin{pmatrix} \mathbf{q}(\mathbf{x}) \\ \nabla\cdot\mathbf{q}(\mathbf{x}) \end{pmatrix} = \underbrace{\begin{pmatrix} -\mathbf{A} & 0 \\ 0 & E - V(\mathbf{x}) \end{pmatrix}}_{\mathbf{Z}(\mathbf{x})} \begin{pmatrix} \nabla\psi(\mathbf{x}) \\ \psi(\mathbf{x}) \end{pmatrix},$$

In all these examples Z has positive semidefinite imaginary part (that often can be made positive definite by a slight rotation in the complex plane)

The analog of the variational principles of Cherkaev and Gibiansky are then the variational principles of

Milton, Seppecher, and Bouchitte (2009)
Milton and Willis (2010)

For acoustics, electromagnetism, elastodynamics

Variational principles for Electromagnetism

Maxwell's equations:

$$\nabla \times \mathbf{E} = i\omega \mathbf{B}, \quad \nabla \times \mathbf{H} = -i\omega \mathbf{D}$$

$$\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}, \quad \mathbf{B} = \boldsymbol{\mu}\mathbf{H},$$

Let

$$\boldsymbol{\mathcal{E}} = \begin{pmatrix} \varepsilon'' + \varepsilon'(\varepsilon'')^{-1}\varepsilon' & \varepsilon'(\varepsilon'')^{-1} \\ (\varepsilon'')^{-1}\varepsilon' & (\varepsilon'')^{-1} \end{pmatrix},$$

When $\boldsymbol{\mu}$ is real : $\quad Y(\mathbf{E}') = \inf_{\underline{\mathbf{E}}'} Y(\underline{\mathbf{E}}'),$

$$Y(\underline{\mathbf{E}}') = \int_{\Omega} \begin{pmatrix} \omega\underline{\mathbf{E}}' \\ -\nabla \times \boldsymbol{\mu}^{-1}(\nabla \times \underline{\mathbf{E}}')/\omega \end{pmatrix} \cdot \boldsymbol{\mathcal{E}} \begin{pmatrix} \omega\underline{\mathbf{E}}' \\ -\nabla \times \boldsymbol{\mu}^{-1}(\nabla \times \underline{\mathbf{E}}')/\omega \end{pmatrix}$$

The infimum is over fields with prescribed tangential components of

$\underline{\mathbf{E}}'$ and $\boldsymbol{\mu}^{-1}\nabla \times \underline{\mathbf{E}}'$ at $\partial\Omega$

Unusual boundary conditions, but can be fixed

Minimization principles for Schrödinger's equation with complex energies

$$E\psi(\mathbf{x}) = -\nabla \cdot \mathbf{A}\nabla\psi(\mathbf{x}) + V(\mathbf{x})\psi(\mathbf{x}) - h(\mathbf{x})\theta_0 \qquad \mathbf{A} = \hbar^2\mathbf{I}/2m$$

Minimize over ψ'

$$W(\psi', p) = \sum_s \int_{\Omega^N} \underbrace{[p(\mathbf{x})]^2 + (E'')^2[\psi'(\mathbf{x})]^2 + 2\theta_0 p(\mathbf{x})h(\mathbf{x})}_{I(p,\psi')}\, d\mathbf{r}$$

where

$$p(\mathbf{x}) = p(\mathbf{x}, \psi') = \nabla \cdot \mathbf{A}\nabla\psi' + (E' - V(\mathbf{x}))\psi',$$

subject to suitable boundary conditions on ψ'

The Desymmetrization of Schrödinger's equation

$$\begin{pmatrix} \mathbf{q}(\mathbf{x}) \\ \nabla\cdot\mathbf{q}(\mathbf{x}) \\ \nabla\cdot\mathbf{v}(\mathbf{x})+S_0 \end{pmatrix} = \begin{pmatrix} -\mathbf{A} & 0 & 0 \\ 0 & E-V(\mathbf{x}) & h(\mathbf{x}) \\ 0 & \overline{h}(\mathbf{x}) & d(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \nabla\psi \\ \psi \\ \theta_0 \end{pmatrix} \qquad \mathbf{A} = \hbar^2\mathbf{I}/2m$$

Replace with:

$\mathbf{J}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\mathbf{E}(\mathbf{x}),$ Let $\mathbf{\Lambda}$ denote appropriate symmetrization operator:

$$\mathbf{\Lambda}\mathbf{J}(\mathbf{x}) = \begin{pmatrix} \mathbf{q}(\mathbf{x}) \\ \nabla\cdot\mathbf{q}(\mathbf{x}) \\ \nabla\cdot\mathbf{v}(\mathbf{x}) \end{pmatrix} \quad \mathbf{L}(\mathbf{x}) = \begin{pmatrix} -\mathbf{A} & 0 & 0 \\ 0 & a(\mathbf{x}_1,\mathbf{x}_2) & g(\mathbf{x}_1,\mathbf{x}_2) \\ 0 & \overline{g}(\mathbf{x}_1,\mathbf{x}_2) & d(\mathbf{x}_1,\mathbf{x}_2) \end{pmatrix} \quad \mathbf{E}(\mathbf{x}) = \begin{pmatrix} \nabla\psi \\ \psi \\ \theta_0 \end{pmatrix}$$

Advantage: Can solve iteratively using FFT, and the FFT operations only need be done on $(\mathbf{x}_1,\mathbf{x}_2)$, i.e. only on two electron co-ordinates not all n electrons.

For electromagnetism, acoustics and elastodynamics, the Dirichlet-to-Neumann map is a Herglotz function of the matrices $\mathbf{Z}_1, \mathbf{Z}_2, \ldots, \mathbf{Z}_n$ of the component materials

For electromagnetism an alternative rigorous proof was obtained by Cassier, Welters, and Milton (talk by Aaron Welters at this meeting).

Some inverse problems for two-component bodies

Electromagnetism:

Suppose $\mu_1 = \mu_2$ is real and frequency independent.

Look for special complex frequencies where

$$\varepsilon_1(\omega) = \varepsilon_2(\omega)$$

Extrapolate (using representation formulas or bounds) measurements at different frequencies, or transient responses, to the neighborhood of these special frequencies

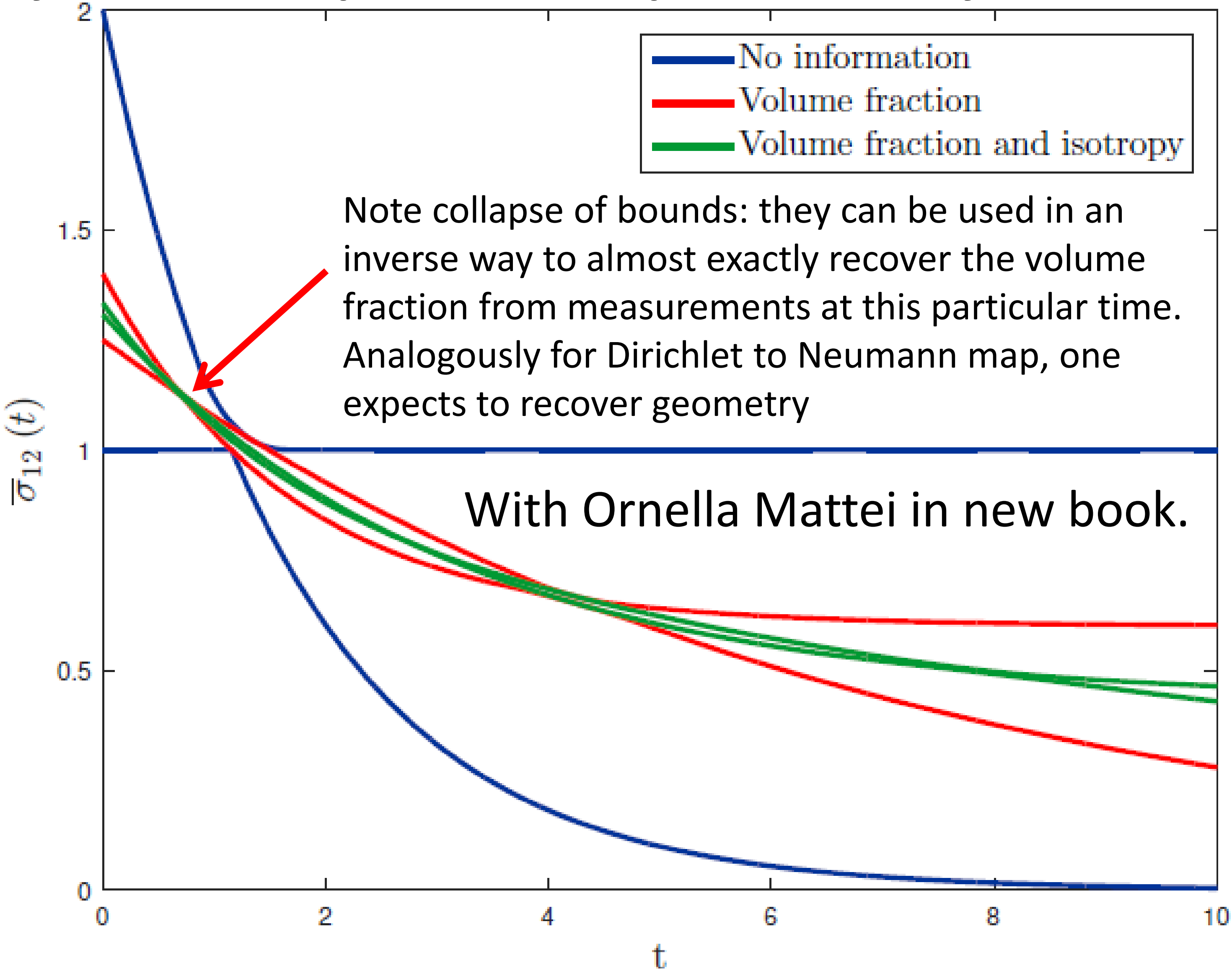
Quasistatic Elastodynamics:

Extrapolate to frequencies where $\mu_1(\omega_0^k) = \mu_2(\omega_0^k)$.

Quasistatic Electromagnetism:

Extrapolate to ratios $\varepsilon_1/\varepsilon_2$ close to 1

Rigorous Upper and Lower Bounds on the Stress Relaxation in cylindrical composites in antiplane elasticity

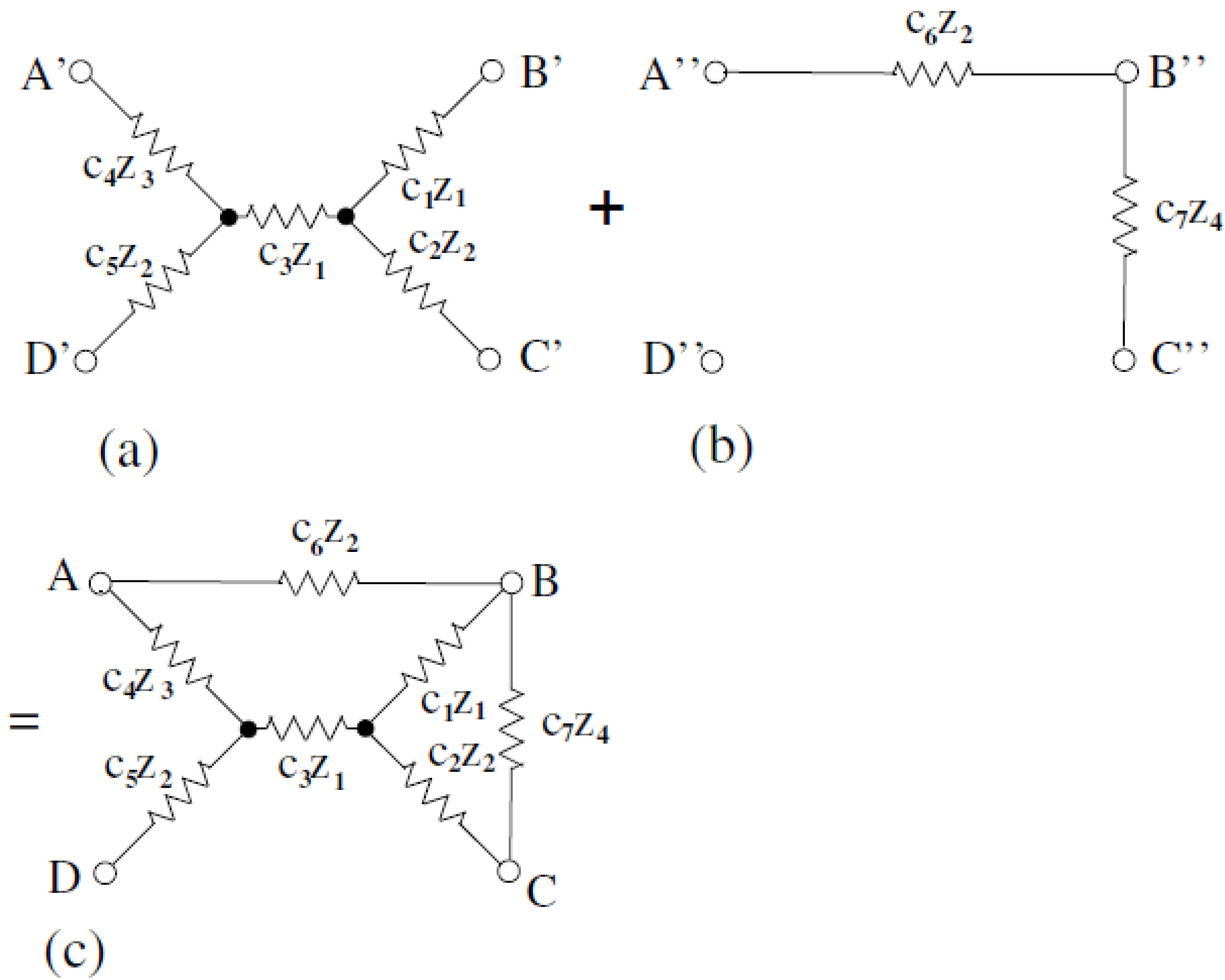


Its beneficial to look in the time domain not just the frequency domain!

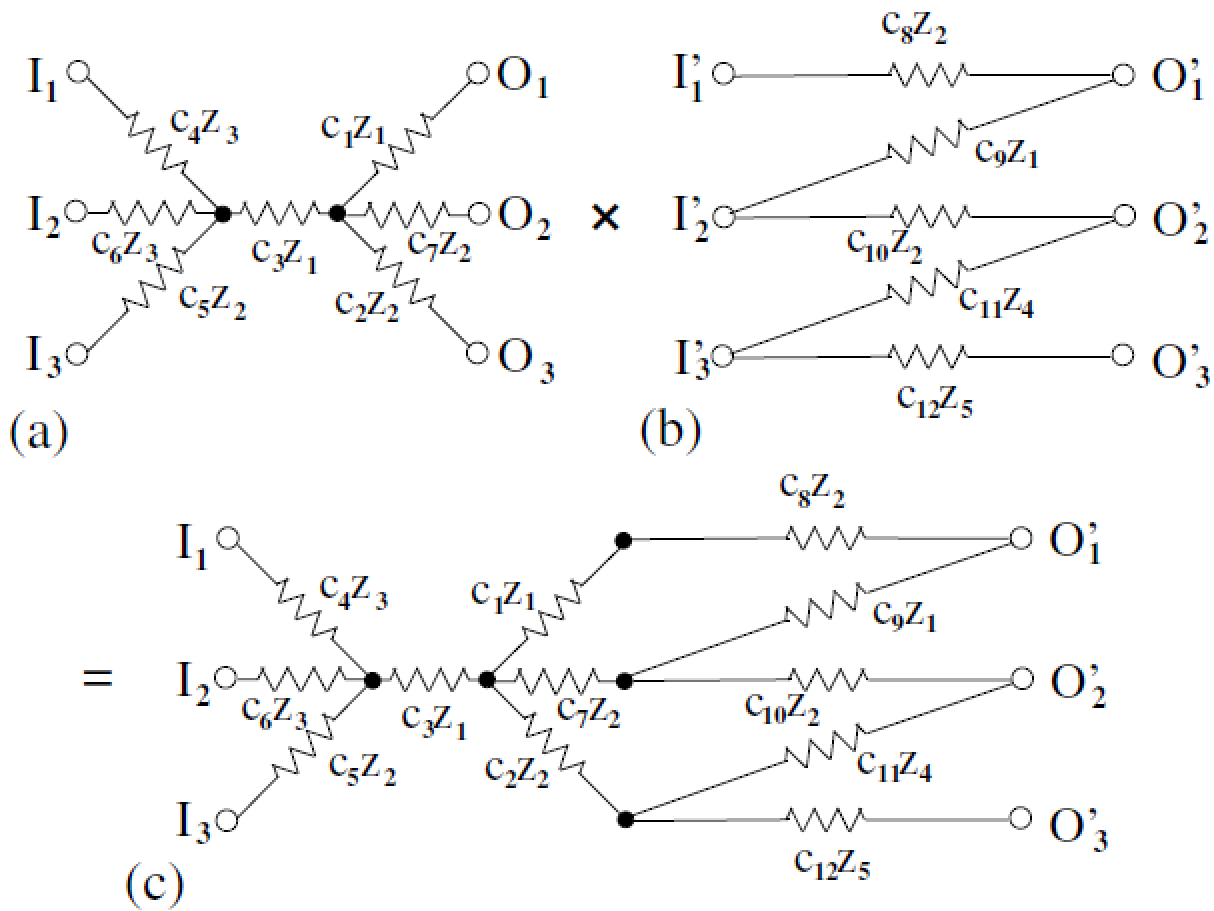
Generalizing the concept of function to

Superfunctions!

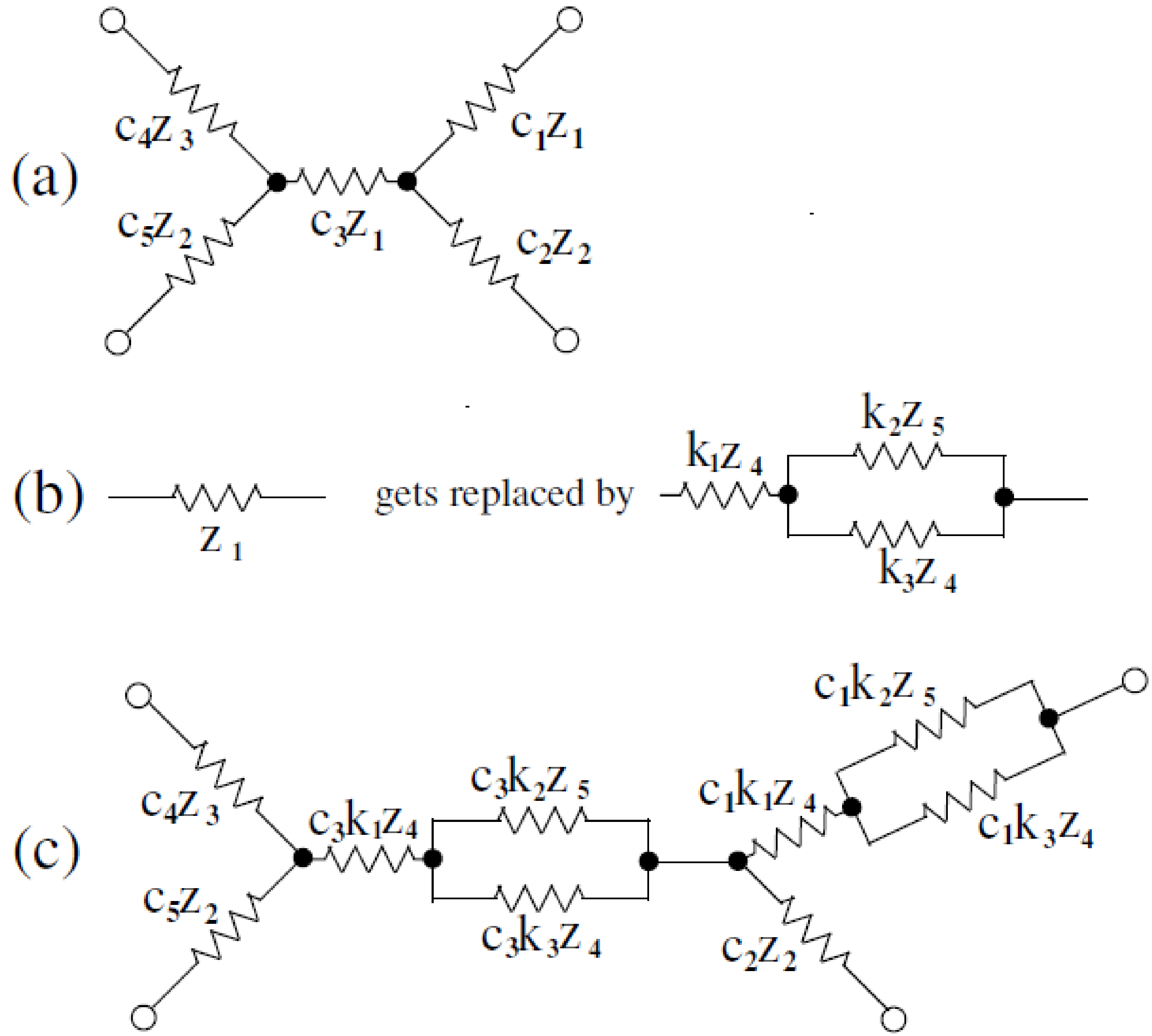
Adding resistor networks



Multiplying resistor networks

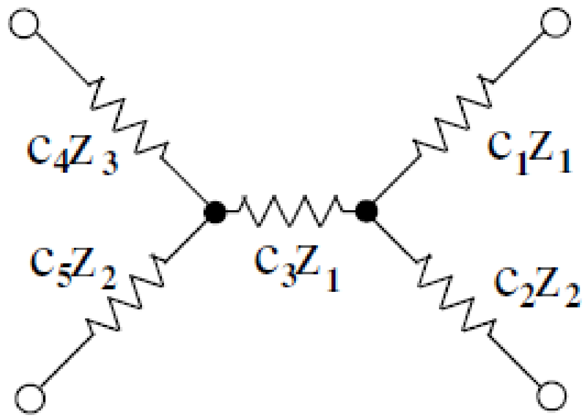


Substitution of networks

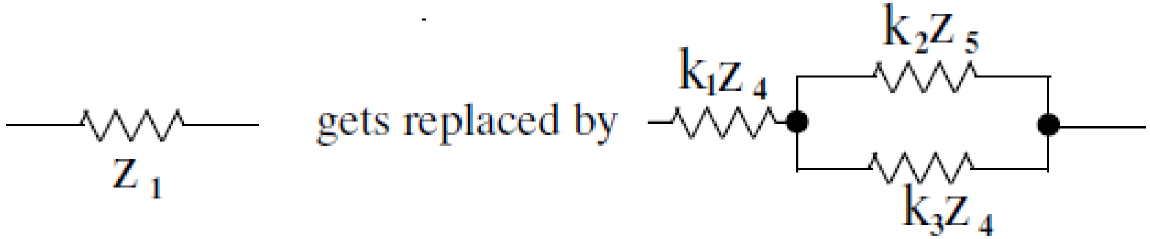


Substitution of networks

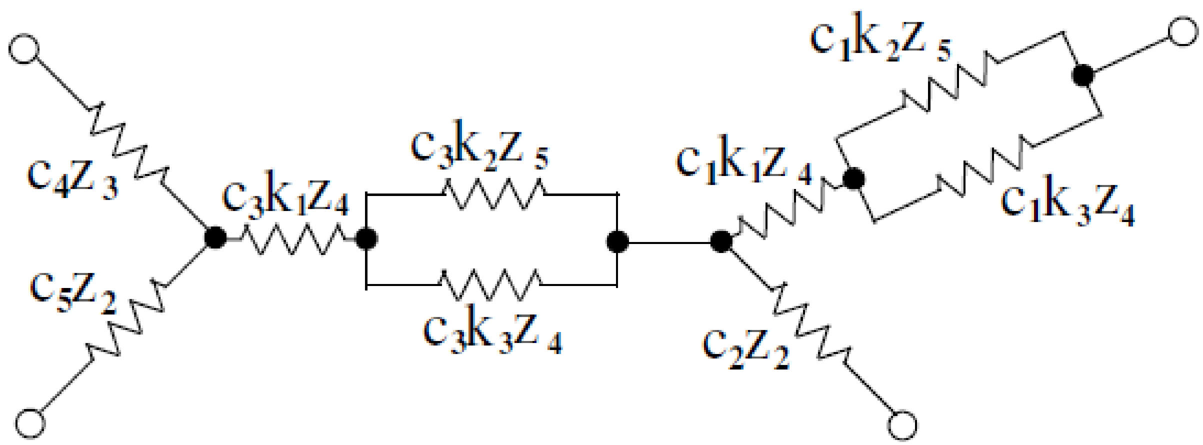
(a)

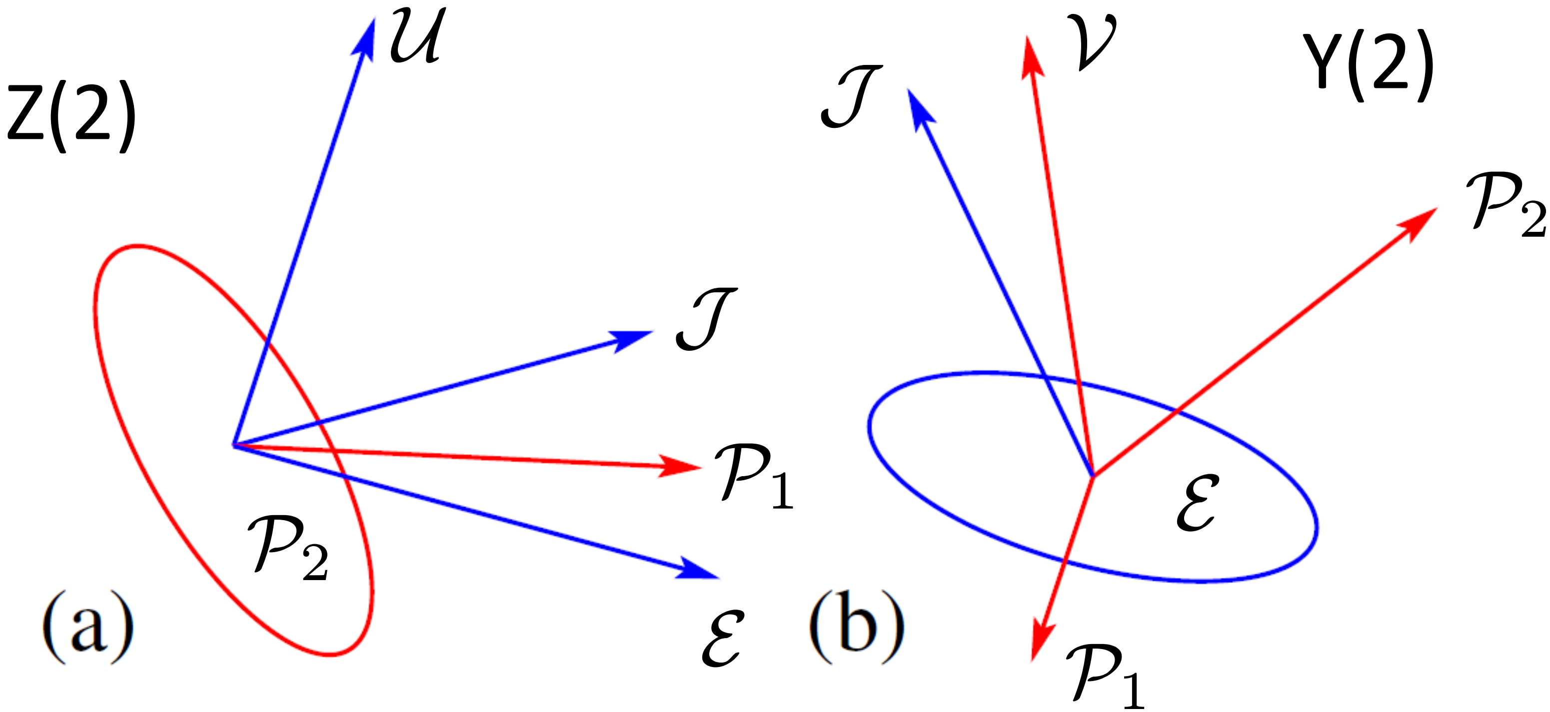


(b)

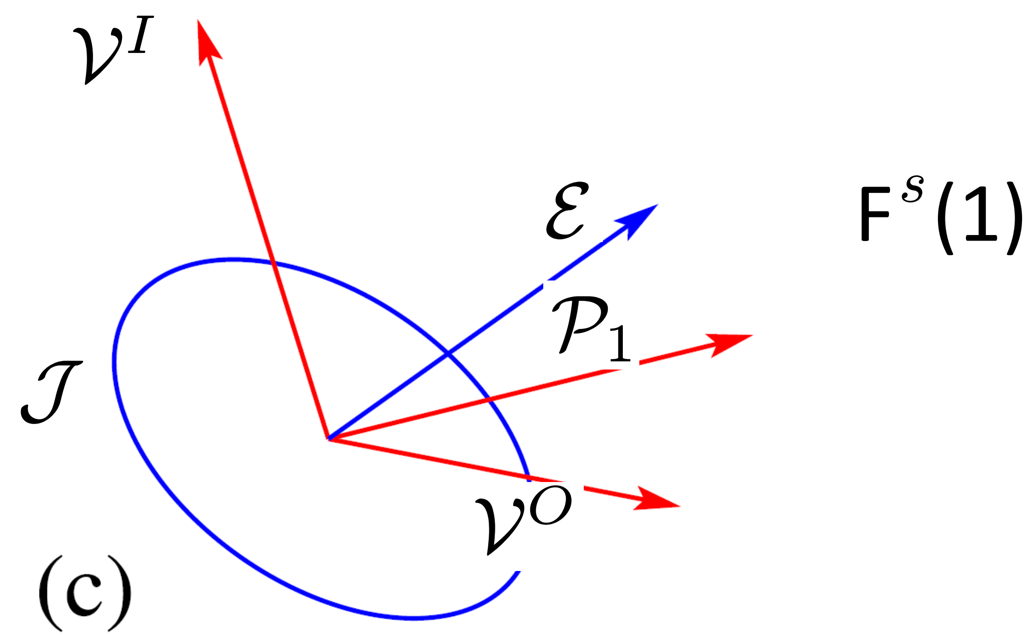


(c)





Key: Allow nonorthogonal Subspace collections. Then we have a whole algebra.



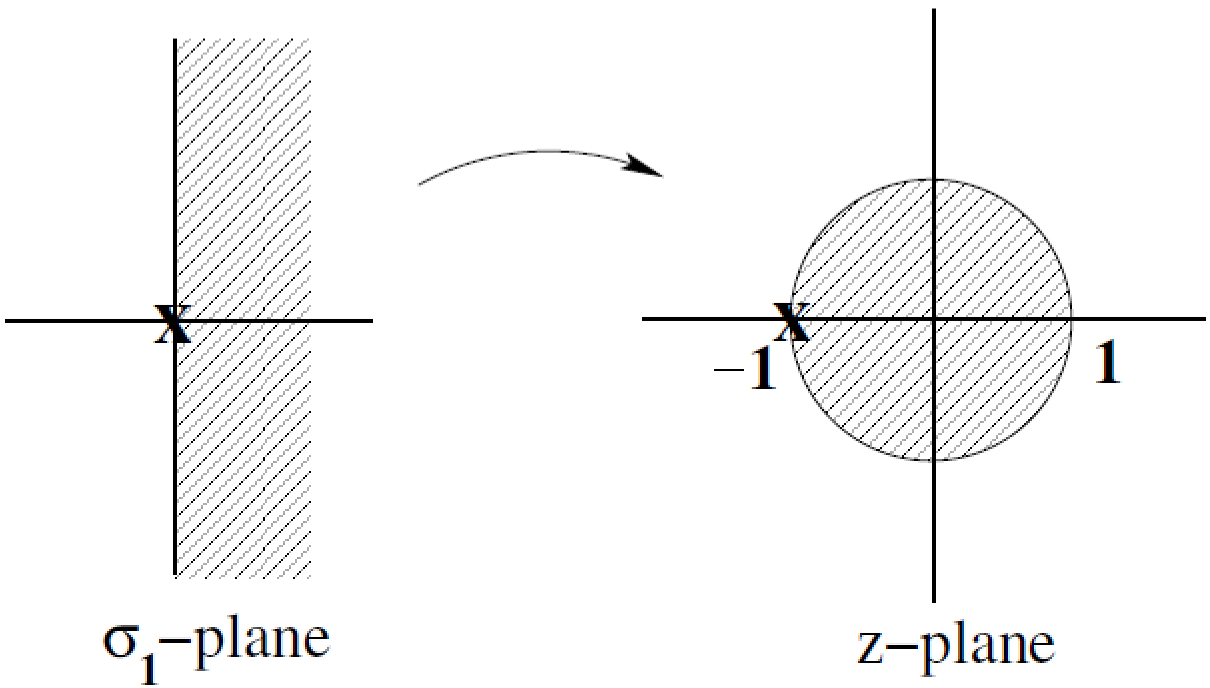
Application: Accelerating some Fast Fourier Transform Methods in two-component composites

Numerical scheme of Moulinec and Suquet (1994)

$$\sigma_*/\sigma_0 = 1 + \sum_{n=1}^{\infty} a_n \left(\frac{\sigma_1 - 1}{\sigma_1 + 1}\right)^n .$$

$$\sigma_* = \sum_{j=0}^{\infty} \Gamma_0 \sigma [\Gamma_1 (\mathbf{I} - \sigma/\sigma_0]^j \Gamma_0$$

$$\sigma_0 = (\sigma_1 + 1)/2$$



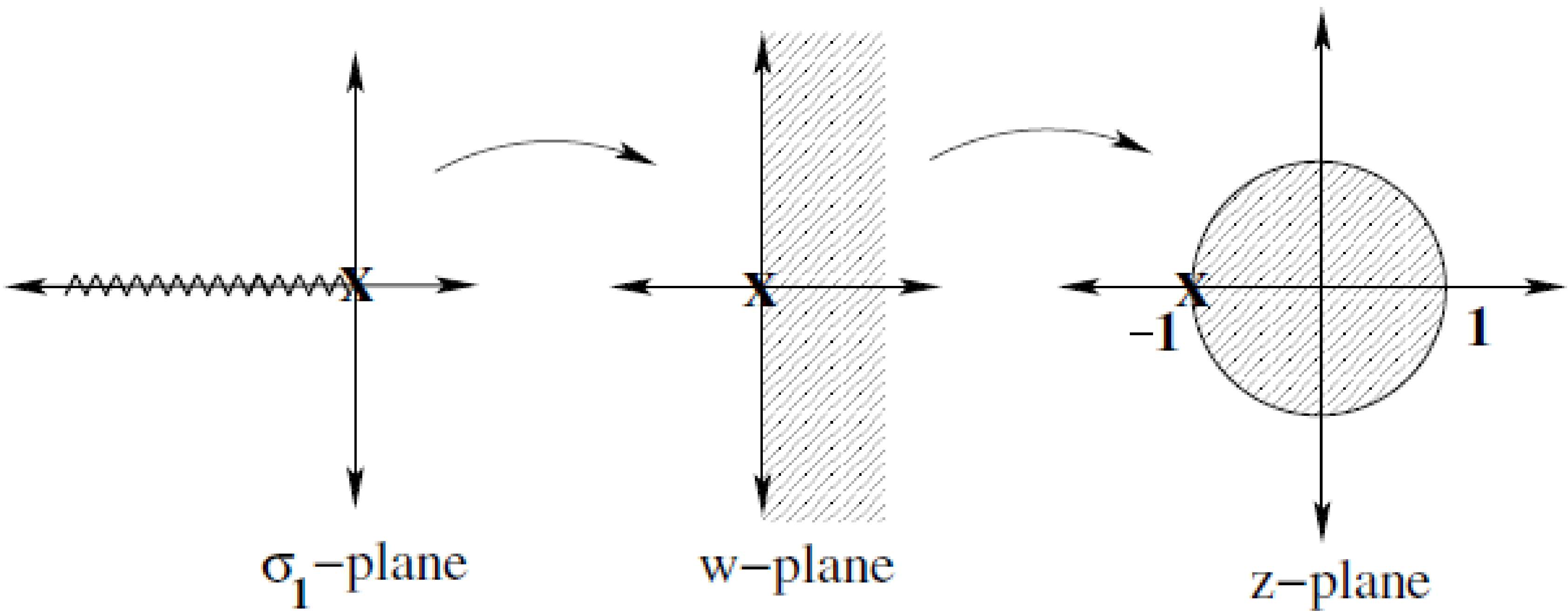
Numerical scheme of Eyre and Milton (1999)

$$\sigma_*/\sqrt{\sigma_1} = 1 + \sum_{n=1}^{\infty} b_n \left(\frac{\sqrt{\sigma_1} - 1}{\sqrt{\sigma_1} + 1} \right)^n .$$

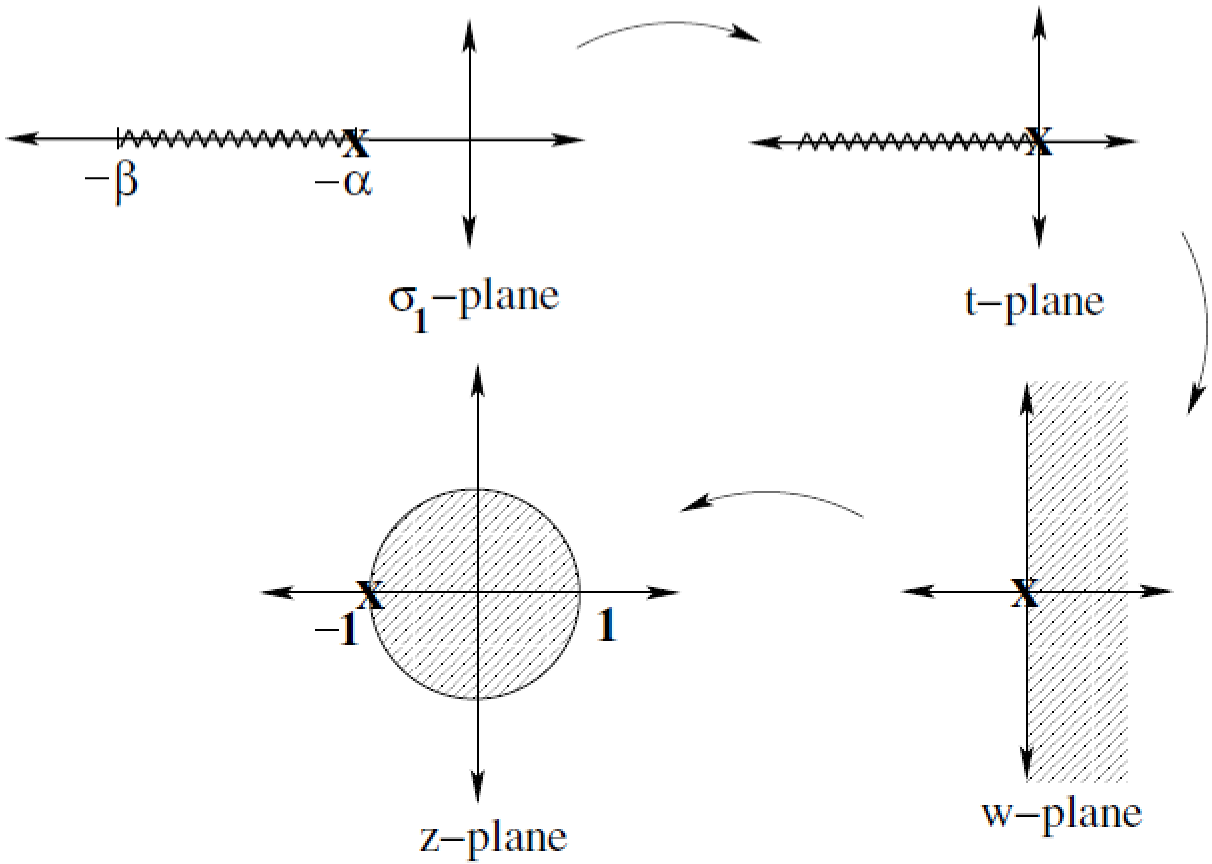
$$\sigma_* = \sigma_0 I + \sigma_0 \sum_{j=0}^{m} \Gamma_0 K (\Upsilon K)^j \Gamma_0,$$

$$\mathbf{K} = \frac{2\sigma_0(\sigma - \sigma_0)}{\sigma + \sigma_0}\mathbf{I}, \quad \Upsilon = (\mathbf{I} - 2\Gamma_1)/(2\sigma_0)$$

$$\sigma_0 = \sqrt{\sigma_1}$$



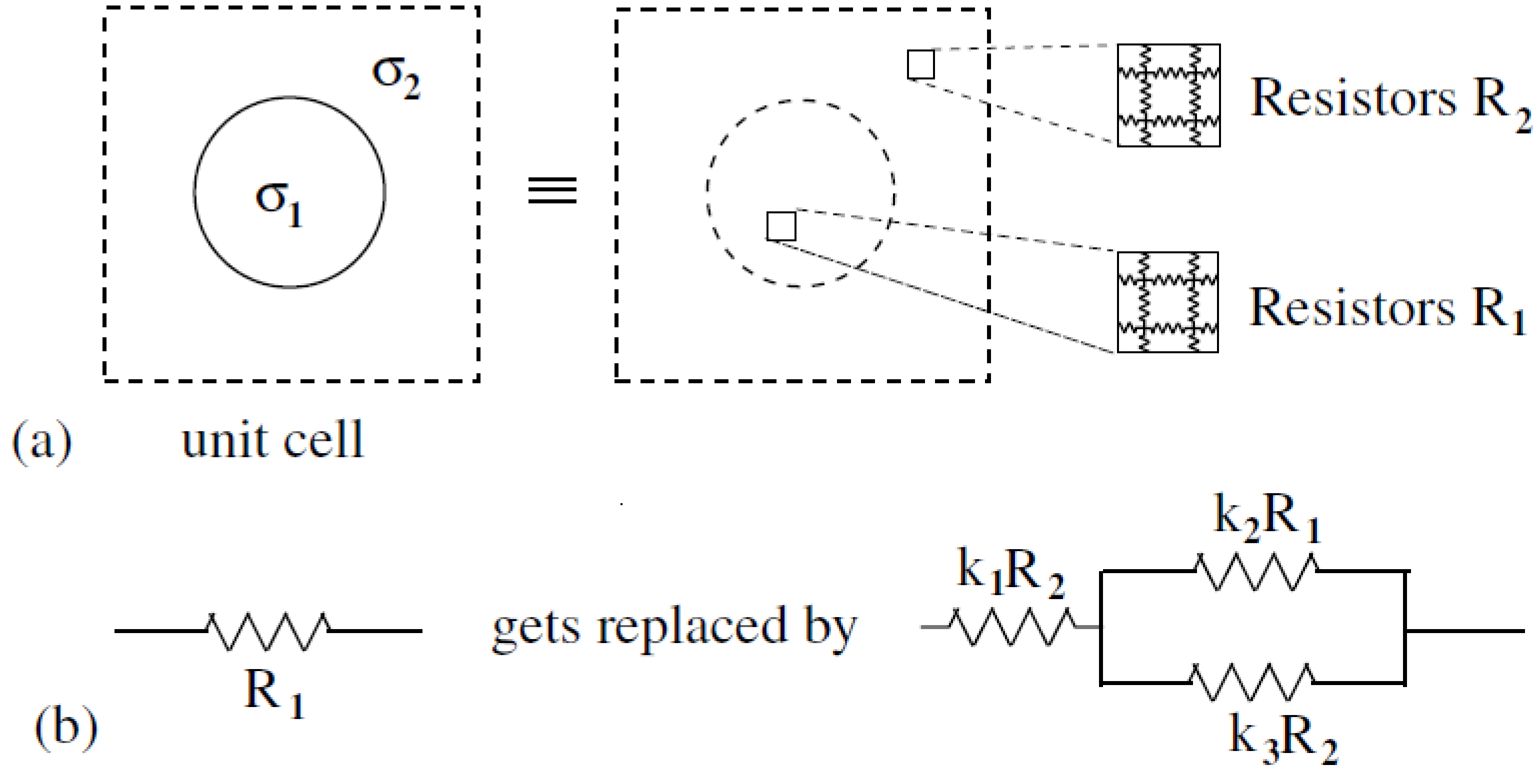
Ideal scheme:



= what iterative scheme??????

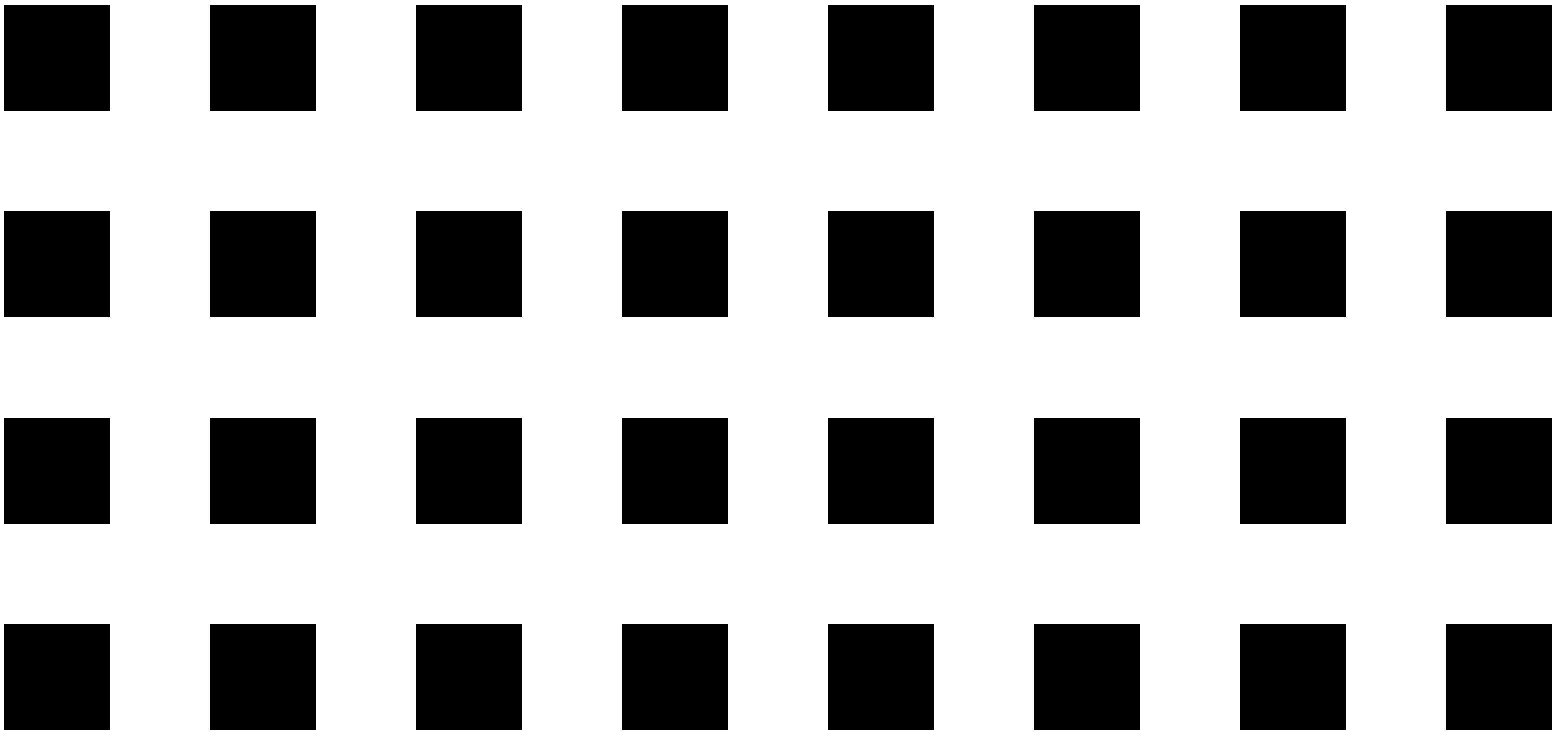
We want to do this transformation at the level of the subspace collection, to recover the fields

At a discrete level



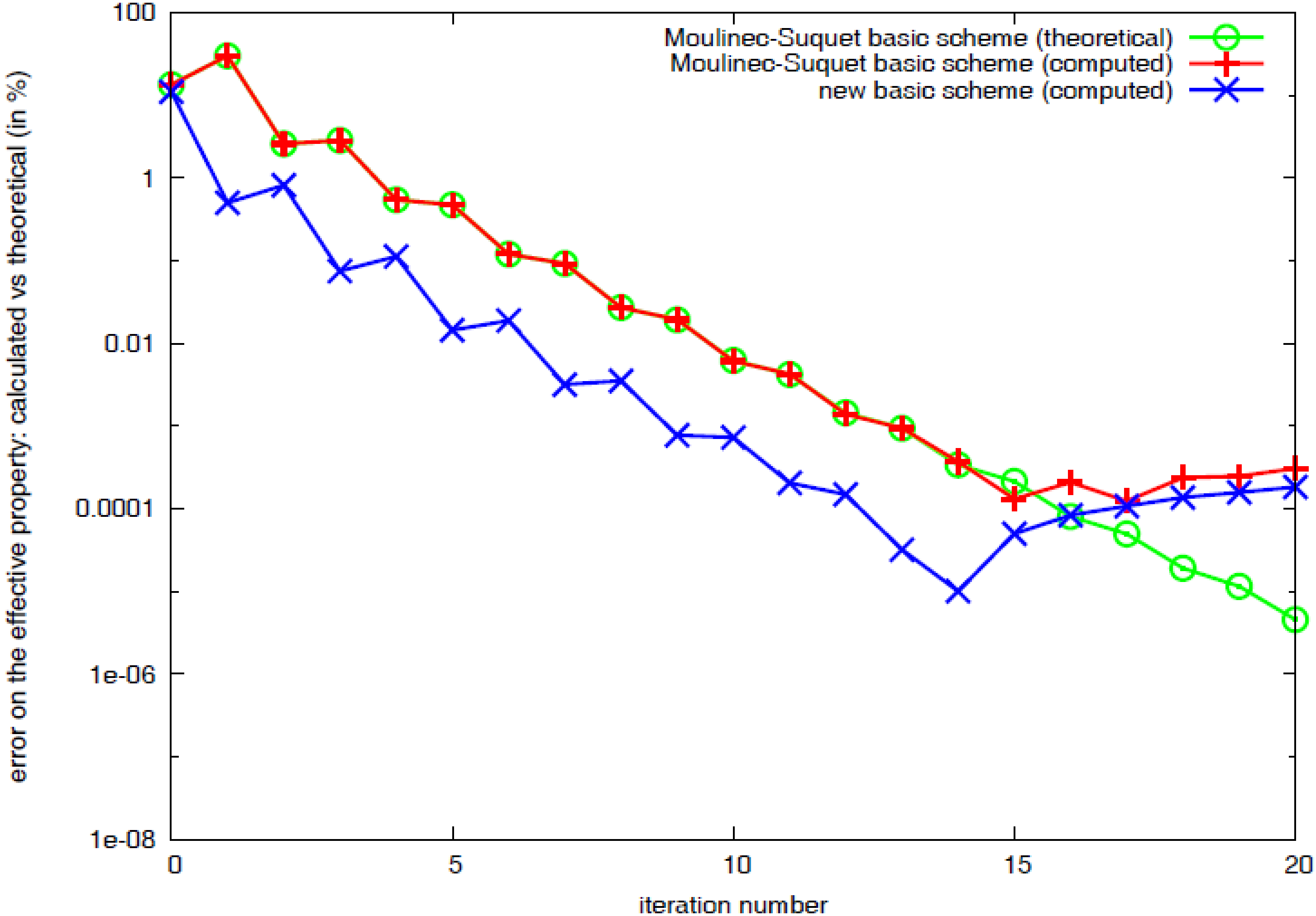
Problem: this substitution shortens the branch cut instead of lengthening it.
Solution:
Substitute non-orthogonal subspace collections

Model example: a square array of squares at 25% volume fraction

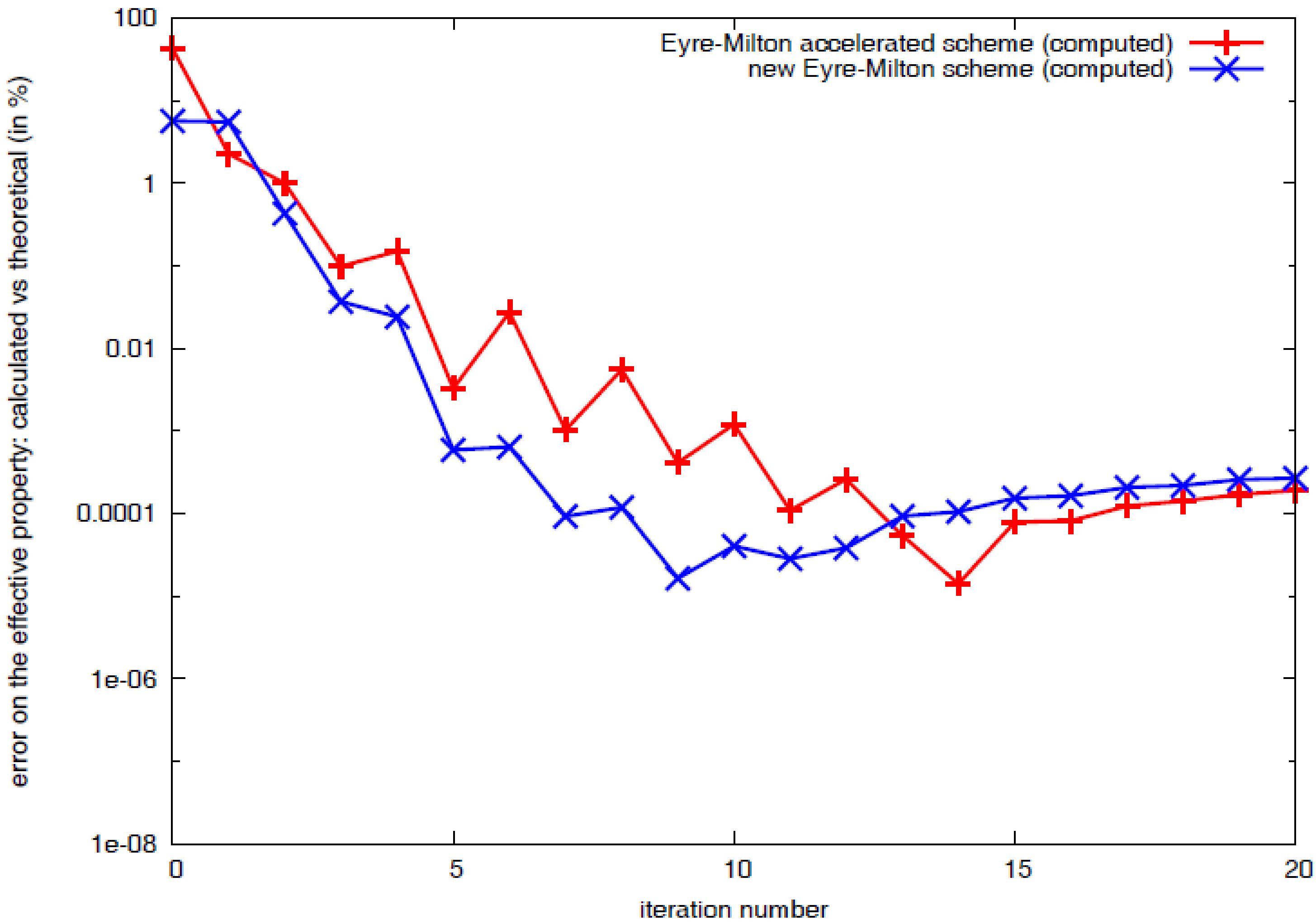


Obnosov's exact formula

$$\sigma_* = \sqrt{(1+3\sigma_1)/(3+\sigma_1)},$$



Moulinec-Suquet basic scheme (theoretical)
Moulinec-Suquet basic scheme (computed)
new basic scheme (computed)
error on the effective property: calculated vs theoretical (in %)
iteration number
100
1
0.01
0.0001
1e-06
1e-08
0
5
10
15
20


Eyre-Milton accelerated scheme (computed)
new Eyre-Milton scheme (computed)
error on the effective property: calculated vs theoretical (in %)
iteration number
100
1
0.01
0.0001
1e-06
1e-08
0
5
10
15
20

With Guevara Vasquez and Onofrei (2011): A complete characterization and synthesis of the dynamic behavior of linear mass-spring networks

Lemma 4. *The response function* $\mathbf{W}(\omega)$ *of any network of springs and masses with* n *terminals is of the form*

$$\mathbf{W}(\omega) = \mathbf{A} - \omega^2 \mathbf{M} + \sum_{i=1}^{p} \frac{\mathbf{C}^{(i)}}{\omega^2 - \omega_i^2} \in \mathbb{R}^{nd \times nd}, \tag{8}$$

where the matrix $\mathbf{M} = \mathrm{diag}(m_1 \mathbf{e}, \ldots, m_n \mathbf{e})$ *is real diagonal with the masses of the boundary nodes in the diagonal, the vector* $\mathbf{e} = [1, \ldots, 1]^T \in \mathbb{R}^d$*, the matrices* $\mathbf{C}^{(i)}$ *are real symmetric positive semidefinite, and the static response*

$$\mathbf{W}(0) = \mathbf{A} - \sum_{i=1}^{p} \omega_i^{-2} \mathbf{C}^{(i)},$$

is real symmetric positive semidefinite and balanced

Furthermore, any such function can be realized by a mass-spring network

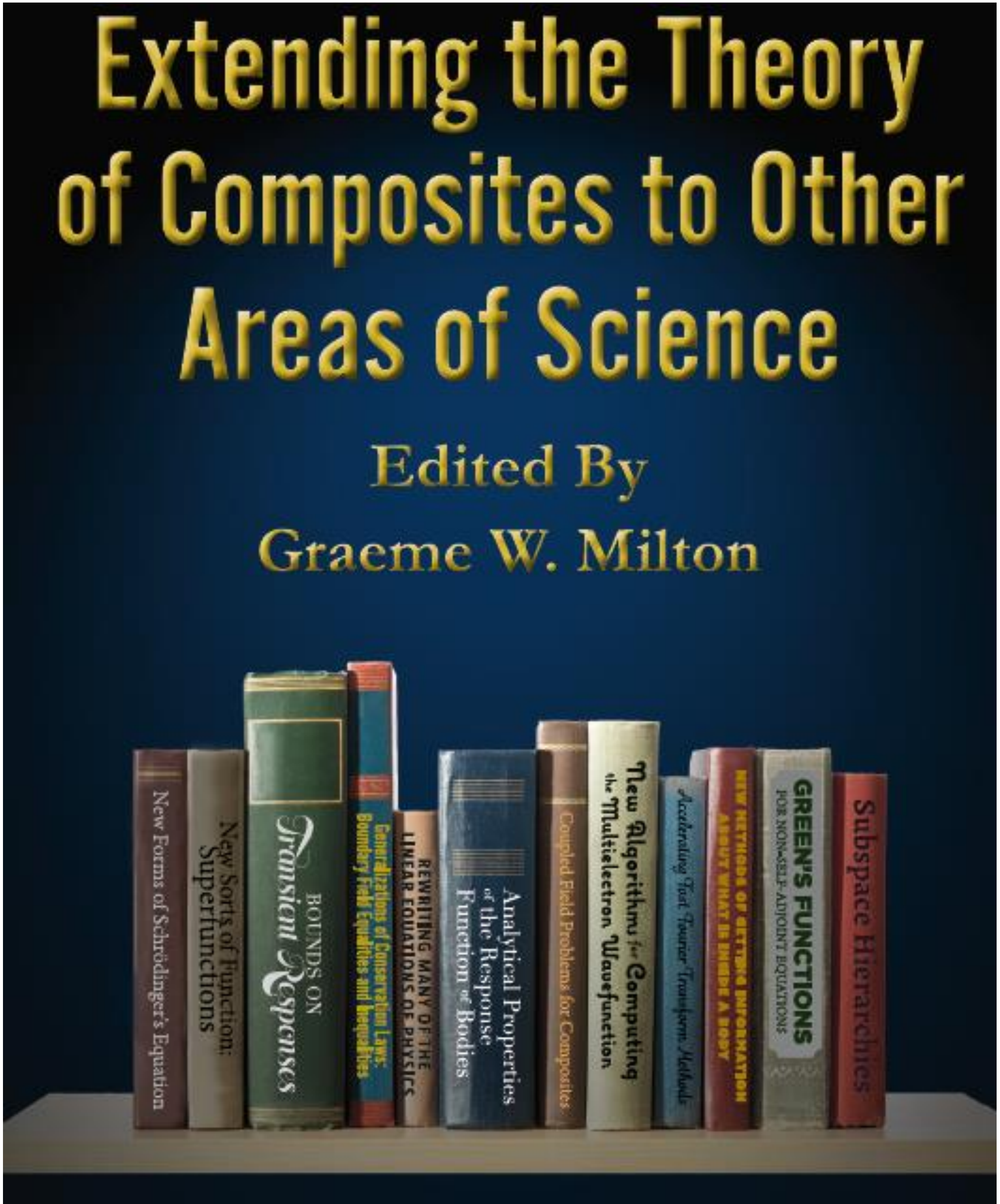
Thank you!

Thank you!

Thank you!

Thank you!

Thank you!



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