ON DEGENERATION OF THE SPECTRAL SEQUENCE FOR THE COMPOSITION OF ZUCKERMAN FUNCTORS

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Introduction. Let G_0 be a connected real semisimple Lie group with finite center. Let K_0 be a maximal compact subgroup of G_0. Denote by $\mathfrak{g}$ the complexification of the Lie algebra of G_0 and by K the complexification of K_0. Then K is a connected reductive complex algebraic group and it acts algebraically on $\mathfrak{g}$. The differential of this action identifies the Lie algebra $\mathfrak{k}$ of K with a subalgebra in $\mathfrak{g}$.

Denote by $\mathcal{U}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Let $\mathcal{M}(\mathfrak{g})$ be the category of $\mathcal{U}(\mathfrak{g})$-modules and $\mathcal{M}(\mathfrak{g}, K)$ the subcategory of Harish-Chandra modules for the pair $(\mathfrak{g}, K)$.

Let T be a complex torus in K and $\mathfrak{t}$ its Lie algebra. Then analogously we can consider the category $\mathcal{M}(\mathfrak{g}, T)$ of Harish-Chandra modules for the pair $(\mathfrak{g}, T)$. Clearly, $\mathcal{M}(\mathfrak{g}, T)$ is a full subcategory of $\mathcal{M}(\mathfrak{g})$ and $\mathcal{M}(\mathfrak{g}, K)$ is a full subcategory of $\mathcal{M}(\mathfrak{g}, T)$. The natural forgetful functors $\mathcal{M}(\mathfrak{g}, T) \to \mathcal{M}(\mathfrak{g})$, $\mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g})$ and $\mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g}, T)$ have right adjoints $\Gamma_T : \mathcal{M}(\mathfrak{g}) \to \mathcal{M}(\mathfrak{g}, T)$, $\Gamma_K : \mathcal{M}(\mathfrak{g}) \to \mathcal{M}(\mathfrak{g}, K)$ and $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g}, T) \to \mathcal{M}(\mathfrak{g}, K)$. These adjoints are called the *Zuckerman functors*. Clearly, $\Gamma_K = \Gamma_{K,T} \circ \Gamma_T$.

Zuckerman functors are left exact and have finite right cohomological dimension. Therefore, one can consider their right derived functors $R^p\Gamma_T$, $R^q\Gamma_K$ and $R^s\Gamma_{K,T}$. They are related by the obvious Grothendieck spectral sequence

$$R^p\Gamma_{K,T}(R^q\Gamma_T(V)) \Rightarrow R^{p+q}\Gamma_K(V)$$

for any V in $\mathcal{M}(\mathfrak{g})$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. Let R be the root system of $(\mathfrak{g}, \mathfrak{h})$ in $\mathfrak{h}^*$. Fix a set of positive roots R^+ in R. Then it determines a nilpotent algebra $\mathfrak{n}$ spanned by the root subspaces corresponding to the positive roots and the corresponding Borel subalgebra $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$.

Let ρ be the half-sum of roots in R^+. Let $\lambda \in \mathfrak{h}^*$ be such that the restriction of $\lambda - \rho$ to $\mathfrak{t}$ is the differential of a character of T. Then the Verma module $M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda+\rho}$ is in $\mathcal{M}(\mathfrak{g}, T)$. By a result of Duflo and Vergne ([3], [6]), we have

$$R^q\Gamma_T(M(\lambda)) = M(\lambda) \otimes \textstyle\bigwedge^q \mathfrak{t}^*.$$

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This implies, together with the above spectral sequence, that we have a spectral sequence

$$R^p\Gamma_{K,T}(M(\lambda)) \otimes \bigwedge^q \mathfrak{t}^* \Rightarrow R^{p+q}\Gamma_K(M(\lambda)).$$

The main result of this note is that this spectral sequence degenerates. In fact, the following formula holds.

Theorem A.

$$R^p\Gamma_K(M(\lambda)) = \bigoplus_{q=0}^{\dim T} R^{p-q}\Gamma_{K,T}(M(\lambda)) \otimes \bigwedge^q \mathfrak{t}^*.$$

This result was suggested by our calculation of the cohomology of standard Harish-Chandra sheaves on the flag variety of $\mathfrak{g}$ and the analysis of the duality theorem of Hecht, Miličić, Schmid and Wolf ([5], [7]). It is an immediate consequence of the following decomposition result for derived Zuckerman functors which strengthens the above Duflo-Vergne formula. For an abelian category $\mathcal{A}$, we denote by $D^b(\mathcal{A})$ its bounded derived category. Also, for an integer p, we denote by $C^\cdot \longmapsto C^\cdot[p]$ the translation functor in $D^b(\mathcal{A})$ which translates a complex $C^\cdot$ by p steps to the left, and by $D : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ the standard embedding functor (see §2). Let $R\Gamma_T : D^b(\mathcal{M}(\mathfrak{g})) \longrightarrow D^b(\mathcal{M}(\mathfrak{g}, T))$ denote the derived Zuckerman functor. Then we have the following result.

Theorem B.

$$R\Gamma_T(D(M(\lambda))) = \bigoplus_{q=0}^{\dim T} D(M(\lambda))[-q] \otimes \bigwedge^q \mathfrak{t}^*.$$

The proof of this result is inspired by some ideas from an analogous result of Deligne [2].

In the first section we review the definition and basic properties of the Zuckerman functors $\Gamma_{K,T}$ and their derived functors including the Duflo-Vergne formula. In the second section we analyze the endomorphism algebra of a bounded complex in a derived category. We establish that a bounded complex (satisfying certain finiteness conditions) is isomorphic to the direct sum of its cohomologies if and only if its endomorphism algebra has maximal possible dimension. In the third section we study the derived functors of the right adjoint to a forgetful functor. In the fourth section we discuss some results in Lie algebra cohomology. Finally, in the last section we prove a decomposition formula for derived Zuckerman functors for tori, which leads immediately to the above theorems.

1. Zuckerman functors. In this section we recall basic definitions and results about Zuckerman functors which are needed in the paper. The details with proofs can be found in [6].

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and K an algebraic group acting on $\mathfrak{g}$ by a morphism $\phi : K \longrightarrow \operatorname{Int}(\mathfrak{g})$ such that its differential $\mathfrak{k} \longrightarrow \mathfrak{g}$ is an injection. In this situation we can identify $\mathfrak{k}$ with a Lie subalgebra of $\mathfrak{g}$. A Harish-Chandra module (π, V) for the pair $(\mathfrak{g}, K)$ is

(i) a $\mathcal{U}(\mathfrak{g})$-module;

(ii) an algebraic K-module, i.e, V is a union of finite dimensional K-invariant subspaces on which K acts algebraically;
(iii) the actions of $\mathfrak{g}$ and K are compatible; i.e.,
  (a) the differential of the K-action agrees with the action of $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$;
  (b)
$$\pi(k)\pi(\xi)\pi(k^{-1})v = \pi(\phi(k)\xi)v$$
  for $k \in K$, $\xi \in \mathfrak{g}$ and all $v \in V$.

A morphism of Harish-Chandra modules is a linear map between Harish-Chandra modules which intertwines the actions of $\mathfrak{g}$ and K. If V and W are two Harish-Chandra modules for $(\mathfrak{g}, K)$, $\operatorname{Hom}_{(\mathfrak{g},K)}(V, W)$ denotes the space of all morphisms between V and W. Let $\mathcal{M}(\mathfrak{g}, K)$ be the category of Harish-Chandra modules for the pair $(\mathfrak{g}, K)$. This is clearly an abelian $\mathbb{C}$-category. It has enough of injective objects, and if K is reductive it also has enough of projective objects.

Let T be a closed algebraic subgroup of K. Then we have a natural forgetful functor $\mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g}, T)$. The Zuckerman functor $\Gamma_{K,T} : \mathcal{M}(\mathfrak{g}, T) \to \mathcal{M}(\mathfrak{g}, K)$ is by definition the right adjoint functor to this forgetful functor.

Let $D^+(\mathcal{M}(\mathfrak{g}, K))$ be the derived category of the abelian category $\mathcal{M}(\mathfrak{g}, K)$ consisting of complexes bounded from below. Let $D^b(\mathcal{M}(\mathfrak{g}, K))$ be its full subcategory of bounded complexes, i.e., the derived category of $\mathcal{M}(\mathfrak{g}, K)$ consisting of bounded complexes.

Since the category $\mathcal{M}(\mathfrak{g}, T)$ has enough of injectives, the Zuckerman functor $\Gamma_{K,T}$ has a right derived functor $R\Gamma_{K,T} : D^+(\mathcal{M}(\mathfrak{g}, T)) \to D^+(\mathcal{M}(\mathfrak{g}, K))$. The right cohomological dimension of $\Gamma_{K,T}$ is finite, so $R\Gamma_{K,T}$ induces a right derived functor between the corresponding bounded derived categories, i.e., $R\Gamma_{K,T} : D^b(\mathcal{M}(\mathfrak{g}, T)) \to D^b(\mathcal{M}(\mathfrak{g}, K))$.

If H is a closed algebraic subgroup of K such that $T \subset H \subset K$, we have

$$R\Gamma_{K,T} = R\Gamma_{K,H} \circ R\Gamma_{H,T}.$$

This leads to the corresponding Grothendieck's spectral sequence

$$R^p\Gamma_{K,H}(R^q\Gamma_{H,T}(V)) \Rightarrow R^{p+q}\Gamma_{K,T}(V)$$

for any Harish-Chandra module V in $\mathcal{M}(\mathfrak{g}, T)$.

Assume that T is reductive. Let L be a Levi factor of H containing T and $\mathfrak{l}$ its Lie algebra. Let $H^\dagger$ be the subgroup of H generated by the identity component of H and T. Let $\operatorname{Ind}_{H^\dagger}^H(1)$ be the space of complex functions on H which are constant on right $H^\dagger$-cosets. It has a natural structure of Harish-Chandra module, with trivial action of $\mathfrak{g}$ and right regular action of H. For any Harish-Chandra module V for the pair $(\mathfrak{l}, T)$, we denote by $H^\cdot(\mathfrak{l}, T; V)$ the relative Lie algebra cohomology for the pair $(\mathfrak{l}, T)$, i.e., $\operatorname{Ext}^\cdot_{(\mathfrak{l},T)}(\mathbb{C}, V)$.

The following result is the Duflo-Vergne formula we mentioned in the introduction ([3], [6], 1.10).

1.1. Lemma. *Let V be a Harish-Chandra module for the pair $(\mathfrak{g}, H)$. Then*

$$R^p\Gamma_{H,T}(V) = H^p(\mathfrak{l}, T; \mathbb{C}) \otimes \operatorname{Ind}_{H^\dagger}^H(1) \otimes V,$$

where the action on $H^p(\mathfrak{l}, T; \mathbb{C})$ is trivial, for all $p \in \mathbb{Z}_+$.

In particular, the above Grothendieck spectral sequence leads to the spectral sequence

$$H^q(\mathfrak{l}, T; \mathbb{C}) \otimes R^p\Gamma_{K,H}(\operatorname{Ind}_{H^\dagger}^H(1) \otimes V) = R^p\Gamma_{K,H}(H^q(\mathfrak{l}, T; \mathbb{C}) \otimes \operatorname{Ind}_{H^\dagger}^H(1) \otimes V) \Rightarrow R^{p+q}\Gamma_{K,T}(V)$$

for any V in $\mathcal{M}(\mathfrak{g}, H)$.

In general, this spectral sequence is not degenerate, as follows from the following example.

1.2. Example. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ be the Lie algebra of two-by-two complex matrices with trace zero and let $K = \mathrm{SL}(2, \mathbb{C})$ be the group of unimodular two-by-two matrices. The group K acts on $\mathfrak{g} = \mathfrak{k}$ by the adjoint action. Let $H \subset K$ be the subgroup consisting of diagonal matrices; so $H \cong \mathbb{C}^*$. Let T be the identity subgroup.

Consider a $(\mathfrak{g}, K)$-module V, i.e., an algebraic representation of K. By 1.1,

$$R^p\Gamma_K(V) = H^p(\mathfrak{k}; \mathbb{C}) \otimes V,$$

and

$$R^p\Gamma_{K,H}(V) = H^p(\mathfrak{k}, H; \mathbb{C}) \otimes V$$

for $p \in \mathbb{Z}_+$. These Lie algebra cohomology spaces are well known: $H^p(\mathfrak{k}; \mathbb{C})$ is equal to $\mathbb{C}$ for $p = 0, 3$ and to zero for all other p, while $H^p(\mathfrak{k}, H; \mathbb{C})$ is equal to $\mathbb{C}$ for $p = 0, 2$ and to zero for all other p. This can either be calculated directly from the definitions, or one can use the following argument. It is obvious that $H^0(\mathfrak{k}; \mathbb{C})$ and $H^0(\mathfrak{k}, H; \mathbb{C})$ are equal to $\mathbb{C}$. By Weyl's Theorem, $H^1(\mathfrak{k}; \mathbb{C})$ and $H^1(\mathfrak{k}, H; \mathbb{C})$ are equal to 0, since these are the self-extensions of $\mathbb{C}$ in the category of $\mathfrak{k}$-modules (respectively $(\mathfrak{k}, H)$-modules). The rest now follows from the Poincaré duality for (relative) Lie algebra cohomology. Also, it is clear that $H^0(\mathfrak{h}, \mathbb{C}) = H^1(\mathfrak{h}, \mathbb{C}) = \mathbb{C}$ and all other $H^p(\mathfrak{h}, \mathbb{C})$ vanish.

So, we see that $R^1\Gamma_K(V) = 0$ cannot have a filtration such that the corresponding graded object is $R^1\Gamma_{K,H}(V) \oplus R^0\Gamma_{K,H}(V) = V$. In other words, the spectral sequence does not degenerate in this case.

A sufficient condition for degeneracy of the above spectral sequence is that

$$R\Gamma_{H,T}(D(V)) = \bigoplus_{q=0}^{\dim(L/T)} D(R^q\Gamma_{H,T}(V))[-q] = \bigoplus_{q=0}^{\dim(L/T)} H^q(\mathfrak{l}, T; \mathbb{C}) \otimes \operatorname{Ind}_{H^\dagger}^H(1) \otimes D(V)[-q].$$

Our main result (5.1) establishes this decomposition in a special case. This explains the degeneracy of the spectral sequence in the introduction. In 5.6 we show that this decomposition is not necessary for the degeneracy of the spectral sequence.

2. Endomorphism algebra of a bounded complex. Let $\mathcal{A}$ be an abelian $\mathbb{C}$-category and $D^b(\mathcal{A})$ the corresponding derived category of bounded complexes. Let $C^{\cdot}$ be a complex in $D^b(\mathcal{A})$. Assume that $H^s(C^{\cdot}) = 0$ for $s > p$ and $s < q$. Then we say that the (*cohomological*) *length* of $C^{\cdot}$ is $\leq p - q + 1$.

We recall the definition of the truncation functors ([4], Ch. IV, §4). If $C^{\cdot}$ is a bounded complex, for $s \in \mathbb{Z}$, we define the truncated complex $\tau_{\leq s}(C^{\cdot})$ as the subcomplex of $C^{\cdot}$ given by

$$\tau_{\leq s}(C^{\cdot})^p = \begin{cases} C^p, & \text{if } p < s \\ \ker d^s, & \text{if } p = s \\ 0, & \text{if } p > s. \end{cases}$$

Let $\alpha : \tau_{\leq s}(C^{\cdot}) \longrightarrow C^{\cdot}$ be the canonical inclusion morphism. Then the morphisms $H^p(\alpha) : H^p(\tau_{\leq s}(C^{\cdot})) \longrightarrow H^p(C^{\cdot})$ are isomorphisms for $p \leq s$ and 0 for $p > s$.

We also define the truncated complex $\tau_{\geq s}(C^{\cdot})$ as a quotient complex of $C^{\cdot}$:

$$\tau_{\geq s}(C^{\cdot})^p = \begin{cases} 0, & \text{if } p < s \\ \operatorname{coker} d^{s-1}, & \text{if } p = s \\ C^p, & \text{if } p > s. \end{cases}$$

Let $\beta : C^{\cdot} \longrightarrow \tau_{\geq s}(C^{\cdot})$ be the canonical projection morphism. Then the morphisms $H^p(\beta) : H^p(C^{\cdot}) \longrightarrow H^p(\tau_{\geq s}(C^{\cdot}))$ are isomorphisms for $p \geq s$ and 0 for $p < s$.

It is evident that $\beta : C^{\cdot} \longrightarrow \tau_{\geq s+1}(C^{\cdot})$ factors through the quotient of $C^{\cdot}$ by $\tau_{\leq s}(C^{\cdot})$, moreover the induced morphism is a quasiisomorphism. Hence, we have the canonical distinguished triangle

$$\tau_{\leq s}(C^{\cdot}) \xrightarrow{\alpha} C^{\cdot} \xrightarrow{\beta} \tau_{\geq s+1}(C^{\cdot}) \xrightarrow{[1]} \tau_{\leq s}(C^{\cdot})[1]$$

in $D^b(\mathcal{A})$.

The endomorphisms of $C^{\cdot}$ form a complex algebra with identity which we denote by $\operatorname{End}_{D^b(\mathcal{A})}(C^{\cdot})$. In this section we study some elementary properties of this algebra.

Clearly,

$$\operatorname{End}_{D^b(\mathcal{A})}(C^{\cdot}) = \operatorname{End}_{D^b(\mathcal{A})}(C^{\cdot}[q]),$$

for any $q \in \mathbb{Z}$, i.e., the endomorphism algebra of a complex and its translate are isomorphic.

Denote by $D : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ the functor which attaches to an object A in $\mathcal{A}$ the complex

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

in $D^b(\mathcal{A})$, where A appears in degree zero.

It is well known that $D : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ is fully faithful ([4], Ch. III, §5, no. 2). Therefore, the endomorphism algebras satisfy

$$\operatorname{End}_{\mathcal{A}}(A) = \operatorname{End}_{D^b(\mathcal{A})}(D(A))$$

for any object A in $\mathcal{A}$. Let $C^\cdot$ be a complex of length ≤ 1. Then there exists $p_0 \in \mathbb{Z}$ such that $H^p(C^\cdot) = 0$ for $p \neq p_0$. Moreover, the complex $C^\cdot$ is isomorphic to $D(H^{p_0}(C^\cdot))[-p_0]$. Therefore, the above formula describes the endomorphism algebras of complexes of length ≤ 1.

Now we consider an arbitrary complex $C^\cdot$ of length $p+1$. By translation, we can assume that $H^s(C^\cdot) = 0$ for $s < 0$ and $s > p$. Moreover, $\tau_{\leq p-1}(C^\cdot)$ is a complex of length $\leq p$ and $\tau_{\geq p}(C^\cdot)$ is a complex of length ≤ 1, hence $\tau_{\geq p}(C^\cdot) = D(H^p(C^\cdot))[-p]$. The distinguished triangle of truncations therefore looks like

$$\tau_{\leq p-1}(C^\cdot) \xrightarrow{\alpha} C^\cdot \xrightarrow{\beta} D(H^p(C^\cdot))[-p] \xrightarrow{[1]} \tau_{\leq p-1}(C^\cdot)[1].$$

Applying the functor $\operatorname{Hom}_{D^b(\mathcal{A})}(-, D(H^p(C^\cdot))[-p])$ to the above distinguished triangle leads to the following long exact sequence

$$\begin{aligned} \dots \longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot)[1], D(H^p(C^\cdot))[-p]) \longrightarrow \operatorname{End}_{D^b(\mathcal{A})}(D(H^p(C^\cdot))[-p]) \\ \longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, D(H^p(C^\cdot))[-p]) \longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot), D(H^p(C^\cdot))[-p]) \longrightarrow \dots . \end{aligned}$$

Since $H^q(\tau_{\leq p-1}(C^\cdot)) = 0$ for $q > p-1$, the linear spaces $\operatorname{Hom}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot), D(H^p(C^\cdot))[-p])$ and $\operatorname{Hom}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot)[1], D(H^p(C^\cdot))[-p])$ are equal to zero. It follows that the above long exact sequence gives an isomorphism

$$\begin{aligned} \operatorname{End}_{\mathcal{A}}(H^p(C^\cdot)) &= \operatorname{End}_{D^b(\mathcal{A})}(D(H^p(C^\cdot))) \\ &= \operatorname{End}_{D^b(\mathcal{A})}(D(H^p(C^\cdot))[-p]) \longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, D(H^p(C^\cdot))[-p]). \end{aligned}$$

This proves the following result.

2.1. Lemma. *The linear map $U \longmapsto D(U)[-p] \circ \beta$ is an isomorphism of* $\operatorname{End}_{\mathcal{A}}(H^p(C^\cdot))$ *onto* $\operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, D(H^p(C^\cdot))[-p])$.

In particular, the morphism $\beta \in \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, D(H^p(C^\cdot))[-p])$ corresponds to the identity in $\operatorname{End}_{\mathcal{A}}(H^p(C^\cdot))$ under this isomorphism.

The composition with $\alpha : \tau_{\leq s-1}(C^\cdot) \longrightarrow C^\cdot$ induces a linear map from $\operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, \tau_{\leq p-1}(C^\cdot))$ into $\operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot))$. If $C^\cdot \cong \tau_{\leq p-1}(C^\cdot) \oplus D(H^p(C^\cdot))[-p]$, this map is clearly surjective. On the other hand, if this map is surjective, there exists a morphism $\gamma \in \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, \tau_{\leq p-1}(C^\cdot))$ such that $\gamma \circ \alpha = 1_{\tau_{\leq p-1}(C^\cdot)}$. Then $\gamma \oplus \beta$ is a morphism of $C^\cdot$ into $\tau_{\leq p-1}(C^\cdot) \oplus D(H^p(C^\cdot))[-p]$. Clearly, $\gamma \oplus \beta$ is a quasiisomorphism, i.e., we have the following splitting criterion.

2.2. Lemma. *The following conditions are equivalent.*

(i) *$C^\cdot$ is isomorphic to*

$$\tau_{\leq p-1}(C^\cdot) \oplus D(H^p(C^\cdot))[-p];$$

(ii) *the linear map $U \longmapsto U \circ \alpha$ of* $\operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, \tau_{\leq p-1}(C^\cdot))$ *into* $\operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot))$ *is surjective.*

Now, we want to establish a sufficient criterion for the finite dimensionality of the endomorphism algebra $\operatorname{End}_{D^b(\mathcal{A})}(C^\cdot)$. We start with an auxiliary result.

2.3. Lemma. *Let A be an object in $\mathcal{A}$ and $D^\cdot$ a complex in $D^b(\mathcal{A})$. Then*

$$\dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(A), D^\cdot) \leq \sum_{q\in\mathbb{Z}_+} \dim \operatorname{Ext}^q_{\mathcal{A}}(A, H^{-q}(D^\cdot)).$$

Proof. The proof is by induction on the length of $D^\cdot$. If $D^\cdot$ is of length ≤ 1, we have $D^\cdot = D(H^r(D^\cdot))[-r]$ for some $r \in \mathbb{Z}$. Therefore,

$$\operatorname{Hom}_{D^b(\mathcal{A})}(D(A), D^\cdot) = \operatorname{Hom}_{D^b(\mathcal{A})}(D(A), D(H^r(D^\cdot))[-r]) = \operatorname{Ext}^{-r}_{\mathcal{A}}(A, H^r(D^\cdot)).$$

Consider now the case of a complex $D^\cdot$ of length ≥ 2. Then, for $s \in \mathbb{Z}$, we have a distinguished triangle of truncations and the corresponding long exact sequence

$$\begin{aligned} \ldots \longrightarrow &\operatorname{Hom}_{D^b(\mathcal{A})}(D(A), \tau_{\leq s-1}(D^\cdot)) \\ &\longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(D(A), D^\cdot) \longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(D(A), \tau_{\geq s}(D^\cdot)) \longrightarrow \ldots. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(A), D^\cdot) \\ &\qquad \leq \dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(A), \tau_{\geq s}(D^\cdot)) + \dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(A), \tau_{\leq s-1}(D^\cdot)). \end{aligned}$$

Clearly, we can pick such s that the lengths of $\tau_{\geq s}(D^\cdot)$ and $\tau_{\leq s-1}(D^\cdot)$ are strictly smaller than the length of $D^\cdot$. Therefore, by the induction assumption

$$\begin{aligned} &\dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(A), D^\cdot) \\ &\quad \leq \sum_{q\in\mathbb{Z}_+} \dim \operatorname{Ext}^q_{\mathcal{A}}(A, H^{-q}(\tau_{\geq s}(D^\cdot))) + \sum_{q\in\mathbb{Z}_+} \dim \operatorname{Ext}^q_{\mathcal{A}}(A, H^{-q}(\tau_{\leq s-1}(D^\cdot))) \\ &\qquad = \sum_{q\in\mathbb{Z}_+} \dim \operatorname{Ext}^{-q}_{\mathcal{A}}(A, H^q(D^\cdot)). \quad \square \end{aligned}$$

The finite dimensionality criterion is given in the following proposition.

2.4. Proposition. *Let $C^\cdot$ be a bounded complex in $D^b(\mathcal{A})$. If $\operatorname{Ext}^q_{\mathcal{A}}(H^{p+q}(C^\cdot), H^p(C^\cdot))$ are finite dimensional for all $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_+$, then $\operatorname{End}_{D^b(\mathcal{A})}(C^\cdot)$ is a finite dimensional algebra. More precisely,*

$$\dim \operatorname{End}_{D^b(\mathcal{A})}(C^\cdot) \leq \sum_{q\in\mathbb{Z}_+} \sum_{p\in\mathbb{Z}} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^{p+q}(C^\cdot), H^p(C^\cdot)).$$

Proof. As before, consider the complex $C^\cdot$ of length $p+1$ with cohomologies vanishing outside the interval $[0, p]$. The long exact sequence corresponding to the functor $\operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, -)$ and the distinguished triangle of truncations is

$$\begin{aligned} \ldots \longrightarrow &\operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, \tau_{\leq p-1}(C^\cdot)) \\ &\longrightarrow \operatorname{End}_{D^b(\mathcal{A})}(C^\cdot) \longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, D(H^p(C^\cdot))[-p]) \longrightarrow \ldots. \end{aligned}$$

Hence, by 2.1, we have

$$\dim \operatorname{End}_{D^b(\mathcal{A})}(C^\cdot) \leq \dim \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, \tau_{\leq p-1}(C^\cdot)) + \dim \operatorname{End}_{\mathcal{A}}(H^p(C^\cdot)).$$

On the other hand, applying the functor $\operatorname{Hom}_{D^b(\mathcal{A})}(-, \tau_{\leq p-1}(C^\cdot))$ to the same distinguished triangle leads to the long exact sequence

$$\begin{aligned} \dots \longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}&(D(H^p(C^\cdot))[-p], \tau_{\leq p-1}(C^\cdot)) \\ &\longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(C^\cdot, \tau_{\leq p-1}(C^\cdot)) \longrightarrow \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot)) \longrightarrow \dots, \end{aligned}$$

and

$$\begin{aligned} \dim \operatorname{Hom}_{D^b(\mathcal{A})}&(C^\cdot, \tau_{\leq p-1}(C^\cdot)) \\ &\leq \dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(H^p(C^\cdot))[-p], \tau_{\leq p-1}(C^\cdot)) + \dim \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot)). \end{aligned}$$

It follows that

$$\begin{aligned} \dim \operatorname{End}_{D^b(\mathcal{A})}(C^\cdot) &\leq \dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(H^p(C^\cdot))[-p], \tau_{\leq p-1}(C^\cdot)) \\ &\quad + \dim \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot)) + \dim \operatorname{End}_{\mathcal{A}}(H^p(C^\cdot)). \end{aligned}$$

By 2.3, we get

$$\begin{aligned} &\dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(H^p(C^\cdot))[-p], \tau_{\leq p-1}(C^\cdot)) \\ &= \dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(H^p(C^\cdot)), \tau_{\leq p-1}(C^\cdot)[p]) \leq \sum_{q \in \mathbb{Z}_+} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^p(C^\cdot), H^{-q}(\tau_{\leq p-1}(C^\cdot)[p])) \\ &= \sum_{q \in \mathbb{Z}_+} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^p(C^\cdot), H^{p-q}(\tau_{\leq p-1}(C^\cdot))) = \sum_{q=1}^{p} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^p(C^\cdot), H^{p-q}(C^\cdot)). \end{aligned}$$

Hence, we have

$$\begin{aligned} \dim \operatorname{End}_{D^b(\mathcal{A})}(C^\cdot) &\leq \sum_{q=1}^{p} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^p(C^\cdot), H^{p-q}(C^\cdot)) \\ &\quad + \dim \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot)) + \dim \operatorname{End}_{\mathcal{A}}(H^p(C^\cdot)) \\ &= \sum_{q=0}^{p} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^p(C^\cdot), H^{p-q}(C^\cdot)) + \dim \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(C^\cdot)). \end{aligned}$$

By induction on the length of the complex $C^\cdot$, we have

$$\begin{aligned} \dim \operatorname{End}_{D^b(\mathcal{A})}(C^\cdot) &\leq \sum_{r \in \mathbb{Z}} \sum_{q \in \mathbb{Z}_+} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^r(C^\cdot), H^{r-q}(C^\cdot)) \\ &= \sum_{q \in \mathbb{Z}_+} \sum_{r \in \mathbb{Z}} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^{r+q}(C^\cdot), H^r(C^\cdot)) < \infty. \quad \square \end{aligned}$$

Let $C^\cdot$ be a complex in $D^b(\mathcal{A})$ such that $\operatorname{Ext}^q_{\mathcal{A}}(H^{p+q}(C^\cdot), H^p(C^\cdot))$ are finite dimensional. Consider the family $\mathcal{C}(C^\cdot)$ of all complexes $D^\cdot$ in $D^b(\mathcal{A})$ such that $H^q(D^\cdot) = H^q(C^\cdot)$ for all $q \in \mathbb{Z}$. Then, by 2.4, $\operatorname{End}_{D^b(\mathcal{A})}(D^\cdot)$ are finite dimensional algebras and

$$\dim \operatorname{End}_{D^b(\mathcal{A})}(D^\cdot) \leq \sum_{q\in\mathbb{Z}_+} \sum_{p\in\mathbb{Z}} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^{p+q}(C^\cdot), H^p(C^\cdot)).$$

for all $D^\cdot$ in $\mathcal{C}(C^\cdot)$.

Clearly, the complex

$$S^\cdot = \bigoplus_{p\in\mathbb{Z}} D(H^p(C^\cdot))[-p]$$

is in $\mathcal{C}(C^\cdot)$. Moreover, we have

$$\begin{aligned}
\operatorname{End}_{D^b(\mathcal{A})}(S^\cdot) &= \bigoplus_{p,q\in\mathbb{Z}} \operatorname{Hom}_{D^b(\mathcal{A})}(D(H^q(C^\cdot))[-q], D(H^p(C^\cdot))[-p]) \\
&= \bigoplus_{p,q\in\mathbb{Z}} \operatorname{Hom}_{D^b(\mathcal{A})}(D(H^q(C^\cdot)), D(H^p(C^\cdot))[q-p]) \\
&= \bigoplus_{p,q\in\mathbb{Z}} \operatorname{Ext}^{q-p}_{\mathcal{A}}(H^q(C^\cdot), H^p(C^\cdot)) = \bigoplus_{q\in\mathbb{Z}_+,\ p\in\mathbb{Z}} \operatorname{Ext}^q_{\mathcal{A}}(H^{p+q}(C^\cdot), H^p(C^\cdot))
\end{aligned}$$

and

$$\dim \operatorname{End}_{D^b(\mathcal{A})}(S^\cdot) = \sum_{q\in\mathbb{Z}_+} \sum_{p\in\mathbb{Z}} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^{p+q}(C^\cdot), H^p(C^\cdot)).$$

Hence, by 2.4, the maximal possible value of the function $D^\cdot \longmapsto \dim \operatorname{End}_{D^b(\mathcal{A})}(D^\cdot)$ on $\mathcal{C}(C^\cdot)$ is attained at $S^\cdot$.

The next result shows that this property characterizes the complex $S^\cdot$.

2.5. Theorem. *For a complex $D^\cdot$ in $\mathcal{C}(C^\cdot)$, the following conditions are equivalent:*

(i) *$D^\cdot$ is isomorphic to*

$$\bigoplus_{p\in\mathbb{Z}} H^p(C^\cdot)[-p];$$

(ii) *the dimension of $\operatorname{End}_{D^b(\mathcal{A})}(D^\cdot)$ is maximal possible.*

If these conditions are satisfied,

$$\dim \operatorname{End}_{D^b(\mathcal{A})}(D^\cdot) = \sum_{q\in\mathbb{Z}_+} \sum_{p\in\mathbb{Z}} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^{p+q}(C^\cdot), H^p(C^\cdot)).$$

Proof. We already know that (i) implies (ii), and that the maximal dimension is given by the above formula. It remains to show that (ii) implies (i). The proof is by induction on the length of $C^\cdot$. As before, we can assume that the length of $C^\cdot$ is $\leq p+1$ and that its cohomologies vanish outside the interval $[0, p]$.

Let $D^{\cdot}$ in $\mathcal{C}(C^{\cdot})$ be a complex such that its endomorphism algebra has maximal possible dimension, i.e.,

$$\dim \operatorname{End}_{D^b(\mathcal{A})}(D^{\cdot}) = \sum_{q\in\mathbb{Z}_+}\sum_{r\in\mathbb{Z}} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^{r+q}(C^{\cdot}), H^r(C^{\cdot})) = \sum_{r=0}^{p}\sum_{q=0}^{r} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^r(C^{\cdot}), H^{r-q}(C^{\cdot})).$$

As we remarked in the proof of 2.4,

$$\sum_{r=0}^{p}\sum_{q=0}^{r} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^r(C^{\cdot}), H^{r-q}(C^{\cdot})) = \dim \operatorname{End}_{D^b(\mathcal{A})}(D^{\cdot}) \leq \dim \operatorname{Hom}_{D^b(\mathcal{A})}(D^{\cdot}, \tau_{\leq p-1}(D^{\cdot})) + \dim \operatorname{End}_{\mathcal{A}}(H^p(C^{\cdot})) \leq \dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(H^p(C^{\cdot}))[-p], \tau_{\leq p-1}(D^{\cdot})) + \dim \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(D^{\cdot})) + \dim \operatorname{End}_{\mathcal{A}}(H^p(C^{\cdot})) \leq \sum_{q=0}^{p} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^p(C^{\cdot}), H^{p-q}(C^{\cdot})) + \dim \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(D^{\cdot})).$$

Hence, we have

$$\sum_{r=0}^{p-1}\sum_{q=0}^{r} \dim \operatorname{Ext}^q_{\mathcal{A}}(H^r(C^{\cdot}), H^{r-q}(C^{\cdot})) \leq \dim \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(D^{\cdot})).$$

By 2.4, we see that this inequality must be an equality and that the dimension of the endomorphism algebra $\operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(D^{\cdot}))$ is maximal possible in $\mathcal{C}(\tau_{\leq p-1}(C^{\cdot}))$. By the induction assumption, we have

$$\tau_{\leq p-1}(D^{\cdot}) \cong \bigoplus_{r=0}^{p-1} D(H^r(C^{\cdot}))[-r].$$

Moreover, since all the above inequalities must be equalities, it follows that

$$\dim \operatorname{Hom}_{D^b(\mathcal{A})}(D^{\cdot}, \tau_{\leq p-1}(D^{\cdot})) = \dim \operatorname{Hom}_{D^b(\mathcal{A})}(D(H^p(C^{\cdot}))[-p], \tau_{\leq p-1}(D^{\cdot})) + \dim \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(D^{\cdot})).$$

Consider the long exact sequence corresponding to the distinguished triangle of truncations and the functor $\operatorname{Hom}_{D^b(\mathcal{A})}(-, \tau_{\leq p-1}(D^{\cdot}))$. It looks like

$$\ldots \longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(D(H^p(C^{\cdot}))[-p], \tau_{\leq p-1}(D^{\cdot})) \longrightarrow \operatorname{Hom}_{D^b(\mathcal{A})}(D^{\cdot}, \tau_{\leq p-1}(D^{\cdot})) \longrightarrow \operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(D^{\cdot})) \longrightarrow \ldots.$$

Hence the above equality is possible only if the linear map from $\operatorname{Hom}_{D^b(\mathcal{A})}(D^{\cdot}, \tau_{\leq p-1}(D^{\cdot}))$ into $\operatorname{End}_{D^b(\mathcal{A})}(\tau_{\leq p-1}(D^{\cdot}))$ is surjective. By 2.2, this implies that

$$D^{\cdot} \cong \tau_{\leq p-1}(D^{\cdot}) \oplus D(H^p(C^{\cdot}))[-p] \cong \bigoplus_{r=0}^{p} D(H^r(C^{\cdot}))[-r]. \quad \square$$

3. A splitting result for adjoint functors. Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories. Assume that the objects of $\mathcal{B}$ are objects in $\mathcal{A}$ and that there is an additive functor $G : \mathcal{B} \longrightarrow \mathcal{A}$ which is the identity on objects. Assume that there exists an additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ which is right adjoint to the functor G, i.e., such that

$$\operatorname{Hom}_{\mathcal{A}}(X, Y) = \operatorname{Hom}_{\mathcal{A}}(G(X), Y) = \operatorname{Hom}_{\mathcal{B}}(X, F(Y))$$

for any object X in $\mathcal{B}$ and Y in $\mathcal{A}$.

Let X be an object in the category $\mathcal{B}$. Then $F(X)$ is an object of $\mathcal{B}$. Also, by the adjointness, the identity morphism $X \longrightarrow X$ determines the canonical morphism $\epsilon_X : X \longrightarrow F(X)$ in $\mathcal{B}$, which is just one of the adjointness morphisms in this special case. The map ϕ from $\operatorname{Hom}_{\mathcal{A}}(X, Y)$ into $\operatorname{Hom}_{\mathcal{B}}(X, F(Y))$ is given by

$$\phi(f) = F(f) \circ \epsilon_X.$$

Conversely, for any Y in $\mathcal{A}$, the identity morphism $F(Y) \longrightarrow F(Y)$ determines the canonical morphism $\eta_Y : F(Y) \longrightarrow Y$ in $\mathcal{A}$, which is the second adjointness morphism. The inverse of the map ϕ is given by

$$\phi^{-1}(g) = \eta_Y \circ G(g).$$

Applying this to $\epsilon_X : X \longrightarrow F(X)$ we get

$$1_X = \phi^{-1}(\epsilon_X) = \eta_X \circ G(\epsilon_X),$$

i.e., the composition

$$X \xrightarrow{G(\epsilon_X)} F(X) \xrightarrow{\eta_X} X$$

is equal to the identity.

Assume now that $\mathcal{B}$ is a full subcategory of $\mathcal{A}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ is the inclusion functor. Then for any X in $\mathcal{B}$, the functors $Y \longmapsto \operatorname{Hom}_{\mathcal{B}}(Y, X) = \operatorname{Hom}_{\mathcal{A}}(Y, X)$ and $Y \longmapsto \operatorname{Hom}_{\mathcal{B}}(Y, F(X))$ from $\mathcal{B}$ into the category of abelian groups are isomorphic, and this isomorphism is induced by ϵ_X. Therefore, ϵ_X is an isomorphism. This implies that η_X is also an isomorphism. Therefore we have the following result.

3.1. Lemma. *Let $\mathcal{B}$ be a full subcategory of $\mathcal{A}$. Then for any object X in $\mathcal{B}$, the natural morphisms $\epsilon_X : X \longrightarrow F(X)$ and $\eta_X : F(X) \longrightarrow X$ are mutually inverse isomorphisms.*

Assume now that $\mathcal{A}$ is an abelian category and $\mathcal{B}$ is a full abelian subcategory. Then, by 3.1, for any object X in $\mathcal{B}$, $F(X) \cong X$. Assume that the category $\mathcal{A}$ has enough injectives. Let $D^+(\mathcal{A})$ and $D^+(\mathcal{B})$ be the corresponding derived categories of complexes bounded from below. Then the embedding of $\mathcal{B}$ into $\mathcal{A}$ defines an exact functor $\iota : D^+(\mathcal{B}) \longrightarrow D^+(\mathcal{A})$ which acts as identity on objects (but, in general, $D^+(\mathcal{B})$ doesn't have to be a subcategory of $D^+(\mathcal{A})$). Also, the functor F has its right derived functor $RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$. Moreover, RF is the right adjoint of ι, i.e., we have the following relation

$$\operatorname{Hom}_{D^+(\mathcal{A})}(X^\cdot, Y^\cdot) = \operatorname{Hom}_{D^+(\mathcal{B})}(X^\cdot, RF(Y^\cdot))$$

for any two complexes $X^\cdot$ in $D^+(\mathcal{B})$ and $Y^\cdot$ in $D^+(\mathcal{A})$.

Let $X^\cdot$ be a complex in $D^+(\mathcal{B})$. Then the adjointness morphisms define canonical morphisms $X^\cdot \xrightarrow{\epsilon_{X^\cdot}} RF(X^\cdot)$ and $RF(X^\cdot) \xrightarrow{\eta_{X^\cdot}} X^\cdot$. Also, the composition of morphisms

$$X^\cdot \xrightarrow{\iota(\epsilon_{X^\cdot})} RF(X^\cdot) \xrightarrow{\eta_{X^\cdot}} X^\cdot$$

is the identity in $D^+(\mathcal{A})$.

Let X be an object in $\mathcal{B}$. Then $D(X)$ is a complex in $D^+(\mathcal{B})$ and the composition of the adjointness morphisms

$$D(X) \xrightarrow{\alpha} RF(D(X)) \xrightarrow{\beta} D(X)$$

is the identity in $D^+(\mathcal{A})$. In particular, the induced map of 0-th cohomologies

$$\begin{array}{ccccc} X & \xrightarrow{H^0(\alpha)} & H^0(RF(D(X))) & \xrightarrow{H^0(\beta)} & X \\ & & \| & & \\ & & F(X) & & \end{array}$$

is the identity. This is the composition of the adjointness morphisms discussed before 3.1. Let C_α be the cone of α in $D^+(\mathcal{A})$. Then we have the distinguished triangle

$$D(X) \xrightarrow{\alpha} RF(D(X)) \xrightarrow{\gamma} C_\alpha \longrightarrow D(X)[1].$$

The corresponding long exact sequence of cohomology is

$$\begin{aligned} 0 \longrightarrow X \xrightarrow{H^0(\alpha)} F(X) \xrightarrow{H^0(\gamma)} H^0(C_\alpha) \longrightarrow 0 \longrightarrow \dots \\ \longrightarrow 0 \longrightarrow R^pF(X) \xrightarrow{H^p(\gamma)} H^p(C_\alpha) \longrightarrow 0 \longrightarrow \dots; \end{aligned}$$

hence we conclude that $H^0(C_\alpha) = 0$ and $H^p(C_\alpha) = R^pF(X)$ for $p \geq 1$.

Now we can consider the morphism $\delta = \beta \oplus \gamma : RF(D(X)) \longrightarrow D(X) \oplus C_\alpha$. Then

$$H^0(\delta) = H^0(\beta) \oplus H^0(\gamma) = H^0(\beta)$$

is an isomorphism of $H^0(RF(D(X))) = F(X)$ onto $H^0(D(X) \oplus C_\alpha) = X$, and

$$H^p(\delta) = H^p(\beta) \oplus H^p(\gamma) = H^p(\gamma)$$

is an isomorphism of $H^p(RF(D(X))) = R^pF(X)$ onto $H^p(D(X) \oplus C_\alpha) = H^p(C_\alpha)$ for $p \geq 1$.

Therefore, we have proved the following result.

3.2. Lemma. *Let X be an object of $\mathcal{B}$. Then $RF(D(X)) \cong D(X) \oplus C_\alpha$ in $D^+(\mathcal{A})$.*

In particular, assume that the right cohomological dimension of F is ≤ 1. Then we have the following result.

3.3. Proposition. *Let X be an object of $\mathcal{B}$. Then*

$$RF(D(X)) \cong D(X) \oplus D(R^1F(X))[-1]$$

in $D(\mathcal{A})$.

Proof. In this case the cone C_α satisfies $H^p(C_\alpha) = 0$ for $p \neq 1$ and $H^1(C_\alpha) = R^1F(X)$. Hence, we have $C_\alpha = D(R^1F(X))[-1]$. □

Of course, since $D^b(\mathcal{B})$ is not a full subcategory of $D^b(\mathcal{A})$ in general, $RF(D(X))$ doesn't have to be isomorphic to $D(X) \oplus D(R^1F(X))[-1]$ in $D^b(\mathcal{B})$.

3.4. Corollary. *Let X be an object of $\mathcal{B}$. Then*

$$\operatorname{End}_{D(\mathcal{B})}(RF(D(X))) = \operatorname{End}_{\mathcal{B}}(X) \oplus \operatorname{Ext}^1_{\mathcal{A}}(R^1F(X), X).$$

Proof. Using the adjunction and 3.3 we see that

$$\begin{aligned}\operatorname{End}_{D(\mathcal{B})}(RF(D(X))) &= \operatorname{Hom}_{D(\mathcal{A})}(RF(D(X)), D(X)) \\ &= \operatorname{Hom}_{D(\mathcal{A})}(D(X) \oplus D(R^1F(X))[-1], D(X)) \\ &= \operatorname{Hom}_{D(\mathcal{A})}(D(X), D(X)) \oplus \operatorname{Hom}_{D(\mathcal{A})}(D(R^1F(X))[-1], D(X)) \\ &= \operatorname{End}_{\mathcal{A}}(X) \oplus \operatorname{Ext}^1_{\mathcal{A}}(R^1F(X), X).\end{aligned}$$

Since $\mathcal{B}$ is a full subcategory of $\mathcal{A}$, $\operatorname{End}_{\mathcal{A}}(X) = \operatorname{End}_{\mathcal{B}}(X)$. □

4. Some remarks on Lie algebra cohomology. In this section we collect some well known facts about Lie algebra cohomology. We include proofs of some of them in cases where we do not know any appropriate reference.

Let $\mathfrak{a}$ be a complex Lie algebra and B an algebraic group acting on $\mathfrak{a}$ by automorphisms. Let $\phi : B \longrightarrow \operatorname{Aut}(\mathfrak{a})$ be the action homomorphism. The differential of ϕ is a Lie algebra homomorphism of the Lie algebra $\mathfrak{b}$ of B into the Lie algebra $\operatorname{Der}(\mathfrak{a})$ of derivations of $\mathfrak{a}$. We assume that the differential of ϕ factors through a monomorphism $\mathfrak{b} \longrightarrow \mathfrak{a}$. Therefore, we can identify $\mathfrak{b}$ with a subalgebra of $\mathfrak{a}$. For such a pair $(\mathfrak{a}, B)$ we can define the category $\mathcal{M}(\mathfrak{a}, B)$ of $(\mathfrak{a}, B)$-modules analogous to the categories of Harish-Chandra modules discussed in §1. Denote by $D^b(\mathcal{M}(\mathfrak{a}, B))$ the bounded derived category of $(\mathfrak{a}, B)$-modules.

First we recall the well known relation between the $\operatorname{Ext}^{\cdot}_{(\mathfrak{a},B)}(-,-)$ groups and relative Lie algebra cohomology groups $H^{\cdot}(\mathfrak{a}, B; -) = \operatorname{Ext}^{\cdot}_{(\mathfrak{a},B)}(\mathbb{C}, -)$.

Let U and V be two $(\mathfrak{a}, B)$-modules. Assume that U is finite dimensional. Then $U \otimes V$ and $\operatorname{Hom}_{\mathbb{C}}(U, V)$ have natural structures of $(\mathfrak{a}, B)$-modules. Therefore, $V \longmapsto U \otimes V$ and

$V \longmapsto \operatorname{Hom}_{\mathbb{C}}(U,V)$ are exact functors from $\mathcal{M}(\mathfrak{a},B)$ into itself. In addition, we have the following adjointness relation for a fixed finite dimensional $(\mathfrak{a},B)$-module W:

$$\operatorname{Hom}_{(\mathfrak{a},B)}(U\otimes W,V)=\operatorname{Hom}_{(\mathfrak{a},B)}(U,\operatorname{Hom}_{\mathbb{C}}(W,V)).$$

This leads to the adjointness relation

$$\operatorname{Hom}_{D^b(\mathcal{M}(\mathfrak{a},B))}(U^{\cdot}\otimes W,V^{\cdot})=\operatorname{Hom}_{D^b(\mathcal{M}(\mathfrak{a},B))}(U^{\cdot},\operatorname{Hom}_{\mathbb{C}}(W,V^{\cdot}))$$

for corresponding functors on $D^b(\mathcal{M}(\mathfrak{a},B))$. Therefore, we have

$$\operatorname{Ext}^p_{(\mathfrak{a},B)}(U\otimes W,V)=\operatorname{Ext}^p_{(\mathfrak{a},B)}(U,\operatorname{Hom}_{\mathbb{C}}(W,V)).$$

In particular, for $U=\mathbb{C}$, we have

$$\operatorname{Ext}^p_{(\mathfrak{a},B)}(W,V)=\operatorname{Ext}^p_{(\mathfrak{a},B)}(\mathbb{C},\operatorname{Hom}_{\mathbb{C}}(W,V))=H^p(\mathfrak{a},B;\operatorname{Hom}_{\mathbb{C}}(W,V)).$$

4.1. Lemma. *Let U and V be two $(\mathfrak{a},B)$-modules. Assume that U is finite dimensional. Then*

$$\operatorname{Ext}^p_{(\mathfrak{a},B)}(U,V)=H^p(\mathfrak{a},B;\operatorname{Hom}_{\mathbb{C}}(U,V))$$

for $p\in\mathbb{Z}_+$.

Assume now that we have such a pair $(\mathfrak{a},B)$ where $\mathfrak{a}$ is an abelian Lie algebra and B is reductive, i.e., its identity component is a torus. Therefore the Lie algebra $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{a}$, invariant under the action of B. Let $\mathfrak{c}=\mathfrak{a}/\mathfrak{b}$. Then $\mathfrak{c}$ is a B-module. The standard complex $N^{\cdot}(\mathfrak{c})=\mathcal{U}(\mathfrak{c})\otimes\bigwedge^{-\cdot}\mathfrak{c}$ is a left resolution of $\mathbb{C}$ in $\mathcal{M}(\mathfrak{a},B)$. On the other hand,

$$\mathcal{U}(\mathfrak{c})\otimes\textstyle\bigwedge^p\mathfrak{c}=\mathcal{U}(\mathfrak{a})\otimes_{\mathcal{U}(\mathfrak{b})}\bigwedge^p\mathfrak{c}$$

for all $p\in\mathbb{Z}$. Hence, by the proof of ([6], 1.4), this is a projective resolution of $\mathbb{C}$ in $\mathcal{M}(\mathfrak{a},B)$. Therefore, we have

$$\begin{aligned}H^p(\mathfrak{a},B;U)&=\operatorname{Ext}^p_{(\mathfrak{a},B)}(\mathbb{C},U)\\&=H^p(\operatorname{Hom}_{(\mathfrak{a},B)}(\mathcal{U}(\mathfrak{a})\otimes_{\mathcal{U}(\mathfrak{b})}\textstyle\bigwedge^{-\cdot}\mathfrak{c},U))=H^p(\operatorname{Hom}_B(\bigwedge^{-\cdot}\mathfrak{c},U)).\end{aligned}$$

Assume now that the action ϕ of B on $\mathfrak{a}$ is trivial. Then

$$H^p(\mathfrak{a},B;U)=H^p(\operatorname{Hom}_{\mathbb{C}}(\textstyle\bigwedge^{-\cdot}\mathfrak{c},U^B))=H^p(\mathfrak{c},U^B)$$

for $p\in\mathbb{Z}_+$.

Let V be an $(\mathfrak{a},B)$-module. For any linear form μ on $\mathfrak{a}$, we define

$$V_\mu=\{v\in V\mid(\xi-\mu(\xi))^pv=0\text{ for any }\xi\in\mathfrak{a}\text{ and sufficiently large }p\in\mathbb{Z}_+\}.$$

Clearly, V_μ is a submodule of V. If V_μ is nonzero, we say that μ is a *weight* of V and that V_μ is a (generalized) *weight subspace* of V corresponding to μ. If V is a finite dimensional module, it is a direct sum of its weight subspaces.

4.2. Lemma. *Let U and V be finite-dimensional modules in $\mathcal{M}(\mathfrak{a}, B)$ such that $U = U_\mu$ for some $\mu \in \mathfrak{a}^*$. Then*

$$\operatorname{Ext}^p_{(\mathfrak{a},B)}(U,V) = \operatorname{Ext}^p_{(\mathfrak{a},B)}(U,V_\mu)$$

for all $p \in \mathbb{Z}_+$.

Proof. By 4.1, we know that

$$\operatorname{Ext}^p_{(\mathfrak{a},B)}(U,V) = H^p(\mathfrak{a}, B; \operatorname{Hom}_{\mathbb{C}}(U,V)).$$

Moreover, from the above formula, we conclude that

$$H^p(\mathfrak{a}, B; \operatorname{Hom}_{\mathbb{C}}(U,V)) = H^p(\mathfrak{c}, \operatorname{Hom}_B(U,V)).$$

Therefore, the result follows immediately from ([1], Exercice 7, §1, Chap. VII.). □

Assume now that $\mathfrak{g}$ is a complex semisimple Lie algebra. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Let T be an algebraic group and $\phi : T \longrightarrow \operatorname{Int}(\mathfrak{g})$ a homomorphism such that its differential is a monomorphism of the Lie algebra $\mathfrak{t}$ of T into $\mathfrak{g}$. Assume that the action of T on $\mathfrak{g}$ leaves $\mathfrak{b}$ invariant. Then the differential of ϕ maps $\mathfrak{t}$ into the normalizer of $\mathfrak{b}$, i.e., into $\mathfrak{b}$. Hence, we can define the categories $\mathcal{M}(\mathfrak{g}, T)$ and $\mathcal{M}(\mathfrak{b}, T)$. Clearly, we have a natural forgetful functor For : $\mathcal{M}(\mathfrak{g}, T) \longrightarrow \mathcal{M}(\mathfrak{b}, T)$. For U in $\mathcal{M}(\mathfrak{b}, T)$, we can define the module $\Phi(U) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} U$, where $\mathfrak{g}$ acts by left multiplication in the first factor and T by the tensor product of the action ϕ with the natural action on U. It is easy to check that such a module is in $\mathcal{M}(\mathfrak{g}, T)$. Moreover, $U \longmapsto \Phi(U)$ defines a functor from $\mathcal{M}(\mathfrak{b}, T)$ into $\mathcal{M}(\mathfrak{g}, T)$. By the Poincaré-Birkhoff-Witt theorem, this functor is exact. Also, one can check that it is right adjoint to the functor For, i.e.,

$$\operatorname{Hom}_{(\mathfrak{g},T)}(\Phi(U), V) = \operatorname{Hom}_{(\mathfrak{b},T)}(U,V)$$

for any Harish-Chandra module V in $\mathcal{M}(\mathfrak{g}, T)$ and U in $\mathcal{M}(\mathfrak{b}, T)$.

Denote by $D^b(\mathcal{M}(\mathfrak{g}, T))$ and $D^b(\mathcal{M}(\mathfrak{b}, T))$ the bounded derived categories of $\mathcal{M}(\mathfrak{g}, T)$ and $\mathcal{M}(\mathfrak{b}, T)$. Then, Φ and For induce the corresponding functors between them and we have the adjointness relation

$$\operatorname{Hom}_{D^b(\mathcal{M}(\mathfrak{g},T))}(\Phi(U^\cdot), V^\cdot) = \operatorname{Hom}_{D^b(\mathcal{M}(\mathfrak{b},T))}(U^\cdot, V^\cdot)$$

for any complex $V^\cdot$ in $D^b(\mathcal{M}(\mathfrak{g}, T))$ and $U^\cdot$ in $D^b(\mathcal{M}(\mathfrak{b}, T))$. In particular, if U is a module in $\mathcal{M}(\mathfrak{b}, T)$ and V a module in $\mathcal{M}(\mathfrak{g}, T)$, we have

$$\begin{aligned}\operatorname{Ext}^p_{(\mathfrak{g},T)}(\Phi(U), V) &= \operatorname{Hom}_{D^b(\mathcal{M}(\mathfrak{g},T))}(\Phi(D(U)), D(V)[p]) \\ &= \operatorname{Hom}_{D^b(\mathcal{M}(\mathfrak{b},T))}(D(U), D(V)[p]) = \operatorname{Ext}^p_{(\mathfrak{b},T)}(U,V)\end{aligned}$$

for any $p \in \mathbb{Z}_+$.

Therefore, we established the following result.

4.3. Lemma. *Let U be a module in $\mathcal{M}(\mathfrak{b},T)$ and V a module in $\mathcal{M}(\mathfrak{g},T)$. Then we have*

$$\operatorname{Ext}^p_{(\mathfrak{g},T)}(\Phi(U),V)=\operatorname{Ext}^p_{(\mathfrak{b},T)}(U,V)$$

for all $p\in\mathbb{Z}_+$.

Assume now that T is reductive. Then, $\phi(T)$ is a reductive subgroup of the Borel subgroup attached to $\mathfrak{b}$, i.e., it is a closed subgroup of some maximal torus H in $\operatorname{Int}(\mathfrak{g})$ which is contained in this Borel subgroup. The Lie algebra $\mathfrak{t}$ of T is contained in the Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which is a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{n}=[\mathfrak{b},\mathfrak{b}]$. Then $\mathfrak{b}$ is a semidirect product of $\mathfrak{h}$ and $\mathfrak{n}$. Let $\mathcal{M}(\mathfrak{h},T)$ be the category of Harish-Chandra modules for the pair $(\mathfrak{h},T)$. Any object U in $\mathcal{M}(\mathfrak{h},T)$ can be viewed as an object in $\mathcal{M}(\mathfrak{b},T)$ on which $\mathfrak{n}$ acts trivially. Therefore, we have an exact functor $i:\mathcal{M}(\mathfrak{h},T)\longrightarrow\mathcal{M}(\mathfrak{b},T)$. Let V be an object in $\mathcal{M}(\mathfrak{b},T)$. Then the space of $\mathfrak{n}$-invariants $V^{\mathfrak{n}}$ is clearly a submodule on which $\mathfrak{n}$ acts trivially. Hence, we can consider it as a functor $\operatorname{Inv}_{\mathfrak{n}}:V\longmapsto V^{\mathfrak{n}}$ from $\mathcal{M}(\mathfrak{b},T)$ into $\mathcal{M}(\mathfrak{h},T)$. It is easy to check that we have the adjointness relation

$$\operatorname{Hom}_{(\mathfrak{b},T)}(i(U),V)=\operatorname{Hom}_{(\mathfrak{h},T)}(U,\operatorname{Inv}_{\mathfrak{n}}(V)).$$

Let $D^b(\mathcal{M}(\mathfrak{h},T))$ be the bounded derived category of $\mathcal{M}(\mathfrak{h},T)$. Then i defines an exact functor $i:D^b(\mathcal{M}(\mathfrak{h},T))\longrightarrow D^b(\mathcal{M}(\mathfrak{b},T))$.

We claim that the right derived functors of the functor $\operatorname{Inv}_{\mathfrak{n}}$ of $\mathfrak{n}$-invariants are the Lie algebra cohomology modules, i.e., that $R^p\operatorname{Inv}_{\mathfrak{n}}(-)=H^p(\mathfrak{n},-)$, $p\in\mathbb{Z}_+$. First, let $\mathfrak{s}=\mathfrak{t}\oplus\mathfrak{n}$. Then $\mathfrak{s}$ is a solvable Lie subalgebra of $\mathfrak{b}$ stable for the action of T. Therefore, we can define the category $\mathcal{M}(\mathfrak{s},T)$. Since the forgetful functor from $\mathcal{M}(\mathfrak{b},T)$ into $\mathcal{M}(\mathfrak{s},T)$ is the right adjoint of the functor $U\longmapsto\mathcal{U}(\mathfrak{b})\otimes_{\mathcal{U}(\mathfrak{s})}U$ from $\mathcal{M}(\mathfrak{s},T)$ into $\mathcal{M}(\mathfrak{b},T)$, it maps injectives into injectives. Hence, the right derived functors of $\operatorname{Inv}_{\mathfrak{n}}$ calculated in $\mathcal{M}(\mathfrak{b},T)$ and $\mathcal{M}(\mathfrak{s},T)$ are isomorphic as representations of T.

Let V be a finite-dimensional algebraic representation of T. Then we can view it as an object in $\mathcal{M}(\mathfrak{s},T)$ with trivial action of $\mathfrak{n}$. As above, for any U in $\mathcal{M}(\mathfrak{s},T)$ we have

$$\operatorname{Hom}_{(\mathfrak{s},T)}(V,U)=\operatorname{Hom}_T(V,\operatorname{Inv}_{\mathfrak{n}}(U)).$$

Moreover, since $\operatorname{Hom}_T(V,-)$ is an exact functor, it follows that

$$\operatorname{Ext}^p_{(\mathfrak{s},T)}(V,U)=\operatorname{Hom}_T(V,R^p\operatorname{Inv}_{\mathfrak{n}}(U)),\ \text{for } p\in\mathbb{Z}_+.$$

By 4.1, we have

$$\operatorname{Ext}^p_{(\mathfrak{s},T)}(V,U)=H^p(\mathfrak{s},T;\operatorname{Hom}_{\mathbb{C}}(V,U)),$$

and finally, by ([6], p. 217), it follows that

$$\operatorname{Ext}^p_{(\mathfrak{s},T)}(V,U)=H^p(\mathfrak{n},\operatorname{Hom}_{\mathbb{C}}(V,U))^T=\operatorname{Hom}_{\mathbb{C}}(V,H^p(\mathfrak{n},U))^T=\operatorname{Hom}_T(V,H^p(\mathfrak{n},U))$$

since the action of $\mathfrak{n}$ on V is trivial. This leads to

$$\operatorname{Hom}_T(V, R^p \operatorname{Inv}_{\mathfrak{n}}(U)) = \operatorname{Hom}_T(V, H^p(\mathfrak{n}, U)), \text{ for } p \in \mathbb{Z}_+,$$

and our assertion follows, since any object in $\mathcal{M}(T)$ is a direct sum of finite dimensional irreducible representations.

Since the functor $\operatorname{Inv}_{\mathfrak{n}}$ has finite right cohomological dimension, it defines a derived functor $R\operatorname{Inv}_{\mathfrak{n}} : D^b(\mathcal{M}(\mathfrak{b}, T)) \to D^b(\mathcal{M}(\mathfrak{h}, T))$ which is the right adjoint of i, i.e., we have

$$\operatorname{Hom}_{D^b(\mathcal{M}(\mathfrak{b},T))}(i(U^\cdot), V^\cdot) = \operatorname{Hom}_{D^b(\mathcal{M}(\mathfrak{h},T))}(U^\cdot, R\operatorname{Inv}_{\mathfrak{n}}(V^\cdot))$$

for $U^\cdot$ in $D^b(\mathcal{M}(\mathfrak{h}, T))$ and $V^\cdot$ in $D^b(\mathcal{M}(\mathfrak{b}, T))$. This leads to the following spectral sequence.

4.4. Lemma. *Let U be a module in $\mathcal{M}(\mathfrak{h}, T)$ and V a module in $\mathcal{M}(\mathfrak{b}, T)$. Then we have the bounded spectral sequence*

$$\operatorname{Ext}^p_{(\mathfrak{h},T)}(U, H^q(\mathfrak{n}, V)) \Rightarrow \operatorname{Ext}^{p+q}_{(\mathfrak{b},T)}(U, V).$$

Combining the last two lemmas we get the following result.

4.5. Proposition. *Let U be a module in $\mathcal{M}(\mathfrak{h}, T)$ and V a module in $\mathcal{M}(\mathfrak{g}, T)$. Then we have the bounded spectral sequence*

$$\operatorname{Ext}^p_{(\mathfrak{h},T)}(U, H^q(\mathfrak{n}, V)) \Rightarrow \operatorname{Ext}^{p+q}_{(\mathfrak{g},T)}(\Phi(U), V).$$

We want to use this result to calculate $\operatorname{Ext}^p_{(\mathfrak{g},T)}(\Phi(U), \Phi(U))$, $p \in \mathbb{Z}_+$, for an indecomposable finite-dimensional module U in $\mathcal{M}(\mathfrak{h}, T)$. Since a finite-dimensional module U is a direct sum of weight subspaces U_μ for $\mu \in \mathfrak{h}^*$, for an indecomposable module U, we have $U = U_\mu$ for some $\mu \in \mathfrak{h}$. We start with the following observation.

4.6. Lemma. *Let U be a finite-dimensional module in $\mathcal{M}(\mathfrak{h}, T)$ such that $U = U_\mu$ for some $\mu \in \mathfrak{h}$. Then*

(i) *the $\mathfrak{n}$-homology modules $H^q(\mathfrak{n}, \Phi(U))$, considered as $\mathfrak{h}$-modules, are direct sums of finite-dimensional weight spaces;*

(ii)
$$H^q(\mathfrak{n}, \Phi(U))_\mu = \begin{cases} U & \text{for } q = 0; \\ 0 & \text{for } q > 0. \end{cases}$$

Proof. Let $\bar{\mathfrak{n}}$ be the nilpotent Lie algebra spanned by the root subspaces corresponding to the negative roots of $(\mathfrak{g}, \mathfrak{h})$. Then, $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ as an $\mathfrak{h}$-module. By the Poincaré-Birkhoff-Witt theorem, $\Phi(U)$ is isomorphic to $\mathcal{U}(\bar{\mathfrak{n}}) \otimes U$ as an $\mathfrak{h}$-module. Therefore, $\Phi(U)$ is a direct sum of finite-dimensional weight spaces as an $\mathfrak{h}$-module. Moreover $\Phi(U)_\lambda \neq 0$ implies that $\mu - \lambda$ is a sum of positive roots (or zero).

The $\mathfrak{n}$-cohomology of $\Phi(U)$ is given as the cohomology of the complex $\mathrm{Hom}(\bigwedge^{\cdot}\mathfrak{n}, \Phi(U))$. With the natural action of $\mathfrak{h}$, this complex is a complex of modules with finite-dimensional weight spaces. Therefore, the cohomology modules have the same property. This establishes (i).

To prove (ii) we just need to remark that the weights of $\mathrm{Hom}(\bigwedge^p \mathfrak{n}, \Phi(U))$ are the differences of a weight of $\Phi(U)$ and a sum of p different positive roots. Therefore, μ appears as a weight only in $\mathrm{Hom}(\bigwedge^0 \mathfrak{n}, \Phi(U)) = \Phi(U)$. Therefore, μ can be a weight of $H^0(\mathfrak{n}, \Phi(U))$ only. On the other hand, $\Phi(U)_\mu \cong U$ by the above remark, and it clearly consists of $\mathfrak{n}$-invariants. □

Now we can summarize these results.

4.7. Theorem. *Let U be an indecomposable finite-dimensional module in $\mathcal{M}(\mathfrak{h}, T)$. Then*

$$\mathrm{Ext}^p_{(\mathfrak{g},T)}(\Phi(U), \Phi(U)) = \mathrm{Ext}^p_{(\mathfrak{h},T)}(U, U).$$

Proof. Assume that $U = U_\mu$ for some $\mu \in \mathfrak{h}^*$. First, by 4.5, we have

$$\mathrm{Ext}^p_{(\mathfrak{h},T)}(U, H^q(\mathfrak{n}, \Phi(U))) \Rightarrow \mathrm{Ext}^{p+q}_{(\mathfrak{g},T)}(\Phi(U), \Phi(U)).$$

Then, by 4.2 and 4.6.(i), it follows that

$$\mathrm{Ext}^p_{(\mathfrak{h},T)}(U, H^q(\mathfrak{n}, \Phi(U))) = \mathrm{Ext}^p_{(\mathfrak{h},T)}(U, H^q(\mathfrak{n}, \Phi(U))_\mu)$$

for all $q \in \mathbb{Z}_+$. Finally, by 4.6.(ii), we see that

$$\mathrm{Ext}^p_{(\mathfrak{h},T)}(U, H^q(\mathfrak{n}, \Phi(U))) = \begin{cases} \mathrm{Ext}^p_{(\mathfrak{h},T)}(U, U) & \text{for } q = 0; \\ 0 & \text{for } q > 0; \end{cases}$$

for all $p \in \mathbb{Z}_+$. Therefore, the above spectral sequence collapses. □

5. Zuckerman functors for tori. As in the last section, we assume that $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{b}$ a Borel subalgebra of $\mathfrak{g}$. Assume that T is a complex torus which acts on $\mathfrak{g}$ by automorphisms which leave $\mathfrak{b}$ invariant and that the differential of this action is an injection of the Lie algebra $\mathfrak{t}$ of T into $\mathfrak{g}$. In this case, $\mathfrak{t}$ is a Lie subalgebra of $\mathfrak{b}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ which contains $\mathfrak{t}$ and which is contained in $\mathfrak{b}$. For any finite dimensional module U in $\mathcal{M}(\mathfrak{h}, T)$, we can define the module $\Phi(U)$ in $\mathcal{M}(\mathfrak{g}, T)$. The main result of this section is the following theorem.

5.1. Theorem. *For any finite dimensional module U in $\mathcal{M}(\mathfrak{h}, T)$, we have*

$$R\Gamma_T(D(\Phi(U))) = \bigoplus_{p=0}^{\dim T} D\left(\Phi(U) \otimes \textstyle\bigwedge^p \mathfrak{t}^*\right)[-p]$$

in $D(\mathcal{M}(\mathfrak{g}, T))$.

If we take for U a one dimensional representation of $(\mathfrak{h}, T)$, we get Theorem B from the introduction.

Clearly, to prove 5.1, we can assume that U is an indecomposable module in $\mathcal{M}(\mathfrak{g}, T)$. Hence, in the following we assume that $U = U_\mu$ for some $\mu \in \mathfrak{h}^*$.

Assume that S is a torus of codimension one in T. Then,

$$H^p(\mathfrak{t}, S; \mathbb{C}) = \bigwedge^p (\mathfrak{t}/\mathfrak{s})^*$$

for any $p \in \mathbb{Z}$, which is $\mathbb{C}$ for $p = 0, 1$ and 0 otherwise. Therefore, by 1.1, we have

$$R^p\Gamma_{T,S}(\Phi(U)) = \begin{cases} 0, & p \neq 0, 1 \\ \Phi(U) & p = 0, 1. \end{cases}$$

By 3.4, we know that

$$\operatorname{End}_{D^b(\mathcal{M}(\mathfrak{g},T))}(R\Gamma_{T,S}(D(\Phi(U)))) = \operatorname{End}_{\mathfrak{g}}(\Phi(U)) \oplus \operatorname{Ext}^1_{(\mathfrak{g},S)}(\Phi(U), \Phi(U)),$$

since $\operatorname{End}_{(\mathfrak{g},S)}(\Phi(U)) = \operatorname{End}_{\mathfrak{g}}(\Phi(U))$. Therefore, we have the following formula for the dimension of the endomorphism algebra of $R\Gamma_{T,S}(D(\Phi(U)))$.

5.2. Lemma.

$$\dim \operatorname{End}_{D^b(\mathcal{M}(\mathfrak{g},T))}(R\Gamma_{T,S}(D(\Phi(U)))) = \dim \operatorname{End}_{\mathfrak{g}}(\Phi(U)) + \dim \operatorname{Ext}^1_{(\mathfrak{g},S)}(\Phi(U), \Phi(U)).$$

By 4.7, we know that $\operatorname{Ext}^p_{(\mathfrak{g},T)}(\Phi(U), \Phi(U))$, $p \in \mathbb{Z}_+$, are finite dimensional linear spaces. Hence, the main results of §2 apply to the complex $R\Gamma_{T,S}(D(\Phi(U)))$ in $D^b(\mathcal{M}(\mathfrak{g}, T))$. In particular, the splitting criterion from 2.5, combined with the following lemma and 5.2, implies that

$$R\Gamma_{T,S}(D(\Phi(U))) \cong D(\Phi(U)) \oplus D(\Phi(U))[-1].$$

5.3. Lemma. *We have*

$$\dim \operatorname{Ext}^q_{(\mathfrak{g},S)}(\Phi(U), \Phi(U)) = \dim \operatorname{Ext}^q_{(\mathfrak{g},T)}(\Phi(U), \Phi(U)) + \dim \operatorname{Ext}^{q-1}_{(\mathfrak{g},T)}(\Phi(U), \Phi(U))$$

for $q \in \mathbb{Z}_+$.

Proof. By 4.7, it is enough to prove that

$$\dim \operatorname{Ext}^q_{(\mathfrak{h},S)}(U, U) = \dim \operatorname{Ext}^q_{(\mathfrak{h},T)}(U, U) + \dim \operatorname{Ext}^{q-1}_{(\mathfrak{h},T)}(U, U)$$

for $q \in \mathbb{Z}_+$. In addition, from the proof of 4.2, we know that

$$\operatorname{Ext}^p_{(\mathfrak{h},T)}(U, U) = H^p(\mathfrak{h}/\mathfrak{t}, \operatorname{End}_{\mathfrak{t}}(U)) \text{ and } \operatorname{Ext}^p_{(\mathfrak{h},S)}(U, U) = H^p(\mathfrak{h}/\mathfrak{s}, \operatorname{End}_{\mathfrak{s}}(U))$$

for $p \in \mathbb{Z}_+$. Clearly, T acts algebraically on $\operatorname{End}_{\mathfrak{s}}(U)$, hence $\operatorname{End}_{\mathfrak{s}}(U)$ is a semisimple $\mathfrak{t}/\mathfrak{s}$-module. Any isotypic component of the $\mathfrak{t}/\mathfrak{s}$-module $\operatorname{End}_{\mathfrak{s}}(U)$ is $\mathfrak{h}/\mathfrak{s}$-invariant. By 4.2, it follows that

$$\operatorname{Ext}^p_{(\mathfrak{h},S)}(U,U) = H^p(\mathfrak{h}/\mathfrak{s}, \operatorname{End}_{\mathfrak{s}}(U)) = H^p(\mathfrak{h}/\mathfrak{s}, H^0(\mathfrak{t}/\mathfrak{s}, \operatorname{End}_{\mathfrak{s}}(U))) = H^p(\mathfrak{h}/\mathfrak{s}, \operatorname{End}_{\mathfrak{t}}(U)).$$

Let $\mathfrak{c} = \mathfrak{h}/\mathfrak{s}$. Denote by $\mathfrak{d}$ the image of $\mathfrak{t}$ in $\mathfrak{c}$. Then $\mathfrak{d}$ is one dimensional. Let $\mathfrak{e}$ be a complement of $\mathfrak{d}$ in $\mathfrak{c}$. Then, by Hochschild-Serre spectral sequence for Lie algebra cohomology, we have

$$H^p(\mathfrak{c}/\mathfrak{e}, H^q(\mathfrak{e}, \operatorname{End}_{\mathfrak{t}}(U))) \Rightarrow H^{p+q}(\mathfrak{c}, \operatorname{End}_{\mathfrak{t}}(U)) = H^{p+q}(\mathfrak{h}/\mathfrak{s}, \operatorname{End}_{\mathfrak{t}}(U)) = \operatorname{Ext}^{p+q}_{(\mathfrak{h},S)}(U,U).$$

Moreover, we have

$$H^q(\mathfrak{e}, \operatorname{End}_{\mathfrak{t}}(U)) = H^q(\mathfrak{h}/\mathfrak{t}, \operatorname{End}_{\mathfrak{t}}(U)) = \operatorname{Ext}^q_{(\mathfrak{h},T)}(U,U).$$

Since $\mathfrak{d}$ acts trivially on $\operatorname{Ext}^q_{(\mathfrak{h},T)}(U,U)$, it follows that

$$H^p(\mathfrak{c}/\mathfrak{e}, H^q(\mathfrak{e}, \operatorname{End}_{\mathfrak{t}}(U))) = H^p(\mathfrak{d}, \operatorname{Ext}^q_{(\mathfrak{h},T)}(U,U)) = H^p(\mathfrak{d}, \mathbb{C}) \otimes \operatorname{Ext}^q_{(\mathfrak{h},T)}(U,U).$$

The algebra $\mathfrak{d}$ is one dimensional, hence $H^p(\mathfrak{d},\mathbb{C}) = 0$ for $p \neq 0,1$ and $H^0(\mathfrak{d},\mathbb{C}) = H^1(\mathfrak{d},\mathbb{C}) = \mathbb{C}$. Therefore, since the differential of the E^2-term has bidegree $(2,-1)$, we see that this spectral sequence degenerates. In particular, for any $q \in \mathbb{Z}_+$, we have

$$\dim \operatorname{Ext}^q_{(\mathfrak{h},T)}(U,U) + \dim \operatorname{Ext}^{q-1}_{(\mathfrak{h},T)}(U,U) = \dim \operatorname{Ext}^q_{(\mathfrak{h},S)}(U,U). \quad \square$$

This leads to the following result.

5.4. Lemma. *Let T be a torus and S its subtorus such that $\mathfrak{h} \supset \mathfrak{t} \supset \mathfrak{s}$. Then we have*

$$R\Gamma_{T,S}(D(\Phi(U))) \cong \bigoplus_{p=0}^{\dim(T/S)} D\left(\Phi(U) \otimes \textstyle\bigwedge^p(\mathfrak{t}/\mathfrak{s})^*\right)[-p]$$

in $D(\mathcal{M}(\mathfrak{g},T))$.

Proof. The proof is by induction on the codimension of S in T. We already established the claim if it is equal to 1.

In general, let T' be a subtorus of T such that $T' \supset S$ and $\dim T' = \dim S + 1$. Then, by the induction assumption, the assertion holds for $R\Gamma_{T,T'}$ and $R\Gamma_{T',S}$. Therefore, we have

$$\begin{aligned}
&R\Gamma_{T,S}(D(\Phi(U))) = R\Gamma_{T,T'}(R\Gamma_{T',S}(D(\Phi(U))) \\
&\quad = R\Gamma_{T,T'}(D(\Phi(U)) \oplus D(\Phi(U))[-1]) = R\Gamma_{T,T'}(D(\Phi(U))) \oplus R\Gamma_{T,T'}(D(\Phi(U)))[-1] \\
&\quad = \bigoplus_{p=0}^{\dim(T/T')} D\left(\Phi(U) \otimes \textstyle\bigwedge^p(\mathfrak{t}/\mathfrak{t}')^*\right)[-p] \oplus \bigoplus_{p=0}^{\dim(T/T')} D\left(\Phi(U) \otimes \textstyle\bigwedge^p(\mathfrak{t}/\mathfrak{t}')^*\right)[-p-1] \\
&\quad = \bigoplus_{p=0}^{\dim(T/T')} D\left(\Phi(U) \otimes \textstyle\bigwedge^p(\mathfrak{t}/\mathfrak{t}')^*\right)[-p] \oplus \bigoplus_{p=1}^{\dim(T/T')+1} D\left(\Phi(U) \otimes \textstyle\bigwedge^{p-1}(\mathfrak{t}/\mathfrak{t}')^*\right)[-p] \\
&\quad = \bigoplus_{p=0}^{\dim(T/S)} D\left(\Phi(U) \otimes \textstyle\bigwedge^p(\mathfrak{t}/\mathfrak{s})^*\right)[-p]. \quad \square
\end{aligned}$$

In particular, if S is trivial, we get 5.1.

Consider now an arbitrary algebraic group K and a torus $T \subset K$ such that $\mathfrak{h} \supset \mathfrak{t}$. Then, we have

$$\begin{aligned}
&R\Gamma_K(D(\Phi(U))) = R\Gamma_{K,T}(R\Gamma_T(D(\Phi(U)))) \\
&\quad = R\Gamma_{K,T}\left(\bigoplus_{p=0}^{\dim T} D\left(\Phi(U) \otimes \textstyle\bigwedge^p \mathfrak{t}^*\right)[-p]\right) = \bigoplus_{p=0}^{\dim T} \left(R\Gamma_{K,T}(\Phi(U)) \otimes \textstyle\bigwedge^p \mathfrak{t}^*\right)[-p].
\end{aligned}$$

5.5. Corollary. *Let K be an algebraic group and $T \subset K$ a torus such that $\mathfrak{h} \supset \mathfrak{t}$. Then we have*

$$R\Gamma_K(D(\Phi(U))) = \bigoplus_{p=0}^{\dim T} \left(R\Gamma_{K,T}(\Phi(U)) \otimes \textstyle\bigwedge^p \mathfrak{t}^*\right)[-p]$$

in $D(\mathcal{M}(\mathfrak{g}, K))$.

For a one dimensional module U, by taking the cohomology in this formula we get Theorem A from the introduction.

5.6. Example. Finally, we give an example that 5.1 is not necessary for the degeneration of the spectral sequence in Theorem A.

Let $G = \mathrm{Int}(\mathfrak{g})$ and let V be an irreducible finite-dimensional representation of G. Then V determines an object in $\mathcal{M}(\mathfrak{g}, T)$, for any one-dimensional torus T in G.

Clearly, $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}$ and $\mathrm{Ext}^1_{\mathfrak{g}}(V, V) = \mathrm{Ext}^1_{(\mathfrak{g},T)}(V, V) = 0$ by the Weyl's theorem. By the result of Duflo and Vergne, $\Gamma_T(V) = V$ and $R^1\Gamma_T(V) = V$. Therefore, by 3.4, we have

$$\mathrm{End}_{D(\mathcal{M}(\mathfrak{g},T))}(R\Gamma_T(D(V))) = \mathrm{End}_{\mathfrak{g}}(V) \oplus \mathrm{Ext}^1_{\mathfrak{g}}(V, V) = \mathbb{C}.$$

On the other hand,

$$\mathrm{End}_{D(\mathcal{M}(\mathfrak{g},T))}(D(V) \oplus D(V)[-1]) = \mathrm{End}_{\mathfrak{g}}(V) \oplus \mathrm{End}_{\mathfrak{g}}(V) \oplus \mathrm{Ext}^1_{(\mathfrak{g},T)}(V, V) = \mathbb{C}^2,$$

hence, $R\Gamma_T(D(V)) \neq D(V) \oplus D(V)[-1]$ in $D(\mathcal{M}(\mathfrak{g}, T))$.

Let $K \supset T$ be another torus in G. Although the decomposition of $R\Gamma_T(D(V))$ fails for the above V, the spectral sequence corresponding to $R\Gamma_K(D(V)) = R\Gamma_{K,T}(R\Gamma_T(D(V)))$ still degenerates; actually, Theorem A from the introduction holds for V. Namely, using 1.1 one can calculate both sides of the equality in Theorem A for V, and see that they are the same.

References

1. N. Bourbaki, *Groupes et algèbres de Lie*, Hermann, Paris.
2. P. Deligne, *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Publ. Math. IHES **35** (1968), 259–278.
3. M. Duflo, M. Vergne, *Sur le functeur de Zuckerman*, C. R. Acad. Sci. Paris **304** (1987), 467–469.
4. S. I. Gelfand, Yu. I. Manin, *Methods of homological algebra*, Springer-Verlag, Berlin, Heidelberg, New York, 1996.
5. H. Hecht, D. Miličić, W. Schmid, J. A. Wolf, *Localization and standard modules for real semisimple Lie groups I: The duality theorem*, Inventiones Math. **90** (1987), 297–332.
6. D. Miličić, P. Pandžić, *Equivariant derived categories, Zuckerman functors and localization*, Geometry and representation theory of real and p-adic Lie groups (J. Tirao, D. Vogan, J.A. Wolf, eds.), Progress in Mathematics, vol. 158, Birkhäuser, Boston, pp. 209–242.
7. D. Miličić, P. Pandžić, *Cohomology of standard Harish-Chandra sheaves*, (in preparation).

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