First-passage times under frequent stochastic resetting

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We determine the full distribution and moments of the first passage time for a wide class of stochastic search processes in the limit of frequent stochastic resetting. Our results apply to any system whose short-time behavior of the search process without resetting can be estimated. In addition to the typical case of exponentially distributed resetting times, we prove our results for a variety of resetting time distributions. We illustrate our results in several examples and show that the errors of our approximations vanish exponentially fast in typical scenarios of diffusive search.

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I. INTRODUCTION

For many decades there has been sustained interest in understanding first passage times (FPTs), which characterize the time it takes a searcher to find a target [1–6]. Search processes of interest include pure diffusion [7,8], anomalous diffusion [9,10], random walks on discrete networks [11,12], run-and-tumble particles [13,14], inactivating searchers [15,16], and so on.

More recently, there has been a strong interest in FPTs of searchers under stochastic resetting, which means that the searcher is reset to elsewhere in the state space at random times [17–21]. Biological approximations of such behavior vary widely in spatial and temporal scale, for example, RNA cleavage during transcription [22] and predator dynamics during foraging [23]. In theoretical treatments of these systems, stochastic resetting can reduce the expected search time [24]. For instance, consider a diffusing searcher on the half line with an absorbing boundary condition at the origin. It is well known that the mean FPT to the origin is infinite. However, if the searcher stochastically resets its position, then the mean FPT is finite [18]. This result can be heuristically explained by the fact that the searcher resets before wandering too far from the origin [see Fig. 1(a)].

Here we study the full distribution and moments of FPTs for a wide class of stochastic search processes in the limit of frequent stochastic resetting. To briefly summarize our results, let T denote the FPT of a stochastic searcher that resets to its initial position (or distribution of initial positions) at random times. We are most interested in the case that the resetting occurs at exponentially distributed times with rate $r > 0$ (i.e., Poissonian resetting), but we prove our results for much more general resetting time distributions. If $p = p(r) \in (0, 1)$ denotes the probability that the searcher finds a target before resetting, then under very mild assumptions on the search process we prove that rp_0T converges in distribution to an exponential random variable with unit rate, which we denote by

$$rp_0T \to_{\text{dist}} \text{Exponential}(1) \quad \text{as } r \to \infty, \tag{1}$$

where $p_0 = p_0(r)$ is any function of r satisfying $p_0 \sim p$ as $r \to \infty$. Throughout this work, $f \sim g$ denotes $f/g \to 1$. Roughly speaking, (1) says that T is approximately exponentially distributed with rate rp_0,

$$\mathbb{P}(T > t) \approx e^{-rp_0t}, \quad \text{if } t \geqslant 0 \text{ and } p_0 \ll 1.$$

In addition to the full distribution in (1), we determine the behavior of all the moments of T,

$$\mathbb{E}[T^m] \sim \frac{m!}{(rp_0)^m} \quad \text{as } r \to \infty, \tag{2}$$

for integers $m \geqslant 1$. To make these results readily applicable, we determine appropriate choices of p_0 based on the short-time distribution of the search process without resetting.

The results in (1) and (2) show that many stochastic search processes with resetting behave similarly once we scale the search time by the resetting rate and the probability of a successful search. This is illustrated in Fig. 1(b), which displays results from numerical solutions of quite disparate search processes that are nevertheless all approximately exponentially distributed with rate rp_0 (the details of these and other examples are given in Sec. III). Further, in typical scenarios of interest for diffusive search, we find that the asymptotic estimates converge exponentially fast. Establishing these results requires knowledge only of the short-time behavior of the search process without resetting. With this information, we determine the asymptotic behavior of p (i.e., we determine p_0) and thus the limiting distribution of the FPT.

The rest of the paper is organized as follows. In Sec. II we prove (1) and (2), and we determine the asymptotic behavior of p under various assumptions on the short-time behavior of the search process without resetting. In Sec. III we apply these results to several scenarios including diffusive and subdiffusive search in one or three spatial dimensions, a random walk on a discrete network, and a run-and-tumble particle. We consider these examples in Secs. III A–III E for

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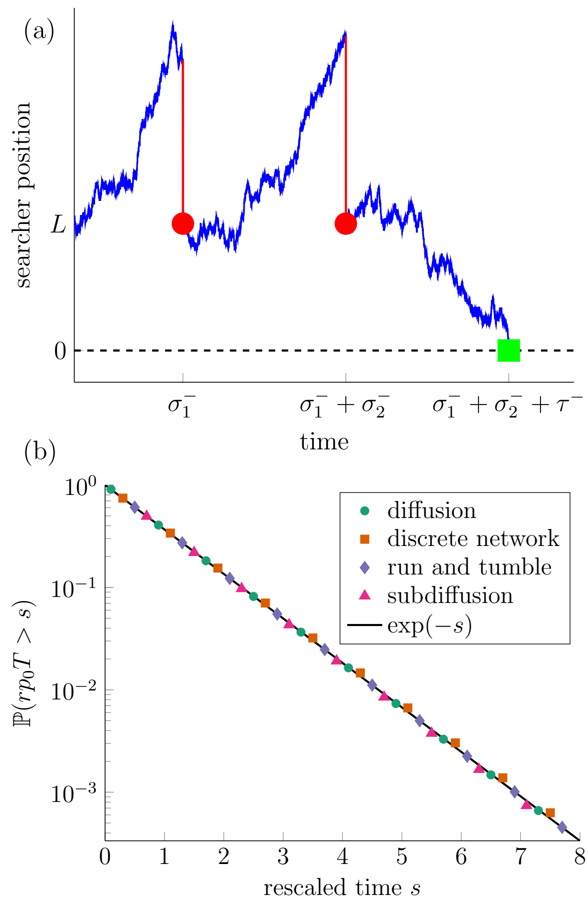

FIG. 1. (a) Diffusive search under stochastic resetting to $x = L > 0$ in one spatial dimension. Red dots indicate resetting times, and the green square indicates the FPT to $x = 0$. The time axis labels correspond to the decomposition of the FPT in (12). (b) FPTs of disparate search processes behave similarly under frequent exponential stochastic resetting. The numerical details of the examples plotted here are described in Sec. III F.

the case of exponential resetting, and we consider diffusive search with sharp, uniform, and gamma resetting distributions in Sec. III G. We conclude with a brief discussion. The Appendix contains proofs and technical points.

II. PROBABILISTIC SETUP AND MAIN RESULTS

In this section we present results on the FPT of a searcher under frequent stochastic resetting. While these results make no explicit reference to the underlying search process (e.g., diffusion or otherwise), we later apply them to a diffusive search process and other stochastic processes.

A. FPTs under frequent stochastic resetting

A stochastic resetting search process can be built from only two ingredients: (1) the search time τ in the absence of resetting and (2) the resetting time σ. We now make assumptions on τ and σ.

Assume τ is any random variable whose cumulative distribution function, $F_\tau(t) := \mathbb{P}(\tau \leqslant t)$, satisfies

$$F_\tau(0) = 0, \tag{3}$$

$$F_\tau(t) > 0 \quad \text{for some } t \in (0, \infty). \tag{4}$$

In words, (3) says that τ is strictly positive and (4) merely excludes the trivial case that τ is always infinite.

Assume σ has mean $\mathbb{E}[\sigma] = 1/r$, where we refer to $r > 0$ as the resetting rate. To construct σ precisely, let $Y > 0$ be any strictly positive random variable that does not depend on r and has unit mean

$$\mathbb{E}[Y] = 1, \tag{5}$$

and a finite moment-generating function in a neighborhood of the origin. That is, assume that there exists a $\delta > 0$ so that

$$\mathbb{E}[e^{zY}] < \infty \quad \text{for all } z \in [-\delta, \delta]. \tag{6}$$

We then define σ as

$$\sigma := Y/r. \tag{7}$$

The probability of the search process ending prior to a resetting event is (i.e., a "successful" search)

$$p = p(r) := \mathbb{P}(\tau < \sigma) = \int_0^\infty S_\sigma(t)\, dF_\tau(t), \tag{8}$$

where $S_\sigma(t) = \mathbb{P}(\sigma > t)$ is the survival probability of σ. To exclude trivial cases, we assume

$$p > 0 \quad \text{for all } r > 0. \tag{9}$$

Most prior studies of stochastic resetting consider exponential (i.e., Poissonian) resetting, which in this framework means that Y is exponential with unit mean, and thus has survival probability

$$S_Y(y) := \mathbb{P}(Y > y) = e^{-y} \quad \text{for } y \geqslant 0. \tag{10}$$

For such exponential resetting, $S_\sigma(t) = e^{-rt}$, p in (8) is the Laplace-Stieltjes transform of $F_\tau(t)$, and (4) ensures (9) is satisfied. Note that if $F_\tau(t)$ is differentiable, then the Laplace-Stieltjes transform of $F_\tau(t)$ is the Laplace transform of $\frac{d}{dt}F_\tau(t)$ [25]. However, the framework in (5)–(7) includes much more general resetting distributions. For example, "sharp reset" [26] in which resetting occurs at a deterministic time $\sigma = 1/r$ fits into this framework by setting

$$S_Y(y) = \begin{cases} 1 & \text{if } y < 1, \\ 0 & \text{if } y \geqslant 1. \end{cases}$$

Many other choices of the resetting time (such as uniform reset and gamma-distributed reset considered in [26]) also fit into this framework (see Sec. III G for examples).

Let $R \in \{0, 1, \dots\}$ denote the number of resets before the searcher finds the target, or the number of "unsuccessful" searches. From (8), we infer that R is a geometric random variable with probability of success $p \in (0, 1)$; that is,

$$\mathbb{P}(R = n) = (1 - p)^n p, \quad \text{for } n \in \{0, 1, \dots\}. \tag{11}$$

The random variable that describes the total search time with resetting, $T > 0$, is thus given by

$$T := \sum_{n=1}^{R} \sigma_n^- + \tau^-, \tag{12}$$

where $\{\sigma_n^-\}_{n\geqslant 1}$ is an independent and identically distributed (iid) sequence of random variables with common survival probability,

$$S_{\sigma^-}(t) := \mathbb{P}(\sigma^- > t) = \mathbb{P}(\sigma > t|\sigma < \tau). \tag{13}$$

Further, τ^- is a random variable defined by survival probability,

$$S_{\tau^-}(t) := \mathbb{P}(\tau^- > t) = \mathbb{P}(\tau > t|\tau < \sigma).$$

In words, the distribution of σ^- is the distribution of σ conditioned on $\sigma < \tau$. Similarly, the distribution of τ^- is the distribution of τ conditioned on $\tau < \sigma$. Hence, the definition of T in (12) is the sum of the unsuccessful search times $(\sigma_1^- + \cdots + \sigma_R^-)$ plus the single successful search time (τ^-). See Fig. 1(a) for an illustration. We emphasize that R, $\{\sigma_n^-\}_{n\geqslant 1}$, and τ^- are independent.

Now that we have defined all pertinent terms, we present the main result and sketch its proof.

Theorem 1. Under the assumptions of Sec. II A, let $p_0 = p_0(r)$ be any function of r satisfying

$$p_0 \sim p \quad \text{as } r \to \infty. \tag{14}$$

Then rp_0T converges in distribution to an exponential random variable with unit rate,

$$rp_0T \to_{\text{dist}} \text{Exponential}(1) \quad \text{as } r \to \infty. \tag{15}$$

Further, for integers $m \geqslant 1$,

$$\mathbb{E}[T^m] \sim \frac{m!}{(rp_0)^m} \quad \text{as } r \to \infty. \tag{16}$$

We now sketch the proof of Theorem 1. The full proof of Theorem 1 and the proofs of the other results in this section are in the Appendix. A short formal calculation that yields (15) in the case of exponential resetting is also given in the Appendix.

Sketch of proof of Theorem 1. In the frequent resetting limit ($r \to \infty$), the searcher resets many times before finally reaching the target. That is, the probability that any given iteration of the search process is "successful" is small,

$$p = \mathbb{P}(\tau < \sigma) \ll 1 \quad \text{for large } r, \tag{17}$$

and therefore we expect many resets (i.e., $R \gg 1$). In addition, the iteration of the search process that finally reaches the target before resetting, denoted by τ^- in (12), must be fast since it is conditioned to be less than a fast resetting time. Hence, τ^- is negligible compared to the sum of $R \gg 1$ realizations of σ^- in (12).

Further, since we have $\sigma < \tau$ with high probability for large r [see (17)], the condition imposed on the survival probability in (13) becomes inconsequential for frequent resetting, and thus $\sigma^- \approx \sigma$. Therefore, (12) reduces to

$$T \approx \sum_{n=1}^{R} \sigma_n = \frac{1}{r}\sum_{n=1}^{R} Y_n \quad \text{for large } r, \tag{18}$$

where $\{Y_n\}_{n\geqslant 1}$ are iid realizations of Y in (5)–(7).

To turn (18) into a statement about moment-generating functions, we multiply (18) by zrp_0, exponentiate, take expectation, and sum over the possible values of R in (11) to obtain that for large r,

$$\begin{aligned}\mathbb{E}[e^{zrp_0T}] &\approx \mathbb{E}[e^{zp_0\sum_{n=1}^{R} Y_n}]\\ &= p\sum_{n=0}^{\infty}(1-p)^n(\mathbb{E}[e^{zp_0Y}])^n\\ &= \frac{p}{1-(1-p)\mathbb{E}[e^{zp_0Y}]}.\end{aligned} \tag{19}$$

In light of (14) and (17), we Taylor expand the exponential function in (19) about the origin and use the assumption in (5) that Y has unit mean to finally obtain

$$\mathbb{E}[e^{zrp_0T}] \approx \frac{1}{1-z} \quad \text{for large } r.$$

The convergence in distribution in (15) follows from noting that $1/(1-z)$ is the moment-generating function of a unit rate exponential random variable. The moment formula in (16) is then natural since the mth moment of an exponential random variable with rate β is $m!/\beta^m$.

B. Asymptotics of the probability p of a successful search under exponential resetting

Applying Theorem 1 to a given system requires knowledge of p. In this section we assume exponential resetting [i.e., $S_\sigma(t) = e^{-rt}$] and consider various assumptions on the cumulative distribution function F_τ. We then determine the resulting asymptotics of the probability p of a successful search, which then yields the distribution and moments of the FPT T via Theorem 1.

1. Diffusion

We first consider the typical short-time behavior of F_τ for a diffusive search when the searcher cannot start arbitrarily close to the target. In this case, $F_\tau(t)$ decays exponentially as $t \to 0^+$ (see Sec. III A for some specific examples or Ref. [27] for a general proof). We begin with a result when we merely know the short-time behavior of F_τ on a logarithmic scale.

Theorem 2. For exponential resetting as in (10), if

$$\lim_{t\to 0^+} t \ln F_\tau(t) = -C < 0, \tag{20}$$

then p in (8) and T in (12) satisfy

$$\begin{aligned}\ln p &\sim -\sqrt{4Cr} \quad \text{as } r \to \infty,\\ \ln(r^m\mathbb{E}[T^m]) &\sim m\sqrt{4Cr} \quad \text{as } r \to \infty.\end{aligned}$$

In applications of interest, the constant C in (20) is a characteristic timescale of diffusive search. Typically, C is given by

$$C = \frac{L^2}{4D} > 0, \tag{21}$$

where $L > 0$ is the shortest distance (in an appropriate distance metric) the searcher must travel to reach the target, and $D > 0$ is a characteristic diffusion coefficient [27].

The next result yields stronger conclusions about the asymptotics of p and T by assuming more detailed information about the short-time behavior of F_τ.

Theorem 3. For exponential resetting as in (10), if

$$F_\tau(t) \sim At^b e^{-C/t} \quad \text{as } t \to 0^+, \tag{22}$$

for $A > 0$, $C > 0$, and $b \in \mathbb{R}$, then p in (8) and T in (12) satisfy

$$p \sim p_0 := A\sqrt{\pi C^{\frac{2b+1}{2}}}\, r^{\frac{1-2b}{4}} \exp(-\sqrt{4Cr}) \quad \text{as } r \to \infty,$$

$$\mathbb{E}[T^m] \sim m!\left[\frac{1}{A\sqrt{\pi C^{\frac{2b+1}{2}}}} r^{\frac{2b-5}{4}} \exp(\sqrt{4Cr})\right]^m \quad \text{as } r \to \infty. \tag{23}$$

In addition to the timescale C in (21), the constants A and b in (22) encode more details about the diffusive search process [28]. Examples of these constants in specific scenarios are given in Sec. III A.

2. Subdiffusion

We now consider the case of subdiffusive search, where subdiffusion is modeled by a fractional Fokker-Planck equation [29]. The results are analogous to the case of diffusion above, except the formulas are more complicated because the short-time behavior of F_τ depends on the subdiffusive exponent. We apply the following two theorems to specific examples in Sec. III E.

Theorem 4. For exponential resetting as in (10), if

$$\lim_{t\to 0^+} t^d \ln F_\tau(t) = -C < 0, \tag{24}$$

for $d > 0$, then p in (8) and T in (12) satisfy

$$\ln p \sim -\gamma r^{d/(d+1)} \quad \text{as } r \to \infty, \tag{25}$$

$$\ln(r^m \mathbb{E}[T^m]) \sim m\gamma r^{d/(d+1)} \quad \text{as } r \to \infty, \tag{26}$$

where $\gamma = \frac{d+1}{d^{d/(d+1)}} C^{\frac{1}{d+1}} > 0$.

For a subdiffusive exponent $\alpha \in (0, 1)$, meaning that the mean-squared displacement of the search process without resetting grows sublinearly in time t according to the power law t^α, we typically have that [30]

$$d = \frac{\alpha}{2-\alpha}, \quad C = (2-\alpha)\alpha^{\alpha/(2-\alpha)}\left(\frac{L^2}{4K_\alpha}\right)^{1/(2-\alpha)}, \tag{27}$$

where the length scale $L > 0$ is as in (21) and K_α is the characteristic subdiffusion coefficient [with dimensions $(\text{length})^2(\text{time})^{-\alpha}$]. Hence, the quantity $\gamma r^{d/(d+1)}$ appearing in (25) and (26) is typically given by

$$\gamma r^{d/(d+1)} = \sqrt{r^\alpha L^2/K_\alpha}.$$

Theorem 5. For exponential resetting as in (10), if

$$F_\tau(t) \sim At^b e^{-C/t^d} \quad \text{as } t \to 0^+, \tag{28}$$

for $A > 0$, $C > 0$, $d > 0$, and $b \in \mathbb{R}$, then p in (8) and T in (12) satisfy

$$p \sim p_0 := \mu r^\beta \exp(-\gamma r^{d/(d+1)}) \quad \text{as } r \to \infty, \tag{29}$$

$$\mathbb{E}[T^m] \sim m!\left[\frac{1}{\mu} r^{-\beta-1} \exp(\gamma r^{d/(d+1)})\right]^m \quad \text{as } r \to \infty, \tag{30}$$

where $\gamma = \frac{d+1}{d^{d/(d+1)}} C^{\frac{1}{d+1}} > 0$ and

$$\mu = \sqrt{\frac{2\pi A^2 (Cd)^{\frac{2b+1}{d+1}}}{d+1}}, \quad \beta = \frac{d-2b}{2d+2}.$$

If $d = 1$ in (24) and (28) (and thus $\alpha = 1$ in (27)), Theorem 4 reduces to Theorem 2 and Theorem 5 reduces to Theorem 3.

3. Other search processes

We now consider the case that $F_\tau(t)$ decays according to a power law as $t \to 0^+$. This can describe the case of (1) diffusive search in which the searcher can start arbitrarily close to the target (see Sec. III B), (2) search by a continuous-time Markov chain on a discrete state space (see Sec. III C), and (3) superdiffusive search (see Ref. [31]).

Proposition 6. For exponential resetting as in (10), if

$$F_\tau(t) \sim At^b \quad \text{as } t \to 0^+, \tag{31}$$

for $A > 0$ and $b > 0$, then p in (8) and T in (12) satisfy

$$p \sim p_0 := \Gamma(b+1)Ar^{-b} \quad \text{as } r \to \infty, \tag{32}$$

$$\mathbb{E}[T^m] \sim m!\left[\frac{1}{A\Gamma(b+1)} r^{b-1}\right]^m \quad \text{as } r \to \infty, \tag{33}$$

where $\Gamma(\beta) := \int_0^\infty z^{\beta-1} e^{-z}\, dz$ denotes the gamma function.

The asymptotics of p in (32) follow from noticing that p is the Laplace-Stieltjes transform of $F_\tau(t)$ and applying a Tauberian theorem (see, for example, Theorem 3 in Sec. 5 of chapter 8 of [25]). The asymptotics of the moments of T in (33) then follow from Theorem 1.

III. EXAMPLES AND NUMERICAL SOLUTIONS

The results in Sec. II give the asymptotics of the FPT T as the resetting rate r increases. In this section we apply these results to several examples and compare them to numerical solutions and simulations. The details of the calculations for these examples are given in the Appendix. We assume exponential resetting as in (10) in Secs. III A–III E and consider nonexponential resetting in Sec. III G.

A. Diffusive search

Consider a searcher that diffuses with diffusivity $D > 0$ in $d \geqslant 1$ spatial dimensions. Assume that the searcher starts at (and is reset to) a position that is distance $L > 0$ from the target. Consider the following three scenarios: (1) $d = 1$ and the target is a single point, (2) $d = 3$ and the target is a sphere of radius $a > 0$, and (3) $d = 3$ and the target is the exterior of a sphere centered at the starting and resetting position (i.e., the FPT is the first time the searcher escapes a sphere of radius $L > 0$).

In each of these three scenarios, the Laplace transform of the distribution of T and all the moments of T can be calculated analytically. Further, the probability p in (8) of a successful search and the corresponding asymptotic form p_0

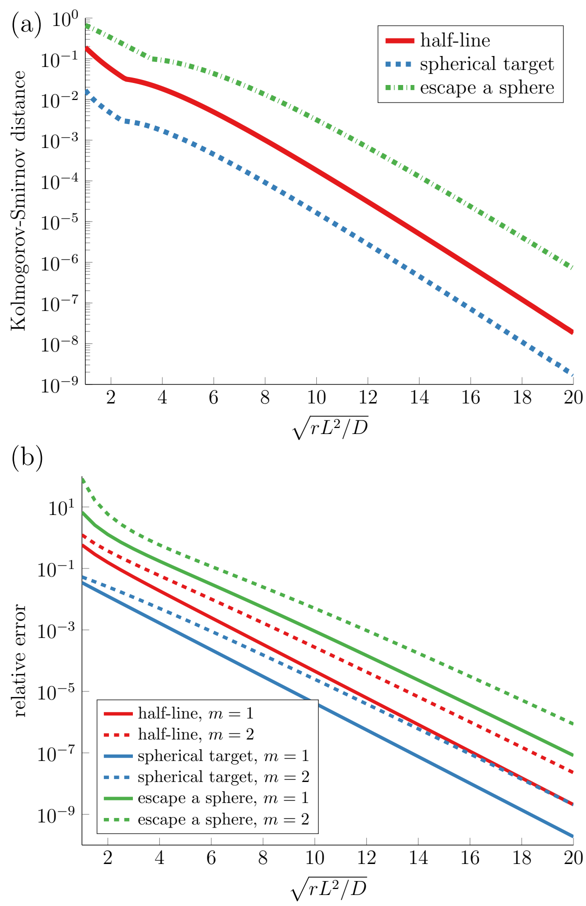

FIG. 2. Diffusive search for a target that is distance $L > 0$ from the initial and resetting position. See Sec. III A for details.

given by Theorem 3 can be computed analytically. In particular, the values of p_0 in (23) in Theorem 3 for these three scenarios are

$$p_0 = \begin{cases} e^{-\sqrt{rL^2/D}} & \text{in scenario (1)}, \\ (1 + L/a)^{-1}\, e^{-\sqrt{rL^2/D}} & \text{in scenario (2)}, \\ \sqrt{4rL^2/D}\, e^{-\sqrt{rL^2/D}} & \text{in scenario (3)}. \end{cases} \tag{34}$$

These calculations are given in the Appendix.

In Fig. 2(a) we plot the convergence in distribution of rp_0T to a unit rate exponential random variable as the dimensionless resetting rate $\sqrt{rL^2/D}$ increases for each of these scenarios. Specifically, we plot the Kolmogorov-Smirnov distance between the distribution of rp_0T and a unit rate exponential distribution, defined as

$$\sup_{x\geqslant 0} |\mathbb{P}(rp_0T > x) - e^{-x}|. \tag{35}$$

In agreement with Theorem 1, rp_0T rapidly converges in distribution to a unit rate exponential random variable. In Fig. 2(b) we plot the relative error between the exact mth moment $\mathbb{E}[T^m]$ and the frequent resetting estimate $\mathbb{E}[T^m] \approx m!/(rp_0)^m$ from Theorems 1 and 3 for $m = 1$ (solid curves) and $m = 2$ (dashed curves). This figure shows that the relative error vanishes exponentially fast as $\sqrt{rL^2/D}$ increases.

These three scenarios share the general features that if the resetting rate r is much faster than the diffusion rate (i.e., $r \gg D/L^2$), then T is approximately exponentially distributed with rate rp_0, where p_0 vanishes exponentially according to $p_0 \approx e^{-\sqrt{rL^2/D}}$ (possibly with a prefactor that depends on the details of the geometry). While these features can be seen explicitly in the three analytically tractable scenarios described above, they characterize diffusive search with frequent resetting much more generally. Indeed, as long as the searcher cannot start (or reset) arbitrarily close to the target (see Sec. III B for a case that this condition excludes), then the FPT distribution without resetting generally satisfies [27]

$$\lim_{t\to 0^+} t \ln F_\tau(t) = -L^2/D < 0,$$

where $D > 0$ is a characteristic diffusivity and $L > 0$ is the shortest distance from the set of initial positions to the target (in an appropriate distance metric). Hence, Theorem 2 yields

$$\ln p \sim -\sqrt{rL^2/D} < 0 \quad \text{as } r \to \infty,$$

and thus the moments of T diverge exponentially according to

$$\ln(r^m \mathbb{E}[T^m]) \sim m\sqrt{rL^2/D} \quad \text{as } r \to \infty.$$

B. Diffusive search with uniform initial condition

We now consider diffusive search in which the starting and resetting positions are not bounded away from the target. Suppose that the searcher diffuses with diffusivity $D > 0$ in one spatial dimension with targets at $x = 0$ and $x = L > 0$, and suppose that the searcher starts and resets to a uniformly distributed position in the interval $[0, L]$. In this case we show in the Appendix A 4 b that the FPT distribution without resetting decays according to the following power law at short time,

$$F_\tau(t) \sim \sqrt{\frac{16Dt}{\pi L^2}} \quad \text{as } t \to 0^+. \tag{36}$$

Hence, for frequent resetting, Theorem 1 implies that T is approximately exponentially distributed with rate rp_0, where Proposition 6 yields

$$p \sim p_0 = \sqrt{4D/(rL^2)} \quad \text{as } r \to \infty. \tag{37}$$

In Fig. 3(a) we plot the Kolmogorov-Smirnov distance as in (35) for this example (solid red curve) as the dimensionless resetting rate $\sqrt{rL^2/D}$ increases. In Fig. 3(a) we also plot the Kolmogorov-Smirnov distance for scenario (3) in Sec. III A (dashed blue curve) except where the searcher starts and resets to uniformly distributed positions in the sphere.

C. Search on a discrete network

Suppose the searcher moves by discrete jumps to adjacent nodes on a discrete network [32]. Specifically, let $X = \{X(t)\}_{t\geqslant 0}$ be a continuous-time Markov chain on a finite or countably infinite state space I. Suppose X starts at (and resets at rate r to) a given state $i_0 \in I$. Consider the FPT to some target set of states $I_{\text{target}} \subset I$ with $i_0 \notin I_{\text{target}}$.

Proposition 1 in [33] implies that the cumulative distribution function of the FPT $\tau := \inf\{t > 0 : X(t) \in I_{\text{target}}\}$

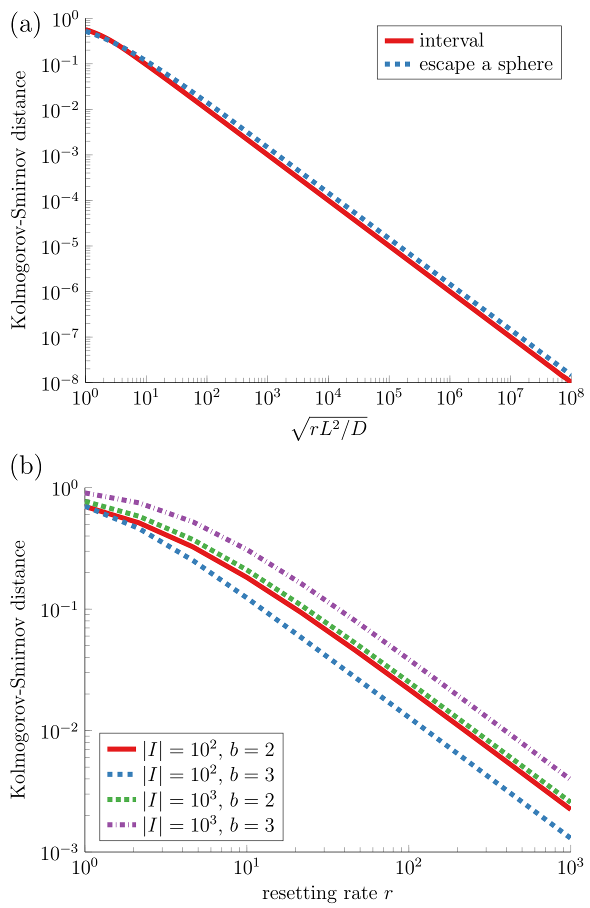

FIG. 3. (a) Diffusive search with uniformly distributed initial and resetting positions. (b) Search on a discrete network with $|I| \gg 1$ states and $b \geqslant 1$ jumps required to reach the target from the starting and resetting position. See Secs. III B and III C for details.

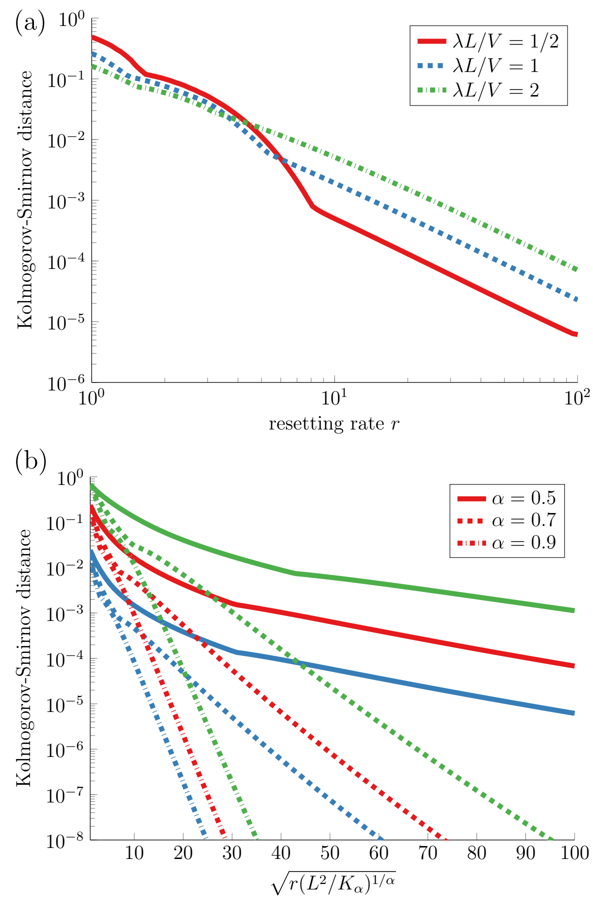

FIG. 4. (a) Run-and-tumble search. (b) Subdiffusive search. See Secs. III D and III E for details.

without resetting decays according to the following power law at short time:

$$F_\tau(t) \sim At^b \quad \text{as } t \to 0^+, \tag{38}$$

where $b \geqslant 1$ is the minimum number of jumps X must take to reach I_{target} from i_0 and $A = \Lambda/b!$, where Λ is the product of the jump rates of X along this shortest path from i_0 to I_{target}. (If there are multiple shortest paths, then Λ is the sum of the products of the jump rates along these paths.) As a technical aside, (38) assumes that the jump rates of X are bounded and $\mathbb{P}(\tau = \infty) \neq 1$ (i.e., there exists a path from i_0 to I_{target}).

Hence, Proposition 6 implies that

$$p \sim p_0 = A\Gamma(b+1)r^{-b} \quad \text{as } r \to \infty,$$

and Theorem 1 yields the distribution and moments of T for frequent resetting. In Fig. 3 we plot the Kolmogorov-Smirnov distance as in (35) for this example as the resetting rate r increases for a few randomly generated networks with the number of states ranging from $|I| = 10^2$ to $|I| = 10^3$. The details of this calculation and these networks are given in the Appendix. The convergence in distribution illustrated in Fig. 3 shows that despite the complexity of these underlying jump processes, the FPT with frequent resetting depends only on the network properties along the shortest path(s) to the target.

D. Run and tumble

Consider a one-dimensional run-and-tumble particle that switches between velocity $V > 0$ and $-V < 0$ at Poissonian rate $\lambda > 0$ [13,14]. Analogous to the first scenario in Sec. III A on diffusion, suppose the target is at $x = 0$ and the searcher starts at and resets to $x = L > 0$. The probability of a successful search satisfies

$$p \sim p_0 = e^{-rL/V}\left(\tfrac{1}{2}e^{-\lambda L/V} + \beta/r\right) \quad \text{as } r \to \infty, \tag{39}$$

where $\beta = \lambda e^{-\frac{\lambda L}{V}}(\lambda L + V)/(4V)$. In Fig. 4(a) we plot the Kolmogorov-Smirnov distance as in (35) for this example as the dimensionless resetting rate rL/V increases for a few different choices of the dimensionless tumbling rate $\lambda L/V$. The details of this calculation are given in the Appendix.

E. Subdiffusive search

Suppose that the searcher moves by subdiffusion in $d \geqslant 1$ spatial dimensions between stochastic resets, meaning that its mean-squared displacement grows in time t according to a sublinear power law t^α for $\alpha \in (0, 1)$. Concretely, suppose that in between stochastic resets, the probability density $p_\alpha(x, t)$ for its position evolves according to the following fractional Fokker-Planck equation [29]:

$$\frac{\partial}{\partial t} p_\alpha(x,t) = {}_0D_t^{1-\alpha} K_\alpha \Delta p_\alpha(x,t), \quad t>0,\ x\in\mathbb{R}^d\backslash U, \tag{40}$$

with absorbing conditions on the target $U\subset\mathbb{R}^d$,

$$p_\alpha = 0, \quad x\in U.$$

In (40), $t>0$ is the time elapsed since the last reset, $K_\alpha>0$ is the generalized diffusivity (with dimensions $(\text{length})^2(\text{time})^{-\alpha}$), and ${}_0D_t^{1-\alpha}$ is the Riemann-Liouville fractional derivative [34],

$${}_0D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\,ds,$$

where $\Gamma(\alpha)$ is the gamma function.

For a given initial distribution of the searcher and a given target, let τ_α denote the FPT without stochastic resetting with corresponding survival probability,

$$S_\alpha(t) := \mathbb{P}(\tau_\alpha > t) = \int_{\mathbb{R}^d\backslash U} p_\alpha(x,t)\,dx. \tag{41}$$

Now, the Laplace transform of the solution p_α to the fractional equation (40) with $\alpha\in(0,1)$ is related to the solution p_1 to the integer order version of (40) [i.e., (40) with $\alpha=1$] via the relation [35]

$$\widetilde{p}_\alpha(x,s) = s^{\alpha-1}\widetilde{p}_1(x,s^\alpha), \quad \alpha\in(0,1], \tag{42}$$

where $\widetilde{f}(s) := \int_0^\infty e^{-st} f(t)\,dt$ denotes the Laplace transform of $f(t)$. Hence, integrating (42) over space and using (41) yields

$$\widetilde{S}_\alpha(s) = s^{\alpha-1}\widetilde{S}_1(s^\alpha), \quad \alpha\in(0,1]. \tag{43}$$

In words, (43) yields the survival probability in Laplace space for subdiffusion, $\widetilde{S}_\alpha$, from the survival probability in Laplace space for normal diffusion, $\widetilde{S}_1$.

Using (43), we now consider the three scenarios analyzed in Sec. III A, but for subdiffusion. Specifically, we consider (1) $d=1$ and the target is a single point, (2) $d=3$ and the target is a sphere of radius $a>0$, and (3) $d=3$ and the target is the exterior of a sphere centered at the starting and resetting position. Using Theorem 5, we obtain that the asymptotic forms p_0 of the probability of a successful search for these three scenarios are given respectively by

$$p_0 = \begin{cases} e^{-\sqrt{r^\alpha L^2/K_\alpha}} & \text{in scenario (1)},\\ (1+L/a)^{-1}\,e^{-\sqrt{r^\alpha L^2/K_\alpha}} & \text{in scenario (2)},\\ \sqrt{4r^\alpha L^2/K_\alpha}\,e^{-\sqrt{r^\alpha L^2/K_\alpha}} & \text{in scenario (3)}.\end{cases}$$

In Fig. 4(b) we plot the Kolmogorov-Smirnov distance as in (35) for these examples as the dimensionless resetting rate $\sqrt{r(L^2/K_\alpha)^{1/\alpha}}$ increases for a few different choices of the subdiffusive exponent $\alpha\in(0,1)$. Scenarios (1), (2), and (3) correspond, respectively, to the red (middle), blue (bottom), and green (top) curves. The values $\alpha=0.5$, $\alpha=0.7$, and $\alpha=0.9$ correspond, respectively, to the solid, dashed, and dot-dashed curves.

F. Examples in Fig. 1(b)

In Fig. 1(b) we plot the FPT distribution for frequent resetting for a variety of different search processes obtained from numerical solutions using the Laplace transform methods described above. We now detail these examples.

The diffusion example in Fig. 1(b) (green circle markers) is scenario (1) in Sec. III A with $rL^2/D=50$. The discrete network example in Fig. 1(b) (orange square markers) is the example in Sec. III C with $|I|=10^2$, $b=3$, and $r=200$. The run-and-tumble example in Fig. 1(b) (purple diamond markers) is the example in Sec. III D with $\lambda L/V=1$ and $r=50$. The subdiffusion example in Fig. 1(b) (pink triangle markers) is scenario (1) in Sec. III E with $r(L^2/K_\alpha)^{1/\alpha}=50$ and $\alpha=0.8$.

G. Nonexponential resetting

In the previous examples, the resetting times were exponentially distributed with rate r. We now illustrate our results for other choices of the resetting time distribution. In particular, we define the resetting time via $\sigma=Y/r$ for the following three choices of the survival probability of Y:

$$S_Y(y) = \begin{cases} 1 & \text{if } y<1,\\ 0 & \text{if } y\geqslant 1.\end{cases} \quad \text{(sharp reset)}, \tag{44}$$

$$S_Y(y) = 1-y/2, \quad y\in[0,2] \quad \text{(uniform reset)}, \tag{45}$$

$$S_Y(y) = (1+2y)e^{-2y}, \quad y\geqslant 0 \quad \text{(gamma reset)}. \tag{46}$$

The choice in (44) yields resetting times which are $\sigma=1/r$ with probability one (so-called sharp reset or sharp restart [26]). The choice in (45) yields resetting times σ which are uniformly distributed on the interval $[0,2/r]$. The choice in (46) yields resetting times σ which have a gamma distribution with shape parameter 2 and scale parameter $1/(2r)$.

We consider these three choices of the resetting time distribution for the case of diffusive search on the half-line [scenario (1) in Sec. III A above]. In Fig. 5(a) we plot the Kolmogorov-Smirnov distance as in (35) (with $p_0=p$) as the dimensionless resetting rate $\sqrt{rL^2/D}$ increases for each of the three resetting time distributions in (44)–(46) as well as an exponential resetting distribution. The distributions of T for each marker in Fig. 5(a) are computed from 10^7 simulated stochastic realizations of T. Similarly, in Fig. 5(b) we plot the mean FPT $\mathbb{E}[T]$ computed from 10^7 simulated stochastic realizations of T (markers) as well as the theoretical prediction $1/(rp)$ of (16) (curves) for varying choices of the resetting time distribution. The computation of p for each of these examples is given in the Appendix.

IV. DISCUSSION

In this work we studied FPTs for stochastic search processes in the limit of frequent stochastic resetting. We determined approximations for the full probability distribution and moments of the FPT, which are exact in the frequent resetting limit. While we generally focused on the case that resetting occurs at exponentially distributed times with rate $r>0$, we proved our results for much more general resetting time distributions. These results depend only on the short-time

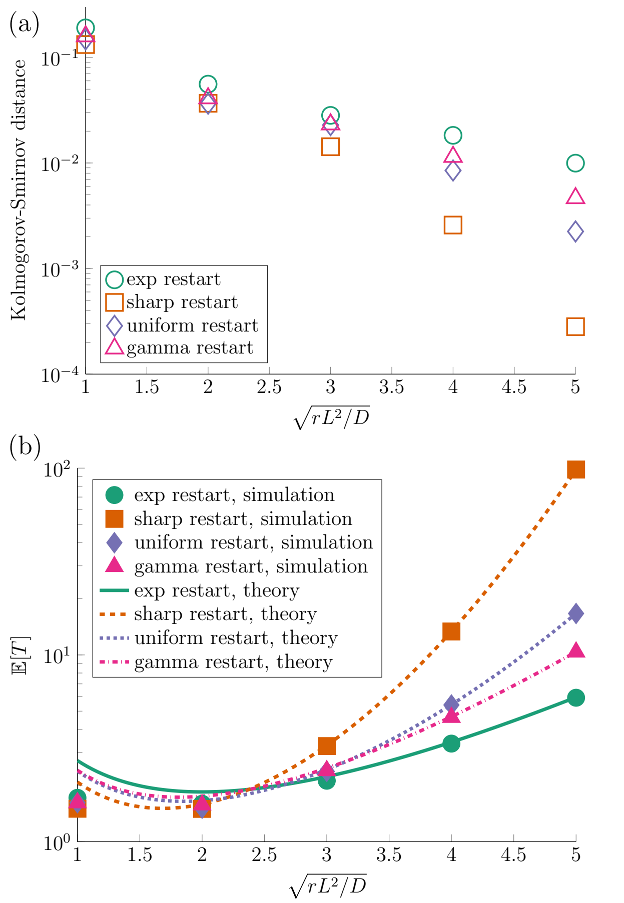

FIG. 5. General resetting time distributions. (a) Kolmogorov-Smirnov distance in (35) between the theoretical limit and stochastic simulations. (b) The mean FPT $\mathbb{E}[T]$ for stochastic simulations (markers) and theory (curves). Section III G for details.

behavior of the search process without resetting and are thus immediately tractable in many settings. In particular, much of the details about the particular search process and geometry are irrelevant for frequent resetting, as illustrated in Sec. III. The relevant information about the search process is encoded into the quantity p, which is the probability of a successful search prior to resetting. We computed approximations $p_0 \approx p$ for a variety of search processes for frequent exponential (Poissonian) resetting.

By considering several specific examples with numerical solutions, we found that the error in these approximations often decays rapidly with the resetting rate r. Indeed for diffusive search in which the searchers cannot start arbitrarily close to the target, we found that these errors vanish exponentially fast as $\sqrt{rL^2/D}$ grows, where D is the diffusivity and L is the shortest distance the searchers must travel to reach the target. Importantly, this exponentially fast convergence held regardless of the spatial dimension. We found similar exponentially fast convergence for subdiffusive processes. For the other search processes we considered, such as run and tumble, we found that the error decayed algebraically. These results show that some resetting search processes for which computing the exact distribution and statistics of the FPT is intractable may be well approximated by simple asymptotic formulas. That is, our results for frequent resetting can be rather good approximations across a broad range of resetting rates.

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APPENDIX

1. Proofs of results in Sec. II A

We begin by proving the main results on distribution and moment convergence of the FPT with stochastic resetting. These proofs make use of subsidiary results, which we formalize as lemmas and whose proofs follow.

Proof of Theorem 1. By Lemma 7, it is enough to prove the theorem for $p_0 = p$. Fix $z \in (-\delta, \delta)$ with δ as in (6). Since σ^- and τ^- are independent, the moment-generating function of prT can be expanded to

$$\mathbb{E}[e^{zprT}] = \mathbb{E}\left[\sum_{k=0}^{\infty} e^{zpr\sum_{n=1}^{k}\sigma_n^-}\mathbb{1}_{R=k}\right]\mathbb{E}[e^{zpr\tau^-}], \tag{A1}$$

where $\mathbb{1}_A$ denotes the indicator function on an event A, so $\mathbb{E}(\mathbb{1}_A) = \mathbb{P}(A)$. Moreover, since R is geometrically distributed with probability of success p and $\{\sigma_n^-\}_{n\geqslant 1}$ are independent and identically distributed,

$$\mathbb{E}\left[\sum_{k=0}^{\infty} e^{zpr\sum_{n=1}^{k}\sigma_n^-}\mathbb{1}_{R=k}\right] = \sum_{k=0}^{\infty}(\mathbb{E}[e^{zpr\sigma^-}])^k(1-p)^k p. \tag{A2}$$

Substituting (A2) into (A1), which is now in the form of a geometric series,

$$\mathbb{E}[e^{zprT}] = \frac{p\mathbb{E}[e^{zpr\tau^-}]}{1-(1-p)\mathbb{E}[e^{zpr\sigma^-}]}. \tag{A3}$$

Using Lemma 10 and Lemma 11 when taking the limit as $r \to \infty$ of (A3) yields

$$\lim_{r\to\infty}\mathbb{E}[e^{zprT}] = \frac{1}{1-z}. \tag{A4}$$

The right-hand side of (A4) is the moment-generating function of an exponential random variable with unit rate. By Lévy's continuity theorem, convergence of moment-generating functions implies convergence in distribution, and so the proof of (15) in Theorem 1 is complete.

To prove the convergence of moments in (16), we show that $\{\mathbb{E}[(prT)^{2m}]\}_{r>0}$ is bounded for sufficiently large r. By the binomial theorem and the definition of T in (12), we have that

$$\mathbb{E}[T^{2m}] = \sum_{k=0}^{2m}\binom{2m}{k}\mathbb{E}\left[\left(\sum_{n=1}^{R}\sigma_n^-\right)^k\right]\mathbb{E}[(\tau^-)^{2m-k}].$$

To bound the moments of τ^-, we apply the Cauchy-Schwarz inequality,

$$\mathbb{E}[(\tau^-)^k] = \frac{\mathbb{E}[\tau^k 1\!\!1_{\tau<\sigma}]}{p} \leqslant \frac{\mathbb{E}[\sigma^k 1\!\!1_{\tau<\sigma}]}{p} \leqslant \frac{\sqrt{\mathbb{E}[\sigma^{2k}]}\sqrt{\mathbb{E}[(1\!\!1_{\tau<\sigma})^2]}}{p} = \frac{\sqrt{\mathbb{E}[Y^{2k}]}}{r^k\sqrt{p}}. \quad \text{(A5)}$$

To bound the moments of $\sum_{n=1}^R \sigma_n^-$, we sum over the possible values of R and use the multinomial theorem to obtain

$$\mathbb{E}\left[\left(\sum_{n=1}^R \sigma_n^-\right)^k\right] = \sum_{j=0}^\infty \mathbb{P}(R=j)\mathbb{E}\left[\left(\sum_{n=1}^j \sigma_n^-\right)^k\right] = \sum_{j=0}^\infty \mathbb{P}(R=j) \sum_{k_1+\dots+k_j=k} \binom{k}{k_1,\dots,k_j} \times \prod_{i=1}^j \mathbb{E}[(\sigma_i^-)^{k_i}],$$

where

$$\binom{k}{k_1,\dots,k_j} = \frac{k!}{k_1!\cdots k_j!}.$$

To bound the moments of σ^-, we first use that (A18) implies that we may take r sufficiently large so that $\mathbb{E}[(\sigma^-)^k] \leqslant 2\mathbb{E}[\sigma^k]$ and then use (7) and (A23) to obtain

$$\mathbb{E}[\sigma^k] = \int_0^\infty kt^{k-1}\mathbb{P}(Y > rt)\,dt \leqslant K\int_0^\infty kt^{k-1}e^{-\delta rt}\,dt = K\mathbb{E}[(\overline{\sigma})^k],$$

where $\overline{\sigma}$ is exponentially distributed with rate $r\delta > 0$ where $\delta > 0$ is as in (6). Therefore, for sufficiently large r we have that

$$\mathbb{E}\left[\left(\sum_{n=1}^R \sigma_n^-\right)^k\right] \leqslant 2K\mathbb{E}\left[\left(\sum_{n=1}^R \overline{\sigma}_n\right)^k\right], \quad \text{(A6)}$$

where $\{\overline{\sigma}_n\}_{n\geqslant 1}$ is an iid sequence of realizations of $\overline{\sigma}$. Since $\overline{\sigma}$ is exponentially distributed with rate $r\delta > 0$ and R is a geometric random variable with parameter p, it follows that $\sum_{n=1}^R \overline{\sigma}_n$ is exponentially distributed with rate $pr\delta$. Therefore, for sufficiently large r we have that

$$\mathbb{E}\left[\left(\sum_{n=1}^R \sigma_n^-\right)^k\right] \leqslant 2K\frac{k!}{(pr\delta)^k}. \quad \text{(A7)}$$

Hence, (A5) and (A7) imply that for sufficiently large r,

$$\begin{aligned}\mathbb{E}[(prT)^{2m}] &= (pr)^{2m}\sum_{k=0}^{2m}\binom{2m}{k}\mathbb{E}\left[\left(\sum_{n=1}^R \sigma_n^-\right)^k\right]\mathbb{E}[(\tau^-)^{2m-k}] \\ &= (pr)^{2m}\mathbb{E}[(\tau^-)^{2m}] + (pr)^{2m}\mathbb{E}\left[\left(\sum_{n=1}^R \sigma_n^-\right)^{2m}\right] \\ &\quad + (pr)^{2m}\sum_{k=1}^{2m-1}\binom{2m}{k}\mathbb{E}\left[\left(\sum_{n=1}^R \sigma_n^-\right)^k\right]\mathbb{E}[(\tau^-)^{2m-k}] \\ &\leqslant p^{2m}\frac{\sqrt{\mathbb{E}[Y^{4m}]}}{\sqrt{p}} + 2K\frac{(2m)!}{\delta^{2m}} \\ &\quad + 2K\delta^{-k}\sum_{k=1}^{2m-1} p^{2m-k-1/2}\binom{2m}{k}k!\sqrt{\mathbb{E}[Y^{2(2m-k)}]}.\end{aligned}$$

Since $2m \geqslant 2$, taking $r \to \infty$ and using that $\lim_{r\to\infty} p = 0$ by Lemma 9 yields

$$\limsup_{r\to\infty} \mathbb{E}[(prT)^{2m}] \leqslant 2K\frac{(2m)!}{\delta^{2m}} < \infty.$$

This implies $\{(prT)^m\}_{r>0}$ is uniformly integrable for sufficiently large r [see, for example, equation (3.18) in [36]]. Combining uniform integrability with the convergence in distribution in (15) in Theorem 1 proves the convergence of moments in (16) [see, for example, Theorem 3.5 in [36]]. ■

The next lemma allows us to prove Theorem 1 with $p_0 = p$.

Lemma 7. Let $\{X_n\}_{n\geqslant 1}$ be any sequence of nonnegative random variables and let $\{a_n\}_{n\geqslant 1}$ be any sequence of real numbers. If

$$a_nX_n \to_{\text{dist}} \text{Exponential}(1) \quad \text{as } n \to \infty, \quad \text{(A8)}$$

and $\lim_{n\to\infty} a_n/b_n = 1$, then

$$b_nX_n \to_{\text{dist}} \text{Exponential}(1) \quad \text{as } n \to \infty.$$

Proof of Lemma 7. Fix $x \geqslant 0$ and let $\varepsilon > 0$. Since $F(x) := 1 - e^{-x}$ is continuous, there exists $\eta > 0$ so that $|F(x) - F(y)| \leqslant \varepsilon$ for all $y \in \mathbb{R}$ such that $|x - y| \leqslant \eta x$. Since $\lim_{n\to\infty} a_n/b_n = 1$, there exists $N_1 \geqslant 1$ such that $1 - \eta \leqslant a_n/b_n \leqslant 1 + \eta$ for all $n \geqslant N_1$. Let $F_n(z) := \mathbb{P}(X_n \leqslant z)$ for $z \in \mathbb{R}$. By (A8), there exists $N_2 \geqslant 1$ so that $|F_n[(1 \pm \eta)x/a_n] - F[(1 \pm \eta)x]| \leqslant \varepsilon$ for all $n \geqslant N_2$.

Since F_n is nondecreasing, we have that for $n \geqslant \max\{N_1, N_2\}$,

$$\begin{aligned}\mathbb{P}(b_nX_n \leqslant x) - F(x) &= F_n[(a_n/b_n)x/a_n] - F(x) \\ &\leqslant F_n[(1+\eta)x/a_n] - F(x) \\ &\leqslant \varepsilon + F[(1+\eta)x] - F(x) \leqslant 2\varepsilon.\end{aligned}$$

Again using that F_n is nondecreasing, we similarly have that for $n \geqslant \max\{N_1, N_2\}$,

$$\begin{aligned}\mathbb{P}(b_nX_n \leqslant x) - F(x) &= F_n[(a_n/b_n)x/a_n] - F(x) \\ &\geqslant F_n[(1-\eta)x/a_n] - F(x) \\ &\geqslant -\varepsilon + F[(1-\eta)x] - F(x) \geqslant -2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete. ■

The following lemma gives the intuitive result that σ^- and σ have equivalent moments for large r.

Lemma 8. Under the assumptions of Sec. II A, we have

$$\mathbb{E}[(\sigma^-)^m] \sim \mathbb{E}[\sigma^m] \quad \text{as } r \to \infty.$$

Proof of Lemma 8. By definition of conditional probability,

$$S_{\sigma^-}(t) = \frac{\mathbb{P}(t < \sigma < \tau)}{\mathbb{P}(\sigma < \tau)}. \quad \text{(A9)}$$

Since σ and τ are independent, the denominator of (A9) is

$$\mathbb{P}(\sigma < \tau) = \mathbb{E}[S_\tau(\sigma)] = \int_0^\infty S_\tau(s)\,dF_\sigma(s), \tag{A10}$$

where $S_\tau(t) := \mathbb{P}(\tau > t)$. Similarly, the numerator of (A9) is

$$\mathbb{P}(t < \sigma < \tau) = \mathbb{E}[S_\tau(\sigma)\mathbb{1}_{\sigma>t}] = \int_t^\infty S_\tau(s)\,dF_\sigma(s) \tag{A11}$$

Substituting the expressions in (A10) and (A11) into (A9),

$$S_{\sigma^-}(t) = \frac{\int_t^\infty S_\tau(s)\,dF_\sigma(s)}{\int_0^\infty S_\tau(s)\,dF_\sigma(s)}. \tag{A12}$$

By (A12),

$$\int_0^\infty mt^{m-1}S_{\sigma^-}(t)\,dt = \frac{\int_0^\infty \int_t^\infty mt^{m-1}S_\tau(s)\,dF_\sigma(s)\,dt}{\int_0^\infty S_\tau(s)\,dF_\sigma(s)}. \tag{A13}$$

Given the nonnegativity of the integrand in the numerator of (A13), Tonelli's theorem implies

$$\int_0^\infty \int_t^\infty mt^{m-1}S_\tau(s)\,dF_\sigma(s)\,dt = \int_0^\infty s^m S_\tau(s)re^{-rs}\,dF_\sigma(s). \tag{A14}$$

Substituting (A14) into (A13) yields

$$\begin{aligned}\int_0^\infty mt^{m-1}S_{\sigma^-}(t)\,dt &= \frac{\int_0^\infty t^m S_\tau(t)\,dF_\sigma(t)}{\int_0^\infty S_\tau(t)\,dF_\sigma(t)}\\ &= \frac{\int_0^\infty t^m S_\tau(t)\,dF_\sigma(t)}{1-p}.\end{aligned} \tag{A15}$$

Changing variables $t = x^{1/m}$ yields

$$\begin{aligned}\mathbb{E}[(\sigma^-)^m] &= \int_0^\infty \mathbb{P}(\sigma^- > x^{1/m})\,dx\\ &= \int_0^\infty mt^{m-1}S_{\sigma^-}(t)\,dt,\end{aligned} \tag{A16}$$

and thus (A15) implies

$$\mathbb{E}[(\sigma^-)^m] = \frac{\int_0^\infty t^m S_\tau(t)\,dF_\sigma(t)}{1-p}. \tag{A17}$$

Let $\varepsilon > 0$. Since $\lim_{r\to\infty} p = 0$ by Lemma 9, we can take r sufficiently large so that $1/(1-p) \leqslant 1+\varepsilon$. Hence, (A17) implies that for sufficiently large r,

$$\begin{aligned}\mathbb{E}[(\sigma^-)^m] &\leqslant (1+\varepsilon)\int_0^\infty t^m S_\tau(t)\,dF_\sigma(t)\\ &\leqslant (1+\varepsilon)\int_0^\infty t^m\,dF_\sigma(t) = (1+\varepsilon)\mathbb{E}[\sigma^m],\end{aligned}$$

since $S_\tau(t) \leqslant 1$ for all t. Therefore,

$$\limsup_{r\to\infty} \frac{\mathbb{E}[(\sigma^-)^m]}{\mathbb{E}[\sigma^m]} \leqslant 1+\varepsilon. \tag{A18}$$

Now we consider the limit infimum. The denominator of (A17) is clearly bounded above by unity. By right continuity of $S_\tau(t)$ and the assumption in (3) that $F_\tau(0) = 0$, there exists $\eta > 0$ such that for $t \in (0,\eta)$, $S_\tau(t) \geqslant 1-\varepsilon$. Thus,

$$\begin{aligned}\int_0^\infty t^m S_\tau(t)\,dF_\sigma(t) &\geqslant \int_0^\eta t^m S_\tau(t)\,dF_\sigma(t)\\ &\geqslant (1-\varepsilon)\int_0^\eta t^m\,dF_\sigma(t)\\ &\geqslant (1-\varepsilon)\mathbb{E}[\sigma^m\mathbb{1}_{\sigma<\eta}]\\ &= (1-\varepsilon)r^{-m}\mathbb{E}[Y^m\mathbb{1}_{Y<r\eta}],\end{aligned}$$

where $\mathbb{1}_A$ denotes the indicator function on an event A. Hence, using $\mathbb{E}[\sigma^m] = r^{-m}\mathbb{E}[Y^m]$ and the monotone convergence theorem we obtain

$$\liminf_{r\to\infty} \frac{\mathbb{E}[(\sigma^-)^m]}{\mathbb{E}[\sigma^m]} \geqslant 1-\varepsilon.$$

Since $\varepsilon \in (0,1)$ is arbitrary, the proof is complete.

We ultimately use convergence of moment-generating functions to conclude Theorem 1. Hence, we use Lemma 9 below to obtain the desired large r behavior of the moment-generating functions for σ^- and τ^-.

Lemma 9. Under the assumptions of Sec. II A, we have

$$\lim_{r\to\infty} p = 0.$$

Proof of Lemma 9. We have

$$p := \mathbb{P}(\tau < \sigma) = \int_0^\infty S_\sigma(t)\,dF_\tau(t) = \int_0^\infty F_\tau(t)\,dF_\sigma(t),$$

where the second equality follows from integration by parts. Let $\varepsilon \in (0,1)$. Since $F_\tau(t)$ is right continuous and since $F_\tau(0) = 0$ by assumption (3), there exists $\eta > 0$ such that $F_\tau(t) < \varepsilon$ for $t \in (0,\eta)$. Hence,

$$\begin{aligned}0 < p &\leqslant \varepsilon\int_0^\eta dF_\sigma(t) + \int_\eta^\infty dF_\sigma(t)\\ &\leqslant \varepsilon + \mathbb{P}(\sigma \geqslant \eta)\\ &= \varepsilon + \mathbb{P}(Y \geqslant r\eta).\end{aligned}$$

Since $\lim_{r\to\infty}\mathbb{P}(Y \geqslant r\eta) = 0$ by (5) and since $\varepsilon \in (0,1)$ is arbitrary, the proof is complete. ■

The remaining lemmas characterize the large r behavior of individual terms in the moment-generating function of prT. Altogether, these results make possible the proof of Theorem 1.

Lemma 10. Under the assumptions of Sec. II A, we have that for all $z \in (-\delta,\delta)$ with $\delta > 0$ as in (6),

$$\mathbb{E}[e^{zpr\sigma^-}] = 1 + zp + o(p) \quad \text{as } r\to\infty.$$

Proof of Lemma 10. It follows immediately from (A17) that

$$\mathbb{E}[(\sigma^-)^m] \leqslant \frac{\mathbb{E}[\sigma^m]}{1-p},$$

and since $\lim_{r\to\infty} p = 0$ by Lemma 9, we have that

$$\mathbb{E}[(\sigma^-)^m] \leqslant 2\mathbb{E}[\sigma^m] \quad \text{for sufficiently large } r.$$

Now, expanding the exponential function and assuming $z \in (-\delta, \delta)$ for δ as in (6) yields

$$\mathbb{E}[e^{zpr\sigma^-}] = \sum_{k=0}^{\infty} \frac{(zpr)^k}{k!}\mathbb{E}[(\sigma^-)^k] = \sum_{k=0}^{\infty} \frac{(zp)^k}{k!}\frac{\mathbb{E}[(\sigma^-)^k]}{\mathbb{E}[\sigma^k]}\mathbb{E}[Y^k], \tag{A19}$$

where we have used that $\sigma = Y/r$. To see the validity of exchanging the expectation with the sum in the first equality in (A19), first note the bound

$$\left|\frac{(zpr)^k}{k!}\mathbb{E}[(\sigma^-)^k]\right| \leqslant \frac{(|z|r)^k}{k!}2\mathbb{E}[\sigma^k] = \frac{|z|^k}{k!}2\mathbb{E}[Y^k] \quad \text{for sufficiently large } r. \tag{A20}$$

Next, Tonelli's theorem implies that

$$\sum_{k=0}^{\infty} \frac{|z|^k}{k!}\mathbb{E}[Y^k] = \mathbb{E}\left[\sum_{k=0}^{\infty} \frac{|z|^k}{k!}Y^k\right] = \mathbb{E}[e^{|z|Y}] < \infty, \tag{A21}$$

by the assumption in (6) that Y has a finite moment-generating function in a neighborhood of the origin. Hence, the bound in (A20) and the finiteness of the sum in (A21) allow us to use the dominated convergence theorem to verify the first equality in (A19).

It follows from (A19) and the assumption in (5) that $\mathbb{E}[Y] = 1$ that

$$\begin{aligned}\frac{1}{p}(1 + zp - \mathbb{E}[e^{zpr\sigma^-}]) &= \frac{1}{p}\left\{1 + zp - \sum_{k=0}^{\infty} \frac{(zp)^k}{k!}\frac{\mathbb{E}[(\sigma^-)^k]}{\mathbb{E}[\sigma^k]}\mathbb{E}[Y^k]\right\} \\ &= z\left(1 - \frac{\mathbb{E}[\sigma^-]}{\mathbb{E}[\sigma]}\right) - \sum_{k=2}^{\infty} \frac{p^{k-1}z^k}{k!}\frac{\mathbb{E}[(\sigma^-)^k]}{\mathbb{E}[\sigma^k]}\mathbb{E}[Y^k].\end{aligned} \tag{A22}$$

Since $\mathbb{E}[\sigma^-] \sim \mathbb{E}[\sigma]$ as $r \to \infty$ by Lemma 8, it remains to show that the sum in (A22) vanishes as $r \to \infty$. Note first that the terms in this sum vanish as $r \to \infty$ since $\lim_{r\to\infty} p = 0$ by Lemma 9. It thus remains to justify exchanging the large r limit with the sum. First, observe that for sufficiently large r we have the bound

$$\left|\frac{p^{k-1}z^k}{k!}\frac{\mathbb{E}[(\sigma^-)^k]}{\mathbb{E}[\sigma^k]}\mathbb{E}[Y^k]\right| \leqslant \frac{|z|^k}{k!}2\mathbb{E}[Y^k],$$

and thus again using (A21) and the dominated convergence theorem allows us to conclude

$$\lim_{r\to\infty}\sum_{k=2}^{\infty} \frac{p^{k-1}z^k}{k!}\frac{\mathbb{E}[(\sigma^-)^k]}{\mathbb{E}[\sigma^k]}\mathbb{E}[Y^k] = \sum_{k=2}^{\infty}\lim_{r\to\infty} \frac{p^{k-1}z^k}{k!}\frac{\mathbb{E}[(\sigma^-)^k]}{\mathbb{E}[\sigma^k]}\mathbb{E}[Y^k] = 0,$$

which completes the proof. ■

Lemma 11. Under the assumptions of Sec. II A, we have that for all $z \in (-\delta, \delta)$ with $\delta > 0$ as in (6),

$$\lim_{r\to\infty}\mathbb{E}[e^{zpr\tau^-}] = 1.$$

Proof of Lemma 11. Since Y has a finite moment-generating function as in (6), its survival probability, $S_Y(y) = \mathbb{P}(Y > y)$, vanishes no slower than exponentially,

$$S_Y(y) \leqslant Ke^{-\delta y} \quad \text{for all } y \in \mathbb{R}, \tag{A23}$$

where $K = \mathbb{E}[e^{\delta Y}] < \infty$. To see this, let $\delta > 0$ be as in (6) and observe that Chebyshev's inequality (see, for example, Theorem 1.6.4 in [37]) implies that for any $y \in \mathbb{R}$,

$$S_Y(y) = \mathbb{P}(Y > y) = \mathbb{P}(e^{\delta Y} > e^{\delta y}) \leqslant e^{-\delta y}\mathbb{E}[e^{\delta Y}] < \infty.$$

Hence, by definition of conditional probability,

$$\begin{aligned}S_{\tau^-}(t) &= \frac{\mathbb{P}(t < \tau < \sigma)}{\mathbb{P}(\tau < \sigma)} \leqslant \min\left\{1, \frac{\mathbb{P}(t < \sigma)}{\mathbb{P}(\tau < \sigma)}\right\} \\ &= \min\left\{1, \frac{1}{p}S_Y(rt)\right\} \leqslant \begin{cases}\frac{K}{p}e^{-r\delta t} & \text{if } t \geqslant C, \\ 1 & \text{if } t < C,\end{cases}\end{aligned}$$

where

$$C = (r\delta)^{-1}\ln(K/p). \tag{A24}$$

Now, if E is exponentially distributed with unit mean, then

$$\mathbb{P}(C + (r\delta)^{-1}E > t) = \begin{cases}\frac{K}{p}e^{-r\delta t} & \text{if } t \geqslant C, \\ 1 & \text{if } t < C,\end{cases}$$

and therefore,

$$S_{\tau^-}(t) \leqslant \mathbb{P}[C + (r\delta)^{-1}E > t] \quad \text{for all } t \in \mathbb{R}.$$

Hence, for $z \in [0, \delta)$, the nondecreasing nature of $f(x) = e^{zx}$ yields

$$1 \leqslant \mathbb{E}[e^{zpr\tau^-}] \leqslant \mathbb{E}[e^{zpr(C+(r\delta)^{-1}E)}] = \frac{e^{zprC}}{1 - zp/\delta}. \tag{A25}$$

For $z \in (-\delta, 0)$, the nonincreasing nature of $f(x) = e^{-|z|x}$ yields

$$1 \geqslant \mathbb{E}[e^{-|z|pr\tau^-}] \geqslant \mathbb{E}[e^{-|z|pr(C+(r\delta)^{-1}E)}] = \frac{e^{zprC}}{1 - zp/\delta}. \tag{A26}$$

Taking $r \to \infty$ in (A25) and (A26) and using the definition of C in (A24) and the fact that $\lim_{r\to\infty} p = 0$ by Lemma 9 completes the proof. ■

2. Derivation of (15) for exponential resetting

We now give a short, formal calculation to derive (15) in the special case of exponential resetting. Let $Q_r(t) := \mathbb{P}(T > t)$ denote the survival probability of T with exponential resetting at rate $r \geqslant 0$ and let $f_r(t) := -\frac{\partial}{\partial t}Q_r$ denote its probability density function. It is straightforward to check that [see, for example, (3.5) in [18]]

$$\widetilde{Q}_r(s) = \frac{\widetilde{Q}_0(r+s)}{1 - r\widetilde{Q}_0(r+s)} = \frac{(r+s)\widetilde{Q}_0(r+s)}{s + r\widetilde{f}_0(r+s)}, \tag{A27}$$

where $\widetilde{g}(s) := \int_0^\infty e^{-st} g(t)\,dt$ denotes Laplace transform and we have used that $\widetilde{f}_r(s) = 1 - s\widetilde{Q}_r(s)$. The initial value theorem implies

$$\lim_{r\to\infty} (r+s)\widetilde{Q}_0(r+s) = Q_0(0^+) = 1, \tag{A28}$$

and therefore

$$\widetilde{Q}_r(s) \sim \frac{1}{s + r\widetilde{f}_0(r+s)} \sim \frac{1}{s + r\widetilde{f}_0(r)} \quad \text{as } r\to\infty. \tag{A29}$$

Taking the inverse Laplace transform of (A29) yields

$$Q_r(t) \sim e^{-r\widetilde{f}_0(r)t} = e^{-rpt} \quad \text{as } r\to\infty,$$

since $\widetilde{f}_0(r) = p$ if τ has a density [see (8)].

3. Proofs of results in Sec. II B

The proofs of Proposition 12 and Theorems 4 and 5 below capture the utility of the results in Theorem 1 given increasingly strong assumptions on the cumulative distribution function of the search process without resetting.

Proposition 12. Assume $C, d > 0$ and $b \in \mathbb{R}$. If $\eta > 0$, then as $r \to \infty$

$$\int_0^\eta e^{-rt} t^b e^{-C/t^d}\,dt \sim \mu r^{\beta-1} \exp(-\gamma r^{d/(d+1)}), \tag{A30}$$

where $\gamma = \frac{d+1}{d^{d/(d+1)}} C^{\frac{1}{d+1}}$ and

$$\mu = \sqrt{\frac{2\pi (Cd)^{\frac{2b+1}{d+1}}}{d+1}}, \quad \beta = \frac{d-2b}{2d+2}.$$

Proof of Proposition 12. One can verify that the exponential factor in the integrand of (A30) is maximized at

$$t^* := \left(\frac{Cd}{r}\right)^{1/(d+1)}.$$

Let $r > 0$ be sufficiently large such that $t^* \in (0, \eta)$. The change of variables $s = t/t^*$ transforms the integral in (A30) into

$$\int_0^\eta e^{-rt} t^b e^{-C/t^d}\,dt = \left(\frac{Cd}{r}\right)^{\frac{b+1}{d+1}} \int_0^{\eta/t^*} f(s) e^{x\phi(s)}\,ds, \tag{A31}$$

where $x = r^{d/(d+1)}$ and

$$f(s) = s^b, \quad \phi(s) = -s(Cd)^{\frac{1}{d+1}} - Cs^{-d}(Cd)^{\frac{-d}{d+1}}.$$

Noting that $s = 1$ corresponds to the maximum of the exponential in (A31), we apply Laplace's method to obtain

$$\int_0^{\eta/t^*} f(s) e^{x\phi(s)}\,ds \sim \frac{\sqrt{2\pi} f(1) e^{x\phi(1)}}{\sqrt{-x\phi''(1)}}, \tag{A32}$$

$x = r^{d/(d+1)} \to \infty$. Simplifying the right-hand side of (A32) completes the proof. ■

Proof of Theorem 2. The proof of Theorem 2 follows from the proof of Theorem 4 below with $d = 1$. ■

Proof of Theorem 4. Let $\varepsilon > 0$. By the assumption in (24), there exists a $\eta > 0$ such that

$$e^{-(C+\varepsilon)/t^d} \leqslant F_\tau(t) \leqslant e^{-(C-\varepsilon)/t^d} \quad \text{for all } t \in (0,\eta). \tag{A33}$$

For any $u, v \in \mathbb{R} \cup \{\infty\}$, define $I_{u,v} := \int_u^v e^{-rt}\,dF_\tau(t)$ so that $p = I_{0,\eta} + I_{\eta,\infty}$. Using integration by parts and the upper bound in (A33) yields

$$\begin{aligned} I_{0,\eta} &= e^{-r\eta} F_\tau(\eta) + \int_0^\eta r e^{-rt} F_\tau(t)\,dt \\ &\leqslant e^{-r\eta} F_\tau(\eta) + \int_0^\eta r e^{-rt} e^{-(C-\varepsilon)/t^d}\,dt. \end{aligned} \tag{A34}$$

Similarly, the lower bound in (A33) yields

$$\int_0^\eta r e^{-rt} e^{-(C+\varepsilon)/t^d}\,dt \leqslant I_{0,\eta}.$$

Now, Proposition 12 implies that as $r \to \infty$,

$$\int_0^\eta r e^{-rt} e^{-(C\pm\varepsilon)/t^d}\,dt \sim \mu_{\pm\varepsilon} r^\beta \exp(-\gamma_{\pm\varepsilon} r^{d/(d+1)}),$$

where

$$\mu_{\pm\varepsilon} = \sqrt{\frac{2\pi[(C\pm\varepsilon)d]^{\frac{2b+1}{d+1}}}{d+1}}, \quad \beta = \frac{d}{2d+2},$$
$$\gamma_{\pm\varepsilon} = \frac{d+1}{d^{d/(d+1)}} (C\pm\varepsilon)^{\frac{1}{d+1}}.$$

It is straightforward to check that $I_{\eta,\infty} = O(re^{-r\eta})$ as $r \to \infty$, and thus

$$\lim_{r\to\infty} \frac{e^{-r\eta} F_\tau(\eta)}{\int_0^\eta r e^{-rt} e^{-(C\pm\varepsilon)/t^d}\,dt} = \lim_{r\to\infty} \frac{I_{\eta,\infty}}{\int_0^\eta r e^{-rt} e^{-(C\pm\varepsilon)/t^d}\,dt} = 0.$$

Therefore,

$$\begin{aligned} \limsup_{r\to\infty} r^{-d/(d+1)} \ln p &= \limsup_{r\to\infty} r^{-d/(d+1)} \ln(I_{0,\eta} + I_{\eta,\infty}) \\ &\leqslant \limsup_{r\to\infty} r^{-d/(d+1)} \ln[e^{-r\eta} F_\tau(\eta) \\ &\quad + \int_0^\eta r e^{-rt} e^{-(C-\varepsilon)/t^d}\,dt + I_{\eta,\infty}] \\ &= -\gamma_{-\varepsilon}. \end{aligned}$$

The analogous calculation on using the lower bound in (A33) finally yields

$$\begin{aligned} -\gamma_{+\varepsilon} &\leqslant \liminf_{r\to\infty} r^{-d/(d+1)} \ln p \\ &\leqslant \limsup_{r\to\infty} r^{-d/(d+1)} \ln p \\ &\leqslant -\gamma_{-\varepsilon}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have proven (25), and (26) follows from (16). ■

Proof of Theorem 3. The proof of Theorem 3 follows from the proof of Theorem 5 below with $d = 1$. ■

Proof of Theorem 5. Let $\varepsilon \in (0,1)$. By the assumption in (28), there exists $\eta > 0$ such that for all $t \in (0,\eta)$,

$$(1-\varepsilon) A t^b e^{-C/t^d} \leqslant F_\tau(t) \leqslant (1+\varepsilon) A t^b e^{-C/t^d}. \tag{A35}$$

For any $u, v \in \mathbb{R} \cup \{\infty\}$, define $I_{u,v} := \int_u^v e^{-rt}\,dF_\tau(t)$ so that p in (8) is $p = I_{0,\infty} = I_{0,\eta} + I_{\eta,\infty}$. Using integration by parts and the assumption in (3), we bound $I_{0,\eta}$ from above,

$$I_{0,\eta} = e^{-r\eta}F_\tau(\eta) + \int_0^\eta re^{-rt}F_\tau(t)\,dt$$
$$\leqslant e^{-r\eta}F_\tau(\eta) + (1+\varepsilon)Ar\int_0^\eta e^{-rt}t^be^{-C/t^d}\,dt. \quad \text{(A36)}$$

By Proposition 12 and the fact that $I_{\eta,\infty}$ and the boundary term in (A36) are $O(re^{-\eta r})$ as $r\to\infty$, we conclude

$$\limsup_{r\to\infty}\frac{I_{0,\infty}}{\mu r^\beta\exp(-\gamma r^{d/(d+1)})} \leqslant 1+\varepsilon. \quad \text{(A37)}$$

The analogous argument for the lower bound yields

$$1-\varepsilon \leqslant \liminf_{r\to\infty}\frac{I_{0,\infty}}{\mu r^\beta\exp(-\gamma r^{d/(d+1)})}. \quad \text{(A38)}$$

Since $\varepsilon\in(0,1)$ is arbitrary, the proof is complete. ■

4. Calculations from examples

We now give the details of the calculations for the examples in Sec. III.

a. Diffusive search

In Sec. III A we considered a searcher that diffuses with diffusivity $D>0$ in $d\geqslant 1$ spatial dimensions and starts at (and is reset to) a position that is distance $L>0$ from the target. In particular, we considered the following three scenarios: (1) $d=1$ and the target is a single point, (2) $d=3$ and the target is a sphere of radius $a>0$, and (3) $d=3$ and the target is the exterior of a sphere centered at the starting and resetting position (i.e., the FPT is the first time the searcher escapes a sphere of radius $L>0$). The FPT distributions without resetting for these three scenarios are given, respectively, by [38]

$$F_\tau(t) = \begin{cases} \mathrm{erfc}\Big(\sqrt{\frac{L^2}{4Dt}}\Big) & \text{in scenario (1)},\\ \big(\frac{a}{a+L}\big)\mathrm{erfc}\Big(\sqrt{\frac{L^2}{4Dt}}\Big) & \text{in scenario (2)},\\ \sqrt{\frac{4L^2}{\pi Dt}}\sum_{j=0}^\infty \exp\big[\frac{-(j+1/2)^2L^2}{Dt}\big] & \text{in scenario (3)},\end{cases}$$

where $\mathrm{erfc}(z)=(2/\sqrt{\pi})\int_z^\infty e^{-u^2}\,du$ denotes the complementary error function. Expanding these expressions as $t\to 0^+$ [and using that $\mathrm{erfc}(z)\sim e^{-z^2}/\sqrt{\pi z^2}$ as $z\to\infty$] and applying Theorem 3 then yields the formulas for p_0 in (34).

To obtain the exact distribution and moments used, we find the Laplace transform $\widetilde{S}(s)$ of the survival probability,

$$S(t) := \mathbb{P}(\tau>t) = 1-F_\tau(t),$$

for these three scenarios. Using that the Laplace transform of $\mathrm{erfc}(\sqrt{c/t})$ is $\int_0^\infty e^{-st}\mathrm{erfc}(\sqrt{c/t})\,dt = e^{-2\sqrt{cs}}/s$ for $c>0$, we obtain that the Laplace transforms of the survival probability for the first two scenarios are

$$\widetilde{S}(s) = \frac{e^{sL^2/D}}{s}, \quad \widetilde{S}(s) = \Big(\frac{a}{a+L}\Big)\frac{e^{sL^2/D}}{s}. \quad \text{(A39)}$$

For the third scenario of escape from a sphere, we first recall that the survival probability conditioned on the starting radius $X(0)=x\in[0,L]$ satisfies the backward Fokker-Planck equation,

$$\partial_t S = (D\partial_{xx}S + (2/x)\partial_x S), \quad x\in(0,L), \quad \text{(A40)}$$

with boundary condition $S=0$ at $x=L$ and unit initial condition. Laplace transforming (A40) yields a linear ordinary differential equation which can be solved to obtain the Laplace transform of the survival probability conditioned on starting at the center of the sphere,

$$\widetilde{S}(s) = \frac{1}{s}[1-(\sqrt{sL^2/D})\mathrm{csch}(\sqrt{sL^2/D})], \quad \text{(A41)}$$

where $\mathrm{csch}(z)=2/(e^z-e^{-z})$.

Having obtained the Laplace transform of the survival probability of the FPT with no resetting, the distribution of the FPT with resetting at rate $r>0$ [i.e., T in (12)] can be computed by numerical Laplace inversion with the general relation [18],

$$\widetilde{S}_{(r)}(s) := \int_0^\infty e^{-st}\mathbb{P}(T>t)\,dt = \frac{\widetilde{S}(r+s)}{1-r\widetilde{S}(r+s)}, \quad s\geqslant 0. \quad \text{(A42)}$$

Further, the moments of T can be obtained using (A42) and the general relation

$$\mathbb{E}[T^m] = m(-1)^{m-1}\frac{d^{m-1}}{ds^{m-1}}\widetilde{S}_{(r)}(s)\Big|_{s=0}.$$

b. Diffusive search with uniform initial conditions

For the example considered in Sec. III B of diffusive search in the interval $[0,L]$ with uniform initial and resetting conditions, recall that the survival probability conditioned on starting at $x\in[0,L]$ satisfies the backward Fokker-Planck equation,

$$\partial_t S = D\partial_{xx}S, \quad x\in(0,L), \quad \text{(A43)}$$

with absorbing boundary conditions $S=0$ at $x\in\{0,L\}$ and unit initial condition. Laplace transforming (A43), dividing by L, and integrating from $x=0$ to $x=L$ yields the Laplace transform of the survival probability conditioned on a uniformly distributed initial position,

$$\widetilde{S}(s) = \frac{1}{s}\bigg\{1-\frac{\tanh[\sqrt{sL^2/(4D)}]}{\sqrt{sL^2/(4D)}}\bigg\}. \quad \text{(A44)}$$

Taking $s\to\infty$ and using Tauberian theorems (see, for example, Theorem 3 in Sec. 5 of chapter 8 of [25]) yields the short-time behavior of $F_\tau(t)$ in (36), which then yields the asymptotic probability p_0 in (37) via Proposition 6. Further, the FPT distribution under stochastic resetting is then obtained via numerical Laplace inversion of (A44) using (A42).

For the example of diffusive exit from a sphere with uniform initial and resetting conditions, we Laplace transform (A40), solve the resulting linear ordinary differential equation analytically, multiply the solution by $(3/L^3)x^2$, and integrate from $x=0$ to $x=L$ to obtain the Laplace transform of the survival probability conditioned on a uniformly distributed initial position,

$$\widetilde{S}(s) = \frac{-3\sqrt{sL^2D}\coth(\sqrt{sL^2/D})+3D+sL^2}{(sL)^2}.$$

As above, taking $s \to \infty$ and using Tauberian theorems yields the short-time behavior of $F_\tau(t)$,

$$F_\tau(t) \sim \sqrt{\frac{36Dt}{\pi L^2}} \quad \text{as } t \to 0^+,$$

which then yields the following asymptotic probability via Proposition 6,

$$p \sim p_0 = 3\sqrt{D/(rL^2)} \quad \text{as } r \to \infty.$$

Further, the FPT distribution under stochastic resetting is then obtained via numerical Laplace inversion of (A44) using (A42).

c. Search on a discrete network

For the example considered in Sec. III C of search on a discrete network, recall that the dynamics of the continuous-time Markov chain X are described by the infinitesimal generator matrix $Q \in \mathbb{R}^{|I|\times|I|}$. [The entry $Q(i,j)$ in the ith column and jth row of Q is the jump rate from state i to state j if $i \neq j$ and the diagonal entries $Q(i,i)$ are chosen so that Q has zero row sums [39].]

To compute the survival probability of the FPT without resetting, $S(t) := \mathbb{P}(\tau > t)$, let the target be a single node, $I_{\text{target}} = i_{\text{target}} \in I$, and let $\widehat{Q}$ denote the matrix obtained by deleting the row and column in Q corresponding to i_{target}. Similarly, for an initial distribution ρ, let $\widehat{\rho}$ denote the vector obtained by deleting the entry in ρ corresponding to i_{target}. Then $S(t)$ is given by the sum of the entries in the vector $e^{W_0 t}\widehat{\rho}$, where W_0 is the transpose of $\widehat{Q}$ and $e^{W_0 t}$ is the matrix exponential [40]. In particular,

$$S(t) = \mathbf{1} \cdot e^{W_0 t}\widehat{\rho}, \tag{A45}$$

where $\cdot$ denotes the dot product and $\mathbf{1} \in \mathbb{R}^{|I|-1}$ is the vector of all ones. Taking the Laplace transform of (A45) then yields

$$\widetilde{S}(s) = \mathbf{1} \cdot (s\,\mathrm{id} - W_0)^{-1}\widehat{\rho}, \tag{A46}$$

where id denotes the identity matrix. The FPT distribution under stochastic resetting is then obtained via numerical Laplace inversion of (A46) using (A42).

The particular continuous-time Markov chains (i.e., choices of Q) used in Fig. 3(b) are created following a method used in [33]. Specifically, we first construct a graph by randomly connecting $|I| \gg 1$ vertices by approximately $5|I|$ edges. We then assign jump rates to each directed edge independently according to a uniform distribution. That is, $Q(i,i) \leqslant 0$ are chosen so that Q has zero row sums and the off-diagonal entries are

$$Q(i,j) = \begin{cases} U_{i,j} & \text{if there is a directed edge from } i \text{ to } j, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{U_{i,j}\}_{i,j\in I}$ are independent uniform random variables on [0,1].

d. Run and tumble

For the example considered in Sec. III D of a run-and-tumble search, the asymptotic probability p_0 in (39) is computed by using (8), integrating by parts, and changing variables to obtain

$$p = \int_0^\infty e^{-rt}\,dF_\tau(t) = re^{-rt_0}\int_0^\infty e^{-rt}F_\tau(t_0+t)\,dt,$$

where $t_0 = L/V > 0$ is the smallest possible value of τ (since the searcher starts distance $L > 0$ from the target and moves at a finite speed V). Next, observe that

$$\begin{aligned} F_\tau(t_0+t) &= \mathbb{P}(\tau \leqslant t_0+t) \\ &= \mathbb{P}(\tau = t_0) + \mathbb{P}(t_0 < \tau < t_0+t), \end{aligned}$$

since $\mathbb{P}(\tau < t_0) = \mathbb{P}(\tau = t_0+t) = 0$ for $t > 0$ and $\mathbb{P}(\tau = t_0) = \frac{1}{2}e^{-\lambda t_0}$ is the probability that the searcher starts in the $-V < 0$ velocity state and does not switch states before time t_0. Hence,

$$p = \frac{1}{2}e^{-(r+\lambda)t_0} + re^{-rt_0}\int_0^\infty e^{-rt}\mathbb{P}(t_0 < \tau < t_0+t)\,dt. \tag{A47}$$

Next, it was shown in Sec. 4.1 of [41] that

$$\mathbb{P}(t_0 < \tau < t_0+t) = \beta t + O(t^2) \quad \text{as } t \to 0^+,$$

where $\beta = \lambda e^{-\frac{\lambda L}{V}}(\lambda L + V)/(4V)$. Therefore applying Proposition 6 to the integral in (A47) yields (39).

To compute the exact FPT distribution, recall that the survival probability $S_\pm(t;x)$ conditioned on starting at $x \geqslant 0$ in the state moving to the right or left (denoted by $+$ or $-$) satisfies the backward Fokker-Planck equation,

$$\begin{aligned} \partial_t S_+ &= V\partial_x S_+ + \lambda(S_- - S_+), \\ \partial_t S_- &= -V\partial_x S_- + \lambda(S_+ - S_-), \end{aligned} \tag{A48}$$

with boundary condition $S_- = 0$ at $x = 0$, far-field condition $S_+ = 1$ as $x \to \infty$, and unit initial conditions. Laplace transforming (A48) yields a pair of linear ordinary differential equations which can be solved to obtain the Laplace transform of the survival probability conditioned on starting at $x = L$ with probability $1/2$ of being in either the $+$ or $-$ state,

$$\widetilde{S}(s) = \frac{1}{2\lambda s}e^{-\frac{L\sqrt{s(2\lambda+s)}}{V}}\left[2\lambda\left(e^{\frac{L\sqrt{s(2\lambda+s)}}{V}} - 1\right) + \sqrt{s(2\lambda+s)} - s\right]. \tag{A49}$$

The FPT distribution under stochastic resetting is then obtained via numerical Laplace inversion of (A49) using (A42).

e. Subdiffusive search

For the examples considered in Sec. III E of subdiffusive search, the exact FPT distribution with resetting is obtained via numerical Laplace inversion of (A39) and (A41) using (A42) and (43).

f. Nonexponential resetting

For the examples considered in Sec. III G, we have $F_\tau(t) = \mathrm{erfc}(\sqrt{\frac{L^2}{4Dt}})$. For sharp reset [see (44)], the probability of a successful search is $p = F_\tau(1/r)$. For uniform reset [see (45)], the probability of a successful search is

$$p = \int_0^\infty S_\sigma(t)\,dF_\tau(t) = \int_0^\infty F_\tau(t)\,dF_\sigma(t)$$
$$= \int_0^{2/r} \mathrm{erfc}\left(\sqrt{\frac{L^2}{4Dt}}\right)\frac{r}{2}\,dt$$
$$= [1 + rL^2/(4D)]\mathrm{erfc}[\sqrt{rL^2/(8D)}] - \sqrt{\frac{rL^2}{2\pi D}}e^{-rL^2/(8D)},$$

where the second equality follows from integration by parts. Similarly, for gamma reset [see (46)], the probability of a successful search is

$$p = \int_0^\infty F_\tau(t)\,dF_\sigma(t) = \int_0^\infty \mathrm{erfc}\left(\sqrt{\frac{L^2}{4Dt}}\right)4r^2te^{-2rt}\,dt$$
$$= e^{-\sqrt{2rL^2/D}}[1 + \sqrt{rL^2/(2D)}].$$

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