HARISH-CHANDRA'S ORTHOGONALITY RELATIONS FOR ADMISSIBLE REPRESENTATIONS

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Abstract. We prove a generalization of Harish-Chandra's character orthogonality relations for discrete series to arbitrary admissible representations for real reductive Lie groups. This result is an analogue of a conjecture by Kazhdan for $\mathfrak{p}$-adic reductive groups proved by Bezrukavnikov, and Schneider and Stuhler.

Introduction

Let G_0 be a connected compact Lie group. Denote by $\mathcal{M}(G_0)$ the category of finite-dimensional representations of G_0. Then $\mathcal{M}(G_0)$ is abelian and semisimple. Denote by $\mathrm{K}(G_0)$ its Grothendieck group. Let U and U' be two finite-dimensional representations of G_0. Denote by $\mathrm{Hom}_{G_0}(U, U')$ the complex vector space of intertwining maps between representations U and U'. Then the map $(U, U') \longmapsto \dim \mathrm{Hom}_{G_0}(U, U')$ extends to a biadditive pairing on $\mathrm{K}(G_0)$, which we call the multiplicity pairing.

For a finite-dimensional representation U of G_0, we denote by Θ_U its character. Let μ_{G_0} be the normalized Haar measure on G_0. Then the map

$$(U, U') \longmapsto \int_{G_0} \Theta_U(g)\overline{\Theta_{U'}(g)}\, d\mu_{G_0}(g)$$

extends to another pairing on $\mathrm{K}(G_0)$. The Schur orthogonality relations for characters of irreducible representations imply that these two pairings are equal.

Let T_0 be a maximal torus in G_0. Denote by $\mathfrak{g}$ and $\mathfrak{t}$ the complexified Lie algebras of G_0 and T_0 respectively. Let R be the root system of the pair $(\mathfrak{g}, \mathfrak{t})$. Let W be the Weyl group of R and $[W]$ its order.

For any root $\alpha \in R$ denote by e^α the corresponding homomorphism of T_0 in the group of complex numbers of absolute value 1. Let

$$D(t) = \prod_{\alpha \in R} (1 - e^\alpha(t))$$

for any $t \in T_0$. Let μ_{T_0} be the normalized Haar measure on T_0. Then we have the Weyl integral formula

$$\int_{G_0} f(g)\, d\mu_{G_0}(g) = \frac{1}{[W]} \int_{T_0} \left(\int_{G_0} f(gtg^{-1})\, d\mu_{G_0}(g) \right) D(t)\, d\mu_{T_0}(t)$$

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for any continuous function f on G_0. In particular, this implies that the above pairing is given by

$$(U,U') \longmapsto \frac{1}{[W]} \int_{T_0} \Theta_U(t)\overline{\Theta_{U'}(t)} D(t)\, d\mu_{T_0}(t).$$

The equality of the above pairings was used by Hermann Weyl to determine the formulas for the restriction to T_0 of the characters of irreducible finite-dimensional representations of G_0.

Assume now that G_0 is a noncompact connected semisimple Lie group with finite center. Let K_0 be its maximal compact subgroup. Assume that the ranks of G_0 and K_0 are equal. Fix a maximal torus T_0 in K_0. Harish-Chandra proved that under this condition the group G_0 has a family of square-integrable irreducible unitary representations called the *discrete series*. Moreover, he determined the distribution characters of these representations using a vast generalization of the ideas from Weyl's construction in the case of compact Lie groups. These characters were given by explicit formulas on the regular elements in T_0 which were analogous to Weyl's character formulas.

Earlier, in his work on distribution characters of representations, Harish-Chandra proved that for an admissible representation U, its character Θ_U is given by real analytic function on the regular set in T_0. More precisely, if R^+ is a set of positive roots in R,

$$\prod_{\alpha \in R^+} (1 - e^{\alpha}(t))\Theta_U(t)$$

extends to a real analytic function on the whole torus T_0. This allows to construct a natural pairing of these characters generalizing the construction for compact groups. For two representations U and U' we put

$$(U,U') \longmapsto \frac{1}{[W_0]} \int_{T_0} \Theta_U(t)\overline{\Theta_{U'}(t)} D(t)\, d\mu_{T_0}(t)$$

where W_0 is the Weyl group of K_0. This is the *elliptic pairing* we discuss in Section 1.

In the case of discrete series characters, Harish-Chandra established an analogue of Schur orthogonality relations for the elliptic pairing [4]. Since the category of square-integrable representations is semisimple, this leads to an analogue of the above statement: the multiplicity pairing and the elliptic pairing are equal on the Grothendieck group of that category.

Harish-Chandra's orthogonality relations played a critical role in his determination of discrete series characters. At that time, it was not clear if they are just a useful tool, or a special case of a more fundamental property of the category of admissible representations of G_0. The main reason for this was that, before pioneering work of Gregg Zuckerman on cohomological induction, the homological algebra of that category was not well understood.

Kazhdan discussed the elliptic pairing in the setting of representation theory of $\mathfrak{p}$-adic reductive groups [9]. He defined an analogue of the multiplicity pairing and conjectured the equality of this pairing with the elliptic pairing in that setting. The main observation was that to deal with nonsemisimplicity of the category of admissible representations, one has to modify the analogue of multiplicity pairing and consider not only dimensions of Hom_{G_0}-spaces but also the higher $\mathrm{Ext}^{\cdot}_{G_0}$-spaces and define the pairing as an alternating sum of their dimensions. His conjecture was proved independently by Bezrukavnikov [1, Thm. 0.20] and Schneider and Stuler [14, Theorem, III.4.21].

In this note we prove a generalization of Harish-Chandra's result for arbitrary admissible representations of real reductive groups. It is the exact equivalent of Kazhdan's conjecture for real groups. The proof is mostly formal in nature. To deal with nonsemisimplicity of the category of representations, we replace it with its derived category. The derived functor RHom defines formally a pairing on the Grothendieck group of the derived category of representations with values in the Grothendieck group of the derived category of vector spaces. Using appropriate finiteness results, following that pairing by the dimension function, we define the analogue of the multiplicity pairing in this setting. It is the *homological pairing* we discuss in Section 2. On Grothendieck group of admissible representations this pairing agrees with the one proposed by Kazhdan.

Since the Grothendieck group is generated by cohomologically induced representations, a Frobenius reciprocity result proved in Section 3 reduces the calculation of this pairing to calculation of Lie algebra homology of nilpotent radicals of Borel subalgebras containing the complexified Lie algebra of T_0 in the complexified Lie algebra of G_0. Finally, to establish the equality of homological and elliptic pairings, we use a very special case of the Osborne conjecture [7]. [1]

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1. Elliptic pairing

1.1. **Groups of Harish-Chandra class.** Let G_0 be a Lie group with finitely many connected components. Let $\mathfrak{g}$ be the complexified Lie algebra of G_0. Assume that $\mathfrak{g}$ is reductive.

Denote by $\mathrm{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$ and $\mathrm{Ad} : G_0 \longrightarrow \mathrm{Aut}(\mathfrak{g})$ the adjoint representation of G_0. Let $\mathrm{Int}(\mathfrak{g})$ be the subgroup of inner automorphisms.

Let G_1 be the derived subgroup of the identity component of G_0.

We say that the group G_0 is of *Harish-Chandra class* (see, for example, [5], [15, II.1]) if the following properties hold:

(HC1) $\mathrm{Ad}(G_0) \subset \mathrm{Int}(\mathfrak{g})$;
(HC2) The center of G_1 is finite.

Fix a maximal compact subgroup K_0 of G_0. Let K be the compexification of K_0. Then K is a reductive complex algebraic group. Let $\mathfrak{k} \subset \mathfrak{g}$ be the complexified Lie algebra of K_0.

1.2. **Categories of $(\mathfrak{g}, K)$-modules.** Fix a group G_0 of Harish-Chandra class and a maximal compact subgroup K_0 of G_0. Denote by $\mathcal{M}(\mathfrak{g}, K)$ the category of objects (π, V) which are $\mathcal{U}(\mathfrak{g})$-modules and algebraic representations of K on V, and the actions π are compatible, i.e.,

(i) the actions of $\mathfrak{k}$ as subset of $\mathcal{U}(\mathfrak{g})$ and as differential of the action of K agree; and
(ii)
$$\pi(k)\pi(\xi)\pi(k^{-1}) = \pi(\mathrm{Ad}(k)\xi)$$
for $k \in K$ and $\xi \in \mathfrak{g}$.

[1] As Hecht and Schmid stated in their paper, that result is already implicit in [13] (but "well-hidden" there).

The objects in this category are $(\mathfrak{g}, K)$-modules. The morphisms are the linear maps intertwining the actions of $\mathcal{U}(\mathfrak{g})$ and K. For any two $(\mathfrak{g}, K)$-modules U and V we denote by $\operatorname{Hom}_{(\mathfrak{g},K)}(U, V)$ the complex vector space of all morphisms of U into V. Clearly, $\mathcal{M}(\mathfrak{g}, K)$ is an abelian category.

We denote by $\mathcal{A}(\mathfrak{g}, K)$ the full subcategory of $\mathcal{M}(\mathfrak{g}, K)$ consisting of all $(\mathfrak{g}, K)$-modules of finite length. The objects in $\mathcal{A}(\mathfrak{g}, K)$ are *Harish-Chandra modules.*

Let V be a $(\mathfrak{g}, K)$-module. Since K is reductive, V is a direct sum of irreducible finite-dimensional representations of K. We say that V is an *admissible* $(\mathfrak{g}, K)$-module if $\operatorname{Hom}_K(F, V)$ is finite-dimensional for any finite-dimensional irreducible representation F of K. By a classical result of Harish-Chandra, any irreducible $(\mathfrak{g}, K)$-module is admissible. Hence, any Harish-Chandra module is admissible.

Let V be a Harish-Chandra module. Denote by $V\check{}$ the K-finite dual of V equipped with the adjoint action of $\mathfrak{g}$ and K. Then $V\check{}$ is the *dual* of V. The functor $V \longmapsto V\check{}$ is an involutive antiequivalence of the category $\mathcal{A}(\mathfrak{g}, K)$.

Let $\mathrm{K}(\mathfrak{g}, K)$ be the Grothendieck group of $\mathcal{A}(\mathfrak{g}, K)$. For any U in $\mathcal{A}(\mathfrak{g}, K)$, we denote by $[U]$ the corresponding element of $\mathrm{K}(\mathfrak{g}, K)$.

For any Harish-Chandra module V, Harish-Chandra defined its *character* Θ_V which is a distribution on G_0. The map $V \longmapsto \Theta_V$ factors through $\mathrm{K}(\mathfrak{g}, K)$. Hence, we can also denote by $\Theta_{[V]}$ the character of the element $[V]$ of $K(\mathfrak{g}, K)$. Clearly, $[V] \longmapsto \Theta_{[V]}$ is a homomorphism of $\mathrm{K}(\mathfrak{g}, K)$ into the additive group of distributions on G_0. A well known regularity theorem of Harish-Chandra states that the distribution $\Theta_{[V]}$ is given by a locally integrable function which is real analytic on the set of regular elements in G_0. Abusing the notation, we denote it by the same letter. More precisely, we have

$$\Theta_{[V]}(f) = \int_{G_0} \Theta_{[V]}(g) f(g)\, d\mu_{G_0}(g)$$

for any compactly supported smooth function f on G_0.

1.3. **Weyl integral formula for the elliptic set.** Assume that the rank of G_0 is equal to the rank of K_0. Let T_0 be a Cartan subgroup of K_0. Then T_0 is also a Cartan subgroup in G_0. Let $\mathfrak{t}$ be the complexified Lie algebra of T_0. Denote by R the root system of $(\mathfrak{g}, \mathfrak{t})$ in $\mathfrak{t}^*$.

The normalizers of T_0 in G_0 and K_0 are equal and we denote them by $N(T_0)$. The quotient $W_0 = N(T_0)/T_0$ is naturally identified with a subgroup W_0 of the Weyl group W of the root system R (cf. [15, Part II, Sec. 1]). Denote by $[W_0]$ the order of W_0.

An element $g \in G_0$ is *elliptic* if $\operatorname{Ad}(g)$ is semisimple and its eigenvalues are complex numbers of absolute value 1. Denote by E the set of all regular elliptic elements in G_0. Also denote by T_0' the set of regular elements in T_0. Clearly, E is an open set in G_0, invariant under conjugation by elements of G_0 and every conjugacy class in E intersects T_0'. Let μ_{G_0} be a Haar measure on G_0. Then there exists a unique W_0-invariant positive measure ν on T_0 such that

$$\int_E f(g)\, d\mu_{G_0}(g) = \int_{T_0} \left(\int_{G_0} f(gtg^{-1})\, d\mu_{G_0}(g) \right) d\nu(t)$$

for any compactly supported continuous function f on G_0. It is evident that the measure ν does not depend on the choice of Haar measure μ_{G_0}.

For any root α in R we denote by e^α the corresponding homomorphism of T_0 into the group of complex numbers of absolute value 1. We put

$$D(t) = \prod_{\alpha \in R} (1 - e^\alpha(t))$$

for $t \in T_0$. Clearly, D is a positive real analytic function on T_0.

Then we have the following formula [15, Part II, Sec. 15, Lemma 17]

$$d\nu(t) = \frac{1}{[W_0]} D(t)\, d\mu_{T_0}(t),$$

where μ_{T_0} is the normalized Haar measure on the group T_0.

1.4. **Elliptic pairing.** Still assuming that the ranks of G_0 and K_0 are equal, let R^+ be a set of positive roots in R. Harish-Chandra proved that the function

$$\Psi_{[V]}(t) = \prod_{\alpha \in R^+} (1 - e^\alpha(t)) \Theta_{[V]}(t), \quad t \in T_0',$$

extends to a real analytic function on T_0 [4].

Therefore, for any two elements $[U]$ and $[V]$ in $\mathrm{K}(\mathfrak{g}, K)$, we can define

$$\begin{aligned} \langle [U] \mid [V] \rangle_{ell} &= \int_{T_0} \Theta_{[U]}(t) \overline{\Theta_{[V]}(t)}\, d\nu(t) \\ &= \frac{1}{[W_0]} \int_{T_0} D(t) \Theta_{[U]}(t) \overline{\Theta_{[V]}(t)}\, d\mu_{T_0}(t) = \frac{1}{[W_0]} \int_{T_0} \Psi_{[U]}(t) \overline{\Psi_{[V]}(t)}\, d\mu_{T_0}(t). \end{aligned}$$

This is clearly a biadditive pairing on $\mathrm{K}(\mathfrak{g}, K)$ with values in $\mathbb{C}$, which we call the *elliptic pairing.*

If the group G_0 has rank greater than its maximal compact subgroup K_0, we define the elliptic pairing on $\mathrm{K}(\mathfrak{g}, K)$ as the zero pairing.

2. Homological pairing

2.1. **Derived categories of $(\mathfrak{g}, K)$-modules.** It is well known that $\mathcal{M}(\mathfrak{g}, K)$ contains enough injective and projective objects [3, Ch. I]. Moreover, for any two $(\mathfrak{g}, K)$-modules U and V we have $\mathrm{Ext}^p_{(\mathfrak{g},K)}(U, V) = 0$ for $p > \dim(\mathfrak{g}/\mathfrak{k})$.

Denote by $D^b(\mathfrak{g}, K)$ the bounded derived category of $\mathcal{M}(\mathfrak{g}, K)$ and $D^b(\mathfrak{g}, K)^\circ$ its opposite category. Then we have the derived bifunctor $\mathrm{RHom}_{(\mathfrak{g},K)}$ from $D^b(\mathfrak{g}, K)^\circ \times D^b(\mathfrak{g}, K)$ into the bounded derived category $D^b(\mathbb{C})$ of complex vector spaces. As it is well known (see, for example, [8, Thm. 13.4.1])

$$H^p(\mathrm{RHom}_{(\mathfrak{g},K)}(U^\cdot, V^\cdot)) = \mathrm{Hom}_{D^b(\mathfrak{g},K)}(U^\cdot, V^\cdot[p]) \tag{2.1}$$

for any two complexes $U^\cdot$, $V^\cdot$ in $D^b(\mathfrak{g}, K)$.

2.2. **A finiteness result.** Denote by $D^b_{adm}(\mathfrak{g}, K)$ the full subcategory of $D^b(\mathfrak{g}, K)$ consisting of complexes with cohomology in $\mathcal{A}(\mathfrak{g}, K)$. Then $D^b_{adm}(\mathfrak{g}, K)$ is a triangulated category with natural t-structure and core $\mathcal{A}(\mathfrak{g}, K)$.

Let $D : \mathcal{A}(\mathfrak{g}, K) \longrightarrow D^b_{adm}(\mathfrak{g}, K)$ be the natural map attaching to a module U the complex $D(U)^\cdot$ such that $D(U)^0 = U$ and $D(U)^p = 0$ for $p \neq 0$.

2.2.1. **Lemma.** *Let $U^\cdot$ and $V^\cdot$ be two objects in $D^b_{adm}(\mathfrak{g}, K)$. Then $\mathrm{RHom}_{(\mathfrak{g},K)}(U^\cdot, V^\cdot)$ is a bounded complex of complex vector spaces with finite-dimensional cohomology.*

Proof. Let U and V be two objects in $\mathcal{A}(\mathfrak{g},K)$. Then they are admissible. Hence, by [3, I.2.8], $\operatorname{Ext}^p_{(\mathfrak{g},K)}(U,V)$ are finite-dimensional for any $p\in\mathbb{Z}_+$.

Therefore, $\operatorname{Hom}_{D^b(\mathfrak{g},K)}(D(U)^\cdot, D(V)^\cdot[p])$ is finite-dimensional for any two modules U and V in $\mathcal{A}(\mathfrak{g},K)$ and $p\in\mathbb{Z}$. By induction on the cohomological length of a bounded complex using standard truncation arguments (cf. [10, Ch. 3, 4.2]), this implies that $\operatorname{Hom}_{D^b(\mathfrak{g},K)}(U^\cdot,V^\cdot)$ is finite-dimensional for any two bounded complexes $U^\cdot$ and $V^\cdot$ in $D^b_{adm}(\mathfrak{g},K)$. By (2.1), this implies the statement of the lemma. □

Therefore, we can consider the bifunctor $\operatorname{RHom}_{(\mathfrak{g},K)}$ from $D^b_{adm}(\mathfrak{g},K)^o\times D^b_{adm}(\mathfrak{g},K)$ into the full subcategory $D^b_{fd}(\mathbb{C})$ of $D^b(\mathbb{C})$ consisting of complexes with finite-dimensional cohomology.

2.3. **Homological pairing.** Since the category $\mathcal{A}(\mathfrak{g},K)$ is not semisimple, to define a natural pairing on its Grothendieck group $\mathrm{K}(\mathfrak{g},K)$ we have to use homological algebra.

We identify the Grothendieck group of the triangulated category $D^b_{adm}(\mathfrak{g},K)$ with $\mathrm{K}(\mathfrak{g},K)$ via the map $[U^\cdot]\longmapsto\sum_{p\in\mathbb{Z}}(-1)^p[H^p(U^\cdot)]$ (see, for example, [10, Ch. 4, Sec. 3.5]). In the same fashion, the Grothendieck group of $D^b_{fd}(\mathbb{C})$ is identified with integers $\mathbb{Z}$ via the map $[A^\cdot]\longmapsto\sum_{p\in\mathbb{Z}}(-1)^p\dim H^p(A^\cdot)$.

Composition of the map $\operatorname{RHom}_{(\mathfrak{g},K)}: D^b_{adm}(\mathfrak{g},K)^o\times D^b_{adm}(\mathfrak{g},K)\longrightarrow D^b_{fd}(\mathbb{C})$ with the natural map of $D^b_{fd}(\mathbb{C})\longrightarrow \mathrm{K}(D^b_{fd}(\mathbb{C}))=\mathbb{Z}$ factors through $\mathrm{K}(\mathfrak{g},K)\times\mathrm{K}(\mathfrak{g},K)$. Hence, it defines a biadditive pairing $\mathrm{K}(\mathfrak{g},K)\times\mathrm{K}(\mathfrak{g},K)\longrightarrow\mathbb{Z}$. We call it the *homological pairing* on $\mathrm{K}(\mathfrak{g},K)$. For $U^\cdot$ and $V^\cdot$ in $D^b_{adm}(\mathfrak{g},K)$ we denote the value of this pairing by $\langle[U^\cdot]\mid[V^\cdot]\rangle_{(\mathfrak{g},K)}$.

2.3.1. **Proposition.** [2] *Let U and U' be two modules in $\mathcal{A}(\mathfrak{g},K)$. Then*

$$\langle[U]\mid[U']\rangle_{(\mathfrak{g},K)}=\sum_{p\in\mathbb{Z}}(-1)^p\dim\operatorname{Ext}^p_{(\mathfrak{g},K)}(U,U').$$

Proof. By (2.1), we have

$$\begin{aligned}\langle[U]\mid[U']\rangle_{(\mathfrak{g},K)}&=\langle[D(U)^\cdot]\mid[D(U')^\cdot]\rangle_{(\mathfrak{g},K)}\\&=\sum_{p\in\mathbb{Z}}(-1)^p\dim H^p(\operatorname{RHom}_{(\mathfrak{g},K)}(D(U)^\cdot,D(U')^\cdot))\end{aligned}$$
$$=\sum_{p\in\mathbb{Z}}(-1)^p\dim\operatorname{Hom}_{D^b(\mathfrak{g},K)}(D(U)^\cdot,D(U')^\cdot[p])=\sum_{p\in\mathbb{Z}}(-1)^p\dim\operatorname{Ext}^p_{(\mathfrak{g},K)}(U,U').$$

□

3. Frobenius reciprocity

3.1. **Frobenius reciprocity for cohomological induction.** Let σ be the Cartan involution corresponding to the maximal compact subgroup K_0 of G_0. Let $\mathfrak{c}$ be a σ-stable Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{c}=\mathfrak{t}\oplus\mathfrak{a}$ is the decomposition into eigenspaces for eigenvalues 1 and -1 of σ. Let T be the subgroup of K which centralizes $\mathfrak{c}$. Then its Lie algebra is identified with $\mathfrak{t}$.

As before, we define the category $\mathcal{M}(\mathfrak{c},T)$ consisting of $(\mathfrak{c},T)$-modules. Clearly, an irreducible $(\mathfrak{c},T)$-module is finite-dimensional. Therefore, in this case $\mathcal{A}(\mathfrak{c},T)$ is the full subcategory of finite-dimensional $(\mathfrak{c},T)$-modules.

[2]Because of this result, this pairing is sometimes called the *Euler-Poincaré pairing*.

Let R be the root system of the pair $(\mathfrak{g}, \mathfrak{c})$ in $\mathfrak{c}^*$. Denote by R^+ a set of positive roots in R. Let

$$\mathfrak{n} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

and

$$\mathfrak{b} = \mathfrak{c} \oplus \mathfrak{n}.$$

Then $\mathfrak{b}$ is a Borel subalgebra in $\mathfrak{g}$. Moreover, T normalizes $\mathfrak{b}$.

Let U be a $(\mathfrak{g}, K)$-module. Then the zeroth Lie algebra homology $H_0(\mathfrak{n}, U)$ is a $(\mathfrak{c}, T)$-module. By abuse of notation, we denote by $H_\bullet(\mathfrak{n}, -)$ the derived functor of Lie algebra homology from $D^b(\mathfrak{g}, K)$ into $D^b(\mathfrak{c}, T)$. Hence, the p-th Lie algebra homology group $H_p(\mathfrak{n}, U)$ of U is $H^{-p}(H_\bullet(\mathfrak{n}, D(U)^\cdot))$.

We consider the forgetful functor from $\mathcal{M}(\mathfrak{g}, K)$ into $\mathcal{M}(\mathfrak{g}, T)$. It has a right adjoint $\Gamma_{K,T}$ – the *Zuckerman functor* from $\mathcal{M}(\mathfrak{g}, T)$ into $\mathcal{M}(\mathfrak{g}, K)$. Its right cohomological dimension is $\leq \dim(K/T)$. We follow the forgetful functor by the forgetful functor from $\mathcal{M}(\mathfrak{g}, T)$ into $\mathcal{M}(\mathfrak{b}, T)$. This functor also has a right adjoint functor P constructed as follows. Consider $\mathcal{U}(\mathfrak{g})$ as a $\mathcal{U}(\mathfrak{b})$-module for left multiplication. Let V be a $(\mathfrak{b}, T)$-module. Then $\operatorname{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathcal{U}(\mathfrak{g}), V)$ has a natural T-action, with T acting on $\mathcal{U}(\mathfrak{g})$ via the adjoint action. Let $\operatorname{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathcal{U}(\mathfrak{g}), V)_{[T]}$ be the largest algebraic submodule of $\operatorname{Hom}_{\mathcal{U}(\mathfrak{b})}(\mathcal{U}(\mathfrak{g}), V)$ for that action of T. Then $\mathcal{U}(\mathfrak{g})$ acts on this module by right multiplication on $\mathcal{U}(\mathfrak{g})$. In this way, one gets the $(\mathfrak{g}, T)$-module $P(V)$. The functor $P : \mathcal{M}(\mathfrak{b}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, T)$ is exact.

Consider now a $(\mathfrak{c}, T)$-module V. We can view it as a $(\mathfrak{b}, T)$-module. This functor has a left adjoint functor $H_0(\mathfrak{n}, -)$.

We define the functor

$$I(\mathfrak{c}, R^+, -) : \mathcal{M}(\mathfrak{c}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$$

as the composition of the functor P followed by the Zuckerman functor $\Gamma_{K,T}$.

The next result is a formal consequence of the above discussion.

3.1.1. **Lemma.** *The functor $I(\mathfrak{c}, R^+, -)$ is a right adjoint of the functor $H_0(\mathfrak{n}, -)$.*

The right derived functors $R^pI(\mathfrak{c}, R^+, -) : \mathcal{M}(\mathfrak{c}, T) \longrightarrow \mathcal{M}(\mathfrak{g}, K)$ are called the *cohomological induction* functors.

Since both functors $I(\mathfrak{c}, R^+, -)$ and $H_0(\mathfrak{n}, -)$ have finite cohomological dimension, their derived functors exist as functors between corresponding bounded derived categories, and a formal consequence of the above lemma is the following version of Frobenius reciprocity [10, Ch. 5, Thm. 1.7.1].

3.1.2. **Proposition.** *The right derived functor $RI(\mathfrak{c}, R^+, -) : D^b(\mathfrak{c}, T) \longrightarrow D^b(\mathfrak{g}, K)$ is a right adjoint of $H_\bullet(\mathfrak{n}, -)$.*

3.2. **Finiteness results.** We also have the following finiteness results.

3.2.1. **Lemma.** *Let $U^\cdot$ be an object in $D^b_{adm}(\mathfrak{g}, K)$. Then $H_\bullet(\mathfrak{n}, U^\cdot)$ is an object in $D^b_{adm}(\mathfrak{c}, T)$.*

Proof. By induction on homological length and standard truncation argument (cf. [10, Ch. 3, 4.2]), we can reduce the proof to the case $U^\cdot = D(U)^\cdot$, where U is a $(\mathfrak{g}, K)$-module of finite length. In this case, it is enough to prove that Lie algebra homology groups $H_p(\mathfrak{n}, U)$, $p \in \mathbb{Z}_+$, are finite-dimensional. This is well-known, a geometric proof can be found, for example, in [11, Ch. 4, Thm. 4.1]. $\square$

3.2.2. **Lemma.** *Let $V^\cdot$ be an object in $D^b_{adm}(\mathfrak{c}, T)$. Then $RI(\mathfrak{c}, R^+, V^\cdot)$ is an object in $D^b_{adm}(\mathfrak{g}, K)$.*

Proof. By induction on homological length and standard truncation argument (cf. [10, Ch. 3, 4.2]), we can reduce the proof to the case $V^\cdot = D(V)^\cdot$, where V is a finite-dimensional $(\mathfrak{c}, T)$-module. Then, by induction in dimension, we can reduce the proof to the case of finite-dimensional irreducible $(\mathfrak{c}, T)$-modules. In this case, by the main result of [6], cohomologies of the complex $RI(\mathfrak{c}, R^+, D(V)^\cdot)$ are duals of cohomologies of holonomic $\mathcal{D}$-modules on the flag variety X of $\mathfrak{g}$. By [11, Ch. 3, Thm. 6.3], these are $(\mathfrak{g}, K)$-modules of finite length. □

3.3. **Homological pairing and cohomological induction.** Frobenius reciprocity and the finiteness results 3.2.1 and 3.2.2 imply the following version of Frobenius reciprocity for the homological pairings on the Grothendieck groups $\mathrm{K}(\mathfrak{g}, K)$ and $\mathrm{K}(\mathfrak{c}, T)$.

3.3.1. **Proposition.** *Let $V^\cdot$ be an object in $D^b_{adm}(\mathfrak{c}, T)$ and $U^\cdot$ an object in $D^b_{adm}(\mathfrak{g}, K)$. Then we have*

$$\langle [U^\cdot] \mid [RI(\mathfrak{c}, R^+, V^\cdot)] \rangle_{(\mathfrak{g},K)} = \langle [H_\bullet(\mathfrak{n}, U^\cdot)] \mid [V^\cdot] \rangle_{(\mathfrak{c},T)}.$$

Proof. Using (2.1) twice, we have

$$\begin{aligned}
\langle [U^\cdot] \mid [RI(\mathfrak{c}, R^+, V^\cdot)] \rangle_{(\mathfrak{g},K)} &= \sum_{p\in\mathbb{Z}} (-1)^p \dim H^p(\mathrm{RHom}_{(\mathfrak{g},K)}(U^\cdot, RI(\mathfrak{c}, R^+, V^\cdot))) \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim \mathrm{Hom}_{D^b(\mathfrak{g},K)}(U^\cdot, RI(\mathfrak{c}, R^+, V^\cdot)[p]) \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim \mathrm{Hom}_{D^b(\mathfrak{g},K)}(U^\cdot, RI(\mathfrak{c}, R^+, V^\cdot[p])) \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim \mathrm{Hom}_{D^b(\mathfrak{c},T)}(H_\bullet(\mathfrak{n}, U^\cdot), V^\cdot[p]) \\
&= \sum_{p\in\mathbb{Z}} (-1)^p \dim H^p(\mathrm{RHom}_{(\mathfrak{c},T)}(H_\bullet(\mathfrak{n}, U^\cdot), V^\cdot)) = \langle [H_\bullet(\mathfrak{n}, U^\cdot)] \mid [V^\cdot] \rangle_{(\mathfrak{c},T)}.
\end{aligned}$$

□

4. Calculation of homological pairing

4.1. **A vanishing result.** In next lemma we assume only that $\mathfrak{a}$ is a nonzero finite-dimensional Lie algebra. Denote by $\mathcal{M}(\mathfrak{a})$ the category of $\mathfrak{a}$-modules. The following vanishing result is well-known, we include a proof for convenience of the reader.

4.1.1. **Lemma.** *Let U and V be two finite-dimensional $\mathfrak{a}$-modules. Then*

$$\sum_{p=0}^{\dim \mathfrak{a}} (-1)^p \dim \mathrm{Ext}^p_{\mathfrak{a}}(U, V) = 0.$$

Proof. By [3, Ch. I], we have

$$\mathrm{Ext}^p_{\mathfrak{a}}(U, V) = H^p(\mathfrak{a}, \mathrm{Hom}_{\mathbb{C}}(U, V))$$

for any $p \in \mathbb{Z}_+$. Therefore, we have

$$\sum_{p=0}^{\dim \mathfrak{a}} (-1)^p \dim \operatorname{Ext}^p_{\mathfrak{a}}(U, V) = \sum_{p=0}^{\dim \mathfrak{a}} (-1)^p \dim H^p(\mathfrak{a}, \operatorname{Hom}_{\mathbb{C}}(U, V)).$$

On the other hand, for any finite-dimensional representation F of $\mathfrak{a}$, we have

$$\begin{aligned}\sum_{p=0}^{\dim \mathfrak{a}} (-1)^p \dim H^p(\mathfrak{a}, F) &= \sum_{p=0}^{\dim \mathfrak{a}} (-1)^p \dim \operatorname{Hom}_{\mathbb{C}}(\bigwedge^p \mathfrak{a}, F) \\ &= \left(\sum_{p=0}^{\dim \mathfrak{a}} (-1)^p \dim \bigwedge^p \mathfrak{a} \right) \cdot \dim F\end{aligned}$$

using the standard complex of Lie algebra cohomology. Finally, we have

$$\sum_{p=0}^{\dim \mathfrak{a}} (-1)^p \dim \bigwedge^p \mathfrak{a} = \sum_{p=0}^{\dim \mathfrak{a}} (-1)^p \binom{\dim \mathfrak{a}}{p} = (1-1)^{\dim \mathfrak{a}} = 0,$$

which implies our assertion. □

Now we return to the situation from Section 3 and consider now the homological pairing on $\mathrm{K}(\mathfrak{c}, T)$ for a σ-stable Cartan subalgebra $\mathfrak{c}$. Assume that $\mathfrak{a} \neq 0$. Let V and V' be two irreducible finite-dimensional $(\mathfrak{c}, T)$-modules. Then, by Schur lemma, $\mathfrak{a}$ acts on V and V' by linear forms $\mu, \mu' \in \mathfrak{a}^*$. The restrictions of V and V' to $(\mathfrak{t}, T)$ are irreducible modules which we denote by the same symbol. By [3, Ch. I], we have

$$\begin{aligned}\operatorname{Ext}^n_{(\mathfrak{c},T)}(V, V') &= \bigoplus_{p+q=n} \operatorname{Ext}^p_{(\mathfrak{t},T)}(V, V') \otimes \operatorname{Ext}^q_{\mathfrak{a}}(\mathbb{C}_\mu, \mathbb{C}_{\mu'}) \\ &= \operatorname{Hom}_{(\mathfrak{t},T)}(V, V') \otimes \operatorname{Ext}^n_{\mathfrak{a}}(\mathbb{C}_\mu, \mathbb{C}_{\mu'}),\end{aligned}$$

for any $n \in \mathbb{Z}_+$, since $\mathcal{A}(\mathfrak{t}, T)$ is semisimple. Therefore, by 4.1.1, we have

$$\langle [V], [V'] \rangle_{(\mathfrak{c},T)} = \dim \operatorname{Hom}_{(\mathfrak{t},T)}(V, V') \cdot \sum_{p=0}^{\dim \mathfrak{a}} (-1)^p \operatorname{Ext}^p_{\mathfrak{a}}(\mathbb{C}_\mu, \mathbb{C}_{\mu'}) = 0.$$

Hence, we get the following elementary vanishing result.

4.1.2. **Lemma.** *If $\mathfrak{a} \neq 0$, the homological pairing on $\mathrm{K}(\mathfrak{c}, T)$ is zero.*

4.2. **Localization and Grothendieck groups of Harish-Chandra modules.** To calculate the homological pairing on $\mathrm{K}(\mathfrak{g}, K)$ we have to invoke the geometric classification of irreducible Harish-Chandra modules. We use freely the notation from [6] and [12].

Let $\mathcal{Z}(\mathfrak{g})$ be the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$. Using Harish-Chandra homomorphism, the maximal ideals in $\mathcal{Z}(\mathfrak{g})$ correspond to the orbits of the Weyl group W in the abstract Cartan algebra $\mathfrak{h}$ of $\mathfrak{g}$. For an orbit θ we denote by I_θ the corresponding maximal ideal in $\mathcal{Z}(\mathfrak{g})$. Let $\mathcal{U}_\theta$ be the quotient of $\mathcal{U}(\mathfrak{g})$ by the two-sided ideal generated by I_θ. We denote by $\mathcal{A}(\mathcal{U}_\theta, K)$ the full subcategory of $\mathcal{A}(\mathfrak{g}, K)$ consisting of Harish-Chandra modules with infinitesimal character corresponding to

I_θ. Let $\mathrm{K}(\mathcal{U}_\theta, K)$ be the Grothendieck group of $\mathcal{A}(\mathcal{U}_\theta, K)$. Then, we have the direct sum decomposition

$$\mathrm{K}(\mathfrak{g}, K) = \bigoplus_\theta \mathrm{K}(\mathcal{U}_\theta, K).$$

By Wigner's lemma [3, Ch. I], the subgroups $\mathrm{K}(\mathcal{U}_\theta, K)$ are mutually orthogonal with respect to the homological pairing. Therefore, we have to calculate it on these subgroups only.

Fix a Weyl group orbit θ. Then there exists a λ in this orbit which is antidominant. Let $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ be the category of coherent K-equivariant $\mathcal{D}_\lambda$-modules on the flag variety X of $\mathfrak{g}$. The objects of $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ are called *Harish-Chandra sheaves*. Since Harish-Chandra sheaves are holonomic [12, Thm. 6.1], they are of finite length.

The functor of global sections $\Gamma(X, -)$ is an exact functor from the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ into $\mathcal{A}(\mathcal{U}_\theta, K)$ since λ is antidominant. More precisely, $\mathcal{A}(\mathcal{U}_\theta, K)$ is equivalent to a quotient category of $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ (cf. [12, 3.8]). Let $\mathrm{K}(\mathcal{D}_\lambda, K)$ be the Grothendieck group of $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$. The above statement implies that $\mathrm{K}(\mathcal{U}_\theta, K)$ is a quotient group of $\mathrm{K}(\mathcal{D}_\lambda, K)$.

It is easy to describe all irreducible Harish-Chandra sheaves. The group K has finitely many orbits in X. Let Q be a K-orbit in X. There exists a finite family of irreducible K-equivariant connections on Q compatible with $\lambda + \rho$. For such connection τ, we denote by $\mathcal{I}(Q, \tau)$ the $\mathcal{D}$-module direct image of τ under the natural inclusion of Q into X. Then $\mathcal{I}(Q, \tau)$ is the *standard Harish-Chandra sheaf* attached to the geometric data (Q, τ). It has a unique irreducible subobject $\mathcal{L}(Q, \tau)$. All irreducible objects in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K)$ are isomorphic to $\mathcal{L}(Q, \tau)$ for some geometric data (Q, τ).

Therefore, the classes $[\mathcal{L}(Q, \tau)]$ form a basis of $\mathrm{K}(\mathcal{D}_\lambda, K)$. Since the other composition factors of $\mathcal{I}(Q, \tau)$ correspond to K-orbits in the boundary of Q, we immediately see that classes of $[\mathcal{I}(Q, \tau)]$ also form a basis of $\mathrm{K}(\mathcal{D}_\lambda, K)$.

We call $\Gamma(X, \mathcal{I}(Q, \tau))$ the *standard Harish-Chandra module* attached to geometric data (Q, τ). The above discussion implies that classes of standard Harish-Chandra modules, for all geometric data (Q, τ), generate $\mathrm{K}(\mathcal{U}_\theta, K)$.

Let $\theta\check{}$ be the orbit of $-\lambda$. Then the duality $U \longmapsto U\check{}$ is an antiequivalence of the category $\mathcal{A}(\mathcal{U}_\theta, K)$ with $\mathcal{A}(\mathcal{U}_{\theta\check{}}, K)$. Therefore, the classes $[\Gamma(X, \mathcal{I}(Q, \tau))\check{}]$ generate $\mathrm{K}(\mathcal{U}_{\theta\check{}}, K)$.

Let x be point in Q. Let $\mathfrak{b}_x$ be corresponding Borel subalgebra in $\mathfrak{g}$. It contains a σ-stable Cartan subalgebra $\mathfrak{c}$ and all such Cartan subalgebras are K-conjugate [12, Lemma 5.3]. Let $\mathfrak{n} = [\mathfrak{b}_x, \mathfrak{b}_x]$. Denote by R^+ the set of positive roots determined by $\mathfrak{n}$.

Let V be the irreducible representation of $(\mathfrak{c}, T)$ on the geometric fiber $T_x(\tau)$ of τ at x. Let Ω_X be the cotangent bundle of X and $T_x(\Omega_X)$ its geometric fiber at x. Then the duality theorem [6, Thm. 4.3] states that

$$\Gamma(X, \mathcal{I}(Q, \tau))\check{} = R^s I(\mathfrak{c}, R^+, V\check{} \otimes T_x(\Omega_X))$$

for $s = \dim(\mathfrak{k} \cap \mathfrak{n})$. In addition, $R^p I(\mathfrak{c}, R^+, V\check{} \otimes T_x(\Omega_X)) = 0$ for $p \neq s$.[3]

[3]Actually, [6, Thm. 4.3] assumes that G_0 is connected semisimple Lie group. In [6, Appendix B] it is explained how to extend it to all groups in the Harish-Chandra class.

By 3.3.1, this immediately implies that

$$
\begin{aligned}
(4.1)\quad \langle [U] \mid [\Gamma(X,\mathcal{I}(Q,\tau))\check{}\,]\rangle_{(\mathfrak{g},K)} &= \langle [U] \mid [R^s I(\mathfrak{c},R^+,V\check{}\otimes T_x(\Omega_X))]\rangle_{(\mathfrak{g},K)} \\
&= (-1)^s \langle [D(U)^{\cdot}] \mid [RI(\mathfrak{c},R^+,D(V\check{}\otimes T_x(\Omega_X))^{\cdot}]\rangle_{(\mathfrak{g},K)} \\
&= (-1)^s \langle [H_\bullet(\mathfrak{n},D(U)^{\cdot})] \mid D(V\check{}\otimes T_x(\Omega_X))^{\cdot}]\rangle_{(\mathfrak{c},T)} \\
&= (-1)^s \sum_{p\in\mathbb{Z}} (-1)^p \langle [H_p(\mathfrak{n},U)] \mid [V\check{}\otimes T_x(\Omega_X)]\rangle_{(\mathfrak{c},T)}
\end{aligned}
$$

for any U in $\mathcal{A}(\mathfrak{g},K)$. Hence, by 4.1.2, the homological pairing on $\mathrm{K}(\mathfrak{g},K)$ vanishes if the orbit Q in the second variable is attached to a Cartan subalgebra $\mathfrak{c}$ with $\mathfrak{a}\neq\{0\}$.

4.3. **Unequal rank case.** If $\operatorname{rank} G_0 > \operatorname{rank} K_0$, any σ-stable Cartan subalgebra in $\mathfrak{g}$ has $\mathfrak{a}\neq\{0\}$. Hence, we see that the homological pairing vanishes on $\mathrm{K}(\mathfrak{g},K)$ i.e., we have the following generalization of 4.1.2.

4.3.1. **Theorem.** *If* $\operatorname{rank} G_0 > \operatorname{rank} K_0$*, the homological pairing vanishes on* $\mathrm{K}(\mathfrak{g},K)$.

4.4. **Symmetry of Euler characteristic of Lie algebra homology.** It remains to treat the case $\operatorname{rank} G_0 = \operatorname{rank} K_0$. In this case, the group G_0 has a compact Cartan subgroup contained in K_0. All such Cartan subgroups are conjugate by K_0.

We fix a compact Cartan subgroup T_0. We denote by $\mathfrak{t}$ the complexification of its Lie algebra and by T the complexification of T_0. We denote by R the root system of $(\mathfrak{g},\mathfrak{t})$ in $\mathfrak{t}^*$.

Clearly, the category $\mathcal{A}(\mathfrak{t},T)$ is just the category of finite-dimensional algebraic representations of T, hence it is semisimple. In addition, $\mathrm{K}(\mathfrak{t},T)$ is a ring with multiplication given by $[V]\cdot[V'] = [V\otimes_{\mathbb{C}} V']$ for finite-dimensional algebraic representations V and V' of T. The ring $\mathrm{K}(\mathfrak{t},T)$ contains as a subring the additive subgroup generated by all characters e^μ of T where μ is a weight in the root lattice of R. Moreover, the subgroup W_0 of the Weyl group W acts naturally on $\mathrm{K}(\mathfrak{t},T)$.

Clearly, the homological pairing on $\mathrm{K}(\mathfrak{t},T)$ is invariant for the action of W_0. Moreover, it is invariant for multiplication by e^μ for any weight μ, i.e., we have

$$\langle A\cdot e^\mu \mid B\cdot e^\mu\rangle_{(\mathfrak{t},T)} = \langle A \mid B\rangle_{(\mathfrak{t},T)}$$

for any A, B in $\mathrm{K}(\mathfrak{t},T)$.

Let R^+ be a set of positive roots in R. Let ρ be the half sum of roots in R^+. Denote by $\mathfrak{n}$ the nilpotent Lie algebra spanned by root subspaces $\mathfrak{g}_\alpha$ for roots $\alpha\in R^+$.

Our calculation is based on remarkable symmetry properties of the Euler characteristic of Lie algebra homology (with respect to $\mathfrak{n}$) of Harish-Chandra modules. They follow from the Osborne conjecture [7].[4] Let U be a Harish-Chandra module and let Θ_U be its character. By the Osborne conjecture, we have

$$\Theta_U = \frac{\sum_{p\in\mathbb{Z}}(-1)^p\Theta_{H_p(\mathfrak{n},U)}}{\prod_{\alpha\in R^+}(1-e^\alpha)}$$

on the regular elements T_0' in the compact Cartan subgroup T_0.

[4] Actually, we need just a special case for compact Cartan subgroups [7, 7.27].

First we need a simple symmetry property of the denominator in this formula. Let $w \in W$, then $\rho - w\rho$ is a sum of all roots in $R^+ \cap (-wR^+)$, hence it defines a character $e^{w\rho-\rho}$ of T_0.

We denote by ϵ the sign representation of W.

4.4.1. **Lemma.** *For any $w \in W$ we have*

$$\prod_{\alpha \in wR^+} (1 - e^\alpha) = \epsilon(w) e^{w\rho-\rho} \prod_{\alpha \in R^+} (1 - e^\alpha).$$

Proof. We have

$$\begin{aligned}\prod_{\alpha \in wR^+} (1 - e^\alpha) &= \prod_{\alpha \in wR^+ \cap R^+} (1 - e^\alpha) \prod_{\alpha \in wR^+ \cap (-R^+)} (1 - e^\alpha) \\ &= \prod_{\alpha \in wR^+ \cap R^+} (1 - e^\alpha) \prod_{\alpha \in (-wR^+) \cap R^+} (1 - e^{-\alpha}) \\ &= \epsilon(w) \prod_{\alpha \in (-wR^+) \cap R^+} e^{-\alpha} \prod_{\alpha \in R^+} (1 - e^\alpha) = \epsilon(w) e^{w\rho-\rho} \prod_{\alpha \in R^+} (1 - e^\alpha).\end{aligned}$$

□

Let $\mathfrak{n}_w$ be the nilpotent Lie algebra spanned by the root subspaces corresponding to the roots in wR^+ for any $w \in W$.

4.4.2. **Lemma.** *Let U be a Harish-Chandra module.*

(i) *For any $w \in W_0$, we have*

$$w\left(\sum_{p \in \mathbb{Z}} (-1)^p [H_p(\mathfrak{n}, U)]\right) = \epsilon(w) \sum_{p \in \mathbb{Z}} (-1)^p [H_p(\mathfrak{n}, U)] \cdot e^{w\rho-\rho}$$

in $\mathrm{K}(\mathfrak{t}, T)$.

(ii) *For any $w \in W$, we have*

$$\sum_{p \in \mathbb{Z}_+} (-1)^p [H_p(\mathfrak{n}, U)] = \epsilon(w) \sum_{p \in \mathbb{Z}_+} (-1)^p [H_p(\mathfrak{n}_w, U)] \cdot e^{\rho-w\rho}.$$

Proof. The proof is based on the Osborne character formula.[5]

(i) Since the character Θ_U is given by a function constant on the conjugacy classes of regular elements, we see that $\Theta_U(t^w) = \Theta_U(t)$ for any $t \in T_0'$ and $w \in W_0$.

By 4.4.1,

$$\begin{aligned}\left(w\left(\sum_{p \in \mathbb{Z}} (-1)^p \Theta_{H_p(\mathfrak{n},U)}\right)\right)(t) &= \Theta_U(t^{w^{-1}}) \prod_{\alpha \in R^+} (1 - e^\alpha(t^{w^{-1}})) \\ &= \Theta_U(t) \prod_{\alpha \in wR^+} (1 - e^\alpha(t)) = \epsilon(w) e^{w\rho-\rho}(t) \Theta_U(t) \prod_{\alpha \in R^+} (1 - e^\alpha(t)) \\ &= \epsilon(w) e^{w\rho-\rho}(t) \sum_{p \in \mathbb{Z}} (-1)^p \Theta_{H_p(\mathfrak{n},U)}(t)\end{aligned}$$

for any $t \in T_0$, and (i) follows.

[5] We use the Osborne conjecture since it leads to a simpler argument. It is possible to circumvent its use and prove 4.4.2 and 4.6.1 by purely algebraic methods. This reduces the use of the Osborne conjecture just to the final identification of pairings in 5.1.1.

(ii) We can calculate Θ_U on T_0' in two different ways

$$\Theta_U = \frac{\sum_{p\in\mathbb{Z}}(-1)^p \Theta_{H_p(\mathfrak{n},U)}}{\prod_{\alpha\in R^+}(1-e^\alpha)} = \frac{\sum_{p\in\mathbb{Z}}(-1)^p \Theta_{H_p(\mathfrak{n}_w,U)}}{\prod_{\alpha\in wR^+}(1-e^\alpha)}.$$

Therefore, we have

$$\left(\sum_{p\in\mathbb{Z}}(-1)^p \Theta_{H_p(\mathfrak{n},U)}\right)\left(\prod_{\alpha\in wR^+}(1-e^\alpha)\right) = \left(\sum_{p\in\mathbb{Z}}(-1)^p \Theta_{H_p(\mathfrak{n}_w,U)}\right)\left(\prod_{\alpha\in R^+}(1-e^\alpha)\right)$$

on T_0. By 4.4.1, this implies

$$\sum_{p\in\mathbb{Z}}(-1)^p \Theta_{H_p(\mathfrak{n},U)} = \epsilon(w)e^{\rho-w\rho}\left(\sum_{p\in\mathbb{Z}}(-1)^p \Theta_{H_p(\mathfrak{n}_w,U)}\right).$$

□

4.5. **Euler characteristic of Lie algebra homology of standard Harish-Chandra modules.** In this section we want to discuss the formulas for the Euler characteristic of Lie algebra homology (with respect to $\mathfrak{n}$) of standard Harish-Chandra modules $\Gamma(X,\mathcal{I}(Q,\tau))$.

Since we are in the equal rank case, by [12, 5.9], an orbit Q is closed if and only if it is attached to the Cartan subalgebra $\mathfrak{t}$. More precisely, any closed orbit Q contains a Borel subalgebra $\mathfrak{b}_w = \mathfrak{t}\oplus\mathfrak{n}_w$ for some $w\in W$ and two such Borel subalgebras $\mathfrak{b}_u$ and $\mathfrak{b}_v$ lie in the same orbit if and only if u and v are in the same right W_0-coset in W.

Let x_w be the point in the flag variety corresponding to the Borel subalgebra $\mathfrak{b}_w$. As explained in [6, p. 303], to each x_w, one attaches a natural isomorphism of the dual $\mathfrak{h}^*$ of the abstract Cartan algebra $\mathfrak{h}$ with $\mathfrak{t}^*$ which we call the *specialization* at x_w. Clearly, the specializations at x and x_w differ by the action of w.

Assume first that $\lambda\in\mathfrak{h}^*$ is regular. Let U be a Harish-Chandra module in $\mathcal{A}(\mathcal{U}_\theta,K)$.

As we remarked before, Lie algebra homology groups $H_p(\mathfrak{n},U)$, $p\in\mathbb{Z}_+$, are finite-dimensional representations of T. Moreover, we have

$$H_p(\mathfrak{n},U) = \bigoplus_{w\in W} H_p(\mathfrak{n},U)_{(w\lambda+\rho)}$$

where $\mathfrak{t}$ acts on $H_p(\mathfrak{n},U)_{(w\lambda+\rho)}$ via the specialization of $w\lambda+\rho$ [11, Ch. 3, Cor. 2.4].

By [11, Ch. 3, Cor. 2.6], the derived geometric fibers $LT_{x_w}(\mathcal{I}(Q,\tau))$ of $\mathcal{I}(Q,\tau)$ at the point x_w correspond to $(\lambda+\rho)$-components of Lie algebra homology (with respect to $\mathfrak{n}_w$) of $\Gamma(X,\mathcal{I}(Q,\tau))$ under the specialization of $\lambda+\rho$ at x_w. Therefore, by 4.4.2.(ii), calculating the $(\lambda+\rho)$-components of Euler characteristic of Lie algebra homology (with respect to $\mathfrak{n}_w$) for all $w\in W$, gives us the formula for Euler characteristic of Lie algebra homology (with respect to $\mathfrak{n}$) of $\Gamma(X,\mathcal{I}(Q,\tau))$.

First we consider the case where Q is not closed in X.

4.5.1. **Lemma.** *Assume that the orbit Q is not closed. Then*

$$\sum_{p\in\mathbb{Z}}(-1)^p[H_p(\mathfrak{n},\Gamma(X,\mathcal{I}(X,Q))))] = 0.$$

Proof. Assume first that λ is regular. Since Q is not closed, the points x_w, $w \in W$, are not in Q.

Let $i_x : \{x\} \longrightarrow X$ and $i_Q : Q \longrightarrow X$ be the natural inclusions. Since the standard Harish-Chandra sheaf $\mathcal{I}(Q,\tau)$ is the D-module direct image $i_{Q,+}(\tau)$, by the base change [2, Ch. VI, 8.5], we see that $i^!_{x_w}(\mathcal{I}(Q,\tau)) = 0$, i.e., $LT_{x_w}(\mathcal{I}(Q,\tau)) = 0$. By the above discussion, this implies that

$$H_p(\mathfrak{n}_w, \Gamma(X, \mathcal{I}(Q,\tau)))_{(\lambda+\rho)} = 0$$

for all $p \in \mathbb{Z}_+$. Hence, we have

$$\sum_{p\in\mathbb{Z}} (-1)^p [H_p(\mathfrak{n}_w, \Gamma(X, \mathcal{I}(Q,\tau)))_{(\lambda+\rho)}] = 0$$

for all $w \in W$. As we remarked, this immediately implies our statement for regular λ. In particular the character of this standard Harish-Chandra module vanishes on T_0'. Since coherent continuation corresponds to twisting the localization by sections of a homogeneous line bundle on X followed by taking global sections [11, Ch. 3, Thm. 7.7], it follows that the character vanishes on T_0' also for singular λ. This in turn implies the statement in general. □

Now we treat the case of closed orbits. We can pick $\mathfrak{n}$ so that the corresponding point x in the flag variety is in Q. Denote by $j_x : \{x\} \longrightarrow Q$ the natural inclusion. Then the geometric fiber $V = T_x(\tau)$ is an irreducible module in $\mathcal{M}(\mathfrak{t}, T)$.

Since Q is closed, $\mathfrak{b} \cap \mathfrak{k}$ is a Borel subalgebra in $\mathfrak{k}$ and $s = \frac{1}{2}\dim(K/T) = \dim Q$.

4.5.2. **Lemma.** *Assume that Q is a closed orbit. Then we have*

$$\sum_{p\in\mathbb{Z}} (-1)^p [H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))] = (-1)^{s+\mathrm{Card}(R^+)} \sum_{w\in W_0} \epsilon(w)[V]^w e^{\rho - w\rho}.$$

Proof. Assume first that λ is regular. By our assumption, the points x_w are in Q if and only if $w \in W_0$. As in the proof of 4.5.1 we conclude that $(\lambda+\rho)$-components of the Euler characteristic of Lie algebra homology with respect to $\mathfrak{n}_w$ (for the specialization of $\lambda+\rho$ at x_w) vanish for w outside W_0. This in turn implies that the $(w\lambda+\rho)$-components of Euler characteristic of Lie algebra homology with respect to $\mathfrak{n}$ (for the specialization at x) vanish for w outside W_0.

Applying base change again [2, Ch. VI, 8.4], we see that

$$i^!_x(\mathcal{I}(Q,\tau)) = j^!_x(\tau) = T_x(\tau)[-\dim Q] = V[-\dim Q].$$

Hence, we have $LT_x(\mathcal{I}(Q,\tau)) = D(V)[\operatorname{codim} Q]$. This immediately implies that

$$H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))_{(\lambda+\rho)} = \begin{cases} V, & \text{if } p = \operatorname{codim} Q; \\ 0, & \text{if } p \neq \operatorname{codim} Q. \end{cases}$$

As we mentioned above, we have $\operatorname{codim} Q = \dim X - s$. Therefore, we see that

$$\sum_{p=0}^{n} (-1)^p [H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))_{(\lambda+\rho)}] = (-1)^{\operatorname{codim} Q}[V] = (-1)^{s+\mathrm{Card}(R^+)}[V].$$

The $(w\lambda+\rho)$-components of Euler characteristic of Lie algebra homology with respect to $\mathfrak{n}$, for $w \in W_0$, are uniquely determined by 4.4.2.(i), i.e., we have

$$\sum_{p=0}^{n} (-1)^p [H_p(\mathfrak{n}, \Gamma(X, \mathcal{I}(Q,\tau)))] = (-1)^{s+\mathrm{Card}(R^+)} \sum_{w\in W_0} \epsilon(w)[V]^w e^{\rho - w\rho}.$$

This completes the proof for regular λ.

The reduction of the general case to the case of regular λ is the same as in the proof of 4.5.1. $\square$

4.6. **Homological pairing in the equal rank case.** As we remarked above the homological pairing $\langle [U] \mid [\Gamma(X, \mathcal{I}(Q,\tau))\check{}\,]\rangle_{(\mathfrak{g},K)}$ could be nonzero only if the second variable is a class attached to a closed orbit Q.

Going back to our calculation of homological pairing in this situation, by (4.1), we have

$$\begin{aligned} \langle [U] \mid [\Gamma(X, \mathcal{I}(Q,\tau))\check{}\,]\rangle_{(\mathfrak{g},K)} &= \sum_{p\in\mathbb{Z}} (-1)^{s+p} \langle [H_p(\mathfrak{n}, U)] \mid [V\check{} \otimes T_x(\Omega_X)]\rangle_{(\mathfrak{t},T)} \\ &= \sum_{p\in\mathbb{Z}} (-1)^{s+p} \langle [H_p(\mathfrak{n}, U)] \mid [V\check{}\,] \cdot e^{2\rho}\rangle_{(\mathfrak{t},T)} \end{aligned} \tag{4.2}$$

Now we want to rewrite the right side of (4.2) in a more symmetric form. First, we have

$$\langle [H_p(\mathfrak{n}, U)] \mid [V\check{}\,] \cdot e^{2\rho}\rangle_{(\mathfrak{t},T)} = \langle [H_p(\mathfrak{n}, U)]^w \mid [V\check{}\,]^w \cdot e^{2w\rho}\rangle_{(\mathfrak{t},T)}$$

for any $w \in W_0$.

Hence, by summing over the group W_0, we get

$$\langle [H_p(\mathfrak{n}, U)] \mid [V\check{}\,] \cdot e^{2\rho}\rangle_{(\mathfrak{t},T)} = \frac{1}{[W_0]} \sum_{w\in W_0} \langle [H_p(\mathfrak{n}, U)]^w \mid [V\check{}\,]^w \cdot e^{2w\rho}\rangle_{(\mathfrak{t},T)}.$$

This implies, by 4.4.2.(i) and (4.2), that

$$\begin{aligned} \langle [U] \mid [\Gamma(X, \mathcal{I}(Q,\tau))\check{}\,]\rangle_{(\mathfrak{g},K)} &= \sum_{p\in\mathbb{Z}} (-1)^{s+p} \langle [H_p(\mathfrak{n}, U)] \mid [V\check{}\,] \cdot e^{2\rho}\rangle_{(\mathfrak{t},T)} \\ &= \sum_{p\in\mathbb{Z}} (-1)^{s+p} \left(\frac{1}{[W_0]} \sum_{w\in W_0} \left\langle [H_p(\mathfrak{n}, U)]^w \mid [V\check{}\,]^w \cdot e^{2w\rho}\right\rangle_{(\mathfrak{t},T)} \right) \\ &= \frac{(-1)^s}{[W_0]} \sum_{w\in W_0} \left(\sum_{p\in\mathbb{Z}} (-1)^p \left\langle [H_p(\mathfrak{n}, U)]^w \mid [V\check{}\,]^w \cdot e^{2w\rho}\right\rangle_{(\mathfrak{t},T)} \right) \\ &= \frac{(-1)^s}{[W_0]} \sum_{w\in W_0} \left\langle w \left(\sum_{p\in\mathbb{Z}} (-1)^p [H_p(\mathfrak{n}, U)] \right) \,\middle|\, [V\check{}\,]^w \cdot e^{2w\rho} \right\rangle_{(\mathfrak{t},T)} \\ &= \frac{(-1)^s}{[W_0]} \sum_{w\in W_0} \epsilon(w) \left\langle \sum_{p\in\mathbb{Z}} (-1)^p [H_p(\mathfrak{n}, U)] \cdot e^{w\rho-\rho} \,\middle|\, [V\check{}\,]^w \cdot e^{2w\rho} \right\rangle_{(\mathfrak{t},T)} \\ &= \frac{(-1)^s}{[W_0]} \left\langle \sum_{p\in\mathbb{Z}} (-1)^p [H_p(\mathfrak{n}, U)] \,\middle|\, \sum_{w\in W_0} \epsilon(w) [V\check{}\,]^w \cdot e^{\rho+w\rho} \right\rangle_{(\mathfrak{t},T)}. \end{aligned} \tag{4.3}$$

To complete our calculation, we need a representation theoretic interpretation of the second sum in the above pairing.

The character of the dual representation $\Gamma(X, \mathcal{I}(Q,\tau))\check{}$ satisfies $\Theta_{\Gamma(X,\mathcal{I}(Q,\tau))\check{}}(t) = \Theta_{\Gamma(X,\mathcal{I}(Q,\tau))}(t^{-1})$ for any $t \in T_0'$. Hence, by 4.5.2 and the Osborne conjecture, we have

$$\Theta_{\Gamma(X,\mathcal{I}(Q,\tau))\check{}} = (-1)^{s+\mathrm{Card}(R^+)} \frac{\sum_{w\in W_0} \epsilon(w)\Theta_{[V\check{}]^w} e^{w\rho-\rho}}{\prod_{\alpha\in R^+}(1-e^{-\alpha})} = (-1)^s \frac{\sum_{w\in W_0} \epsilon(w)\Theta_{[V\check{}]^w} e^{w\rho+\rho}}{\prod_{\alpha\in R^+}(1-e^{\alpha})}$$

on T_0'. Moreover, using the Osborne conjecture again, we have

$$\sum_{p=0}^{n}(-1)^p[H_p(\mathfrak{n},\Gamma(X,\mathcal{I}(Q,\tau))\check{})] = (-1)^s \sum_{w\in W_0} \epsilon(w)[V\check{}]^w \cdot e^{\rho+w\rho}.$$

Plugging this into (4.3), for any Harish-Chandra module U, we finally get

$$\langle [U] \mid [\Gamma(X,\mathcal{I}(Q,\tau))\check{}]\rangle_{(\mathfrak{g},K)} = \frac{1}{[W_0]} \sum_{p\in\mathbb{Z}} (-1)^{p+q} \langle H_p(\mathfrak{n},U) \mid H_q(\mathfrak{n},\Gamma(X,\mathcal{I}(Q,\tau))\check{})\rangle_{(\mathfrak{t},T)}. \tag{4.4}$$

In the calculation leading to (4.4) the choice of the Lie algebra $\mathfrak{n}$ (or equivalently set of positive roots R^+) was specific for the orbit Q. On the other hand, the symmetry established in 4.4.2.(ii), implies that the formula holds for any set of positive roots wR^+ in R. Therefore, the formula holds for any $\mathfrak{n}$ attached to the Cartan subalgebra $\mathfrak{t}$.

This finally leads to the following result which completely determines the homological pairing on $\mathrm{K}(\mathfrak{g},K)$.

4.6.1. **Theorem.** *Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{k}$ and R^+ a set of positive roots in the root system R of $(\mathfrak{g},\mathfrak{t})$. Let $\mathfrak{n}$ be the Lie algebra spanned by root subspaces corresponding to R^+. Then we have*

$$\langle [U] \mid [U']\rangle_{(\mathfrak{g},K)} = \frac{1}{[W_0]} \sum_{p,q\in\mathbb{Z}} (-1)^{p+q} \dim \mathrm{Hom}_{(\mathfrak{t},T)}(H_p(\mathfrak{n},U), H_q(\mathfrak{n},U'))$$

for any two Harish-Chandra modules U and U'.

Proof. It is enough to check it on duals $\Gamma(X,\mathcal{I}(Q,\tau))\check{}$ of standard Harish-Chandra modules in the second variable for any geometric data (Q,τ).

If Q is closed, the formula is established in (4.4).

Otherwise, if Q is not closed, by 4.5.1 and the Osborne conjecture, the character of the standard module $\Gamma(X,\mathcal{I}(Q,\tau))$ vanishes on T_0'. This in turn implies that the character of its dual vanishes on T_0'. Hence, we have

$$\sum_{p\in\mathbb{Z}}(-1)^p[H_p(\mathfrak{n},\Gamma(X,\mathcal{I}(Q,\tau))\check{})] = 0$$

and the right side of the above formula vanishes. Since a σ-stable Cartan subalgebra attached to Q has $\mathfrak{a} \neq \{0\}$, the left side vanishes by (4.1) (as we already remarked). □

5. Proof of the equality of pairings

5.1. **Equality of two pairings.** Finally, we prove our main result.

5.1.1. **Theorem.** *The elliptic and homological pairing on $\mathrm{K}(\mathfrak{g},K)$ agree.*

Proof. In the case of nonequal rank, the elliptic pairing on $\mathrm{K}(\mathfrak{g}, K)$ is zero by definition. The homological pairing is zero by 4.3.1.

Assume that the $\operatorname{rank} G_0 = \operatorname{rank} K_0$. Then, by 4.6.1 and the orthogonality relations for the compact group T_0, for any two Harish-Chandra modules U and U' we have

$$\langle [U] \mid [U'] \rangle_{(\mathfrak{g},K)} = \frac{1}{[W_0]} \sum_{p,q \in \mathbb{Z}} (-1)^{p+q} \int_{T_0} \Theta_{H_p(\mathfrak{n},U)}(t) \overline{\Theta_{H_q(\mathfrak{n},U')}(t)}\, d\mu_{T_0}(t).$$

By the Osborne conjecture, we see that

$$\langle [U] \mid [U'] \rangle_{(\mathfrak{g},K)} = \frac{1}{[W_0]} \int_{T_0} \Psi_U(t) \overline{\Psi_{U'}(t)}\, d\mu_{T_0}(t) = \langle [U] \mid [U'] \rangle_{ell}.$$

□

5.2. **Harish-Chandra's orthogonality relations for discrete series.** Assume now that $\operatorname{rank} G_0 = \operatorname{rank} K_0$, i.e., the group G_0 admits discrete series. Let U and U' be two discrete series representations. As it is well-known, in this situation, we have $\operatorname{Ext}^p_{(\mathfrak{g},K)}(U, U') = 0$ for $p > 0$. Therefore, by 2.3.1, we see that

$$\langle [U] \mid [U'] \rangle_{(\mathfrak{g},K)} = \dim \operatorname{Hom}_{(\mathfrak{g},K)}(U, U').$$

Hence, our main result implies immediately Harish-Chandra's orthogonality relations.

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