Lectures on Representation Theory

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Contents

CHAPTER I

Representations of finite groups

1. Category of representations of finite groups

1.1. Category of group representations. Let G be a group. Let V be a vector space over $\mathbb{C}$. Denote by $\mathrm{GL}(V)$ the general linear group of V, i.e., the group of all linear automorphisms of V.

A *representation* (π, V) of G on the vector space V is a group homomorphism $\pi : G \longrightarrow \mathrm{GL}(V)$. A *morphism* $\phi : (\pi, V) \longrightarrow (\nu, U)$ of representation (π, V) into (μ, U) is a linear map $\phi : V \longrightarrow U$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "\pi(g)"] \arrow[d, "\phi"'] & V \arrow[d, "\phi"] \\ U \arrow[r, "\nu(g)"'] & U \end{tikzcd}$$

commutes for all $g \in G$. Morphisms of representations are also called *intertwining maps.* The set of all morphisms of (π, V) into (ν, U) is denoted by $\mathrm{Hom}_G(V, U)$.

It is easy to check that all representations of G form a category $\mathcal{R}\mathrm{ep}(G)$ of representations of G.

An isomorphism $\phi : (\pi, V) \longrightarrow (\nu, U)$ in this category is a morphism of representations which is a linear isomorphism of the vector space V with U. If two representations are isomorphic, we say that they are *equivalent.*

Let (π, V) and (ν, U) be two representations of G. Let ϕ and ψ be two morphisms in $\mathrm{Hom}_G(V, U)$. Then $\phi + \psi$ is also a morphism of (π, V) into (ν, U), and $\mathrm{Hom}_G(V, U)$ is an abelian group. Moreover, for $\alpha \in \mathbb{C}$, we can define a morphism $\alpha\phi$ by $(\alpha\phi)(v) = \alpha\phi(v)$ for any $v \in V$. In this way, $\mathrm{Hom}_G(V, U)$ is a vector space over $\mathbb{C}$.

If (π, V) and (ν, U) are two representations of G, we can define the representation $\pi \oplus \nu$ of G on $V \oplus U$ such that $(\pi \oplus \nu)(g)(v, u) = (\pi(g)v, \nu(g)u)$ for all $g \in G$, $v \in V$ and $u \in U$. The representation $\pi \oplus \nu$ is called the *direct sum* of π and ν.

Let (π, V) be a representation of G. Let U be a subspace of V which is invariant for G, i.e., $\pi(g)(U) \subset U$ for all $g \in G$. Then the linear maps $\pi(g)$ restricted to U define linear maps $\nu(g)$, $g \in G$. Clearly, (ν, U) is a representation of G. We call it the *subrepresentation* of π on U.

Let $\phi : (\pi, V) \longrightarrow (\nu, U)$ is a morphism of representations. Then, $\ker\phi \subset V$ is a G-invariant subspace of V. Hence, $\ker\phi$ is a subrepresentation of (π, V). Also, $\mathrm{im}\,\phi$ is a G-invariant subspace of U, so $\mathrm{im}\,\phi$ is a subrepresentation of (ν, U).

Let (π, V) be a representation of G. Let U be an invariant subspace of V. For each $g \in G$ we define a linear operator $\rho(g)$ on the quotient space V/U by

$\rho(g)(v+U) = \pi(g)v+U$ for any $g \in G$. Then $(\rho, V/U)$ is a *quotient representation* of (π, V).

Clearly, the category $\mathcal{R}\mathrm{ep}(G)$ is an abelian category.

If the vector space V is equipped with an inner product $(\cdot \mid \cdot)$ and all linear operators $\pi(g)$, $g \in G$, are unitary with respect to this inner product structure, we say that the representation (π, V) is *unitary*.

1.2. Representations of finite groups. Let G be a group. We say that G is a *finite* group, if G is a finite set.

In this section we assume that the group G is finite. We put $[G] = \mathrm{Card}(G)$.

A representation (π, V) of G is *finite-dimensional* if V is a finite-dimensional vector space. We put $\dim \pi = \dim_{\mathbb{C}} V$.

1.2.1. Lemma. *Let (π, V) be a representation of G. Let $v \in V$, $v \neq 0$. Then there exists a finite-dimensional subrepresentation (ν, U) of (π, V) such that $v \in U$.*

Proof. Let U be the vector subspace of V generated by vectors $\pi(g)v$, $g \in G$. Then U is G-invariant and finite-dimensional. Moreover, $v = \pi(1)v$ is in U. □

A representation (π, V) of G is called *irreducible* if $V \neq 0$ and the only G-invariant subspaces in V are $\{0\}$ and V.

1.2.2. Theorem. *Let (π, V) be an irreducible representation of G. Then π is finite-dimensional.*

Proof. Let $v \in V$, $v \neq 0$. By 1.2.1, V contains a finite-dimensional G-invariant subspace U such that $v \in U$. If π is irreducible, we must have $U = V$ and V is finite-dimensional. □

1.2.3. Corollary. *Every nonzero representation (π, V) of G contains an irreducible subrepresentation.*

The main result on representations of finite groups is the following observation.

1.2.4. Theorem (Mascke). *Let (π, V) be a representation of G. Let (ν, U) be a subrepresentation of (π, V). Then there exists a subrepresentation (ρ, W) of (π, V) such that $\pi = \nu \oplus \rho$.*

Proof. Let P be a projector of V onto U. Consider the linear map

$$Q = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1})\, P\, \pi(g)$$

on V. Clearly, since U is G-invariant, $Q(V) \subset U$. Moreover, for any $u \in U$, we have

$$Qu = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1})\, P\, \pi(g)u = \frac{1}{[G]} \sum_{g \in G} u = u.$$

Therefore, $U = \operatorname{im} Q$ and $Q^2 = Q$. It follows that Q is a projection onto U along $\ker Q$.

In addition, we have

$$Q\, \pi(h) = \frac{1}{[G]} \sum_{g \in G} \pi(g^{-1})\, P\, \pi(gh) = \frac{1}{[G]} \sum_{g \in G} \pi(hg^{-1})\, P\, \pi(g) = \pi(h)Q$$

for all $h \in G$, i.e., Q is a morphism of (π, V) into (ν, U). Hence $W = \ker Q$ is a G-invariant subspace and $V = U \oplus W$. □

Therefore, the category $\mathcal{R}\mathrm{ep}(G)$ is semisimple.

1.3. Schur Lemma. Let (ν, U) and (π, V) be two irreducible representations of G. Let φ be a morphism of ν into π. Then, $\ker \varphi$ is a subrepresentation of ν. Since ν is irreducible, $\ker \varphi$ is equal to either $\{0\}$ or U. In the latter case, we see that $\varphi = 0$. In the first case, φ is injective. It follows that $\operatorname{im} \varphi$ cannot be $\{0\}$. Since $\operatorname{im} \varphi$ is a subrepresentation of π, It follows that it must be equal to V, and φ is an isomorphism.

This implies the following result.

1.3.1. Proposition. *Let (ν, U) and (π, V) be two irreducible representations of G. Assume that π and ν are not isomorphic. Then* $\mathrm{Hom}_G(U, V) = \{0\}$.

In addition we have the following result.

1.3.2. Theorem (Schur Lemma). *Let (π, V) be an irreducible representation of G. Then* $\mathrm{Hom}_G(V, V) = \mathbb{C}I$.

Proof. Let φ be an endomorphism of π. Since V is finite-dimensional, φ has an eigenvalue $\lambda \in \mathbb{C}$. Therefore, $\psi = \varphi - \lambda I$ is an endomorphism of π which is not injective. By the above discussion, it must be equal to 0. Hence, we have $\varphi = \lambda I$. □

1.4. Regular representation. Let G be a finite group. Denote by $\mathbb{C}[G]$ the space of all complex valued functions on G. clearly, $\dim \mathbb{C}[G] = [G]$. The vector space $\mathbb{C}[G]$ has a structure of inner product space with the inner product

$$(f \mid f') = \frac{1}{[G]} \sum_{g \in G} f(g) \overline{f'(g)}$$

for $f, f' \in \mathbb{C}[G]$.

for $g \in G$ and $f \in \mathbb{C}[G]$ define the function $R(g)f$ by $(R(g)f)(h) = f(gh)$ for any $h \in G$. Clearly, $R(g) : f \longmapsto R(g)f$ is a linear map on $\mathbb{C}[G]$.

Moreover, for $g, h \in G$, we have

$$(R(gh)f)(k) = f(kgh) = (R(h)f)(kg) = (R(g)R(h)f)(k)$$

for any $k \in K$. Therefore $R(gh) = R(g)R(h)$. Clearly, $R(1) = I$. It follows that $(R, \mathcal{C}[G])$ is a representation of G. We call it the *(right) regular representation* of G.

1.4.1. Lemma. *The right regular representation is unitary.*

Proof. Clearly, for $g \in G$, we have

$$(R(g)f \mid R(g)f') = \frac{1}{[G]} \sum_{h \in G} f(hg) \overline{f'(hg)} = \frac{1}{[G]} \sum_{h \in G} f(h) \overline{f'(h)} = (f \mid f')$$

for any $f, f' \in \mathbb{C}[G]$. Therefore, $R(g)$, $g \in G$, are unitary operators. □

The following property of regular representation is critical.

1.4.2. Lemma. *Let $g \in G$, $g \neq 1$. Then $R(g) \neq I$.*

Proof. Denote by δ_h the function on G which is 1 at point $h \in G$ and zero everywhere else. Then we have

$$(R(g)\delta_1)(h) = \delta_1(hg) = \delta_{g^{-1}}(h)$$

for any $h \in G$, i.e., $R(g)\delta_1 = \delta_{g^{-1}} \neq \delta_1$. □

Since R is a direct sum of irreducible representations of G, this result has a following consequence.

1.4.3. THEOREM. *Let $g \in G$, $g \neq 1$. Then there exists an irreducible representation π of G such that $\pi(g) \neq I$.*

In other words, irreducible representations of G separate points in G.

1.5. Abelian finite groups. Let G be a finite group. Let π be an one-dimensional representation of G. Then $\pi(g) = \lambda(g)I$, where $\lambda : G \longrightarrow \mathbb{C}^*$ is group homomorphism of G into the multiplicative group of complex numbers different than zero. This implies that $g \longmapsto |\lambda(g)|$ is a homomorphism of G into the multiplicative group of positive real numbers $\mathbb{R}^*$. Since 1 is the only element of that group of finite order, we conclude that $|\lambda(g)| = 1$, i.e., λ is a homomorphism of G into the group of complex numbers of absolute value equal to 1. We call such homomorphisms the *characters* of G.

Assume that G is abelian finite group. Let (π, V) be an irreducible representation of G. Let $g \in G$. Then

$$\pi(g)\pi(h) = \pi(gh) = \pi(hg) = \pi(h)\pi(g)$$

for all $h \in G$. Therefore, by Schur Lemma, we see that $\pi(g) = \lambda(g)I$ for some complex number $\lambda(g) \neq 0$. By the above discussion, λ is a character of G. This in turn implies that $\dim \pi = 1$.

1.5.1. PROPOSITION. *Let G be a finite group. Then the following conditions are equivalent.*

(i) *G is abelian;*
(i) *all irreducible representations of G are one-dimensional.*

PROOF. We already proved that (i) implies (ii).

Assume that all irreducible representations are one-dimensional. Let $g, h \in G$. Consider the element $a = ghg^{-1}h^{-1}$. Let π be an irreducible representation of G. Then π is one-dimensional and

$$\pi(a) = \pi(ghg^{-1}h^{-1}) = \pi(g)\pi(h)\pi(g)^{-1}\pi(h)^{-1} = I$$

since $\pi(g)$ and $\pi(h)$ commute. By 1.4.3, this implies that $a = 1$, i.e., $ghg^{-1}h^{-1} = 1$. It follows that $gh = hg$ for all $g, h \in G$, i.e., G is abelian. □

Hence, all irreducible representations of an abelian finite group are characters.

Let ϕ and ψ be two characters of G. Then we have

$$\phi(g)(\phi \mid \psi) = \frac{1}{[G]} \sum_{h \in G} \phi(gh)\overline{\psi(h)} = \frac{1}{[G]} \sum_{h \in G} \phi(h)\overline{\psi(g^{-1}h)} = \psi(g)(\phi|\psi)$$

for any $g \in G$. Hence, if ϕ and ψ are different, they are orthogonal to each other. Moreover, for a character ϕ we have

$$\|\phi\|^2 = (\phi \mid \phi) = \frac{1}{[G]} \sum_{g \in G} \phi(g)\overline{\phi(g)} = \frac{1}{[G]} \sum_{g \in G} \phi(g)\phi(g^{-1}) = 1.$$

Hence, the characters form an orthonormal family of functions in $\mathbb{C}[G]$. Moreover, we have the following result.

1.5.2. PROPOSITION. *Characters form an orthonormal basis of $\mathbb{C}[G]$.*

PROOF. Since irreducible representations of G are characters, R is an direct sum of characters. This implies that there is a basis e_i, $1 \le i \le [G]$, and characters ϕ_i, $1 \le i \le [G]$, such that $R(g)e_i = \phi_i(g)e_i$ for any $g \in G$. This in turn implies that

$$e_i(g) = (R(g)e_i)(1) = \phi_i(g)e_i(1)$$

for all $g \in G$. Since e_i is a nonzero vector, we must have $e_i(1) \neq 0$. Hence e_i is proportional to ϕ_i. Therefore, $\mathbb{C}[G]$ is spanned by characters. □

Let $\hat{G}$ be the set of all characters of G. Let ϕ, ψ be two characters of G. Define their product as $(\phi \cdot \psi)(g) = \phi(g) \cdot \psi(g)$ for all $g \in G$. This defines a binary operation on $\hat{G}$. It is easy to check that $\hat{G}$ is an abelian group with this operation. By the above result, $\hat{G}$ is finite and $[\hat{G}] = \dim \mathbb{C}[G] = [G]$. We call $\hat{G}$ the *dual group* of G.

Applying the above discussion twice, we get

$$[\hat{\hat{G}}] = [\hat{G}] = [G].$$

Let $g \in G$. Then the map $\phi \longmapsto \phi(g)$ is a character of $\hat{G}$. This defines a map α from G into $\hat{\hat{G}}$. Moreover,

$$\alpha(gh)(\phi) = \phi(gh) = \phi(g)\phi(h) = \alpha(g)(\phi)\alpha(h)(\phi) = (\alpha(g) \cdot \alpha(h))(\phi)$$

for all $\phi \in \hat{G}$, i.e., $\alpha : G \longrightarrow \hat{\hat{G}}$ is a group morphism.

Assume that $\alpha(g) = 1$. Then $\alpha(g)(\phi) = \phi(g) = 1$ for all $\phi \in \hat{G}$. By 1.4.3, it follows that $g = 1$. Therefore, α is an injective morphism. Hence, $\alpha : G \longrightarrow \hat{\hat{G}}$ is a group isomorphism.

1.5.3. THEOREM. *Let G be an abelian finite group and $\hat{G}$ its dual group. Then*

(i) $[\hat{G}] = [G]$;

(ii) $\alpha : G \longrightarrow \hat{\hat{G}}$ *is an isomorphism.*

This is a special case of *Pontryagin duality.*

Since characters form an orthonormal basis of $\mathbb{C}[G]$, any function f in $\mathbb{C}[G]$ can be written as

$$f = \sum_{\phi \in \hat{G}} (f \mid \phi)\phi.$$

By Bessel equality, we have

$$\|f\|^2 = \sum_{\phi \in \hat{G}} |(f \mid \phi)|^2.$$

We define the *Fourier transform* $\mathcal{F}f$ of f as the function on $\hat{G}$ given by

$$(\mathcal{F}f)(\phi) = \frac{1}{[G]} \sum_{g \in G} f(g)\overline{\phi(g)}, \quad \phi \in \hat{G}.$$

Therefore, the inverse Fourier transform is given by

$$f(g) = \sum_{\phi \in \hat{G}} (\mathcal{F}f)(\phi)\phi(g), \quad g \in G.$$

The above equality then implies that

$$\|f\|^2 = \sum_{\phi \in \hat{G}} |(\mathcal{F}f)(\phi)|^2.$$

This is a special case of *Plancherel theorem*.

1.6. Unitarity. Let (π, V) be a finite-dimensional representation of G. Let $\langle \cdot \mid \cdot \rangle$ be an inner product on V.

Put

$$(u \mid v) = \frac{1}{[G]} \sum_{g \in G} \langle \pi(g)u \mid \pi(g)v \rangle.$$

Clearly, $(u, v) \longmapsto (u \mid v)$ is a linear in first and antilinear in the second variable. Moreover, we have $(u \mid v) = \overline{(v \mid u)}$. In addition,

$$(v \mid v) = \frac{1}{[G]} \sum_{g \in G} \langle \pi(g) \mid \pi(g)v \rangle \geq 0$$

for any $v \in V$. If $(v \mid v) = 0$, we have $\langle \pi(g)v \mid \pi(g)v \rangle = 0$ for all $g \in G$. In particular $\langle v \mid v \rangle = 0$, and $v = 0$. Hence, $(\cdot \mid \cdot)$ is an inner product on V.

1.6.1. Lemma. *Inner product $(\cdot \mid \cdot)$ is G-invariant.*

Proof. Let $g \in G$. Then we have

$$(\pi(g)u \mid \pi(g)v) = \frac{1}{[G]} \sum_{h \in G} \langle \pi(hg)u \mid \pi(hg)v \rangle = \frac{1}{[G]} \sum_{h \in G} \langle \pi(h)u \mid \pi(h)v \rangle = (u \mid v).$$

□

Therefore, there exists an inner product on V such that (π, V) is a unitary representation.

1.7. Orthogonality relations. Let (ν, U) and (π, V) be two irreducible representations of G. Let $A : U \longrightarrow V$ be a linear map. Define

$$B = \frac{1}{[G]} \sum_{g \in G} \pi(g) A \nu(g^{-1}).$$

Then, B is also a linear map from U into V.

Let $g \in G$. Then

$$\pi(g)B = \frac{1}{[G]} \sum_{h \in G} \pi(gh) A \nu(h^{-1}) = \frac{1}{[G]} \sum_{h \in G} \pi(h) A \nu(h^{-1}g) = B\nu(g).$$

Hence, it follows that $B \in \operatorname{Hom}_G(U, V)$.

If ν and π are not equivalent, by Schur Lemma, we have $B = 0$.

1.7.1. Lemma. *Let (ν, U) and (π, V) be two inequivalent irreducible representations of G. Then*

$$\frac{1}{[G]} \sum_{g \in G} \pi(g) A \nu(g^{-1}) = 0$$

for any linear operator $A : U \longrightarrow V$.

Consider now an irreducible representation (π, V) and a linear map $A : V \longrightarrow V$. Let

$$B = \frac{1}{[G]} \sum_{g \in G} \pi(g) A \pi(g^{-1}).$$

Then $B \in \operatorname{Hom}_G(V, V)$. By Schur Lemma, we conclude that $B = \lambda I$ for some $\lambda \in \mathbb{C}$.

Moreover, we have

$$\operatorname{tr} B = \frac{1}{[G]} \sum_{g \in G} \operatorname{tr}(\pi(g) A \pi(g^{-1})) = \frac{1}{[G]} \sum_{g \in G} \operatorname{tr} A = \operatorname{tr} A.$$

This implies the following result.

1.7.2. Lemma. *Let (π, V) be an irreducible representation of G. Then*

$$\frac{1}{[G]} \sum_{g \in G} \pi(g) A \pi(g^{-1}) = \frac{\operatorname{tr} A}{\dim \pi} I$$

for any linear operator $A : V \longrightarrow V$.

By 1.6.1, we can assume that U and V are equipped with G-invariant inner products. Let $(e_i; 1 \leq i \leq \dim \nu)$ and $(f_j; 1 \leq j \leq \dim \pi)$, be two orthonormal bases of U and V respectively. Denote by $\nu(g)_{pq}$ and $\pi(g)_{rs}$ the matrix coefficients of $\nu(g)$ and $\pi(g)$ respectively. Then we first observe that

$$\sum_{s=1}^{\dim \pi} \sum_{p=1}^{\dim \nu} \frac{1}{[G]} \sum_{g \in G} \pi(g)_{rs} A_{sp} \nu(g^{-1})_{pq} = 0,$$

where A_{sp} are matrix coefficients of A. Since A is arbitrary, we conclude that

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rs} \nu(g^{-1})_{pq} = 0$$

for all p, q, r, s. Clearly, since $(\nu(g^{-1})_{pq})$ is a unitary matrix, we have $\nu(g^{-1})_{pq} = \overline{\nu(g)_{qp}}$ for all p, q. Hence, we conclude that

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rs} \overline{\nu(g)_{pq}} = 0$$

for all p, q, r, s.

Let (π, V) be an irreducible representation of G. Denote by $M(\pi)$ the vector subspace of $\mathbb{C}[G]$ spanned by matrix coefficients of π. This subspace is independent of choice of the basis of V. Moreover, it depends only on the equivalence class of π.

1.7.3. Proposition. *Let (π, V) be an irreducible representation of G. Then the subspace $M(\pi)$ is an invariant subspace of the regular representation $(R, \mathbb{C}[G])$.*

Proof. Let $(e_1, e_2, \dots, e_n)$ be a basis of V. Denote by $g \longmapsto \pi(g)_{ij}$, $1 \leq i, j \leq n$, the matrix coefficients of π in this basis. Then $M(\pi)$ is spanned by these functions.

Let $1 \leq p, q \leq n$. Put $f(g) = \pi(g)_{pq}$ for $g \in G$. Then we have

$$(R(g)f)(h) = f(hg) = \pi(hg)_{pq} = \sum_{s=1}^{n} \pi(h)_{ps} \pi(g)_{sq}$$

for all $h \in G$. Therefore, $R(g)f$ is a linear combination of matrix coefficients of π, i.e., $R(g)f \in M(\pi)$. It follows that $M(\pi)$ is invariant for $R(g)$. $\square$

The above calculation proves the following result.

1.7.4. Proposition. *Let ν and π be two inequivalent irreducible representations of G. Then $M(\nu) \perp M(\pi)$.*

Consider now an irreducible representation (π, V). As above, we have

$$\sum_{s=1}^{\dim \pi} \sum_{p=1}^{\dim \pi} \frac{1}{[G]} \sum_{g \in G} \pi(g)_{rs} A_{sp} \pi(g^{-1})_{pq} = \frac{\operatorname{tr} A}{\dim \pi} \delta_{rq}.$$

By selecting A such that $A_{kl} = 1$ for some $k \neq l$, and all other entries are 0, we get

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{lq} = 0.$$

If we select A such that $A_{kk} = 1$ for some k, and all other entries are 0, we get

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{kq} = \frac{1}{\dim \pi} \delta_{rq}.$$

Therefore, we have

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rk} \pi(g^{-1})_{lq} = \frac{1}{\dim \pi} \delta_{kl} \delta_{rq}$$

and

$$\frac{1}{[G]} \sum_{g \in G} \pi(g)_{rk} \overline{\pi(g)_{ql}} = \frac{1}{\dim \pi} \delta_{kl} \delta_{rq}$$

for all $1 \leq k, l, q, r \leq \dim \pi$. These are *Schur orthogonality relations.* This implies that $(\pi(g)_{ij}; 1 \leq i.j \leq \dim \pi)$ is an orthogonal basis of $M(\pi)$.

1.7.5. THEOREM. *Let* (π, V) *be an irreducible representation of* G. *Then* $\dim M(\pi) = (\dim \pi)^2$.

The next result describes the structure of regular representation.

1.7.6. THEOREM. *We have*

$$\mathbb{C}[G] = \bigoplus_{\pi \in \hat{G}} M(\pi).$$

PROOF. By 1.7.3, the subspaces $M(\pi)$, $\pi \in \hat{G}$, are invariant subspaces of $(R, \mathbb{C}[G])$. Therefore, their orthogonal sum $M = \bigoplus_{\pi \in \hat{G}} M(\pi)$ is an invariant subspace in $(R, \mathbb{C}[G])$.

Let $M^{\perp}$ be the orthogonal complement of M. Then $M^{\perp}$ is also an invariant subspace since R is unitary. Assume that $M^{\perp}$ is different from $\{0\}$. Then it contains an irreducible representation (ν, U) of G by 1.2.3. Let $(f_1, f_2, \dots, f_m)$ be a basis of U. Then we have

$$\nu(g) f_i = \sum_{j=1}^{m} \pi(g)_{ji} f_j.$$

Therefore, we have

$$f_i(g) = (R(g) f_i)(1) = (\nu(g) f_i)(1) = \sum_{j=1}^{m} \nu(g)_{ji} f_j(1)$$

for all $g \in G$. Hence, we have $f_i \in M(\nu) \subset M$. Therefore, f_i is orthogonal on itself, and $f_i = 0$. This contradicts our choice. It follows that $M^{\perp} = \{0\}$, i.e., $M = \mathbb{C}[G]$. $\square$

This has the following consequence.

1.7.7. Corollary. *We have*

$$[G] = \sum_{\pi \in \hat{G}} (\dim(\pi))^2.$$

1.8. Characters and central functions. Let (π, V) be a finite-dimensional representation of G. Define the function $\mathrm{ch}(\pi) : G \longrightarrow \mathbb{C}$ by

$$\mathrm{ch}(\pi)(g) = \mathrm{tr}\,\pi(g)$$

for $g \in G$. The function $\mathrm{ch}(\pi)$ on G is called the *character* of π. The character of π depends only on the equivalence class of π.

1.8.1. Example. Let $(R, \mathbb{C}[G])$ be the regular representation of G. For any $g \in G$, define the function δ_g which is equal 1 at g and 0 everywhere else. Clearly, $(\delta_g, g \in G)$ is a basis of $\mathbb{C}[G]$.

Let $g \in G$. Then we have

$$(R(g)\delta_h)(k) = \delta_h(kg) = \begin{cases} 1, & \text{if } k = hg^{-1}; \\ 0, & \text{if } k \neq hg^{-1} \end{cases} = \delta_{hg_{-1}}(k)$$

for all $k \in G$. Hence $R(g)\delta_h = \delta_{hg^{-1}}$ for all $h \in G$. It follows that the matrix of $R(g)$ has nonzero coefficients on the diagonal if and only if $g = 1$. Hence we see that $\mathrm{tr}\,R(g) = 0$ if $g \neq 1$ and $\mathrm{tr}\,R(1) = \dim(R) = [G]$. Therefore, we have $\mathrm{ch}(R) = [G] \cdot \delta_1$.

Moreover, if $\pi = \nu \oplus \rho$ we have

$$\mathrm{ch}(\pi) = \mathrm{ch}(\nu) + \mathrm{ch}(\rho).$$

Hence, the character map defines a homomorphism of the Grothendieck group of $\mathcal{R}\mathrm{ep}_{fd}(G)$ into functions on G.

1.8.2. Theorem.
(i) *Let (π, V) and (ν, U) be two irreducible representations of G. If π is not equivalent to ν we have $(\mathrm{ch}(\pi) \mid \mathrm{ch}(\nu)) = 0$.*
(ii) *Let (π, V) be irreducible representation of G. Then we have $(\mathrm{ch}(\pi) \mid \mathrm{ch}(\pi)) = 1$.*

Proof. This follows immediately from Schur orthogonality relations. □

Therefore, $(\mathrm{ch}(\pi); \pi \in \hat{G})$ is an orthonormal family of functions in $\mathbb{C}[G]$.

Moreover we see that

$$\dim \mathrm{Hom}_G(U, V) = (\mathrm{ch}(\nu) \mid \mathrm{ch}(\pi))$$

for any two finite-dimensional representations of G.

Clearly, if $g, h \in G$ we have

$$\mathrm{ch}(\pi)(hgh^{-1}) = \mathrm{tr}(\pi(hgh^{-1})) = \mathrm{tr}(\pi(h)\pi(g)\pi(h)^{-1}) = \mathrm{tr}(\pi(g) = \mathrm{ch}(\pi)(g).$$

Hence, characters are constant on conjugacy classes in G.

This has the following consequence.

1.8.3. Proposition. *Let (π, V) be an irreducible representation of G. Let f be a matrix coefficient of π. Then*

$$\frac{1}{[G]} \sum_{h \in G} f(hgh^{-1}) = \frac{f(1)}{\dim \pi} \mathrm{ch}(\pi)(g)$$

for any $g \in G$.

Proof. Clearly, both sides of the equality are linear forms in f on the space $M(\pi)$. Therefore, it is enough to check the equality on a basis of $M(\pi)$.

By 1.6.1 we can assume that π is unitary. Let $(e_i; 1 \le i \le \dim \pi)$, be an orthonormal basis of V. Let $g \longmapsto \pi(g)_{ij}$ the matrix coefficients of π in that basis. Then they are a a basis of $M(\pi)$.

For these functions we have

$$\begin{aligned}\frac{1}{[G]} \sum_{h \in G} \pi(hgh^{-1})_{ij} &= \frac{1}{[G]} \sum_{h \in G} \left(\sum_{k=1}^{\dim \pi} \sum_{l=1}^{\dim \pi} \pi(h)_{ik} \pi(g)_{kl} \pi(h^{-1})_{lj} \right) \\ &= \sum_{k=1}^{\dim \pi} \sum_{l=1}^{\dim \pi} \pi(g)_{kl} \left(\frac{1}{[G]} \sum_{h \in G} \pi(h)_{ik} \overline{\pi(h)_{jl}} \right) = \frac{1}{\dim \pi} \sum_{k=1}^{\dim \pi} \sum_{l=1}^{\dim \pi} \pi(g)_{kl} \delta_{ij} \delta_{kl} \\ &= \frac{1}{\dim \pi} \sum_{k=1}^{\dim \pi} \pi(g)_{kk} \delta_{ij} = \frac{1}{\dim \pi} \operatorname{ch}(\pi)(g) \delta_{ij} = \frac{1}{\dim \pi} \operatorname{ch}(\pi)(g) \pi(1)_{ij}.\end{aligned}$$

using Schur orthogonality relations. □

We say that a function f on G is *central* if it is constant on conjugacy classes in G. Denote by $C(G)$ the vector subspace of $\mathbb{C}[G]$ consisting of all central functions. Clearly, the dimension of $C(G)$ is equal to the number of conjugacy classes in G.

By 1.8.2, $(\operatorname{ch}(\pi); \pi \in \hat{G})$ is an orthonormal family of functions in $C(G)$.

1.8.4. Theorem. $(\operatorname{ch}(\pi); \pi \in \hat{G})$ *is an orthonormal basis of* $C(G)$.

Proof. We already know that $(\operatorname{ch}(\pi); \pi \in \hat{G})$ is an orthonormal family in $C(G)$.

Let f be a central function on G orthogonal on all characters $\operatorname{ch}(\pi)$, $\pi \in \hat{G}$. Let $\phi \in M(\pi)$, then we have

$$\begin{aligned}(\phi \mid f) &= \frac{1}{[G]} \sum_{g \in G} \phi(g) \overline{f(g)} = \frac{1}{[G]} \sum_{g \in G} \left(\frac{1}{[G]} \sum_{h \in G} \phi(g) \overline{f(h^{-1}gh)} \right) \\ &= \frac{1}{[G]} \sum_{h \in G} \left(\frac{1}{[G]} \sum_{g \in G} \phi(g) \overline{f(h^{-1}gh)} \right) = \frac{1}{[G]} \sum_{g \in G} \left(\frac{1}{[G]} \sum_{h \in G} \phi(hgh^{-1}) \right) \overline{f(g)},\end{aligned}$$

since f is a central function. By 1.8.3, it follows that

$$(\phi \mid f) = \frac{f(1)}{\dim \pi} \frac{1}{[G]} \sum_{g \in G} \operatorname{ch}(\pi)(g) \overline{f(g)} = \frac{f(1)}{\dim \pi} (\operatorname{ch}(\pi) | f) = 0.$$

Hence f is orthogonal to $M(\pi)$ for all $\pi \in \hat{G}$. By 1.7.6, it follows that f is orthogonal to $\mathbb{C}[G]$. Hence $f = 0$. Therefore, $(\operatorname{ch}(\pi); \pi \in \hat{G})$ is a maximal orthonormal family in $C(G)$, i.e., it is an orthonormal basis. □

Therefore, $\dim C(G)$ is equal to $\operatorname{Card}(\hat{G})$. This implies the following result.

1.8.5. Corollary. $\operatorname{Card}(\hat{G})$ *is equal to the number of conjugacy classes in* G.

2. Frobenius reciprocity

2.1. Restriction functor. Let G be a a finite group. Let H be the a subgroup of G. Denote by $\mathcal{R}\mathrm{ep}(G)$, resp. $\mathcal{R}\mathrm{ep}(H)$, the categories of representations of G, resp. H.

Let (π, V) be a representation in $\mathcal{R}\mathrm{ep}(G)$. Denote by ν the restriction of function $\pi : G \longrightarrow \mathrm{GL}(V)$ to H. Then (ν, V) is a representation in $\mathcal{R}ep(H)$. This representation is called the *restriction* of π to H and denoted by $\mathrm{Res}_H^G(\pi)$ (when there is no ambiguity we shall just write $\mathrm{Res}(\pi)$).

Clearly, Res_H^G is an exact functor form the abelian category $\mathcal{R}\mathrm{ep}(G)$ into the abelian category $\mathcal{R}\mathrm{ep}(H)$.

2.2. Induction functor. Let (ν, U) be a representation of H. Denote by $V = \mathrm{Ind}(U)$ the space of all functions $F : G \longrightarrow U$ such that $F(hg) = \nu(h)F(g)$ for all $h \in H$ and $g \in G$. Let F be the function in V and $g \in G$. Then the function $\rho(g)F : G \longrightarrow U$ defined by $(\rho(g)F)(g') = F(g'g)$ for all $g' \in G$, satisfies

$$(\rho(g)F)(hg') = F(hg'g) = \nu(h)F(g'g) = \nu(h)(\rho(g)F)(g')$$

for all $h \in H$ and $g' \in G$. Therefore $\rho(g)F$ is a function in V.

Clearly $\rho(g)$ is a linear operator on V for any $g \in G$. Moreover, $\rho(1)$ is the identity on V. For any F in V we have

$$(\rho(gg')F)(g'') = F(g''gg') = (\rho(g')F)(g''g) = (\rho(g)(\rho(g')F))(g'')$$

for all $g'' \in G$, i.e., we have

$$\rho(gg')F = \rho(g)(\rho(g')F)$$

for $g, g' \in G$. Therefore, $\rho(gg') = \rho(g)\rho(g')$ for any $g, g' \in G$ and ρ is a representation of G on V.

The representation (ρ, V) of G is called the *induced representation* and denoted by $\mathrm{Ind}_H^G(\nu)$.

If H is the identity subgroup and ν is the trivial representation, the corresponding induced representation is the regular representation of G.

Let (ν, U) and (ν', U') be two representations of H and ϕ a morphism of ν into ν'. Let F be a function in $\mathrm{Ind}(U)$. Then $\Phi(F)(g) = \phi(F(g))$ for all $g \in G$ is a function from G into U'. Moreover, we have

$$\Phi(F)(hg) = \phi(F(hg)) = \phi(\nu(h)F(g)) = \nu'(h)\phi(F(g)) = \nu'(h)\Phi(F)(g)$$

for all $h \in H$ and $g \in G$. Hence, $\Phi(F)$ is in $\mathrm{Ind}(U')$. Clearly, Φ is a linear map from $\mathrm{Ind}(U)$ into $\mathrm{Ind}(U')$.

Moreover, we have

$$(\rho'(g)\Phi(F))(g') = \Phi(F)(g'g) = \phi(F(g'g)) = \phi((\rho(g)F)(g')) = \Phi(\rho(g)F)(g')$$

for all $g' \in G$. Therefore, $\rho'(g) \circ \Phi = \Phi \circ \rho(g)$ for all $g \in G$, and Φ is a morphism of $\mathrm{Ind}_H^G(\nu)$ into $\mathrm{Ind}_H^G(\nu')$. We put $\mathrm{Ind}_H^G(\phi) = \Phi$. It is straightforward to check that in this way Ind_H^G becomes an additive functor from $\mathcal{R}\mathrm{ep}(H)$ into $\mathcal{R}\mathrm{ep}(G)$.

We call $\mathrm{Ind}_H^G : \mathcal{R}\mathrm{ep}(H) \longrightarrow \mathcal{R}\mathrm{ep}(G)$ the *induction functor*.

The next result is a functorial form of *Frobenius reciprocity*.

2.2.1. THEOREM. *The induction functor* $\mathrm{Ind}_H^G : \mathcal{R}\mathrm{ep}(H) \longrightarrow \mathcal{R}\mathrm{ep}(G)$ *is a right adjoint functor of the restriction functor* $\mathrm{Res}_H^G : \mathcal{R}\mathrm{ep}(G) \longrightarrow \mathcal{R}\mathrm{ep}(H)$.

Proof. Let (ν, U) a representation of H. Consider the induced representation $\mathrm{Ind}_H^G(\nu)$ of G. The evaluation map $e : \mathrm{Ind}(U) \longrightarrow U$ given by $e(F) = F(1)$ for $F \in \mathrm{Ind}(U)$, satisfies

$$e(\rho(h)F)(1) = (\rho(h)F)(1) = F(h) = \nu(h)F(1) = \nu(h)e(F)$$

for all $F \in \mathrm{Ind}(U)$, i.e., e is a morphism of representations of H.

Let (π, V) be a representation of G. Let $\Psi : V \longrightarrow \mathrm{Ind}(U)$ be a morphism of representations of G. Then the composition $e \circ \Psi$ is a morphism of $\mathrm{Res}_H^G(\pi)$ into ν. Denote the linear map $\Psi \longmapsto e \circ \Psi$ from $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$ into $\mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$ by A.

Let $\phi : V \longrightarrow U$ be a morphism of representations of H. Let $v \in V$. Then we consider the function $F_v : G \longrightarrow U$ given by $F_v(g) = \phi(\pi(g)v)$ for any $g \in G$. First, for $h \in H$, we have

$$F_v(hg) = \phi(\pi(hg)v) = \phi(\pi(h)\pi(g)v) = \nu(h)\phi(\pi(g)v) = \nu(h)F_v(g)$$

for all $g \in G$. Hence F_v is a function in $\mathrm{Ind}(U)$. Consider the map $\Phi : V \longrightarrow \mathrm{Ind}(U)$ defined by $\Phi(v) = F_v$. Clearly,

$$\begin{aligned}\Phi(v + v')(g) = F_{v+v'}(g) = \phi(\pi(g)(v + v')) &= \phi(\pi(g)v) + \phi(\pi(g)v') \\ &= F_v(g) + F_{v'}(g) = \Phi(v)(g) + \Phi(v')(g)\end{aligned}$$

for any $g \in G$, hence we have $\Phi(v+v') = \Phi(v)+\Phi(v')$ for all $v, v' \in V$. In addition,

$$\Phi(\alpha v)(g) = \alpha\phi(\pi(g)v) = \alpha\Phi(v)(g)$$

for all $g \in G$, hence we have $\Phi(\alpha v) = \alpha\Phi(v)$ for all $\alpha \in \mathbb{C}$ and $v \in V$. It follows that Φ is a linear map from V into $\mathrm{Ind}(U)$. Moreover, we have

$$\Phi(\pi(g)v)(g') = \phi(\pi(g')\pi(g)v) = \phi(\pi(g'g)v) = \Phi(v)(g'g) = (\rho(g)\Phi(v))(g')$$

for all $g' \in V$. Hence, we have $\Phi(\pi(g)v) = \rho(g)\Phi(v)$ for all $g \in G$ and $v \in V$. Therefore, Φ is a morphism of representations (π, V) and $\mathrm{Ind}_H^G(\nu)$ of G. Denote the map $\phi \longmapsto \Phi$ from $\mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$ into $\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$ by B.

Clearly, for $\phi \in \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu)$, we have

$$((A \circ B)(\phi))(v) = (A(\Phi))(v) = \Phi(v)(1) = F_v(1) = \phi(v)$$

for all $v \in V$. Therefore, $A \circ B$ is the identity map.

In addition, for $\Psi \in \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$, we have

$$\begin{aligned}(((B \circ A)(\Psi))(v))(g) = (B(A(\Psi))(v))(g) &= A(\Psi)(\pi(g)v) \\ &= (\Psi(\pi(g)v))(1) = (\rho(g)\Psi(v))(1) = \Psi(v)(g)\end{aligned}$$

for all $g \in G$. Hence, we have $((B\circ A)(\Psi))(v) = \Psi(v)$ for all $v \in V$, i.e., $(B\circ A)(\Psi) = \Psi$ for all Ψ and $B \circ A$ is also the identity map. ☐

By Maschke's theorem, $\mathcal{R}ep(H)$ is semisimple, and every short exact sequence splits. Therefore we have the following result.

2.2.2. Theorem. *The induction functor* $\mathrm{Ind}_H^G : \mathcal{R}\mathrm{ep}(H) \longrightarrow \mathcal{R}\mathrm{ep}(G)$ *is exact.*

2.3. Induction in stages. Let K be a subgroup of H. Then we have $\mathrm{Res}_K^G = \mathrm{Res}_K^H \circ \mathrm{Res}_H^G$ as functors from $\mathcal{R}\mathrm{ep}(G)$ into $\mathcal{R}\mathrm{ep}(K)$. Since induction functors are right adjoints, this immediately implies the following result which is called the *induction in stages*.

2.3.1. THEOREM. *Let H be a subgroup of G and K a subgroup of H. Then the functors Ind_K^G and $\mathrm{Ind}_H^G \circ \mathrm{Ind}_K^H$ are isomorphic.*

2.4. Frobenius Reciprocity. Obviously, the restriction functor Res_H^G maps finite-dimensional representations into finite dimensional representations. From the following result we see that the induction functor Ind_H^G does the same.

2.4.1. PROPOSITION. *Let (ν, U) be a finite-dimensional representation of H. Then*

$$\dim \mathrm{Ind}_H^G(\nu) = \mathrm{Card}(H\backslash G) \cdot \dim(\nu).$$

PROOF. Let C be a right H-coset in G. Let g_C be an element in C. Then the functions

$$F_{C,v}(g) = \begin{cases} \nu(gg_C^{-1})v & \text{for } g \in Hg_C; \\ 0 & \text{for } g \notin Hg_C; \end{cases}$$

span $\mathrm{Ind}(U)$. If $e_1, e_2, \dots, e_m$ is a basis of U, the family F_{C,e_i}, $C \in H\backslash G$, $1 \le i \le m$, is a basis of $\mathrm{Ind}(U)$. □

Let (π, V) be an irreducible representation of G and ν an irreducible representation of H. Then $\mathrm{Ind}_H^G(\nu)$ is finite-dimensional by 2.4.1 and a direct sum of irreducible representations of G. The multiplicity of π in this direct sum is $\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu))$ by Schur Lemma. By 2.2.1, we conclude that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G(\nu)) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\mathrm{Res}_H^G(\pi), \nu).$$

The latter expression is the multiplicity of ν in $\mathrm{Res}_H^G(\pi)$.

This leads to the following version of Frobenius reciprocity for representations of finite groups.

2.4.2. THEOREM. *Let π be an irreducible representation of G and ν an irreducible representation of H. Then the multiplicity of π in $\mathrm{Ind}_H^G(\nu)$ is equal to the multiplicity of ν in $\mathrm{Res}_H^G(\pi)$.*

2.5. An example. Let S_3 be the symmetric group in three letters. We shall show how above results allow us to construct irreducible representations of S_3.

The order of S_3 is $3! = 6$. It contains the normal subgroup A_3 consisting of all even permutations which is of order 3. The quotient group S_3/A_3 consists of two elements.

The identity element is $(1\ 2\ 3)$. The other two even permutations are $(2\ 3\ 1)$ and $(3\ 1\ 2)$. We have $(2\ 1\ 3)^2 = 1$ and

$$(2\ 1\ 3)(2\ 3\ 1)(2\ 1\ 3) = (3\ 1\ 2).$$

Hence nontrivial even permutations form a conjugacy class.

The odd permutations are $(2\ 1\ 3)$, $(1\ 3\ 2)$ and $(3\ 2\ 1)$. Since $(2\ 1\ 3)(1\ 3\ 2)(2\ 1\ 3) = (3\ 1\ 2)$, $(1\ 3\ 2)$ and $(3\ 2\ 1)$ are conjugate. On the other hand, $(1\ 3\ 2)^2 = 1$ and $(1\ 3\ 2)(2\ 3\ 1)(1\ 3\ 2) = (3\ 2\ 1)$, and $(2\ 3\ 1)$ and $(3\ 2\ 1)$ are conjugate. Therefore all odd permutations form a conjugacy class. It follows that S_3 has three conjugacy classes. Therefore S_3 has three irreducible representations.

Clearly, two irreducible representations of S_3 are the trivial representation and the sign representation. Since $1^2 + 1^2 + 2^2 = 6$, by Burnside theorem, the third irreducible representation π is two-dimensional. By 1.8.1, the character of regular representation is 6 at the identity element and 0 on all other elements. By Burnside theorem the character of π is one half of the difference of the characters of regular representation and the direct sum of trivial and sign representation. The latter character is 2 on even elements and 0 on odd elements. Therefore, the character of π is 2 at the identity, -1 on nontrivial even elements and 0 at odd elements. It follows that the character of π is supported on A_3.

The group A_3 is cyclic with three elements. It has two nontrivial one-dimensional representations. If we pick a generator $a = (2\ 3\ 1)$ of A_3 one character maps a into $e^{i\frac{2\pi}{3}}$ and the other maps a to $e^{-i\frac{2\pi}{3}}$. We call the first one ν. By a direct calculation we see that $(2\ 1\ 3)a(2\ 1\ 3) = a^{-1}$. The restriction of π to A_3 is a direct sum of two characters of A_3. Since we know that $\operatorname{ch}(\pi)(a) = -1$ we see that it must be

$$\nu(a) + \nu(a)^{-1} = e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}} = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1.$$

Therefore, $\operatorname{Res}^{S_3}_{A_3}(\pi) = \nu \oplus \nu^{-1}$.

By Frobenius reciprocity, we have

$$\dim_{\mathbb{C}} \operatorname{Hom}_{S_3}(\pi, \operatorname{Ind}^{S_3}_{A_3}(\nu)) = \dim \operatorname{Hom}_{A_3}(\operatorname{Res}^{S_3}_{A_3}(\pi), \nu) = 1.$$

Hence, π is a equivalent to a subrepresentation of $\operatorname{Ind}^{S_3}_{A_3}(\nu)$. Since their dimensions are equal, we have $\pi \cong \operatorname{Ind}^{S_3}_{A_3}(\nu)$. Analogously, we prove that $\pi \cong \operatorname{Ind}^{S_3}_{A_3}(\nu^{-1})$.

Therefore we proved that the dual of S_3 consists of the classes of the trivial representation, sign representation and the induced representation $\operatorname{Ind}^{S_3}_{A_3}(\nu) \cong \operatorname{Ind}^{S_3}_{A_3}(\nu^{-1})$.

2.6. Characters of induced representations. Let (ν, U) be a finite-dimensional representation of H. Let $(e_i; 1 \leq i \leq n)$ be a basis of U. In the proof of 2.4.1, we constructed a basis $(F_{C,i}; C \in H\backslash G,\ 1 \leq i \leq n)$ of $\operatorname{Ind}(U)$. Let $C \in H\backslash G$ and $1 \leq i \leq n$. Let $g \in G$. Then

$$(\rho(g)F_{C,i})(g') = F_{C,i}(g'g)$$

for all $g' \in G$, i.e., $\rho(g)F_{C,i}$ is supported on the coset $D = C \cdot g^{-1}$. Therefore, it is a linear combination of $F_{D,j}$, $1 \leq j \leq n$, i.e.,

$$\rho(g)F_{C,i} = \sum_{j=1}^{n} c_j F_{D,j}.$$

Hence, $\rho(g)F_{C,i}$ is a linear combination of $F_{C,j}$, $1 \leq j \leq n$, if and only if $D = C$, i.e., g_C and $g_C g$ are in the same H-coset. This implies that $g_C g = h g_C$ for some $h \in H$, i.e., $g_C g g_C^{-1} = h \in H$. Conversely, if $g_C g g_C^{-1} \in H$ for some C, we have

$$C = Hg_C = Hg_C g = C \cdot g$$

and g_C and $g_C g$ are in the same H-coset. Moreover, we have

$$\begin{aligned}(\rho(g)F_{C,i})(g_C) = F_{C,i}(g_C g) = F_{C,i}(h g_C) = \nu(h)F_{C,i}(g_C) \\ = \nu(h)e_i = \sum_{j=1}^{n} \nu(h)_{ji} e_j = \sum_{j=1}^{n} \nu(h)_{ji} F_{C,j}(g_C).\end{aligned}$$

This in turn implies that

$$\rho(g)F_{C,i} = \sum_{j=1}^{n} \nu(h)_{ji} F_{C,j}$$

if $C \cdot g^{-1} = C$. Therefore, the matrix of $\rho(g)$ has a nonzero diagonal entry in the basis $(F_{C,i}, C \in H\backslash G, 1 \le i \le n)$, only if $C = C \cdot g$ and then these entries are $\nu(h)_{jj}$, $1 \le j \le n$. This implies that

$$\begin{aligned}
\operatorname{ch}(\operatorname{Ind}_H^G(\nu))(g) &= \sum_{C\cdot g = C} \operatorname{ch}(\nu)(h) = \sum_{C \cdot g = C} \operatorname{ch}(\nu)(g_C g g_C^{-1}) \\
&= \sum_{g_C g g_C^{-1} \in H} \operatorname{ch}(\nu)(g_C g g_C^{-1}) = \frac{1}{[H]} \sum_{h \in H} \sum_{g_C g g_C^{-1} \in H} \operatorname{ch}(\nu)(h g_C g g_C^{-1} h^{-1}) \\
&= \frac{1}{[H]} \sum_{g' g g'^{-1} \in H} \operatorname{ch}(\nu)(g' g g'^{-1}).
\end{aligned}$$

We extend the character of ν to a function χ_ν on G which vanishes outside H. Then we get the following result.

2.6.1. THEOREM. *The character of induced representation* $\operatorname{Ind}_H^G(\nu)$ *is equal to*

$$\operatorname{ch}(\operatorname{Ind}_H^G(\nu))(g) = \frac{1}{[H]} \sum_{g' \in G} \chi_\nu(g' g g'^{-1}).$$

Therefore the character of the induced representation is proportional to the average of the function χ_ν on the equivalence classes in G.

In particular we have the following result.

2.6.2. COROLLARY. *The character of* $\operatorname{Ind}_H^G(\nu)$ *is supported in the union of conjugacy classes in G which intersect H.*

The result is particularly simple if H is a normal subgroup of G.

2.6.3. COROLLARY. *Let H be a normal subgroup of G. Then:*

(i) *the support of the character of* $\operatorname{Ind}_H^G(\nu)$ *is in H;*

(ii) *we have*

$$\operatorname{ch}(\operatorname{Ind}_H^G(\nu))(h) = \frac{1}{[H]} \sum_{g \in G} \operatorname{ch}(\nu)(g h g^{-1})$$

for any $h \in H$.

2.7. An example. Consider again the representation $\pi \cong \operatorname{Ind}_{A_3}^{S_3}(\nu)$. By the above formula, its character vanishes outside of A_3 and is equal to

$$\operatorname{ch}(\pi)(h) = \frac{1}{3} \sum_{g \in S_3} \nu(g h g^{-1})$$

for $h \in A_3$. If $h = 1$, we see that

$$\operatorname{ch}(\pi)(1) = \frac{6}{3} = 2.$$

If $h = a$, we have $gag^{-1} = a$ for $g \in A_3$. If g is not in A_3, it is in the other A_3-coset. Therefore, it is in the coset represented by (2 1 3). By the calculation done before, $gag^{-1} = a^{-1}$ for $g \notin A_3$. Therefore, we have

$$\operatorname{ch}(\pi)(a) = \frac{1}{3}\sum_{g \in S_3} \nu(gag^{-1}) = \nu(a) + \nu(a^{-1}) = -1.$$

This agrees with the calculation of the character of π done before.

2.8. Characters and Frobenius reciprocity. Now we are going to give a proof of 2.4.2 based on character formula for the induced representation and the orthogonality relations.

We denote by $(\cdot \mid \cdot)_G$ the inner product on $\mathbb{C}[G]$ and by $(\cdot \mid \cdot)_H$ the inner product on $\mathbb{C}[H]$. Let π be a finite-dimensional representation of G and ν a finite-dimensional representation of H. Then we have

$$(\operatorname{ch}(\operatorname{Ind}_H^G(\nu)) \mid \operatorname{ch}(\pi))_G = \frac{1}{[G]}\sum_{g \in G} \operatorname{ch}(\operatorname{Ind}_H^G(\nu))(g)\overline{\operatorname{ch}(\pi)(g)}$$

$$= \frac{1}{[G][H]}\sum_{g \in G}\left(\sum_{g' \in G} \chi_\nu(g'gg'^{-1})\overline{\operatorname{ch}(\pi)(g)}\right) = \frac{1}{[H]}\sum_{g' \in G}\frac{1}{[G]}\left(\sum_{g \in G} \chi_\nu(g'gg'^{-1})\overline{\operatorname{ch}(\pi)(g)}\right)$$

$$= \frac{1}{[H]}\sum_{g' \in G}\frac{1}{[G]}\left(\sum_{g \in G} \chi_\nu(g)\overline{\operatorname{ch}(\pi)(g)}\right) = \frac{1}{[H]}\sum_{h \in H} \operatorname{ch}(\nu)(h)\overline{\operatorname{ch}(\pi)(h)}$$

$$= (\operatorname{ch}(\nu) \mid \operatorname{ch}(\operatorname{Res}_H^G(\pi)))_H.$$

CHAPTER II

Representations of compact groups

1. Haar measure on compact groups

1.1. Compact groups. Let G be a group. We say that G is a *topological group* if G is equipped with hausdorff topology such that the multiplication $(g, h) \longmapsto gh$ from the product space $G \times G$ into G and the inversion $g \longmapsto g^{-1}$ from G into G are continuous functions.

Let G and H be two topological groups. A *morphism of topological groups* $\varphi : G \longrightarrow H$ is a group homomorphism which is also continuous.

Topological groups and morphisms of topological groups for the category of topological groups.

Let G be a topological group. Let G^{opp} be the topological space G with the multiplication $(g, h) \longmapsto g \star h = h \cdot g$. Then G^{opp} is also a topological group which we call the *opposite group* of G. Clearly, the inverse of an element $g \in G$ is the same as the inverse in G^{opp}. Moreover, the map $g \longmapsto g^{-1}$ is an isomorphism of G with G^{opp}. Clearly, we have $(G^{opp})^{opp} = G$.

A topological group G is *compact*, if G is a compact space. The opposite group of a compact group is compact.

We shall need the following fact. Let G be a topological group. We say that a function $\phi : G \longrightarrow \mathbb{C}$ is *right (resp. left) uniformly continuous* on G if for any $\epsilon > 0$ there exists an open neighborhood U of 1 such that $|\phi(g) - \phi(h)| < \epsilon$ for any $g, h \in G$ such that $gh^{-1} \in U$ (resp. $g^{-1}h \in U$). Clearly, a right uniformly continuous function on G is left uniformly continuous function on G^{opp}.

1.1.1. Lemma. *Let G be a compact group. Let ϕ be a continuous function on G. Then ϕ is right and left uniformly continuous on G.*

Proof. By the above discussion, it is enough to prove that ϕ is right uniformly continuous.

Let $\epsilon > 0$. Let consider the set $A = \{(g, g') \in G \times G \mid |\phi(g) - \phi(g')| < \epsilon\}$. Then A is an open set in $G \times G$. Let U be an open neighborhood of 1 in G and $B_U = \{(g, g') \in G \times G \mid g'g^{-1} \in U\}$. Since the function $(g, g') \longmapsto g'g^{-1}$ is continuous on $G \times G$ the set B_U is open. It is enough to show that there exists an open neighborhood U of 1 in G such that $B_U \subset A$.

Clearly, B_U are open sets containing the diagonal Δ in $G \times G$. Moreover, under the homomorphism κ of $G \times G$ given by $\kappa(g, g') = (g, g'g^{-1})$, $g, g' \in G$, the sets B_U correspond to the sets $G \times U$. In addition, the diagonal Δ corresponds to $G \times \{1\}$. Assume that the open set A corresponds to O.

By the definition of product topology, for any $g \in G$ there exist neighborhoods U_g of 1 and V_g of g such that $V_g \times U_g$ is a neighborhood of $(g, 1)$ contained in O. Clearly, $(V_g; g \in G)$ is an open cover of G. Since G is compact, there exists a finite subcovering $(V_{g_i}; 1 \leq i \leq n)$ of G. Put $U = \bigcap_{i=1}^n U_{g_i}$. Then U is an open neighborhood of 1 in G. Moreover, $G \times U$ is an open set in $G \times G$ contained in O. Therefore $B_U \subset A$. $\square$

Therefore, we can say that a continuous function on G is uniformly continuous.

1.2. A compactness criterion. Let X be a compact space. Denote by $C(X)$ the space of all complex valued continuous functions on X. Let $\|f\| = \sup_{x \in X} |f(x)|$ for any $f \in C(X)$. Then $f \longmapsto \|f\|$ is a norm on $C(X)$, $C(X)$ is a Banach space.

Let $\mathcal{S}$ be a subset of $C(X)$.

We say that $\mathcal{S}$ is *equicontinuous* if for any $\epsilon > 0$ and $x \in X$ there exists a neighborhood U of x such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and $f \in \mathcal{S}$.

We say that $\mathcal{S}$ is *pointwise bounded* if for any $x \in X$ there exists $M > 0$ such that $|f(x)| \leq M$ for all $f \in \mathcal{S}$.

The aim of this section is to establish the following theorem.

1.2.1. Theorem (Arzelà-Ascoli). *Let $\mathcal{S}$ be a pointwise bounded, equicontinuous subset of $C(X)$. Then the closure of $\mathcal{S}$ is a compact subset of $C(X)$.*

Proof. We first prove that $\mathcal{S}$ is bounded in $C(X)$. Let $\epsilon > 0$. Since $\mathcal{S}$ is equicontinuous, for any $x \in X$, there exists an open neighborhood U_x of x such that $y \in U_x$ implies that $|f(y) - f(x)| < \epsilon$ for all $f \in \mathcal{S}$. Since X is compact, there exists a finite set of points $x_1, x_2, \ldots, x_n \in X$ such that $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$ cover X.

Since $\mathcal{S}$ is pointwise bounded, there exists $M \geq 2\epsilon$ such that $|f(x_i)| \leq \frac{M}{2}$ for all $1 \leq i \leq n$ and all $f \in \mathcal{S}$. Let $x \in X$. Then $x \in U_{x_i}$ for some $1 \leq i \leq n$. Therefore, we have

$$|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| < \frac{M}{2} + \epsilon \leq M$$

for all $f \in S$. It follows that $\|f\| \leq M$ for all $f \in \mathcal{S}$. Hence $\mathcal{S}$ is contained in a closed ball of radius M centered at 0 in $C(X)$.

Now we prove that $\mathcal{S}$ is contained in a finite family of balls of fixed small radius centered in elements of $\mathcal{S}$. We keep the choices from the first part of the proof. Let $D = \{z \in \mathbb{C} \mid |z| \leq M\}$. Then D is compact. Consider the compact set D^n. It has natural metric given by $d(z, y) = \max_{1 \leq i \leq n} |z_i - y_i|$. There exist points $\alpha_1, \alpha_2, \ldots, \alpha_m$ in D^n such that the balls $B_i = \{\beta \in D^n \mid d(\alpha_i, \beta) < \epsilon\}$ cover D^n.

Denote by Φ the map from $\mathcal{S}$ into D^n given by $f \longmapsto (f(x_1), f(x_2), \ldots, f(x_n))$. Then we can find a subfamily of the above cover of D^n consisting of balls intersecting $\Phi(\mathcal{S})$. After a relabeling, we can assume that these balls are B_i for $1 \leq i \leq k$. Let $f_1, f_2, \ldots, f_k$ be functions in $\mathcal{S}$ such that $\Phi(f_i)$ is in the ball B_i for any $1 \leq i \leq k$. Denote by C_i the open ball of radius 2ϵ centered in $\Phi(f_i)$. Let $\beta \in B_i$. Then we have $d(\beta, \alpha_i) < \epsilon$ and $d(\Phi(f_i), \alpha_i) < \epsilon$. Hence, we have $d(\beta, \Phi(f_i)) < 2\epsilon$, i.e., $B_i \subset C_i$. It follows that $\Phi(\mathcal{S})$ is contained in the union of $C_1, C_2, \ldots, C_k$.

Differently put, for any function $f \in \mathcal{S}$, there exists $1 \leq i \leq k$ such that $|f(x_j) - f_i(x_j)| < 2\epsilon$ for all $1 \leq j \leq n$.

Let $x \in X$. Then $x \in U_{x_j}$ for some $1 \leq j \leq n$. Hence, we have

$$|f(x) - f_i(x)| \leq |f(x) - f(x_j)| + |f(x_j) - f_i(x_j)| + |f_i(x_j) - f_i(x)| < 4\epsilon,$$

i.e., $\|f - f_i\| < 4\epsilon$.

Now we can prove the compactness of the closure $\bar{\mathcal{S}}$ of $\mathcal{S}$. Assume that $\bar{\mathcal{S}}$ is not compact. Then there exists an open cover $\mathcal{U}$ of $\bar{\mathcal{S}}$ which doesn't contain a finite subcover. By the above remark, $\bar{\mathcal{S}}$ can be covered by finitely many closed balls $\{f \in C(X) \mid \|f - f_i\| \leq 1\}$ with $f_i \in \mathcal{S}$. Therefore, there exists a set K_1

which is the intersection of $\bar{\mathcal{S}}$ with one of the closed balls and which is not covered by a finite subcover of $\mathcal{U}$. By induction, we can construct a decreasing family $K_1 \supset K_2 \supset \cdots \supset K_n \supset \cdots$ of closed subsets of $\bar{\mathcal{S}}$ which are contained in closed balls of radius $\frac{1}{n}$ centered in some point of $\mathcal{S}$, such that none of K_n is covered by a finite subcover of $\mathcal{U}$.

Let $(F_n; n \in \mathbb{N})$ be a sequence of functions such that $F_n \in K_n$ for all $n \in \mathbb{N}$. Then $F_p, F_q \in K_n$ for all p, q greater than n. Since K_n are contained in closed balls of radius $\frac{1}{n}$, $\|F_p - F_q\| \leq \frac{2}{n}$ for all p, q greater than n. Hence, (F_n) is a Cauchy sequence in $C(X)$. Therefore, it converges to a function $F \in C(X)$. This function is in $\bar{\mathcal{S}}$ and therefore in one element V of the open cover $\mathcal{U}$. Therefore, for sufficiently large n, there exists a closed ball of radius $\frac{2}{n}$ centered in F which is contained in V. Since F is also in K_n, we see that K_n is in V. This clearly contradicts our construction of K_n. It follows that $\bar{\mathcal{S}}$ must be compact. □

1.3. Haar measure on compact groups. Let $\mathcal{C}_{\mathbb{R}}(G)$ be the space of real valued functions on G. For any function $f \in \mathcal{C}_{\mathbb{R}}(G)$ we define the *maximum* $M(f) = \max_{g\in G} f(g)$ and *minimum* $m(f) = \min_{g\in G} f(g)$. Moreover, we denote by $V(f) = M(f) - m(f)$ the *variation* of f.

Clearly, the function f is constant on G if and only if $V(f) = 0$.

Let $f, f' \in \mathcal{C}_{\mathbb{R}}(G)$ be two functions such that $\|f - f'\| < \epsilon$. Then

$$f(g) - \epsilon < f'(g) < f(g) + \epsilon$$

for all $g \in G$. This implies that

$$m(f) - \epsilon < f'(g) < M(f) + \epsilon$$

for all $g \in G$, and

$$m(f) - \epsilon < m(f') < M(f') < M(f) + \epsilon.$$

Hence

$$V(f') = M(f') - m(f') < M(f) - m(f) + 2\epsilon = V(f) + 2\epsilon,$$

i.e., $V(f') - V(f) < 2\epsilon$. By symmetry, we also have $V(f) - V(f') < 2\epsilon$. It follows that $|V(f) - V(f')| < 2\epsilon$.

Therefore, we have the following result.

1.3.1. Lemma. *The variation V is a continuous function on $\mathcal{C}_{\mathbb{R}}(G)$.*

Let $f \in \mathcal{C}_{\mathbb{R}}(G)$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ a finite sequence of points in G. We define the *(right) mean value* $\mu(f, \mathbf{a})$ of f with respect to $\mathbf{a}$ as

$$\mu(f, \mathbf{a})(g) = \frac{1}{n} \sum_{i=1}^{n} f(g a_i)$$

for all $g \in G$. Clearly, $\mu(f, \mathbf{a})$ is a continuous real function on G.

If f is a constant function, $\mu(f, \mathbf{a}) = f$.

Clearly, mean value $f \longmapsto \mu(f, \mathbf{a})$ is a linear map. Moreover, we have the following result.

1.3.2. Lemma. (i) *The linear map $f \longmapsto \mu(f, \mathbf{a})$ is continuous. More precisely, we have*

$$\|\mu(f, \mathbf{a})\| \leq \|f\|$$

for any $f \in C_{\mathbb{R}}(G)$;

(ii)
$$M(\mu(f,\mathbf{a})) \leq M(f)$$
for any $f \in C_{\mathbb{R}}(G)$*;*

(iii)
$$m(\mu(f,\mathbf{a})) \geq m(f)$$
for any $f \in C_{\mathbb{R}}(G)$*;*

(iv)
$$V(\mu(f,\mathbf{a})) \leq V(f)$$
for any $f \in C_{\mathbb{R}}(G)$*.*

Proof. (i) Clearly, we have
$$\|\mu(f,\mathbf{a})\| = \max_{g\in G} |\mu(f,\mathbf{a})| \leq \frac{1}{n} \sum_{g\in G} \max_{g\in G} |f(ga_i)| = \|f\|.$$

(ii) We have
$$M(\mu(f,\mathbf{a})) = \frac{1}{n} \max_{g\in G} \left(\sum_{i=1}^{n} f(ga_i) \right) \leq \frac{1}{n} \sum_{i=1}^{n} \max_{g\in G} f(ga_i) = M(f).$$

(iii) We have
$$m(\mu(f,\mathbf{a})) = \frac{1}{n} \min_{g\in G} \left(\sum_{i=1}^{n} f(ga_i) \right) \geq \frac{1}{n} \sum_{i=1}^{n} \min_{g\in G} f(ga_i) = m(f).$$

(iv) By (ii) and (iii), we have
$$V(\mu(f,\mathbf{a})) = M(\mu(f,\mathbf{a})) - m(\mu(f,\mathbf{a})) \leq M(f) - m(f) = V(f).$$
□

Denote by $\mathcal{M}_f$ the set of mean values of f for all finite sequences in G.

1.3.3. Lemma. *The set of functions* $\mathcal{M}_f$ *is uniformly bounded and equicontinuous.*

Proof. By 1.3.2 (ii) and (iii), it follows that
$$m(f) \leq m(\mu(f,\mathbf{a})) \leq \mu(f,\mathbf{a})(g) \leq M(\mu(f,\mathbf{a})) \leq M(f).$$
This implies that $\mathcal{M}_f$ is uniformly bounded.

Now we want to prove that $\mathcal{M}_f$ is equicontinuous. First, by 1.1.1, the function f is uniformly continuous. Hence, for any $\epsilon > 0$, there exists an open neighborhood U of 1 in G such that $|f(g) - f(h)| < \epsilon$ if $gh^{-1} \in U$. Since, this implies that $(ga_i)(ha_i)^{-1} = gh^{-1} \in U$ for any $1 \leq i \leq n$, we see that
$$|\mu(f,\mathbf{a})(g) - \mu(f,\mathbf{a})(h)| = \frac{1}{n} \left| \sum_{i=1}^{n} (f(ga_i) - f(ha_i)) \right| \leq \frac{1}{n} \sum_{i=1}^{n} |f(ga_i) - f(ha_i)| < \epsilon$$
for $g \in hU$. Hence, the family $\mathcal{M}_f$ is equicontinuous. □

By 1.2.1, we have the following consequence.

1.3.4. Lemma. *The set* $\mathcal{M}_f$ *of all right mean values of* f *has compact closure in* $\mathcal{C}_{\mathbb{R}}(G)$.

We need another result on variation of mean value functions. Clearly, if f is a constant function $\mu(f,\mathbf{a}) = f$ for any $\mathbf{a}$.

1.3.5. Lemma. *Let f be a function in $\mathcal{C}_{\mathbb{R}}(G)$. Assume that f is not a constant. Then there exists $\mathbf{a}$ such that $V(\mu(f,\mathbf{a})) < V(f)$.*

Proof. Since f is not constant, we have $m(f) < M(f)$. Let C be such that $m(f) < C < M(f)$. Then there exists an open set V in G such that $f(g) \leq C$ for all $g \in V$. Since the right translates of V cover G, by compactness of G we can find $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ such that $(Va_i^{-1}, 1 \leq i \leq n)$ is an open cover of G. For any $g \in Va_i^{-1}$ we have $ga_i \in V$ and $f(ga_i) \leq C$. Hence, we have

$$\mu(f,\mathbf{a})(g) = \frac{1}{n}\sum_{j=1}^{n} f(ga_j) = \frac{1}{n}\left(f(ga_i) + \sum_{j\neq i} f(ga_j)\right)$$
$$\leq \frac{1}{n}(C + (n-1)M(f)) < M(f).$$

On the other hand, by 1.3.2.(iii) we know that $m(\mu(f,\mathbf{a})) \geq m(f)$ for any $\mathbf{a}$. Hence we have

$$V(\mu(f,\mathbf{a})) = M(\mu(f,\mathbf{a})) - m(\mu(f,\mathbf{a})) < M(f) - m(f) = V(f).$$

□

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ be two finite sequences in G. We define $\mathbf{a} \cdot \mathbf{b} = (a_i b_j; 1 \leq i \leq n, 1 \leq j \leq m)$.

1.3.6. Lemma. *We have*

$$\mu(\mu(f,\mathbf{b}),\mathbf{a}) = \mu(f, \mathbf{b}\cdot\mathbf{a}).$$

Proof. We have

$$\mu(\mu(f,\mathbf{b}),\mathbf{a}) = \frac{1}{m}\sum_{i=1}^{m}\mu(f,\mathbf{b})(ga_i) = \frac{1}{nm}\sum_{j=1}^{m}\sum_{i=1}^{n} f(ga_ib_j) = \mu(f,\mathbf{a}\cdot\mathbf{b}).$$

□

1.3.7. Lemma. *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$, the closure $\mathcal{M}_f$ contains a constant function on G.*

Proof. By 1.3.4, we know that $\overline{\mathcal{M}_f}$ is compact. Since, by 1.3.1, the variation V is continuous on $\mathcal{C}_{\mathbb{R}}(G)$, it attains its minimum α at some $\varphi \in \overline{\mathcal{M}_f}$.

Assume that φ is not a constant. By 1.3.5, there exists $\mathbf{a}$ such that $V(\mu(\varphi,\mathbf{a})) < V(\varphi)$. Let $\alpha - V(\mu(\varphi,\mathbf{a})) = \epsilon > 0$.

Since V and $\mu(\cdot,\mathbf{a})$ are continuous maps by 1.3.1 and 1.3.2.(i), this implies that there is $\mathbf{b}$ such that $|V(\mu(\varphi,\mathbf{a})) - V(\mu(\mu(f,\mathbf{b}),\mathbf{a}))| < \frac{\epsilon}{2}$. Therefore, we have

$$V(\mu(\mu(f,\mathbf{b}),\mathbf{a})) \leq V(\mu(\varphi,\mathbf{a})) + \frac{\epsilon}{2} = \alpha - \frac{\epsilon}{2}.$$

By 1.3.6, we have

$$V(\mu(f,\mathbf{a}\cdot\mathbf{b})) < \alpha - \frac{\epsilon}{2}$$

contrary to our choice of α.

It follows that φ is a constant function. In addition $\alpha = 0$. □

Consider now left mean values of a function $f \in \mathcal{C}_{\mathbb{R}}(G)$. We define the left mean value of f with respect to $\mathbf{a} = (a_1, a_2, \dots, a_n)$ as the function

$$\nu(f, \mathbf{a})(g) = \frac{1}{n} \sum_{i=1}^{n} f(a_i g)$$

for $g \in G$. We denote my $\mathcal{N}_f$ the set of all left mean values of f.

Let G^{opp} be the compact group opposite to G. Then the left mean values of f on G are the right mean values of f on G^{opp}.

Hence, from 1.3.7, we deduce the following result.

1.3.8. Lemma. *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$, the closure $\mathcal{N}_f$ contains a constant function on G.*

By direct calculation we get the following result.

1.3.9. Lemma. *For any $f \in C_{\mathbb{R}}(G)$ we have*

$$\nu(\mu(f, \mathbf{a}), \mathbf{b}) = \mu(\nu(f, \mathbf{b}), \mathbf{a})$$

for any two finite sequences $\mathbf{a}$ and $\mathbf{b}$ in G.

Proof. We have

$$\begin{aligned}\nu(\mu(f, \mathbf{a}), \mathbf{b})(g) = \frac{1}{m} \sum_{j=1}^{m} \mu(f, \mathbf{a})(b_j g) &= \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} f(b_j g a_i) \\ &= \frac{1}{n} \sum_{i=1}^{n} \nu(f, \mathbf{b})(g a_i) = \mu(\nu(f, \mathbf{b}), \mathbf{a})(g)\end{aligned}$$

for any $g \in G$. □

Putting together these results, we finally get the following.

1.3.10. Proposition. *For any $f \in \mathcal{C}_{\mathbb{R}}(G)$, the closure $\mathcal{M}_f$ contains a unique function constant on G.*

This function is also the unique constant function in $\mathcal{N}_f$.

Proof. Let φ and ψ be two constant functions such that φ is in the closure of $\mathcal{M}_f$ and ψ is in the closure of $\mathcal{N}_f$. For any $\epsilon > 0$ we have $\mathbf{a}$ and $\mathbf{b}$ such that $\|\mu(f, \mathbf{a}) - \varphi\| < \frac{\epsilon}{2}$ and $\|\nu(f, \mathbf{b}) - \psi\| < \frac{\epsilon}{2}$.

On the other hand, we have

$$\begin{aligned}\|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| &= \|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \nu(\varphi, \mathbf{b})\| \\ &= \|\nu(\mu(f, \mathbf{a}) - \varphi, \mathbf{b})\| \le \|\mu(f, \mathbf{a}) - \varphi\| < \frac{\epsilon}{2}.\end{aligned}$$

In the same way. we also have

$$\begin{aligned}\|\mu(\nu(f, \mathbf{b}), \mathbf{a}) - \psi\| &= \|\mu(\nu(f, \mathbf{b}), \mathbf{a}) - \mu(\psi, \mathbf{a})\| \\ &= \|\mu(\nu(f, \mathbf{b}) - \psi, \mathbf{a})\| \le \|\nu(f, \mathbf{b}) - \psi\| < \frac{\epsilon}{2}.\end{aligned}$$

By 1.3.9, this immediately yields

$$\|\varphi - \psi\| \le \|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| + \|\mu(\nu(f, \mathbf{b}), \mathbf{a}) - \psi\| < \epsilon.$$

This implies that $\varphi = \psi$. Therefore, any constant function in the closure of $\mathcal{M}_f$ has to be equal to ψ. □

The value of the unique constant function in the closure of $\mathcal{M}_f$ is denoted by $\mu(f)$ and called *the mean value* of f on G. In this way, we get a function $f \longmapsto \mu(f)$ on $\mathcal{C}_{\mathbb{R}}(G)$.

Let $\gamma : \mathcal{C}_{\mathbb{R}}(G) \longrightarrow \mathbb{R}$ be a linear form. We say that γ is *positive* if for any $f \in \mathcal{C}_{\mathbb{R}}(G)$ such that $f(g) \geq 0$ for any $g \in G$ we have $\gamma(f) \geq 0$.

1.3.11. LEMMA. *The function μ is a positive linear form on $\mathcal{C}_{\mathbb{R}}(G)$.*

To prove this result we need some preparation.

1.3.12. LEMMA. *Let $f \in C_{\mathbb{R}}(G)$. Then, for any $\mathbf{a}$ we have*

$$\mu(\mu(f, \mathbf{a})) = \mu(f).$$

PROOF. Let $\mu(f) = \alpha$. Let φ be the function equal to α everywhere on G. Fix $\epsilon > 0$. Then there exists a finite sequence $\mathbf{b}$ such that

$$\|\nu(f, \mathbf{b}) - \varphi\| < \epsilon.$$

This implies that

$$\|\nu(f - \varphi, \mathbf{b})\| = \|\nu(f, \mathbf{b}) - \nu(\varphi, \mathbf{b})\| = \|\nu(f, \mathbf{b}) - \varphi\| < \epsilon.$$

This, by 1.3.2.(i), implies that

$$\|\mu(\nu(f - \varphi, \mathbf{b}), \mathbf{a})\| \leq \|\nu(f - \varphi, \mathbf{b})\| < \epsilon$$

for any finite sequence $\mathbf{a}$.

By 1.3.9, we have

$$\|\nu(\mu(f - \varphi, \mathbf{a}), \mathbf{b})\| = \|\mu(\nu(f - \varphi, \mathbf{b}), \mathbf{a})\| < \epsilon,$$

and

$$\|\nu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| = \|\nu(\mu(f, \mathbf{a}) - \varphi, \mathbf{b})\| = \|\nu(\mu(f - \varphi, \mathbf{a}), \mathbf{b})\| < \epsilon.$$

Therefore, if we fix $\mathbf{a}$, we see that φ is in the closure of $\mathcal{N}_{\mu(f,\mathbf{a})}$. By 1.3.10, this proves our assertion. □

Let f and f' be two functions in $C_{\mathbb{R}}(G)$. Let $\alpha = \mu(f)$ and $\beta = \mu(f')$. Denote by φ and ψ the corresponding constant functions. Let $\epsilon > 0$.

Clearly. there exists $\mathbf{a}$ such that

$$\|\mu(f, \mathbf{a}) - \varphi\| < \frac{\epsilon}{2}.$$

This, by 1.3.2.(ii) implies, that we have

$$\|\mu(\mu(f, \mathbf{a}), \mathbf{b}) - \varphi\| = \|\mu(\mu(f, \mathbf{a}) - \varphi, \mathbf{b})\| < \frac{\epsilon}{2}$$

for arbitrary $\mathbf{b}$. By 1.3.6, this in turn implies that

$$\|\mu(f, \mathbf{a} \cdot \mathbf{b}) - \varphi\| < \frac{\epsilon}{2}.$$

On the other hand, by 1.3.12, we have $\mu(\mu(f', \mathbf{a})) = \mu(f') = \beta$. Therefore, there exists a finite sequence $\mathbf{b}$ such that

$$\|\mu(\mu(f', \mathbf{a}), \mathbf{b}) - \psi\| < \frac{\epsilon}{2}.$$

This, by 1.3.6, implies that

$$\|\mu(f', \mathbf{a} \cdot \mathbf{b}) - \psi\| < \frac{\epsilon}{2}.$$

Hence, we have

$$\|\mu(f+f',\mathbf{a}\cdot\mathbf{b})-(\varphi+\psi)\|\leq\|\mu(f,\mathbf{a}\cdot\mathbf{b})-\varphi\|+\|\mu(f',\mathbf{a}\cdot\mathbf{b})-\psi\|<\epsilon.$$

Therefore, $\varphi+\psi$ is in the closure of $\mathcal{M}_{f+f'}$. It follows that

$$\mu(f+f')=\alpha+\beta=\mu(f)+\mu(f'),$$

i.e., μ is additive.

Let $c\in\mathbb{R}$ and $f\in C_{\mathbb{R}}(G)$. Then $\mu(cf,\mathbf{a})=c\mu(f,\mathbf{a})$ for any $\mathbf{a}$. Therefore, $\mathcal{M}_{cf}=c\mathcal{M}_f$. This immediately implies that $\mu(cf)=c\mu(f)$. Therefore μ is a linear form.

Assume that f is a function in $C_{\mathbb{R}}(G)$ such that $f(g)\geq 0$ for all $g\in G$. Then $\mu(f,\mathbf{a})(g)\geq 0$ for any $\mathbf{a}$ and $g\in G$. Hence, any function $\phi\in\mathcal{M}_f$ satisfies $\phi(g)\geq 0$ for all $g\in G$. This immediately implies that $\phi(g)\geq 0$, $g\in G$, for any ϕ in the closure of $\mathcal{M}_f$. It follows that $\mu(f)\geq 0$. Hence, we μ is a positive linear form. This completes the proof of 1.3.11.

Clearly, $\mu(1)=1$. Let $f\in\mathcal{C}_{\mathbb{R}}(G)$. Then we have

$$-\|f\|\leq f(g)\leq\|f\|$$

for any $g\in G$. Since μ is a positive linear form, we see that

$$-\|f\|=\mu(-\|f\|)\leq\mu(f)\leq\mu(\|f\|)=\|f\|.$$

Therefore, we have

$$|\mu(f)|\leq\|f\|$$

for any $f\in\mathcal{C}_{\mathbb{R}}(G)$. In particular, μ is a continuous linear form on $\mathcal{C}_{\mathbb{R}}(G)$.

By Riesz representation theorem, the linear form $\mu: C_{\mathbb{R}}(G)\longrightarrow\mathbb{R}$ defines a regular positive measure μ on G such that

$$\mu(f)=\int_G f\,d\mu.$$

Clearly, we have

$$\mu(G)=\int_G d\mu=\mu(1)=1.$$

so we say that μ is *normalized*.

Denote by R (resp. L) the *right regular representation* (resp. *left regular representation* of G on $C(G)$ given by $(R(g)f)(h)=f(hg)$ (resp. $(L(g)f)(h)=f(g^{-1}h)$) for any $f\in C(G)$ and $g,h\in G$.

1.3.13. Lemma. *Let $f\in C_{\mathbb{R}}(G)$ and $g\in G$. Then*

$$\mu(R(g)f)=\mu(L(g)f)=\mu(f).$$

Proof. Let $\mathbf{g}=(g)$. Clearly, we have

$$\mu(f,\mathbf{g})(h)=f(hg)=(R(g)f)(h)$$

for all $h\in G$, i.e., $R(g)f=\mu(f,\mathbf{g})$. By 1.3.12, we have

$$\mu(R(g)f)=\mu(\mu(f,\mathbf{g}))=\mu(f).$$

This statement for G^{opp} implies the other equality. □

We say that the linear form μ is *biinvariant*, i.e., *right invariant* and *left invariant*.

The above result implies that the measure μ is *biinvariant*, i.e., we have the following result.

1.3.14. Lemma. *Let A be a measurable set in G. Then gA and Ag are also measurable and*

$$\mu(gA) = \mu(Ag) = \mu(A)$$

for any $g \in G$.

Proof. Since $C_{\mathbb{R}}(G)$ is dense in $L^1(\mu)$, the invariance from 1.3.13 holds for any function $f \in L^1(\mu)$. Applying it to the characteristic function of the set A implies the result. $\square$

A normalized biinvariant positive measure μ on G is called a *Haar measure* on G.

We proved the existence part of the following result.

1.3.15. Theorem. *Let G be a compact group. Then there exists a unique Haar measure μ on G.*

Proof. We constructed a Haar measure on G.

It remains to prove the uniqueness. Let ν be another Haar measure on G. Then, by left invariance, we have

$$\int_G \mu(f, \mathbf{a})\, d\nu = \frac{1}{n} \sum_{i=1}^{n} \int_G f(ga_i)\, d\nu(g) = \int_G f\, d\nu$$

for any $\mathbf{a}$. Hence the integral with respect to ν is constant on $\mathcal{M}_f$. By continuity, it is also constant on its closure. Therefore, we have

$$\int_G f\, d\nu = \mu(f) \int_G d\nu = \mu(f) = \int_G f\, d\mu$$

for any $C_{\mathbb{R}}(G)$. This in turn implies that $\nu = \mu$. $\square$

1.3.16. Lemma. *Let μ be the Haar measure on G. Let U be a nonempty open set in G. Then $\mu(U) > 0$.*

Proof. Since U is nonempty, $(Ug; g \in G)$ is an open cover of G. It contains a finite subcover $(Ug_i; 1 \leq i \leq n)$. Therefore we have

$$1 = \mu(G) = \mu\left(\bigcup_{i=1}^{n} Ug_i\right) \leq \sum_{i=1}^{n} \mu(Ug_i) = \sum_{i=1}^{n} \mu(U) = n\, \mu(U)$$

by 1.3.14. This implies that $\mu(U) \geq \frac{1}{n}$. $\square$

1.3.17. Lemma. *Let f be a continuous function on G. Then*

$$\int_G f(g^{-1})\, d\mu(g) = \int_G f(g)\, d\mu(g).$$

Proof. Clearly, it is enough to prove the statement for real-valued functions. Therefore, we can consider the linear form $\nu : f \longmapsto \int_G f(g^{-1})\, d\mu(g)$. Clearly, this a positive continuous linear form on $C_{\mathbb{R}}(G)$. Moreover,

$$\begin{aligned}\nu(f) = \int_G f(h^{-1})\, d\mu(h) &= \int_G f((hg)^{-1})\, d\mu(h) \\ &= \int_G f(g^{-1}h^{-1})\, d\mu(h) = \int_G (L(g)f)(h^{-1})\, d\mu(h) = \nu(L(g)f)\end{aligned}$$

and

$$\nu(f) = \int_G f(h^{-1})\, d\mu(h) = \int_G f((g^{-1}h)^{-1})\, d\mu(h)$$
$$= \int_G f(h^{-1}g)\, d\mu(h) = \int_G (R(g)f)(h^{-1})\, d\mu(h) = \nu(R(g)f)$$

for any $g \in G$. Hence, this linear form is left and right invariant. By the uniqueness of the Haar measure we get the statement. □

2. Algebra of matrix coefficients

2.1. Finite-dimensional topological vector spaces. Let E be a vector space over $\mathbb{C}$. We say that E is a *topological vector space* over $\mathbb{C}$, if it is also equipped with a topology such that the functions $(u, v) \longmapsto u + v$ from $E \times E$ into E, and $(\alpha, u) \longmapsto \alpha u$ from $\mathbb{C} \times E$ into E are continuous.

A morphism $\varphi : E \longrightarrow F$ of topological vector space E into F is a continuous linear map from E to F.

We say that E is a hausdorff topological vector space if the topology of E is hausdorff.

Let E be a normed vector space over $\mathbb{C}$ with norm $\|\cdot\|$. Then the norm defines a metric $d(u, v) = \|u-v\|$, $u, v \in E$, on E. This metric defines a hausdorff topology on E, and E is a hausdorff topological vector space.

In particular, the vector space $\mathbb{C}^n$ with the euclidean norm

$$\|c\| = \left(\sum_{i=1}^{n} |c_i|^2\right)^{\frac{1}{2}}$$

for $c \in \mathbb{C}^n$, is a hausdorff topological vector space.

Let $A : \mathbb{C}^n \longrightarrow \mathbb{C}^m$ be the linear map given by the matrix $(A_{ij}; 1 \le i \le m, 1 \le j \le n)$. Then A is continuous.

2.1.1. Lemma. *Let E be a topological vector space over $\mathbb{C}$. Then the following conditions are equivalent:*

(i) *E is hausdorff;*
(ii) *$\{0\}$ is a closed set in E.*

Proof. Assume that E is hausdorff. Let $v \in E$, $v \neq 0$. Then there exist open neighborhoods U of 0 and V of v such that $U \cap V = \emptyset$. In particular, $V \subset E - \{0\}$. Hence, $E - \{0\}$ is an open set. This implies that $\{0\}$ is closed.

Assume now that $\{0\}$ is closed in E. Then $E - \{0\}$ is an open set. Let u and v be different vectors in E. Then $u - v \neq 0$. Since the function $(x, y) \longmapsto x + y$ is continuous, there exist open neighborhoods U of u and V of v such that $U - V \subset E - \{0\}$. This in turn implies that $U \cap V = \emptyset$. □

The main result of this section is the following claim. It states that hausdorff finite-dimensional topological vector spaces have unique topology.

2.1.2. Proposition. *Let E be a finite-dimensonal hausdorff topological vector space over $\mathbb{C}$. Let $v_1, v_2, \dots, v_n$ be a basis of E. Then the linear map $\mathbb{C}^n \longrightarrow E$ given by*

$$(c_1, c_2, \dots, c_n) \longmapsto \sum_{i=1}^{n} c_i v_i$$

is an isomorphism of topological vector spaces.

PROOF. Clearly, the map

$$\phi(c) = \sum_{i=1}^{n} c_i v_i,$$

for all $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$, is a continuous linear isomorphism of $\mathbb{C}^n$ onto E. Therefore, it is enough to show that that map is also open.

Let $B_1 = \{c \in \mathbb{C}^n \mid \|c\| < 1\}$ be the open unit ball in $\mathbb{C}^n$.

Let $S = \{z \in \mathbb{C}^n \mid \|z\| = 1\}$ be the unit sphere in $\mathbb{C}^n$. Then, S is a bounded and closed set in $\mathbb{C}^n$. Hence it is compact. This implies that $\phi(S)$ is a compact set in E. Since 0 is not in S, 0 is not in $\phi(S)$. Since E is hausdorff, $\phi(S)$ is closed and $E - \phi(S)$ is an open neighborhood of 0 in E. By continuity of multiplication by scalars at $(0,0)$, there exists $\epsilon > 0$ and an open neighborhood U of 0 in E such that $zU \subset E - \phi(S)$, i.e., $zU \cap \phi(S) = \emptyset$ for all $|z| \leq \epsilon$.

Let $v \in U - \{0\}$. Then we have

$$v = \sum_{i=1}^{n} c_i v_i.$$

Let $c = (c_1, c_2, \dots, c_n) \in \mathbb{C}^n$. Then, $\frac{1}{\|c\|} c \in S$, and $\frac{1}{\|c\|} v \in \phi(S)$. By our construction, we must have $\frac{1}{\|c\|} > \epsilon$. Hence, we have $\|c\| < \frac{1}{\epsilon}$, i.e., $c \in B_{\frac{1}{\epsilon}}$. This in turn yields $v \in \phi\left(\frac{1}{\epsilon} B_1\right) = \frac{1}{\epsilon}\phi(B_1)$. Therefore, we have

$$\epsilon U \subset \phi(B_1).$$

Hence, $\phi(B_1)$ is a neighborhood of 0 in E.

Let O be an open set in $\mathbb{C}^n$. Let $v \in O$. Then there exist an open ball of radius r centered in v contained in O, i.e., $v + rB_1 \subset O$. This implies that

$$\phi(v) + r\phi(B_1) = \phi(v + rB_1) \subset \phi(O),$$

and $\phi(v) + r\phi(B_1)$ is a neighborhood of $\phi(v)$ in E. Hence $\phi(v)$ is an interior point in $\phi(O)$. It follows that $\phi(O)$ is open and ϕ is an open map. $\square$

2.1.3. COROLLARY. *Let E and F be two finite-dimensional hausdorff topological vector spaces over $\mathbb{C}$. Then any linear map $A : E \longrightarrow F$ is continuous.*

PROOF. Let $u_1, u_2, \dots, u_n$ be a basis of E and $\phi(c) = \sum_{i=1}^{n} c_i u_i$, for $c \in \mathbb{C}^n$. Also, let $v_1, v_2, \dots, v_m$ be a basis of F and $\psi(d) = \sum_{i=1}^{m} d_i v_i$, for $d \in \mathbb{C}^m$. By 2.1.2, ϕ and ψ are isomorphisms of topological vector spaces. Consider the commutative diagram

$$\begin{tikzcd} \mathbb{C}^n \arrow[r, "\phi"] \arrow[d, "B"'] & E \arrow[d, "A"] \\ \mathbb{C}^m \arrow[r, "\psi"'] & F \end{tikzcd}.$$

As we remarked before, the linear map B is continuous. Hence, A must be continuous. $\square$

Combining 2.1.2 and 2.1.3, we get the following result.

2.1.4. Theorem. *The forgetful functor from the category of finite-dimensional hausdorff topological vector spaces into the category of finite-dimensional vector spaces is an equivalence of categories.*

Let E be a topological vector space and F a vector subspace of E. Then F is a topological vector space with the induced topology. Moreover, if E is hausdorff, F is also hausdorff.

2.1.5. Corollary. *Let E be a hausdorff topological vector space over $\mathbb{C}$. Let F be a finite-dimensional vector subspace of E. Then F is closed in E.*

Proof. Clearly, the topology of E induces a structure of hausdorff topological vector space on F. Let $v_1, v_2, \ldots, v_n$ be a basis of F. Assume that F is not closed. Let w be a vector in the closure of F which is not in F. Then w is linearly independent of $v_1, v_2, \ldots, v_n$. Let F' be the direct sum of F and $\mathbb{C}w$. Then F' is a $(n+1)$-dimensional hausdorff topological vector space. By 2.1.2, we know that

$$(c_1, c_2, \ldots, c_n, c_{n+1}) \longmapsto \sum_{i=1}^{n} c_i v_i + c_{n+1} w$$

is an isomorphism of the topological vector space $\mathbb{C}^{n+1}$ onto F'. This isomorphism maps $\mathbb{C}^n \times \{0\}$ onto F. Therefore, F is closed in F', and w is not in the closure of F. Hence, we have a contradiction. □

2.2. Some results about Banach spaces. Let E be a normed vector space. For $v \in V$ and $r > 0$ we denote by $B_r(v) = \{u \in V \mid \|u - v\| < r\}$ the open ball in E of radius r centered in v.

Let E and F be two normed vector spaces and $T : E \longrightarrow F$ a linear map. The map T is *bounded* if the set $\{\|T(u)\| \mid u \in B_1(0)\}$ is bounded. In this case, we put $\|T\| = \sup_{u \in B_1(0)} \|T(u)\|$ and we say that $\|T\|$ is the *norm* of T. Clearly, $\|T(u)\| \leq \|T\| \|u\|$ for any $u \in E$. Therefore, we have

$$\|T(u) - T(u')\| \leq \|T(u - u')\| \leq \|T\| \|u - u'\|$$

for $u, u' \in E$, and the map $T : E \longrightarrow F$ is continuous.

2.2.1. Lemma. *Let $T : E \longrightarrow F$ be a linear map. Then the following conditions are equivalent:*

(i) *T is continuous;*
(ii) *T is bounded.*

Proof. We proved that (ii) implies (i).

Assume that T is continuous. Then T is continuous at 0 and $T(0) = 0$. Hence there exists a neighborhood U of 0 such that $T(U) \subset B_1(0)$ in F. Moreover, there exists $\epsilon > 0$ such that $B_\epsilon(0) \subset U$. Hence, we have $T(B_\epsilon(0)) \subset B_1(0)$ and $T(B_1(0)) \subset B_{\frac{1}{\epsilon}}(0)$. □

2.2.2. Lemma. *Let T be a continuous linear map from normed space E into normed space F. Let $v \in E$ and $r > 0$. Then we have*

$$r\|T\| \leq \sup_{u \in B_r(v)} \|T(u)\|.$$

Proof. Let $u \in B_r(v)$. Then $w = u - v$ satisfies $\|w\| < r$. Then $T(v+w) - T(v-w) = 2T(w)$ and

$$2\|T(w)\| \le \|T(v+w)\| + \|T(v-w)\| \le 2 \sup_{u \in B_r(v)} \|T(u)\|.$$

Therefore, we have

$$\|T(w)\| \le \sup_{u \in B_r(v)} \|T(u)\|$$

for all $w \in B_r(0)$. Hence, we have

$$r\|T\| = \sup_{w \in B_r(0)} \|T(w)\| \le \sup_{u \in B_r(v)} \|T(u)\|.$$

□

2.2.3. Theorem (Banach-Steinhaus). *Let E be a Banach space and $\mathcal{F}$ a family of continuous linear maps from E into normed space F. Assume that $\{\|Tv\|; T \in \mathcal{F}\}$ is a bounded set for any $v \in E$. Then $\{\|T\|; T \in \mathcal{F}\}$ is bounded.*

Proof. Assume that this is false. Then there exists a sequence $\{T_n; n \in \mathbb{N}\}$ in $\mathcal{F}$ such that $\|T_n\| \ge 4^n$ for $n \in \mathbb{N}$.

By 2.2.2 we can construct a sequence $\{v_n; n \in \mathbb{N}\}$, such that $v_1 = 0$ and $\|v_n - v_{n-1}\| < \frac{1}{3^n}$ and

$$\|T_n v_n\| > \frac{2}{3}\frac{1}{3^n}\|T_n\|$$

for $n > 1$.

Then, for $m > n$, we have

$$\begin{aligned}\|v_m - v_n\| = \left\| \sum_{i=n+1}^{m} (v_i - v_{i-1}) \right\| &\le \sum_{i=n+1}^{m} \|v_i - v_{i-1}\| \\ &\le \sum_{i=n+1}^{m} \frac{1}{3^i} \le \sum_{i=n+1}^{\infty} \frac{1}{3^i} = \frac{1}{3^{n+1}} \frac{1}{1-\frac{1}{3}} = \frac{1}{2}\frac{1}{3^n}.\end{aligned}$$

Hence, $\{v_n; n \in \mathbb{N}\}$ is a Cauchy sequence. Since E is complete, there exist $v \in E$ such that $v = \lim v_n$. Moreover, it follows that $\|v - v_n\| \le \frac{1}{2}\frac{1}{3^n}$ for all $n \in \mathbb{N}$.

By triangle inequality, we see that

$$\|T_n v\| = \|T_n(v - v_n) + T_n v_n\| \ge \|T_n v_n\| - \|T_n(v - v_n)\|.$$

On the other hand, we have

$$\|T_n v_n\| > \frac{2}{3}\frac{1}{3^n}\|T_n\|$$

and

$$\|T_n(v - v_n)\| \le \|T_n\|\|v - v_n\| \le \frac{1}{2}\frac{1}{3^n}\|T_n\|$$

for all $n \in \mathbb{N}$. Therefore, it follows that

$$\|T_n v_n\| - \|T_n(v - v_n)\| > \frac{2}{3}\frac{1}{3^n}\|T_n\| - \frac{1}{2}\frac{1}{3^n}\|T_n\| = \frac{1}{6}\frac{1}{3^n}\|T_n\| \ge \frac{1}{6}\left(\frac{4}{3}\right)^n$$

for all $n \in \mathbb{N}$. This implies that $\|T_n v\| \ge \frac{1}{6}\left(\frac{4}{3}\right)^n$ for $n \in \mathbb{N}$, contradicting the assumption that $\{\|Tv\|; T \in \mathcal{F}\}$ is bounded. □

2.3. Representations on topological vector spaces. Let G be a compact group. Let E be a hausdorff topological vector space over $\mathbb{C}$. We denote by $\mathrm{GL}(E)$ the group of all automorphisms of E.

If E is a finite-dimensional hausdorff topological vector space, by 2.1.4, any linear automorphism of E is automatically an automorphism of topological vector spaces. Therefore $\mathrm{GL}(E)$ is just the group of all linear automorphisms of E as before.

A (continuous) representation of G on E is a group homomorphism $\pi : G \longrightarrow \mathrm{GL}(E)$ such that $(g, v) \longmapsto \pi(g)v$ is continuous from $G \times E$ into E.

2.3.1. Lemma. *Let E be a Banach space and $\pi : G \longrightarrow \mathrm{GL}(E)$ a homomorphism such that $g \longmapsto \pi(g)v$ is continuous function from G into E for all $v \in E$. Then (π, E) is a representation of G on E.*

Proof. Assume that the function $g \longmapsto \pi(g)v$ is continuous for any $v \in V$. Then the function $g \longmapsto \|\pi(g)v\|$ is continuous on G. Since G is compact, there exists M such that $\|\pi(g)v\| < M$ for all $g \in G$. By 2.2.3, we see that the function $g \longmapsto \|\pi(g)\|$ is bounded on G.

Pick $C > 0$ such that $\|\pi(g)\| \leq C$ for all $g \in G$. Then we have

$$\begin{aligned}\|\pi(g)v - \pi(g')v'\| &= \|(\pi(g)v - \pi(g')v) + \pi(g')(v - v')\| \\ &\leq \|\pi(g)v - \pi(g')v\| + \|\pi(g')\|\|v - v'\| \leq \|\pi(g)v - \pi(g')v\| + C\|v - v'\|\end{aligned}$$

for all $g, g' \in G$ and $v, v' \in E$. This clearly implies the continuity of the function $(g, v) \longmapsto \pi(g)v$. $\square$

Moreover, since the topology of E is described by the euclidean norm and E is a Banach space with respect to it, by 2.3.1, the only additional condition for a representation of G is the continuity of the function $g \longmapsto \pi(g)v$ for any $v \in E$. This implies the following result.

2.3.2. Lemma. *Let E be a finite-dimensional hausdorff topological vector space and π a homomorphism of G into $\mathrm{GL}(E)$. Let $v_1, v_2, \ldots, v_n$ be a basis of E.*

(i) *(π, E) is a representation of G on E;*
(ii) *all matrix coefficients of $\pi(g)$ with respect to the basis $v_1, v_2, \ldots, v_n$ are continuous functions on G.*

2.4. Algebra of matrix coefficients. Let G be a compact group. The Banach space $C(G)$ is an commutative algebra with pointwise multiplication of functions, i.e., $(\psi, \phi) \longmapsto \psi \cdot \phi$ where $(\psi \cdot \phi)(g) = \psi(g)\phi(g)$ for any $g \in G$.

First, we remark the following fact.

2.4.1. Lemma. *R and L are representations of G on $C(G)$.*

Proof. Clearly, we have

$$\|R(g)\phi\| = \max_{h \in G} |(R(g)\phi)(h)| = \max_{h \in G} |\phi(hg)| = \max_{h \in G} |\phi(h)| = \|\phi\|.$$

Hence $R(g)$ is a continuous linear map on $C(G)$. Its inverse is $R(g^{-1})$, so $R(g) \in \mathrm{GL}(C(G))$.

By 2.3.1, it is enough to show that the function $g \longmapsto R(g)\phi$ is continuous for any function $\phi \in C(G)$.

By 1.1.1, ϕ is uniformly continuous, i.e., there exists a neighborhood U of 1 in G such that $g^{-1}g' \in U$ implies $|\phi(hg) - \phi(hg')| < \epsilon$ for all $h \in G$. Hence, we have

$$\|R(g)\phi - R(g')\phi\| = \max_{h\in G} |(R(g)\phi)(h) - (R(g')\phi)(h)| = \max_{h\in G} |\phi(hg) - \phi(hg')| < \epsilon$$

for $g' \in gU$. Hence, the function $g \longrightarrow R(g)\phi$ is continuous.

The proof for L is analogous. □

We say that the function $\phi \in C(G)$ is right (resp. left) *G-finite* if the vectors $\{R(g)\phi; g \in G\}$ (resp. $\{L(g)\phi; g \in G\}$) span a finite-dimensional subspace of $C(G)$.

2.4.2. Lemma. *Let $\phi \in C(G)$. The following conditions are equivalent.*

(i) *ϕ is left G-finite;*
(ii) *ϕ is right G-finite;*
(iii) *there exist n and functions $a_i, b_i \in C(G)$, $1 \le i \le n$, such that*

$$\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h)$$

for all $g, h \in G$.

Proof. Let ϕ be a right G-finite function. Then ϕ is in a finite-dimensional subspace F invariant for R. The restriction of the representation R to F is continuous. Let $a_1, a_2, \dots, a_n$ be a basis of F. Then, by 2.3.2, there exist $b_1, b_2, \dots, b_n \in C(G)$ such that $R(g)\phi = \sum_{i=1}^{n} b_i(g)a_i$. Therefore we have

$$\phi(hg) = \sum_{i=1}^{n} b_i(g)a_i(h) = \sum_{i=1}^{n} a_i(h)b_i(g)$$

for all $h, g \in G$. Therefore (iii) holds.

If (iii) holds,

$$R(g)\phi = \sum_{i=1}^{n} a_i(g)b_i$$

and ϕ is right G-finite.

Since the condition (iii) is symmetric, the equivalence of (i) and (iii) follows by applying the above argument to the opposite group of G. □

Therefore, we can call ϕ just a *G-finite* function in $C(G)$. Let $R(G)$ be the subset of all G-finite functions in $C(G)$.

2.4.3. Proposition. *The set $R(G)$ is a subalgebra of $C(G)$.*

Proof. Clearly, a multiple of a G-finite function is a G-finite function.

Let ϕ and ψ be two G-finite functions. Then, by 2.4.2, there exists functions $a_i, b_i, c_i, d_i \in C(G)$ such that

$$\phi(gh) = \sum_{i=1}^{n} a_i(g)b_i(h) \text{ and } \psi(gh) = \sum_{i=1}^{m} c_i(g)d_i(h)$$

for all $g, h \in G$. This implies that

$$(\phi + \psi)(gh) = \sum_{i=1}^{n} a_i(g)b_i(h) + \sum_{i=1}^{m} c_i(g)d_i(h)$$

and

$$(\phi \cdot \psi)(gh) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i(g) c_j(g) b_i(h) d_j(h) = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i \cdot c_j)(g)(b_i \cdot d_i)(h)$$

for all $g, h \in G$. Hence, $\phi + \psi$ and $\phi \cdot \psi$ are G-finite. $\square$

Clearly, $R(G)$ is an invariant subspace for R and L.

The main result of this section is the following observation. Let V be a finite-dimensional complex linear space and π a continuous homomorphism of G into $\mathrm{GL}(V)$, i.e., (π, V) is a *representation* of G. For $v \in V$ and $v \in V^*$ we call the continuous function $g \longmapsto c_{v,v*}(g) = \langle \pi(g)v, v^* \rangle$ a *matrix coefficient* of (π, V).

2.4.4. Theorem. *Let $\phi \in C(G)$. Then the following statements are equivalent:*

(i) *ϕ is in $R(G)$;*

(ii) *ϕ is a matrix coefficient of a finite-dimensional representation of G.*

Proof. Let (π, V) be a finite-dimensional representation of G. Let $v \in V$ and $v^* \in V^*$. By scaling v^* if necessary, we can assume that v is a vector in a basis of V and v^* a vector in the dual basis of V^*. Then, $c_{v,v^*}(g)$ is a matrix coefficient of the matrix of $\pi(g)$ in the basis of V. The rule of matrix multiplication implies that (iii) from 2.4.2 holds for c_{v,v^*}. Hence ϕ is G-finite.

Assume that ϕ is G-finite. Then, by 2.4.2, we have $R(g)\phi = \sum_{i=1}^{n} a_i(g) b_i$ where $a_i, b_i \in C(G)$. We can also assume that b_i are linearly independent. Let V be the subspace of $R(G)$ spanned by $b_1, b_2, \dots, b_n$. Then V is a G-invariant subspace. Let $v = \phi$ and $v^* \in V^*$ such that $b_i(1) = \langle b_i, v^* \rangle$. Then

$$\langle R(g)v, v^* \rangle = \sum_{i=1}^{n} a_i(g) \langle b_i, v^* \rangle = \sum_{i=1}^{n} a_i(g) b_i(1) = \phi(g),$$

i.e., ϕ is a matrix coefficient of the restriction of R to V. $\square$

Therefore, we call $R(G)$ the *algebra of matrix coefficients* of G.

We also have the following stronger version of 2.4.2

2.4.5. Corollary. *Let $\phi \in R(G)$. Then there exist n and functions $a_i, b_i \in R(G)$, $1 \le i \le n$, such that*

$$\phi(gh) = \sum_{i=1}^{n} a_i(g) b_i(h)$$

for all $g, h \in G$.

Proof. Since ϕ is a matrix coefficient of a finite-dimensional representation by 2.4.4, the statement follows from the formula for the product of two matrices. $\square$

Moreover, $R(G)$ has the following properties. For a function $\phi \in C(G)$ we denote by $\bar{\phi}$ the function $g \longmapsto \overline{f(g)}$ on G; and by $\hat{\phi}$ the function $g \longmapsto f(g^{-1})$.

2.4.6. Lemma. *Let $\phi \in R(G)$. Then*

(i) *the function $\bar{\phi}$ is in $R(G)$;*

(ii) *the function $\hat{\phi}$ is in $R(G)$.*

Proof. Obvious by 2.4.2. $\square$

3. Some results from functional analysis

3.1. Compact operators. Let E be a Hilbert space and $T : E \longrightarrow E$ a continuous linear operator.

We say that T is a *compact operator* if T is a continuous linear operator which maps the unit ball in E into a relatively compact set.

3.1.1. LEMMA. *Compact operators for a two-sided ideal in the algebra of all continuous linear operators on E.*

PROOF. Let S and T be compact operators. Let B be the unit ball in E. Then the images of B in E under T and S have compact closure. Hence, the image of $B \times B$ under $S \times T : E \times E \longrightarrow E \times E$ has compact closure. Since the addition is a continuous map from $E \times E$ into E, the image of B under $S+T$ also has compact closure. Therefore, $S+T$ is a compact operator.

If S is a bounded linear operator and T a compact operator, the image of B under T has compact closure. Since S is continuous, the image of B under ST also has compact closure. Hence, ST is compact.

Analogously, the image of B under S is a bounded set since S is bounded. Therefore, the image of B under TS has compact closure and TS is also compact. $\square$

3.2. Compact selfadjoint operators. Let E be a Hilbert space. Let $T : E \longrightarrow E$ be a nonzero compact selfadjoint operator.

3.2.1. THEOREM. *Either $\|T\|$ or $-\|T\|$ is an eigenvalue of T.*

First we recall a simple fact.

3.2.2. LEMMA. *Let u and v be two nonzero vectors in E such that $|(u|v)| = \|u\| \cdot \|v\|$. Then u and v are colinear.*

PROOF. Let λv be the orthogonal projection of u to v. Then $u = \lambda v + w$ and w is perpendicular to v. This implies that $\|u\|^2 = |\lambda|^2 \|v\|^2 + \|w\|^2$. On the other hand, we have $\|u\| \cdot \|v\| = |(u|v)| = |\lambda| \|v\|^2$, i.e., $|\lambda| = \frac{\|u\|}{\|v\|}$. Hence, it follows that

$$\|u\|^2 = |\lambda|^2 \|v\|^2 + \|w\|^2 = \|u\|^2 + \|w\|^2,$$

i.e., $\|w\|^2 = 0$ and $w = 0$. $\square$

Now we can prove the theorem. By rescaling T, we can assume that $\|T\| = 1$.

Let B be the unit ball in E. By our assumption, we know that

$$1 = \|T\| = \sup_{v \in B} \|Tv\|.$$

Therefore, there exists a sequence of vectors $v_n \in B$ such that $\lim_{n\to\infty} \|Tv_n\| = 1$. Since T is compact, by going to a subsequence, we can also assume that $\lim_{n\to\infty} Tv_n = u$. This implies that

$$1 = \lim_{n\to\infty} \|Tv_n\| = \|u\|.$$

Moreover, we have $\lim_{n\to\infty} T^2 v_n = Tu$. Hence, we have

$$\begin{aligned} 1 = \|T\| \cdot \|u\| \geq \|Tu\| = \lim_{n\to\infty} \|T^2 v_n\| \geq \limsup_{n\to\infty} (\|T^2 v_n\| \cdot \|v_n\|) \\ \geq \limsup_{n\to\infty} (T^2 v_n | v_n) = \lim_{n\to\infty} (Tv_n | Tv_n) = \lim_{n\to\infty} \|Tv_n\|^2 = 1. \end{aligned}$$

It follows that
$$\|Tu\| = 1.$$
Moreover, we have
$$1 = \|Tu\|^2 = (Tu|Tu) = (T^2u|u) \le \|T^2u\|\|u\| \le \|T^2\|\|u\|^2 \le \|T\|^2\|u\|^2 = 1.$$
This finally implies that
$$(T^2u|u) = \|T^2u\|\|u\|.$$
By 3.2.2, it follows that T^2u is proportional to u, i.e. $T^2u = \lambda u$. Moreover, we have
$$\lambda = \lambda(u|u) = (T^2u|u) = \|Tu\|^2 = 1.$$
It follows that $T^2u = u$.

Therefore, the linear subspace F of E spanned by u and Tu is T-invariant. Either $Tu = u$ or $v = \frac{1}{2}(u - Tu) \neq 0$. In the second case, we have $Tv = -v$.

This completes the proof of the existence of eigenvalues.

We need another fact.

3.2.3. Lemma. *Let T be a compact selfadjoint operator. Let λ be an eigenvalue different from 0. Then the eigenspace of λ is finite-dimensional.*

Proof. Assume that the corresponding eigenspace V is infinite-dimensional. Then there would exist an orthonormal sequence $(e_n, n \in \mathbb{N})$ in F. Clearly, then the sequence $(Te_n, n \in \mathbb{N})$ would consist of mutually orthogonal vectors of length $|\lambda|$, hence it could not have compact closure in V, contradicting the compactness of T. Therefore, V cannot be infinite-dimensional. □

3.3. An example. Denote by μ the Haar measure on G. Let $L^2(G)$ be the Hilbert space of square-integrable complex valued functions on G with respect to the Haar measure μ. We denote its norm by $\|\cdot\|_2$. Clearly, we have
$$\|f\|_2^2 = \int_G |f(g)|^2\,d\mu(g) \le \|f\|^2$$
for any $f \in C(G)$. Hence the inclusion $C(G) \longrightarrow L^2(G)$ is a continuous map.

3.3.1. Lemma. *The continuous linear map $i : C(G) \longrightarrow L^2(G)$ is injective.*

Proof. Let $f \in C(G)$ be such that $i(f) = 0$. This implies that $\|f\|_2 = 0$. On the other hand, the function $g \longmapsto |f(g)|$ is a nonnegative continuous function on G. Assume that M is the maximum of this function on G. If we would have $M > 0$, there would exist a nonempty open set $U \subset G$ such that $|f(g)| \ge \frac{M}{2}$ for $g \in U$. Therefore, we would have
$$\|f\|_2^2 = \int_G |f(g)|^2\,d\mu(g) \ge \frac{M^2}{4}\mu(U) > 0,$$
by 1.3.16. Therefore, we must have $M = 0$. □

Since the measure of G is 1, by Cauchy-Schwartz inequality, we have
$$\int_G |\phi(g)|\,d\mu(g) = \int_G 1 \cdot |\phi(g)|\,d\mu(g) \le \|1\|_2 \cdot \|\phi\|_2 = \|\phi\|_2$$
for any $\phi \in L^2(\mu)$. Hence, $L_2(G) \subset L_1(G)$, where $L_1(G)$ is the Banach space of integrable functions on G.

Let f be a continuous function on G. For any $\phi \in L^2(G)$, we put

$$(R(f)\phi)(g) = \int_G f(h)\phi(gh)d\mu(h)$$

for $g \in G$.

By 1.1.1, f is uniformly continuous on G. This implies that for any $\epsilon > 0$ there exists a neighborhood U of 1 in G such that $g'g^{-1} \in U$ implies $|f(g) - f(g')| < \epsilon$. Therefore, for arbitrary $h \in G$, we see that for $(g'^{-1}h)(g^{-1}h)^{-1} = g'^{-1}g \in U$ and we have

$$|f(g^{-1}h) - f(g'^{-1}h)| < \epsilon.$$

This in turn implies that

$$\begin{aligned}
|(R(f)\phi)(g) - (R(f)\phi)(g')| &= \left|\int_G f(h)\phi(gh)d\mu(h) - \int_G f(h)\phi(g'h)\,d\mu(h)\right| \\
= \left|\int_G (f(g^{-1}h) - f(g'^{-1}h))\phi(h)\,d\mu(h)\right| &= \int_G |f(g^{-1}h) - f(g'^{-1}h)|\,|\phi(h)|\,d\mu(h) \\
&< \epsilon \cdot \int_G |\phi(h)|\,d\mu(h) \leq \epsilon \cdot \|\phi\|_2
\end{aligned}$$

for any $g' \in Ug$ and ϕ in $L^2(G)$. This proves that functions $R(f)\phi$ are in $C(G)$ for any $\phi \in L^2(G)$.

Moreover, by the invariance of Haar measure, we have

$$\begin{aligned}
|(R(f)\phi)(g)| \leq \int_G |f(h)||\phi(gh)|\,d\mu(h) &\leq \|f\| \int_G |\phi(gh)|\,d\mu(h) \\
&\leq \|f\| \int_G |\phi(h)|\,d\mu(h) \leq \|f\| \cdot \|\phi\|_2,
\end{aligned}$$

it follows that

$$\|R(f)\phi\| \leq \|f\| \cdot \|\phi\|_2$$

for any $\phi \in L^2(G)$. Hence, $R(f)$ is a bounded linear operator from $L^2(G)$ into $C(G)$.

Hence the set $\mathcal{S} = \{R(f)\phi \mid \|\phi\|_2 \leq 1\}$ is bounded in $C(G)$.

Clearly, the composition of $R(f)$ with the natural inclusion $i : C(G) \longrightarrow L^2(G)$ is a continuous linear map from $L^2(G)$ into itself which will denote by the same symbol. Therefore, the following diagram of continuous maps

$$\begin{tikzcd}
L^2(G) \arrow[r, "R(f)"] \arrow[d, "R(f)"'] & L^2(G) \\
C(G) \arrow[ur, "i"'] &
\end{tikzcd}$$

is commutative.

We already remarked that $\mathcal{S}$ is a bounded set in $C(G)$. Hence, $\mathcal{S}$ is a pointwise bounded family of continuous functions. In addition, by the above formula

$$|(R(f)\phi)(g) - (R(f)\phi)(g')| < \epsilon,$$

for all $g' \in Ug$ and ϕ in the unit ball in $L^2(G)$. Hence, the set $\mathcal{S}$ is equicontinuous.

Hence we proved the following result.

3.3.2. Lemma. *The set $\mathcal{S} \subset C(G)$ is pointwise bounded and equicontinuous.*

By 1.2.1, the closure of the set $\mathcal{S}$ in $C(G)$ is compact. Since $i : C(G) \longrightarrow L^2(G)$ is continuous, $\mathcal{S}$ has compact closure in $L^2(G)$. Therefore, we have the following result.

3.3.3. LEMMA. *The linear operator $R(f) : L^2(G) \longrightarrow L^2(G)$ is compact.*

Put $f^*(g) = \overline{f(g^{-1})}$, $g \in G$. Then $f^* \in C(G)$.

3.3.4. LEMMA. *For any $f \in C(G)$ we have*

$$R(f)^* = R(f^*).$$

PROOF. For $\phi, \psi \in L^2(G)$, we have, by 1.3.17,

$$(R(f)\phi \mid \psi) = \int_G (R(f)\phi)(g)\overline{\psi(g)}\, d\mu(g) = \int_G \left(\int_G f(h)\phi(gh)\, d\mu(h)\right) \overline{\psi(g)}\, d\mu(g)$$
$$= \int_G f(h) \left(\int_G \phi(gh)\overline{\psi(g)}\, d\mu(g)\right)\, d\mu(h) = \int_G f(h) \left(\int_G \phi(g)\overline{\psi(gh^{-1})}\, d\mu(g)\right)\, d\mu(h)$$
$$= \int_G \phi(g) \left(\overline{\int_G \overline{f(h)}\psi(gh^{-1})\, d\mu(h)}\right)\, d\mu(g)$$
$$= \int_G \phi(g) \left(\overline{\int_G f^*(h^{-1})\psi(gh^{-1})\, d\mu(h)}\right)\, d\mu(g)$$
$$= \int_G \phi(g) \left(\overline{\int_G f^*(h)\psi(gh)\, d\mu(h)}\right)\, d\mu(g) = (\phi \mid R(f^*)\psi).$$

$\square$

3.3.5. COROLLARY. *The operator $R(f^*)R(f) = R(f)^*R(f)$ is a positive compact selfadjoint operator on $L^2(G)$.*

4. Peter-Weyl theorem

4.1. L^2 version. Let $\phi \in L^2(G)$. Let $g \in G$. We put $(R(g)\phi)(h) = \phi(hg)$ for any $h \in G$. Then we have

$$\|R(g)\phi\|_2^2 = \int_G |(R(g)\phi)(h)|^2\, d\mu(h) = \int_G |\phi(hg)|^2\, d\mu(h) = \int_G |\phi(h)|^2\, d\mu(h) = \|\phi\|_2^2.$$

Therefore, $R(g)$ is a continuous linear operator on $L^2(G)$. Clearly it is in $\mathrm{GL}(L^2(G))$. Moreover, $R(g)$ is unitary.

Clearly, for any $g \in G$, the following diagram

$$\begin{array}{ccc} C(G) & \xrightarrow{R(g)} & C(G) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ L^2(G) & \xrightarrow[R(g)]{} & L^2(G) \end{array}$$

is commutative.

Analogously, we define $(L(g)\phi)(h) = \phi(g^{-1}h)$ for $h \in G$. Then $L(g)$ is a unitary operator on $L^2(G)$ which extends from $C(G)$.

Clearly, $R(g)$ and $L(h)$ commute for any $g, h \in G$.

4.1.1. LEMMA. *L and R are unitary representations of G on $L^2(G)$.*

Proof. It is enough to discuss R. The proof for L is analogous.

Let $g \in G$ and $\phi \in L^2(G)$. We have to show that $h \longmapsto R(h)\phi$ is continuous at g. Let $\epsilon > 0$. Since $C(G)$ is dense in $L^2(G)$, there exists $\psi \in C(G)$ such that $\|\phi - \psi\|_2 < \frac{\epsilon}{3}$. Since R is a representation on $C(G)$, there exists a neighborhood U of g such that $h \in U$ implies $\|R(h)\psi - R(g)\psi\| < \frac{\epsilon}{3}$. This in turn implies that $\|R(h)\psi - R(g)\psi\|_2 < \frac{\epsilon}{3}$. Therefore we have

$$\begin{aligned}\|R(h)\phi - R(g)\phi\|_2 &\leq \|R(h)(\phi - \psi)\|_2 + \|R(h)\psi - R(g)\psi\|_2 + \|R(g)(\psi - \phi)\|_2 \\ &\leq 2\|\phi - \psi\|_2 + \|R(h)\psi - R(g)\psi\|_2 < \epsilon\end{aligned}$$

for any $h \in U$. □

Let f be a continuous function on G. By 3.3.3, $R(f)$ is a compact operator on $L^2(G)$.

Let $\phi \in L^2(G)$. Then

$$\begin{aligned}(R(f)L(g)\phi)(h) &= \int_G f(k)(L(g)\phi)(hk)\, d\mu(k) \\ &= \int_G f(k)\phi(g^{-1}hk)\, d\mu(k) = (R(f)\phi)(g^{-1}h) = (L(g)R(f)\phi)(h)\end{aligned}$$

for all $g, h \in G$. Therefore, $R(f)$ commutes with $L(g)$ for any $g \in G$.

Let F be the eigenspace of $R(f^*)R(f)$ for eigenvalue $\lambda > 0$. Then F is finite-dimensional by 3.2.3.

4.1.2. Lemma. (i) *Let $\phi \in F$. Then ϕ is a continuous function.*
(ii) *The vector subspace F of $C(G)$ is in $R(G)$.*

Proof. (i) The function ϕ is in the image of $R(f^*)$. Hence it is a continuous function.

(ii) By (i), $F \subset C(G)$. As we remarked above, the operator $R(f^*)R(f)$ commutes with the representation L. Therefore, the eigenspace F is invariant subspace for L. Let ϕ be a function in F. Since F is invariant for L, ϕ is G-finite. Hence, $\phi \in R(G)$. □

4.1.3. Lemma. *The subspace $R(G)$ is invariant for $R(f)$.*

Proof. Let $\phi \in R(G)$. By 2.4.5 we have

$$(R(f)\phi)(g) = \int_G f(h)\phi(gh)\, d\mu(h) = \sum_{i=1}^n a_i(g) \int_G f(h) b_i(h)\, d\mu(h)$$

for any $g \in G$, i.e., $R(f)\phi$ is a linear combination of a_i, $1 \leq i \leq n$. □

Let $E = R(G)^\perp$ in $L^2(G)$. Then, by 4.1.3, $R(G)$ is invariant for selfadjoint operator $R(f^*)R(f)$. This in turn implies that E is also invariant for this operator. Therefore the restriction of this operator to E is a positive selfadjoint compact operator. Assume that its norm is greater than 0. Then, by 3.2.1, the norm is an eigenvalue of this operator, and there exists a nonzero eigenvector $\phi \in E$ for that eigenvalue. Clearly, ϕ is an eigenvector for $R(f^*)R(f)$ too. By 4.1.2, ϕ is also in $R(G)$. Hence, we have $\|\phi\|_2^2 = (\phi \mid \phi) = 0$, and $\phi = 0$ in $L^2(G)$. Hence, we have a contradiction.

Therefore, the operator $R(f^*)R(f)$ is 0 when restricted to E. Hence

$$0 = (R(f^*)R(f)\psi|\psi) = \|R(f)\psi\|_2^2$$

for any $\psi \in E$. It follows that $R(f)\psi = 0$. Since $R(f)\psi$ is a continuous function, we have

$$0 = (R(f)\psi)(1) = \int_G f(g)\psi(g)d\mu(g),$$

i.e., ψ is orthogonal to $\bar{f}$.

Since $f \in C(G)$ was arbitrary and $C(G)$ is dense in $L^2(G)$, it follows that $\psi = 0$. This implies that $E = \{0\}$.

This completes the proof of the following result.

4.1.4. Theorem (Peter-Weyl). *The algebra $R(G)$ is dense in $L^2(G)$.*

4.2. Continuous version. Let $g \in G$. Assume that $g \neq 1$. Then there exists an open neighborhood U of 1 such that U and Ug are disjoint. There exists positive function ϕ in $C(G)$ such that $\phi|_U = 0$ and $\phi|_{Ug} = 1$. This implies that

$$\begin{aligned}\|R(g)\phi - \phi\|^2 &= \int_G |\phi(hg) - \phi(h)|^2 d\mu(h) \\ &= \int_U |\phi(hg) - \phi(h)|^2 d\mu(h) + \int_{G-U} |\phi(hg) - \phi(h)|^2 d\mu(h) \geq \mu(U).\end{aligned}$$

Therefore $R(g) \neq I$. Since by 4.1.4, $R(G)$ is dense in $L^2(G)$, $R(g)|_{R(G)}$ is not the identity operator.

This implies the following result.

4.2.1. Lemma. *Let $g, g' \in G$ and $g \neq g'$. Then there exists a function $\phi \in R(G)$ such that $\phi(g) \neq \phi(g')$.*

Proof. Let $h = g^{-1}g' \neq 1$. Then there exists $\psi \in R(G)$ such that $R(h)\psi \neq \psi$. Hence, we have $R(g)\psi \neq R(g')\psi$. It follows that $\psi(hg) \neq \psi(hg')$ for some $h \in G$. Therefore, the function $\phi = L(h^{-1})\psi$ has the required property. □

In other words, $R(G)$ separates points in G. By Stone-Weierstrass theorem, we have the following result which is a continuous version of Peter-Weyl theorem.

4.2.2. Theorem (Peter-Weyl). *The algebra $R(G)$ is dense in $C(G)$.*

Another consequence of 4.2.1 is the following result.

4.2.3. Lemma. *Let U be an open neighborhood of 1 in G. Then there exists a finite-dimensional representation (π, V) of G such that $\ker \pi \subset U$.*

Proof. The complement $G - U$ of U is a compact set. Since $R(G)$ separates the points of G, for any $g \in G - U$ there exists a function $\phi_g \in R(G)$ and an open neighborhood U_g of g such that $\phi_g(h) \neq \phi_g(1)$ for $h \in U_g$. Since $G - U$ is compact, there exists a finite set $g_1, g_2, \cdots, g_m$ in $G - U$ such that $U_{g_1}, U_{g_2}, \cdots, U_{g_m}$ form an open cover of $G - U$ and $\phi_{g_i}(h) \neq \phi_{g_i}(1)$ for $h \in U_{g_i}$. Let π_i be a finite-dimensional representation of G with matrix coefficient ϕ_{g_i}. Then $\pi_i(h) \neq I$ for $h \in U_{g_i}$, $1 \leq i \leq n$. Let π be the direct sum of π_i. Then $\pi(h) \neq I$ for $h \in G - U$, i.e., $\ker \pi \subset U$. □

4.3. Matrix groups. Let G be a topological group. We say that G *has no small subgroups* if there exists a neighborhood U of $1 \in G$ such that any subgroup of G contained in U is trivial.

4.3.1. Lemma. *Let V be a finite-dimensional complex vector space. Then the group* $\mathrm{GL}(V)$ *has no small subgroups.*

Proof. Let $\mathcal{L}(V)$ be the space of all linear endomorphisms of V. Then $\exp : \mathcal{L}(V) \longrightarrow \mathrm{GL}(V)$ given by

$$\exp(T) = \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

defines a holomorphic map. Its differential at 0 is the identity map I on $\mathcal{L}(V)$. Hence, by the inverse function theorem, it is a local diffeomorphism.

Let U be an open neighborhood of 1 in $\mathrm{GL}(V)$ and V the open ball around 0 in $\mathcal{L}(V)$ of radius ϵ (with respect to the linear operator norm) such that $\exp : V \longrightarrow U$ is a diffeomorphism. Let V' be the open ball of radius $\frac{\epsilon}{2}$ around 0 in $\mathcal{L}(V)$. Then $U' = \exp(V')$ is an open neighborhood of 1 in $\mathrm{GL}(V)$. Let H be a subgroup of $\mathrm{GL}(V)$ contained in U'. Let $S \in H$. Then $S = \exp(T)$ for some $T \in V'$. Hence, we have $S^2 = \exp(T)^2 = \exp(2T) \in H$. Moreover, $S^2 \in H$ and $S^2 = \exp(T')$ for some $T' \in V'$. It follows that $\exp(T') = \exp(2T)$ for $2T, T' \in V$. Since exp is injective on V, we must have $2T = T'$. Hence, $T \in \frac{1}{2}V'$. It follows that $H \subset \exp\left(\frac{1}{2}V'\right)$. By induction we get that $H \subset \exp\left(\frac{1}{2^n}V'\right)$ for any $n \in \mathbb{N}$. This implies that $H = \{1\}$. □

A compact subgroup of $\mathrm{GL}(V)$ we call a compact *matrix* group.

4.3.2. Theorem. *Let G be a compact group. Then the following conditions are equivalent:*

(i) *G has no small subgroups;*
(ii) *G is isomorphic to a compact matrix group.*

Proof. (i)⇒(ii) Let U be an open neighborhood of $1 \in G$ such that it contains no nontrivial subgroups of G. By 4.2.3, there exists a finite-dimensional representation (π, V) of G such that $\ker \pi \subset U$. This clearly implies that $\ker \pi = \{1\}$, and $\pi : G \longrightarrow \mathrm{GL}(V)$ is an injective homomorphism. Since G is compact, π is homoeomprphism of G onto $\pi(G)$. Therefore, G is isomorphic to the compact subgroup $\pi(G)$ of $\mathrm{GL}(V)$.

(ii)⇒(i) Assume that G is a compact subgroup of $\mathrm{GL}(V)$. By 4.3.1, there exists an open neighborhood U of 1 in $\mathrm{GL}(V)$ such that it contains no nontrivial subgroups. This implies that $G \cap U$ contains on nontrivial subgroups of G. □

4.3.3. Remark. For a compact matrix group G, since matrix coefficients of the natural representation separate points in G, 4.2.1 obviously holds. Therefore, in this situation, Stone-Weierstrass theorem immediately implies the second version of Peter-Weyl theorem, which in turn implies the first one.

4.3.4. Remark. By Cartan's theorem [**1**], any compact matrix group is a Lie group. On the other hand, by [**1**] any Lie group has no small subgroups. Hence, compact Lie groups have no small subgroups and therefore they are compact matrix groups.

4.3.5. Remark. Let $T = \mathbb{R}/\mathbb{Z}$. Then T is a compact abelian group. Let G be the product of inifinite number of copies of T. Then G is a compact abelian group. By the definition of product topology, any neghborhood of 1 contains a nontrivial subgroup.

Let G be an arbitrary compact group. Let (π, V) be a finite-dimensional representation. Put $N = \ker \pi$. Then N is a compact normal subgroup of G and G/N equipped with the quotient topology is a compact group. Clearly, G/N is a compact matrix group.

Let $\mathcal{S}$ be the family of all compact normal subgroups N of G such that G/N is a compact matrix group. Clearly, N, N' in $\mathcal{S}$ implies $N \cap N' \in \mathcal{S}$. Therefore, $\mathcal{S}$ ordered by inclusion is a directed set. One can show that G is a projective limit of the system G/N, $N \in \mathcal{S}$. Therefore, any compact group is a projective limit of compact matrix groups. By the above remark, this implies that any compact group is a projective limit of compact Lie groups.

Bibliography

[1] Dragan Miličić, *Lectures on Lie Groups*, unpublished manuscript.