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A New Derivation of Robin Boundary Conditions through Homogenization of a Stochastically Switching Boundary*

Sean D. Lawley† and James P. Keener‡

Abstract. We give a new derivation of Robin boundary conditions and interface jump conditions for the diffusion equation in one dimension. To derive a Robin boundary condition, we consider the diffusion equation with a boundary condition that randomly switches between a Dirichlet and a Neumann condition. We prove that, in the limit of infinitely fast switching rate with the proportion of time spent in the Dirichlet state, denoted by ρ, approaching zero, the mean of the solution satisfies a Robin condition, with conductivity parameter determined by the rate at which ρ approaches zero. We carry out a similar procedure to derive an interface jump condition by considering the diffusion equation with a no flux condition in the interior of the domain that is randomly imposed/removed. Our results also provide the effective deterministic boundary condition for a randomly switching boundary.

Key words. Robin boundary conditions, interface jump conditions, stochastic hybrid systems, piecewise deterministic Markov process, random PDEs

AMS subject classifications. 35R60, 35A99, 37H99, 60J99, 92C40

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1. Introduction.

The Robin boundary condition (also known as third type, impedance, radiation, convective, partially absorbing, or reactive condition) for a partial differential equation (PDE) enjoys a long history in science and engineering applications. While the Dirichlet boundary condition specifies the value of the solution and the Neumann boundary condition specifies the value of the derivative of the solution, the Robin boundary condition specifies a relationship between the solution and its derivative. It arises in a variety of physical situations and is often used as a way to homogenize complex boundary dynamics. In biology and chemistry, the Robin condition for the diffusion equation is often invoked to represent reactive or semipermeable boundaries. Similarly, interface jump conditions are widely used in applications and are typically needed when the medium of diffusion has sharply changing properties.

In this paper, we give a new derivation of Robin boundary conditions and interface jump conditions for the diffusion equation in one dimension. To derive a Robin boundary condition, we consider the diffusion equation with a boundary condition that randomly switches between a Dirichlet and a Neumann condition. We prove that, in the limit of infinitely fast switching rate with the proportion of time spent in the Dirichlet state, denoted by ρ, approaching zero,

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†Department of Mathematics, University of Utah, Salt Lake City, UT 84112 (lawley@math.utah.edu). This author was supported by NSF grant DMS-RTG 1148230.

‡Department of Mathematics and Department of Bioengineering, University of Utah, Salt Lake City, UT 84112 (keener@math.utah.edu). This author was supported by NSF grants DMS 1122297 and DMS-RTG 1148230.

the mean of the solution satisfies a Robin condition, with conductivity parameter determined by the rate at which ρ approaches zero. Furthermore, our results give the effective or homogenized deterministic boundary condition for a randomly switching boundary, thus providing a tractable boundary condition for applications involving a switching boundary and giving a precise answer to the following question: "What is the average of a Dirichlet and a Neumann condition?" In addition, we carry out a similar procedure to derive an interface jump condition. That is, we consider the diffusion equation with a no flux condition in the interior of the domain that is randomly imposed/removed and show that the mean of the solution approximately satisfies an interface jump condition.

From a theoretical standpoint, our results are of interest as we provide a completely new derivation and thus interpretation of the classical Robin boundary and interface jump conditions. We show that, in fact, any Robin condition *is* the mean of a condition that randomly switches between Dirichlet and Neumann. In comparison, a number of other works derive Robin boundary conditions by homogenizing a mixed boundary value problem, where the boundary contains alternating Dirichlet and Neumann conditions [2], [3], [9], [10], [11], [12], [18]. In such problems, the Dirichlet and Neumann conditions alternate *in space* as different conditions are imposed on different pieces of the boundary. In our derivation, a Robin condition is produced through Dirichlet and Neumann conditions that alternate *in time* instead of space.

Furthermore, other works derive Robin boundary conditions based on the stochastic trajectories of individual diffusing particles. In such settings, when a particle is near the boundary, it is envisioned as being either absorbed with some probability or otherwise reflected. Erban and Chapman [8] and Singer et al. [21] have given mathematical justification and precision to this derivation. In contrast, in our derivation the fate of a particle at the boundary is determined by the state of a Markov process.

There is one recurring theme in derivations of Robin conditions which our method further highlights. In the derivations mentioned above involving mixed boundary value problems that impose either Dirichlet or Neumann conditions on different pieces of the boundary, a Robin condition is recovered only if (a) the size of each piece of the boundary goes to zero, and (b) the proportion of the boundary with a Neumann condition goes to one. A similar phenomena occurs in the derivations mentioned above based on the trajectories of individual particles that, when near the boundary, are either absorbed with some probability or otherwise reflected. Such derivations start with a discretized random walk for a particle's trajectory, and a Robin condition is recovered only if (a) the discrete step size goes to zero, and (b) the probability of reflection goes to one. In our derivation, in order to recover a Robin condition we must take (a) the rate of boundary switching to infinity, and (b) the proportion of time in the Neumann state to one. In all three derivations, if (a) holds but (b) does not, then the resulting condition is pure Dirichlet.

From an applied perspective, our results provide a homogenization technique for a problem with a switching boundary. In [7], the authors give the boundary value problems satisfied by the mean of a PDE with a switching boundary, but finding explicit solutions to these problems is often impossible, and thus our results provide a needed tractable alternative. Indeed, a growing number of applied problems, especially in biology, couple diffusion with a switching boundary. For example, several works—often in the context of biochemical reactions—study the escape of diffusing molecules in the presence of a switching boundary in which the molecules can exit only when the boundary is in a particular state [5], [6], [16], [23], [24]. Similarly, other works study the escape of diffusing molecules when the molecules themselves switch states and can exit only in a particular state [1], [4], [19], [20], [22]. In addition, the membrane voltage fluctuations of a single neuron due to diffusion of ions through stochastically gated channels continues to be an important problem [13], [17]. Furthermore, the diffusion equation with a switching boundary has been proposed as a model for other very diverse biological processes such as the modulation of neurotransmitter concentration in the brain and insect respiration [15]. Indeed, our results apply naturally to insect respiration, as the switching is much faster than the other timescales in that problem (see [15] for more details).

The rest of the paper is organized as follows. In section 2, we set up the switching PDEs and outline our main results. In section 3, we give the boundary value problems that the means of the switching PDEs satisfy. In section 4, we prove that the mean of the process with a randomly switching boundary condition approximately satisfies a Robin boundary condition. In section 5, we prove that the mean of the process with a randomly imposed no flux condition in the interior of the domain approximately satisfies an interface jump condition.

2. Main results.

Let $J_t \in \{0, 1\}$ be a continuous time Markov jump process with jump rate from state 0 to state 1 given by $k_+ := D(1-\rho)/(\varepsilon L^2)$ and jump rate from state 1 to state 0 given by $k_- := D\rho/(\varepsilon L^2)$:

$$0 \underset{k_-}{\overset{k_+}{\rightleftharpoons}} 1.$$

The dimensionless parameter $0 < \rho < 1$ specifies the proportion of time J_t spends in the 0 state, and $\varepsilon > 0$ is a dimensionless parameter that scales the jump rate. Suppose J_t starts in its invariant distribution: $\mathbb{P}(J_0 = 0) = \rho$. The process J_t controls the state of the boundary (or interface) in the PDEs we consider.

2.1. Robin condition.

Let $u^\varepsilon(x,t) : [0, L] \times [0, \infty) \to \mathbb{R}^2$ satisfy the diffusion equation with an absorbing condition, $u^\varepsilon = 0$, at $x = 0$ and a condition at $x = L$ that switches between a Dirichlet condition, $u^\varepsilon = \eta > 0$, and a Neumann condition, $\partial_x u^\varepsilon = \delta > 0$, as J_t jumps between 0 and 1. That is, u^ε satisfies

$$\begin{aligned} \partial_t u^\varepsilon &= D\Delta u^\varepsilon, \quad x \in (0, L),\ t > 0, \\ u^\varepsilon(0, t) &= 0, \end{aligned} \tag{2.1}$$

with switching boundary condition at $x = L$:

$$(1 - J_t)(u^\varepsilon(L, t) - \eta) + J_t(\partial_x u^\varepsilon(L, t) - \delta) = 0. \tag{2.2}$$

We show that the mean of this stochastic process approximately satisfies a Robin boundary condition at $x = L$. That is, if $v^\varepsilon(x,t) := \mathbb{E}[u^\varepsilon(x,t)]$, then v^ε satisfies (2.1) and

$$|v^\varepsilon(x,t) - w^\varepsilon(x,t)| \to 0 \quad \text{uniformly in } x \text{ as } \varepsilon \to 0,$$

where w^ε satisfies (2.1) with the Robin condition at $x = L$:

$$\rho\big(w^\varepsilon(L,t) - \eta\big) + L\sqrt{\varepsilon}(1-\rho)\big(\partial_x w^\varepsilon(L,t) - \delta\big) = 0. \tag{2.3}$$

Thus, we obtain a general Robin condition

$$w^\varepsilon(L,t) + C\partial_x w^\varepsilon(L,t) = \eta$$

for any $C > 0$ as the mean of a randomly switching boundary condition by setting

$$\rho = \Big(\frac{C}{L\sqrt{\varepsilon}} + 1\Big)^{-1},$$

$\delta = 0$, and taking $\varepsilon \to 0$. In this case, the switching rate goes to infinity and the proportion of time in the Neumann state goes to one.

We also remark on the form of the conditions in (2.2) and (2.3). While $\mathbb{E}[J_t] = 1 - \rho$, naively taking the expected value of the condition in (2.2) does not yield the correct effective condition in (2.3). The effective condition in (2.3) is the expected value of (2.2) with a correcting factor of $L\sqrt{\varepsilon}$.

2.2. Interface condition. Similarly, we can derive a general interface jump condition as the mean of a solution to a PDE with a randomly imposed no flux condition in the interior of the domain. Suppose u^ε satisfies the diffusion equation on $[0, L]$ with deterministic boundary conditions, but with a no flux condition at $x = L/2$ that is imposed only when $J_t = 1$. That is, u^ε satisfies

$$\begin{aligned} \partial_t u^\varepsilon &= D\Delta u^\varepsilon, \quad x \in (0, L/2) \cup (L/2, L),\ t > 0, \\ u^\varepsilon(0,t) &= 0, \quad \text{and} \quad u^\varepsilon(L,t) = \eta, \end{aligned} \tag{2.4}$$

with a randomly imposed no flux condition at $x = L/2$:

$$J_t \partial_x u^\varepsilon(L/2, t) = 0.$$

We show that the mean of this stochastic process approximately satisfies an interface jump condition at $x = L/2$. That is, if $v^\varepsilon(x,t) := \mathbb{E}[u^\varepsilon(x,t)]$, then v^ε satisfies (2.4) and

$$|v^\varepsilon(x,t) - w^\varepsilon(x,t)| \to 0 \quad \text{uniformly in } x \text{ as } \varepsilon \to 0,$$

where w^ε satisfies (2.4) with an interface jump condition at $x = L/2$:

$$w_+^\varepsilon - w_-^\varepsilon = 2L\sqrt{\varepsilon}\Big(\frac{1-\rho}{\rho}\Big)\partial_x w_+^\varepsilon,$$

where $w_+^\varepsilon := \lim_{x \to L/2+} w^\varepsilon(x)$ and $w_-^\varepsilon := \lim_{x \to L/2-} w^\varepsilon(x)$. Thus, we obtain a general interface jump condition

$$w_+^\varepsilon - w_-^\varepsilon = C\partial_x w_+^\varepsilon$$

for any $C > 0$ as the mean of a solution with a randomly imposed no flux condition in the interior of the domain by setting

$$\rho = \Big(\frac{C}{2L\sqrt{\varepsilon}} + 1\Big)^{-1}$$

and taking $\varepsilon \to 0$.

3. First moment equations. For convenience, from now on we suppress the explicit ε dependence in our functions; i.e., we now write u for u^ε, w for w^ε, and so on.

3.1. Boundary switching. Since we can always define new time and space variables, $\frac{D}{L^2}t$ and $\frac{1}{L}x$, we henceforth take $L = D = 1$ without loss of generality. Under that rescaling, a Neumann condition of δ becomes $L\delta$, so let u be the stochastic process defined in section 2.1 with $L = D = 1$, and let the Neumann condition δ be replaced by $L\delta$. Suppose u has initial condition $u(x, 0) = \varphi(x)$ for some given $\varphi \in L^2[0, 1]$.

Define the deterministic functions

$$v_0(x,t) := \mathbb{E}[u(x,t)1_{J_t=0}] \quad \text{and} \quad v_1(x,t) := \mathbb{E}[u(x,t)1_{J_t=1}],$$

where $1_{\{\cdot\}}$ denotes the indicator function. Observe that

$$\mathbb{E}[u(x,t)] = v_0(x,t) + v_1(x,t).$$

The following proposition gives the boundary value problem that v_0 and v_1 satisfy. It follows immediately from [7].

Proposition 3.1. *The functions*

$$\begin{pmatrix} v_0(x,t) \\ v_1(x,t) \end{pmatrix} : [0,1] \times [0,\infty) \to \mathbb{R}^2$$

satisfy

$$\partial_t \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \Delta \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} + \frac{1}{\varepsilon} \begin{pmatrix} \rho - 1 & \rho \\ 1 - \rho & -\rho \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \quad x \in (0,1),\ t > 0, \tag{3.1}$$

subject to boundary conditions

$$v_0(0,t) = v_1(0,t) = 0, \quad v_0(1,t) = \rho\eta, \quad \textit{and} \quad \partial_x v_1(1,t) = (1-\rho)L\delta$$

and initial conditions $v_0(x,0) = \rho\varphi(x)$ *and* $v_1(x,0) = (1-\rho)\varphi(x)$.

3.2. Interface switching. Let u be the stochastic process defined in section 2.2 with $L = D = 1$ subject to the initial condition $u(x, 0) = \varphi(x)$ for some given $\varphi \in L^2[0, 1]$.

As above, we define the deterministic functions as

$$v_0(x,t) := \mathbb{E}[u(x,t)1_{J_t=0}] \quad \text{and} \quad v_1(x,t) := \mathbb{E}[u(x,t)1_{J_t=1}].$$

The following proposition is directly analogous to Proposition 3.1 and is included as an example in [14]. Equations (3.2) and (3.4) follow from an interchange of limits, and (3.3) is immediate. This interchange amounts to checking the hypotheses of the dominated convergence theorem, which follow from standard estimates for the heat equation.

Proposition 3.2. *The functions*

$$\begin{pmatrix} v_0(x,t) \\ v_1(x,t) \end{pmatrix} : [0,0.5) \cup (0.5,1] \times [0,\infty) \to \mathbb{R}^2$$

satisfy

$$\partial_t \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \Delta \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} + \frac{1}{\varepsilon} \begin{pmatrix} \rho - 1 & \rho \\ 1-\rho & -\rho \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \quad x \in (0, 0.5) \cup (0.5, 1),\ t > 0, \tag{3.2}$$

subject to boundary conditions

$$v_0(0,t) = v_1(0,t) = 0, \quad v_0(1,t) = \rho\eta, \quad \textit{and} \quad v_1(1,t) = (1-\rho)\eta, \tag{3.3}$$

and interface conditions

$$v_{0+} = v_{0-}, \quad \partial_x v_{0+} = \partial_x v_{0-}, \quad \textit{and} \quad \partial_x v_{1+} = \partial_x v_{1-} = 0, \tag{3.4}$$

where $f_{\pm} := \lim_{x \to 0.5\pm} f(x)$, *and* $v_0(x,0) = \rho\varphi(x)$ *and* $v_1(x,0) = (1-\rho)\varphi(x)$.

4. The mean approximately satisfies a Robin condition. Since the matrix in (3.1) has zero column sums, the mean $\mathbb{E}[u] = v_0 + v_1$ satisfies the diffusion equation. The following theorem determines the Robin condition that the mean approximately satisfies at $x = 1$.

Theorem 4.1. *Suppose* $w(x,t) : [0,1] \times [0,\infty) \to \mathbb{R}$ *satisfies*

$$\partial_t w = \Delta w, \quad x \in (0,1),\ t > 0,$$

with boundary conditions

$$w(0,t) = 0, \quad \rho\big(w(1,t) - \eta\big) + \sqrt{\varepsilon}(1-\rho)\big(w_x(1,t) - L\delta\big) = 0 \tag{4.1}$$

and initial condition $w(x,0) = \varphi(x)$. *Let* v_0 *and* v_1 *be as in Proposition* 3.1. *Then for each* $t \geq 0$ *and* $q > 0$, *there exists an* M *and an* $\varepsilon_0 > 0$ *such that*

$$|v_0(x,t) + v_1(x,t) - w(x,t)| \leq M\varepsilon^{3/2-q} \tag{4.2}$$

for all $0 < \varepsilon < \varepsilon_0$ *and* $x \in [0,1]$.

Since the coefficient in the Robin condition in (4.1) is multiplied by $\sqrt{\varepsilon}$, it vanishes in the limit that $\varepsilon \to 0$. But, if ρ, the proportion of time in the Dirichlet state, is $O(\sqrt{\varepsilon})$, then the Robin condition is recovered. Indeed, if we let ρ depend on ε, then we can obtain any coefficient in the Robin boundary condition.

Theorem 4.2. *Let* $C > 0$, *and suppose* $w(x,t) : [0,1] \times [0,\infty) \to \mathbb{R}$ *satisfies*

$$\partial_t w = \Delta w, \quad x \in (0,1),\ t > 0,$$

with boundary conditions

$$w(0,t) = 0, \quad w(1,t) + Cw_x(1,t) = \eta \tag{4.3}$$

and initial condition $w(x,0) = \varphi(x)$. *Assume* v_0 *and* v_1 *are as in Proposition* 3.1, *with*

$$\rho = \left(\frac{C}{\sqrt{\varepsilon}} + 1\right)^{-1} \quad \textit{and} \quad \delta = 0.$$

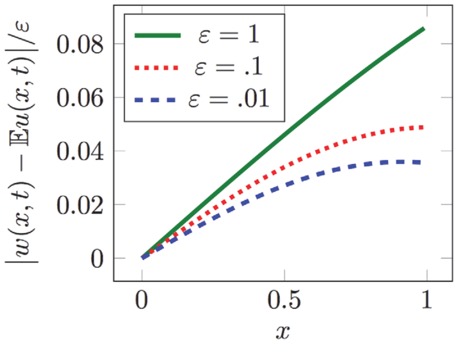

Figure 1. *Using the notation of Theorem 4.2, we plot* $\varepsilon^{-1}|w(x,t)-\mathbb{E}u(x,t)| = \varepsilon^{-1}|w(x,t)-v_0(x,t)-v_1(x,t)|$ *as a function of* x *for different values of* ε *with* $t = \eta = 1$*. Plots for* $\varepsilon = 10^{-3}$, 10^{-4}*, and* 10^{-5} *are not shown, as they are indistinguishable from the plot for* $\varepsilon = 10^{-2}$*. This suggests that the convergence rate in (4.4) is sharp for* $q = 0$ *and some* $M < .04$*.*

Then for each $t \geq 0$ *and* $q > 0$*, there exists an* M *and an* $\varepsilon_0 > 0$ *such that*

$$|v_0(x,t) + v_1(x,t) - w(x,t)| \leq M\varepsilon^{1-q} \tag{4.4}$$

for all $0 < \varepsilon < \varepsilon_0$ *and* $x \in [0,1]$*.*

Figure 1 illustrates Theorem 4.2 numerically and suggests that the convergence rate in (4.4) is sharp for $q = 0$ and some $M < .04$.

We prove Theorems 4.1 and 4.2 using a series of lemmas. Our strategy is as follows. First, we make a change of coordinates in $\mathbb{R}^2$ to decompose v_0 and v_1 into a pair of functions, α and β, with the property that $\alpha = v_0 + v_1$ and β is a function whose most significant variation is in a boundary layer near $x = L$. Then we show that the steady states of α and w are close to each other. To complete the argument, we show that the transient parts of α and w are close to each other by looking at their spectral decompositions. More specifically, if α^d and w^d are the transient parts of α and w, then Lemma 4.3 below gives

$$|v_0 + v_1 - w| = |\alpha^d - w^d| + O(\varepsilon^n) \quad \text{for each } n.$$

The remaining lemmas collect the necessary estimates on $|\alpha^d - w^d|$ to complete the proof.

Lemma 4.3. *If* v_0 *and* v_1 *are as in Proposition 3.1,* w *is as in Theorems 4.1 and 4.2, and we define* α *and* β *by the equation*

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \alpha \begin{pmatrix} \rho \\ 1-\rho \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

then for each n*, we have that*

$$\alpha(x,t) - w(x,t) = \alpha^d(x,t) - w^d(x,t) + O(\varepsilon^n), \tag{4.5}$$

where α^d *and* β^d *satisfy*

$$\begin{aligned} &\alpha^d_t = \alpha^d_{xx} \quad \text{and} \quad \beta^d_t = \beta^d_{xx} - \frac{1}{\varepsilon}\beta^d && x \in (0,1),\ t > 0, \\ &\alpha^d = \beta^d = 0 && \text{at } x = 0, \\ &\rho\alpha^d + \beta^d = 0 \quad \text{and} \quad (1-\rho)\alpha^d_x - \beta^d_x = 0 && \text{at } x = 1, \end{aligned} \tag{4.6}$$

and w^d satisfies

$$w^d_t = w^d_{xx}, \quad x \in (0,1),\ t > 0,$$
$$w^d(0,t) = 0, \quad w^d(1,t) + \sqrt{\varepsilon}\left(\frac{1-\rho}{\rho}\right) w^d_x(1,t) = 0. \tag{4.7}$$

Further, if we denote the initial conditions by

$$\varphi^\alpha(x) = \alpha^d(x,0) \quad \textit{and} \quad \varphi^w(x) = w^d(x,0),$$

then for each n, we have that

$$\varphi^\alpha(x) = \varphi^w(x) + O(\varepsilon^n). \tag{4.8}$$

Proof. It is straightforward to check that α and β can be decomposed into a sum of the following steady state and decay parts:

$$\alpha(x,t) = C_1 x + \alpha^d(x,t),$$
$$\beta(x,t) = C_2 \sinh(\gamma x) + \beta^d(x,t), \tag{4.9}$$

where $\gamma = 1/\sqrt{\varepsilon}$,

$$C_1 = \frac{\eta + L\delta\frac{1-\rho}{\rho\gamma}\tanh(\gamma)}{1 + \frac{1-\rho}{\rho\gamma}\tanh(\gamma)}, \quad \text{and} \quad C_2 = \frac{(1-\rho)(C_1 - L\delta)}{\gamma\cosh(\gamma)},$$

and that the decay parts, α^d and β^d, satisfy the boundary value problem in (4.6).

Further, one can quickly check that w can be decomposed into the following sum of steady state and decay parts:

$$w(x,t) = Bx + w^d(x,t), \quad \text{where} \quad B = \frac{\eta + L\delta\frac{1-\rho}{\rho\gamma}}{1 + \frac{1-\rho}{\rho\gamma}}, \tag{4.10}$$

and that w^d satisfies the boundary value problem given in (4.7). It is easy to check that $|B - C_1| = O(\varepsilon^n)$ for each n, and thus (4.5) and (4.8) hold, even in the case that $\rho = O(\sqrt{\varepsilon})$. ∎

Now that we have established that $|v_0 + v_1 - w| = |\alpha - w| = |\alpha^d - w^d| + O(\varepsilon^n)$, it remains to bound $|\alpha^d - w^d|$. We do this by analyzing the spectral decompositions of α^d and w^d. First, we verify that the differential operator associated with the boundary value problem for α^d is self-adjoint.

Lemma 4.4. *Let $\mathcal{L}$ be the operator corresponding to the boundary value problem in (4.6). That is, define*

$$\mathcal{L}\begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} f'' \\ g'' - g/\varepsilon \end{pmatrix}$$

for (f,g) *in the domain of* $\mathcal{L}$ *given by*

$$\{f, g \in H^2(0,1) : f(0) = g(0) = \rho f(1) + g(1) = (1-\rho)f'(1) - g'(1) = 0\}. \tag{4.11}$$

Then $\mathcal{L}$ *is self-adjoint on the direct sum* $L^2[0,1] \oplus L^2[0,1]$ *with the inner product*

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix} \right\rangle := \kappa \int_0^1 f(x)\hat{f}(x)\, dx + (1-\kappa)\int_0^1 g(x)\hat{g}(x)\, dx, \tag{4.12}$$

where

$$\kappa = 1 - \frac{1}{1+\rho(1-\rho)}. \tag{4.13}$$

Furthermore, $\mathcal{L}$ *is negative definite.*

Proof. We first check that $\mathcal{L}$ is self-adjoint. Let (f,g) and $(\hat{f},\hat{g})$ be in the domain of $\mathcal{L}$, and observe that integrating by parts twice gives

$$\begin{aligned}\left\langle \mathcal{L}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \mathcal{L}\begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix} \right\rangle &+ \kappa\big[f'(1)\hat{f}(1) - f(1)\hat{f}'(1)\big] \\ &+ (1-\kappa)\big[g'(1)\hat{g}(1) - g(1)\hat{g}'(1)\big],\end{aligned}$$

where we have used that (f,g) and $(\hat{f},\hat{g})$ vanish at $x=0$. Further, by our choice of the domain of $\mathcal{L}$ and the factor κ, we have that

$$\begin{aligned}&\kappa\big[f'(1)\hat{f}(1) - f(1)\hat{f}'(1)\big] + (1-\kappa)\big[g'(1)\hat{g}(1) - g(1)\hat{g}'(1)\big] \\ &\quad = (\kappa - (1-\kappa)(1-\rho)\rho)\big[f'(1)\hat{f}(1) - f(1)\hat{f}'(1)\big] = 0.\end{aligned} \tag{4.14}$$

Hence, $\mathcal{L}$ is symmetric.

To show that $\mathcal{L}$ is self-adjoint, we need only show that the domain of the adjoint of $\mathcal{L}$ is contained in the domain of $\mathcal{L}$. Let (f_0, f_1) be in the domain of the adjoint of $\mathcal{L}$. Thus, if φ and ψ are both in $C_0^\infty(0,1)$ (and thus in the domain of $\mathcal{L}$), then by the definition of adjoint there exists a pair of $L^2[0,1]$ functions g_0 and g_1 such that

$$\kappa\int_0^1 \Delta\varphi f_0\, dx + (1-\kappa)\int_0^1 \Delta\psi f_1\, dx = \kappa\int_0^1 \varphi g_0\, dx + (1-\kappa)\int_0^1 \psi g_1\, dx.$$

Taking $\varphi \equiv 0$ shows that f_1 has a weak second derivative, and taking $\psi \equiv 0$ shows that f_0 has a weak second derivative. Since f_0 and f_1 are both functions of only one variable, they trivially have all their weak derivatives of order two, and thus have weak first derivatives and thus are in $H^2(0,1)$. Showing that (f_0, f_1) satisfy the correct boundary conditions follows immediately from the definition of adjoint, integration by parts twice, and our choice of κ.

To check that $\mathcal{L}$ is negative definite, let (f,g) be in the domain of $\mathcal{L}$, and observe that integrating by parts once gives

$$\begin{aligned}\left\langle \mathcal{L}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle = \kappa&\left[f'(1)f(1) - \int_0^1 (f'(x))^2\, dx\right] \\ &+ (1-\kappa)\left[g'(1)g(1) - \int_0^1 (g'(x))^2\, dx - \frac{1}{\varepsilon}\int_0^1 (g(x))^2\, dx\right],\end{aligned}$$

where we have used that (f,g) vanishes at $x=0$. As in (4.14), the boundary terms cancel since (f,g) is in the domain of $\mathcal{L}$ and by the choice of κ. Hence,

$$\left\langle \mathcal{L}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle = -\int_0^1 \left[(f'(x))^2 + (g'(x))^2 + (g(x))^2\right] dx \leq 0.$$

If the above integral is 0, then $f'(x) = g'(x) = g(x) = 0$ for all $x \in [0,L]$ since (f,g) is in the domain of $\mathcal{L}$, and thus they are both continuously differentiable. But then $f(x) = 0$ for all $x \in [0,L]$ since $f' = 0$ for all $x \in [0,L]$ and $\rho f(1) = -g(1) = 0$. Hence, $\mathcal{L}$ is negative definite. ■

The next two lemmas collect information about the spectral decompositions of w^d and α^d.

Lemma 4.5. *Let $w^d(x,t)$ be as in Lemma 4.3. Then, for each $t > 0$, it is given by the uniformly convergent series*

$$w^d(x,t) = \sum_{n\geq 1} b_n e^{-\bar{\mu}_n^2 t} \sin(\bar{\mu}_n x), \tag{4.15}$$

where $0 < \bar{\mu}_1 < \bar{\mu}_2 < \cdots$ are the solutions of

$$\tan(\bar{\mu}) = -\frac{1-\rho}{\gamma\rho}\bar{\mu}, \tag{4.16}$$

where $\gamma = 1/\sqrt{\varepsilon}$, and

$$b_n = \int_0^1 \varphi^w(x) \sin(\bar{\mu}_n x)\, dx \Big/ \int_0^1 \sin^2(\bar{\mu}_n x)\, dx. \tag{4.17}$$

Further, we have that

$$|b_n| \leq 4\|\varphi^w\|_{L^2}. \tag{4.18}$$

Finally, if $n \neq k$, then

$$|\bar{\mu}_n - \bar{\mu}_k| \geq \pi/2. \tag{4.19}$$

Proof. Deriving the formulas in (4.16) and (4.17) is routine. The bound on $|b_n|$ in (4.18) follows from Holder's inequality and the bound

$$\int_0^1 \sin^2(\bar{\mu}_n x)\, dx = 0.5 - \frac{\sin(2\bar{\mu}_n)}{4\bar{\mu}_n} \geq 1/4,$$

since $\bar{\mu}_n \geq \bar{\mu}_1 \geq \pi/2$ because the left-hand side of (4.16) is positive for $0 < \bar{\mu} < \pi/2$ and the right-hand side of (4.16) is negative for $\bar{\mu} > 0$.

To see why (4.19) holds, observe that for each positive integer n, the interval $I_n := (n\pi + \pi/2, (n+1)\pi + \pi/2)$ contains at most one solution to (4.16) since $\tan(\mu)$ is a strictly increasing function of μ on I_n and $-(1-\rho)/(\gamma\rho)\mu$ is a strictly decreasing function of μ. Further, a solution to (4.16) in I_n must occur in the left half of I_n since $-(1-\rho)/(\gamma\rho)\mu < 0$ and $\tan(\mu)$ is negative only on the left half of I_n. Thus, if $n \neq k$, then $\bar{\mu}_n$ must be in the left half of some I_i, and $\bar{\mu}_k$ in the left half of some I_j with $i \neq j$. But all such points are separated by at least $\pi/2$, and thus (4.19) holds.

The uniform convergence of (4.15) for $t > 0$ follows immediately from (4.18) and (4.19). ■

Lemma 4.6. *Let $\alpha^d(x,t)$ be as in Lemma 4.3. Then, for each $t > 0$, it is given by the uniformly convergent series*

$$\alpha^d(x,t) = \sum_{n\geq 1} c_n e^{-\mu_n^2 t} \sin(\mu_n x),$$

where $0 < \mu_1 < \mu_2 < \cdots$ are the solutions to either

$$\tan(\mu) = -\frac{1-\rho}{\sqrt{\gamma^2-\mu^2}\rho} \tanh(\sqrt{\gamma^2-\mu^2})\mu \tag{4.20}$$

or

$$\tan(\mu) = -\frac{1-\rho}{\sqrt{\mu^2-\gamma^2}\rho} \tan(\sqrt{\mu^2-\gamma^2})\mu, \tag{4.21}$$

depending on whether $\mu_n^2 < \gamma^2$ or $\mu_n^2 > \gamma^2$, where $\gamma = 1/\sqrt{\varepsilon}$. If ε is such that $\tan(\gamma) = -((1-\rho)/\rho)\gamma$, then there is one value $\mu_n = \gamma$. Further,

$$c_n = d_n \int_0^1 \varphi^\alpha(x) \sin(\mu_n x)\, dx, \tag{4.22}$$

where

$$d_n = \left(\frac{\mu_n \rho \sin^2(\mu_n)(\sinh(2\lambda_n) - 2\lambda_n)}{\lambda_n(\rho-1)\sinh^2(\lambda_n)(\sin(2\mu_n) - 2\mu_n)} + 1 \right)^{-1}, \tag{4.23}$$

if $\mu_n^2 < \gamma^2$, where $\lambda_n = \sqrt{\gamma^2 - \mu_n^2}$. Further, for all $n \geq 1$, we have that

$$|c_n| \leq 4\|\varphi^\alpha\|_{L^2}. \tag{4.24}$$

Further, if $n \neq k$ are such that $\max\{\mu_n^2, \mu_k^2\} < \gamma^2$, then

$$|\mu_n - \mu_k| \geq \pi/2. \tag{4.25}$$

Finally, if n is a positive integer, then there are at most two solutions to (4.21) on the interval $(n\pi + \pi/2, (n+1)\pi + \pi/2)$.

Proof. Since $\mathcal{L}$ is self-adjoint and negative definite, we seek eigenvalues $-\mu^2 < 0$ and eigenfunctions $\bar{\alpha}$ and $\bar{\beta}$, satisfying

$$\begin{aligned} \bar{\alpha}'' + \mu^2 \bar{\alpha} &= 0, \\ \bar{\beta}'' + (\mu^2 - 1/\varepsilon)\bar{\beta} &= 0. \end{aligned} \tag{4.26}$$

If $\mu^2 - 1/\varepsilon < 0$, then the eigenfunctions are of the form $\bar{\alpha} = A\sin(\mu x)$ and $\bar{\beta} = B\sinh(\lambda x)$, where

$$\lambda = \sqrt{\gamma^2 - \mu^2}, \tag{4.27}$$

with $\gamma = 1\sqrt{\varepsilon}$. In order for the eigenfunctions to satisfy the boundary conditions, we need

$$\rho A\sin(\mu) + B\sinh(\lambda) = 0, \tag{4.28}$$

$$(1-\rho)\mu A\cos(\mu) - B\lambda\cosh(\lambda) = 0. \tag{4.29}$$

Since we seek nonzero A and B, we need the determinant of the matrix in (4.28) and (4.29) to be zero. Equivalently, we need

$$-\rho\sin(\mu)\lambda\cosh(\lambda) - (1-\rho)\mu\cos(\mu)\sinh(\lambda) = 0,$$

which can be written as

$$\tan(\mu) = -\frac{1-\rho}{\lambda\rho}\tanh(\lambda)\mu = -\frac{1-\rho}{\rho\sqrt{\gamma^2-\mu^2}}\tanh(\sqrt{\gamma^2-\mu^2})\mu. \tag{4.30}$$

For each $\varepsilon > 0$, this equation has a finite number of solutions μ satisfying $\mu^2 - 1/\varepsilon < 0$. Let $\mu_n^2 < \gamma^2$ denote the nth positive solution to (4.30) and λ_n be defined from (4.27) with respect to μ_n. Hence, the eigenfunctions are of the form $\alpha_n := A_n\sin(\mu_n x)$ and $\beta_n := B_n\sinh(\lambda_n x)$.

Now, we want the eigenfunctions (α_n, β_n) to have norm equal to one with respect to the inner product defined in (4.12). That is, we want

$$\kappa\int_0^1 A_n^2\sin^2(\mu_n x)\,dx + (1-\kappa)\int_0^1 B_n^2\sinh^2(\lambda_n x)\,dx = 1, \tag{4.31}$$

where κ is defined in (4.13). Since (4.28) implies that $A_n = -B_n\sinh(\lambda)/(\rho\sin(\mu))$, it follows from (4.31) that

$$B_n = \frac{1}{K_n} \quad \text{and} \quad A_n = \frac{-\sinh(\lambda_n)}{K_n\rho\sin(\mu_n)}, \tag{4.32}$$

where

$$K_n = \sqrt{\kappa\left(\frac{\sinh(\lambda_n)}{\rho\sin(\mu_n)}\right)^2\left[\frac{1}{2} - \frac{\sin(2\mu_n)}{4\mu_n}\right] + (1-\kappa)\left[\frac{\sinh(2\lambda_n)}{4\lambda_n} - \frac{1}{2}\right]}, \tag{4.33}$$

which comes from explicitly evaluating the integrals in (4.31).

It is easy to see from (4.26) that all the normalized eigenfunctions for α^d are of the form $A_n\sin(\mu_n x)$, even for $\mu_n^2 \geq \gamma^2$. So, putting this together, we have that

$$\alpha^d(x,t) = \sum_{n\geq 1} a_n e^{-\mu_n^2 t} A_n\sin(\mu_n x), \tag{4.34}$$

where the a_n's are chosen to meet the initial conditions. To find a_n, we note, by the self-adjointness of $\mathcal{L}$, that its normalized eigenfunctions form an orthonormal basis for the direct sum $L^2[0,1]$ with $L^2[0,1]$. Thus, for each $f, g \in L^2[0,1]$, we have that

$$\begin{pmatrix} f \\ g \end{pmatrix} = \sum_{n=1}^{\infty} \left\langle \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix},$$

where the inner product $\langle \cdot, \cdot \rangle$ is defined by (4.12). Since this holds for each $g \in L^2[0,1]$, it must hold for $g = 0$. Thus, for n such that $\mu_n^2 < \gamma^2$, we have from (4.34) that

$$a_n = A_n \kappa \int_0^1 \varphi^\alpha(x) \sin(\mu_n x)\, dx. \tag{4.35}$$

Since $c_n = a_n A_n$, (4.22) and (4.23) follow.

To obtain the bound on $|c_n|$ in (4.24), observe that for all $n \geq 1$, we have that

$$\kappa A_n^2 \int_0^1 \sin^2(\mu_n x)\, dx + (1-\kappa) \int_0^1 \beta_n^2(x)\, dx = 1,$$

where β_n is the nth normalized eigenfunction. Thus,

$$A_n^2 \leq \left(\kappa \int_0^1 \sin^2(\mu_n x)\, dx \right)^{-1}.$$

Equation (4.24) then follows from the fact that $c_n = a_n A_n$, equation (4.35), and the same argument that gave (4.18) in Lemma 4.5.

The proof of (4.25) is similar to the proof of (4.19). To see that there are at most two solutions to (4.21) on the interval $I_n := (n\pi + \pi/2, (n+1)\pi + \pi/2)$ for positive integers n, observe that $\tan(\mu)/\mu$ is a strictly increasing function of μ on I_n and $-(1-\rho)/\rho \tan(\sqrt{\mu^2 - \gamma^2})/\sqrt{\mu^2 - \gamma^2}$ is a strictly decreasing function of μ on I_n except for one point of discontinuity.

The uniform convergence of (4.34) for each $t > 0$ follows immediately from (4.24) and the fact that there are at most two solutions to (4.21) on the interval $(n\pi + \pi/2, (n+1)\pi + \pi/2)$. ■

To bound the difference between α^d and w^d, we bound the difference between their eigenvalues, μ_n and $\bar{\mu}_n$, for μ_n sufficiently less than $1/\varepsilon$.

Lemma 4.7. *Let $p > 0$ and n be such that $\max\{\mu_n, \bar{\mu}_n\} < \varepsilon^{-p/2}$, where μ_n and $\bar{\mu}_n$ are as in Lemmas 4.5 and 4.6. If ρ is independent of ε, as is the case in Theorem 4.1, then*

$$|\bar{\mu}_n - \mu_n| = O(\varepsilon^{3/2 - 3p/2}). \tag{4.36}$$

If $\rho = O(\sqrt{\varepsilon})$, as is the case in Theorem 4.2, then

$$|\bar{\mu}_n - \mu_n| = O(\varepsilon^{1 - 3p/2}). \tag{4.37}$$

Proof. First, observe from (4.20) and (4.16) that

$$\begin{aligned}\frac{\tan(\bar{\mu}_n)}{\bar{\mu}_n} - \frac{\tan(\mu_n)}{\mu_n} &= -\frac{1-\rho}{\gamma\rho} + \frac{(1-\rho)}{\rho\sqrt{\gamma^2-\mu_n^2}}\tanh(\sqrt{\gamma^2-\mu_n^2}) \\ &= -\frac{1-\rho}{\gamma\rho}\left(1 - \frac{\tanh(\sqrt{\gamma^2-\mu_n^2})}{\sqrt{1-\mu_n^2/\gamma^2}}\right). \qquad (4.38)\end{aligned}$$

Let $0 < p < 1$ and notice that if $\max\{\mu_n, \bar{\mu}_n\} < \varepsilon^{-p/2}$, then

$$\begin{aligned}\Big|1 - \frac{\tanh(\sqrt{\gamma^2-\mu^2})}{\sqrt{1-\mu^2/\gamma^2}}\Big| &\le \Big|1 - \frac{1}{\sqrt{1-\mu^2/\gamma^2}}\Big| \\ &\quad + \Big|\frac{1}{\sqrt{1-\mu^2/\gamma^2}}\Big|\Big|1 - \tanh(\sqrt{\gamma^2-\mu^2})\Big| \\ &= O(\varepsilon^{1-p})\end{aligned}$$

if ρ is independent of ε. Hence by (4.38), we have that if $\max\{\mu_n, \bar{\mu}_n\} < \varepsilon^{-p/2}$, then

$$\Big|\frac{\tan(\bar{\mu}_n)}{\bar{\mu}_n} - \frac{\tan(\mu_n)}{\mu_n}\Big| = O(\varepsilon^{3/2-p}). \qquad (4.39)$$

Let $f(\mu) = \tan(\mu)/\mu$, and observe that

$$1/f'(\mu) = \frac{\mu^2}{\mu\sec^2(\mu) - \tan(\mu)} \le \mu.$$

Thus, by the mean value theorem and (4.39), we have that if $\max\{\mu_n, \bar{\mu}_n\} < \varepsilon^{-p/2}$, then

$$|\bar{\mu}_n - \mu_n| \le \varepsilon^{-p/2}\Big|\frac{\tan(\bar{\mu}_n)}{\bar{\mu}_n} - \frac{\tan(\mu_n)}{\mu_n}\Big| = O(\varepsilon^{3/2-3p/2}).$$

The case where $\rho = O(\sqrt{\varepsilon})$ is similar. ■

Now that we have a bound on the difference between the eigenvalues of α^d and w^d for eigenvalues less than $\varepsilon^{-p/2}$, we are ready to bound the difference between the spectral decompositions of α^d and w^d. For the terms in the sum corresponding to eigenvalues less than $\varepsilon^{-p/2}$, we use that the eigenvalues must be close. The other terms in the sum are transcendentally small in ε for each $t > 0$ since they contain a factor of the form $e^{-\mu^2 t}$ with $\mu \ge \varepsilon^{-p/2}$.

Lemma 4.8. *Let $t \ge 0$ and $p > 0$. If ρ is independent of ε, as is the case in Theorem 4.1, then there exists an M and an ε_0 such that the functions $\alpha^d(x,t)$ and $w^d(x,t)$ defined in Lemma 4.3 satisfy*

$$|\alpha^d(x,t) - w^d(x,t)| \le M\varepsilon^{3/2-2p}$$

for all $0 < \varepsilon < \varepsilon_0$ and $x \in [0,1]$. If $\rho = O(\sqrt{\varepsilon})$, as is the case in Theorem 4.2, then the same statement holds but with $\varepsilon^{3/2-2p}$ replaced by ε^{1-2p}.

Proof. Suppose ρ is independent of ε. Using Lemmas 4.5 and 4.6, we have that

$$\begin{aligned}|\alpha^d(x,t) - w^d(x,t)| \le &\sum_{\max\{\mu_n,\bar{\mu}_n\}<\varepsilon^{-p/2}} |c_n e^{-\mu_n^2 t}\sin(\mu_n x) - b_n e^{-\bar{\mu}_n^2 t}\sin(\bar{\mu}_n x)| \\ &+ \sum_{\max\{\mu_n,\bar{\mu}_n\}\ge\varepsilon^{-p/2}} |c_n e^{-\mu_n^2 t}\sin(\mu_n x) - b_n e^{-\bar{\mu}_n^2 t}\sin(\bar{\mu}_n x)| \\ &=: S_1 + S_2.\end{aligned}$$

It is straightforward to see that S_2 must be $O(\varepsilon^n)$ for every n. This follows from the bounds on c_n and b_n and the fact that there are at most two eigenvalues on each interval $(n\pi + \pi/2, (n+1)\pi + \pi/2)$, all shown in Lemmas 4.5 and 4.6. Hence, we need only bound S_1 to complete the proof.

By the triangle inequality and the mean value theorem, we have that

$$\begin{aligned}&|c_n e^{-\mu_n^2 t}\sin(\mu_n x) - b_n e^{-\bar{\mu}_n^2 t}\sin(\bar{\mu}_n x)| \\ &\quad\le |c_n - b_n| + |b_n||e^{-\mu_n^2 t}\sin(\mu_n x) - e^{-\bar{\mu}_n^2 t}\sin(\bar{\mu}_n x)| \\ &\quad\le |c_n - b_n| + |b_n|(t+1)|\mu_n - \bar{\mu}_n|.\end{aligned} \tag{4.40}$$

Now by (4.19) and (4.25), there are order $\varepsilon^{-p/2}$ terms in the sum S_1. Thus, using Lemma 4.7, (4.18) in Lemma 4.5, and the bound in (4.40), we have that

$$S_1 \le O(\varepsilon^{3/2-2p}) + \sum_{\max\{\mu_n,\bar{\mu}_n\}<\varepsilon^{-p/2}} |c_n - b_n|. \tag{4.41}$$

Hence, it remains only to bound $|c_n - b_n|$. By (4.17) and the triangle inequality, we have that

$$\begin{aligned}|c_n - b_n| &\le \left|c_n - \frac{\int_0^1 \varphi^\alpha \sin(\mu_n x)\,dx}{\int_0^1 \sin^2(\mu_n x)\,dx}\right| + \left|\frac{\int_0^1 \varphi^\alpha \sin(\mu_n x)\,dx}{\int_0^1 \sin^2(\mu_n x)\,dx} - \frac{\int_0^1 \varphi^w \sin(\bar{\mu}_n x)\,dx}{\int_0^1 \sin^2(\bar{\mu}_n x)\,dx}\right| \\ &=: T_1 + T_2.\end{aligned}$$

By (4.8), to show that $T_2 = O(\varepsilon^{3/2-2p})$ we need only show that $T_3 = O(\varepsilon^{3/2-2p})$, where T_3 is T_2, with φ^w replaced by φ^α. This bound on T_3 follows from applying the mean value theorem to $f(y) := \int_0^1 \varphi^\alpha(x)\sin(yx)\,dx / \int_0^1 \sin^2(yx)\,dx$, equation (4.36), and the fact that $\min\{\mu_n, \bar{\mu}_n\} \ge \pi/2$ for all n, which follows immediately from the definitions of μ_n and $\bar{\mu}_n$ in (4.16), (4.20), and (4.21).

Now we have that

$$T_1 \le |d_n - 1|\left|\frac{\int_0^1 \varphi^\alpha \sin(\mu_n x)\,dx}{\int_0^1 \sin^2(\mu_n x)\,dx}\right|, \tag{4.42}$$

where d_n is as in (4.23) in Lemma 4.6. In addition, the same argument that gave (4.18) gives that

$$\left|\frac{\int_0^1 \varphi^\alpha \sin(\mu_n x)\,dx}{\int_0^1 \sin^2(\mu_n x)\,dx}\right| \le 4\|\varphi^\alpha\|_{L^2}. \tag{4.43}$$

Further, it follows from (4.23) that

$$(4.44)\qquad \begin{aligned}|d_n - 1| &= \Big|\frac{\lambda_n(\rho-1)\sinh^2(\lambda_n)(\sin(2\mu_n)-2\mu_n)}{\mu_n\rho\sin^2(\mu_n)(\sinh(2\lambda_n)-2\lambda_n)} + 1\Big|^{-1}\\ &\leq \Big|\frac{\mu_n\rho\sin^2(\mu_n)(\sinh(2\lambda_n)-2\lambda_n)}{\lambda_n(\rho-1)\sinh^2(\lambda_n)(\sin(2\mu_n)-2\mu_n)}\Big|.\end{aligned}$$

Since $\mu_n > \pi/2$ for all n, we have that

$$(4.45)\qquad \Big|\frac{1}{\sin(2\mu_n)-2\mu_n}\Big| \leq \frac{1}{\pi}.$$

Further, it is straightforward to see that

$$(4.46)\qquad \Big|\frac{\sinh(2\lambda_n)-2\lambda_n}{\sinh^2(\lambda_n)}\Big| < 3.$$

Thus, using (4.45) and (4.46) in (4.44), we have

$$(4.47)\qquad |d_n - 1| \leq \frac{3}{\pi}\Big|\frac{\mu_n\rho\sin^2(\mu_n)}{\lambda_n(\rho-1)}\Big|.$$

Now, using (4.20), we have that

$$(4.48)\qquad \begin{aligned}\sin^2(\mu_n) &= \Big(\frac{(1-\rho)}{\rho\sqrt{\gamma^2-\mu_n^2}}\tanh(\sqrt{\gamma^2-\mu_n^2})\cos(\mu_n)\mu_n\Big)^2\\ &\leq \Big(\frac{1-\rho}{\rho}\Big)^2\frac{\mu_n^2}{\gamma^2-\mu_n^2} \leq \Big(\frac{1-\rho}{\rho}\Big)^2\frac{\varepsilon^{-p}}{\varepsilon^{-1}-\varepsilon^{-p}} = O(\varepsilon^{1-p}).\end{aligned}$$

Since $1/\lambda_n = O(\varepsilon^{0.5})$ and $\mu_n \leq \varepsilon^{-p/2}$, using (4.47) and (4.48) gives

$$(4.49)\qquad |d_n - 1| = O(\varepsilon^{3/2-3p/2}).$$

Combining this with (4.42) and (4.43) gives that $T_1 = O(\varepsilon^{3/2-3p/2})$. Hence, from equation (4.41) we have that $S_1 = O(\varepsilon^{3/2-2p})$ since there are order $\varepsilon^{-p/2}$ terms in the sum. Thus, for every $0 < p < 1$, we have that

$$|\alpha^d(x,t) - w^d(x,t)| = O(\varepsilon^{3/2-2p}).$$

The case where $\rho = O(\sqrt{\varepsilon})$ is similar. ∎

5. The mean approximately satisfies an interface jump condition. In this section, we show that the mean of the process with a randomly imposed no flux condition at $x = 0.5$ approximately satisfies an interface jump condition. The results and proofs in this section are directly analogous to those in section 4. We use the notation $w_+ := \lim_{x\to 0.5+} w(x)$ and $w_- := \lim_{x\to 0.5-} w(x)$ throughout this section.

Theorem 5.1. *Suppose* $w(x,t) : [0,0.5)\cup(0.5,1]\times[0,\infty)\to\mathbb{R}$ *satisfies*

$$\partial_t w = \Delta w, \quad x \in (0,0.5)\cup(0.5,1),\ t > 0,$$

with boundary conditions

$$w(0,t) = 0, \quad w(1,t) = \eta, \tag{5.1}$$

interface conditions

$$\partial_x w_+ = \partial_x w_- \quad \textit{and} \quad w_+ - w_- = 2\sqrt{\varepsilon}\Big(\frac{1-\rho}{\rho}\Big)\partial_x w_+,$$

and initial condition $w(x,0) = \varphi(x)$. *Let* v_0 *and* v_1 *be as in Proposition* 3.2. *Then, for each* $t \geq 0$ *and* $q > 0$, *there exists an* M *and an* $\varepsilon_0 > 0$ *such that*

$$|v_0(x,t) + v_1(x,t) - w(x,t)| \leq M\varepsilon^{3/2-q} \tag{5.2}$$

for all $0 < \varepsilon < \varepsilon_0$ *and* $x \in (0, 0.5) \cup (0.5, 1)$.

If we let ρ depend on ε, then we can obtain any coefficient in the interface jump condition.

Theorem 5.2. *Let* $C > 0$ *and suppose* $w(x,t) : [0, 0.5) \cup (0.5, 1] \times [0, \infty) \to \mathbb{R}$ *satisfies*

$$\partial_t w = \Delta w, \quad x \in (0, 0.5) \cup (0.5, 1),\ t > 0,$$

with boundary conditions

$$w(0,t) = 0, \quad w(1,t) = \eta,$$

interface conditions

$$\partial_x w_+ = \partial_x w_- \quad \textit{and} \quad w_+ - w_- = C\partial_x w_+,$$

and initial condition $w(x,0) = \varphi(x)$. *Assume* v_0 *and* v_1 *are as in Proposition* 3.2 *with*

$$\rho = \Big(\frac{C}{2\sqrt{\varepsilon}} + 1\Big)^{-1}.$$

Then, for each $t \geq 0$ *and* $q > 0$, *there exists an* M *and an* $\varepsilon_0 > 0$ *such that*

$$|v_0(x,t) + v_1(x,t) - w(x,t)| \leq M\varepsilon^{1-q}$$

for all $0 < \varepsilon < \varepsilon_0$ *and* $x \in (0, 0.5) \cup (0.5, 1)$.

We prove Theorems 5.1 and 5.2 using a series of lemmas. The lemmas are all directly analogous to those in section 4.

Lemma 5.3. *If* v_0 *and* v_1 *are as in Proposition* 3.2, w *is as in Theorems* 5.1 *and* 5.2, *and we define* α *and* β *by the equation*

$$\begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \alpha \begin{pmatrix} \rho \\ 1-\rho \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

then, for each n, *we have that*

$$\alpha(x,t) - w(x,t) = \alpha^d(x,t) - w^d(x,t) + O(\varepsilon^n), \tag{5.3}$$

where α^d and β^d satisfy

$$
\begin{aligned}
\alpha^d_t &= \alpha^d_{xx} \quad \text{and} \quad \beta^d_t = \beta^d_{xx} - \frac{1}{\varepsilon}\beta^d, && x \in (0,1),\ t>0,\\
\alpha^d &= \beta^d = 0 && \text{at } x = 0,1,\\
\partial_x \alpha^d_+ &= \partial_x \alpha^d_- \quad \text{and} \quad \partial_x \beta^d_+ = \partial_x \beta^d_- && \text{at interface } x = 0.5,\\
-\rho(\alpha^d_+ - \alpha^d_-) &= \beta^d_+ - \beta^d_- \quad \text{and} \quad (1-\rho)\partial_x \alpha^d_+ = \partial_x \beta^d_+ && \text{at interface } x = 0.5,
\end{aligned}
\tag{5.4}
$$

and w^d satisfies

$$
\begin{aligned}
w^d_t &= w^d_{xx}, \quad x \in (0,0.5)\cup(0.5,1), && t>0,\\
w^d &= 0 && \text{at } x=0,1,\\
\partial_x w^d_+ &= \partial_x w^d_- \quad \text{and} \quad w^d_+ - w^d_- = 2\sqrt{\varepsilon}\Big(\frac{1-\rho}{\rho}\Big)\partial_x w^d_+ && \text{at interface } x = 0.5.
\end{aligned}
$$

Further, if we denote the initial conditions by

$$\varphi^\alpha(x) = \alpha^d(x,0) \quad \text{and} \quad \varphi^w(x) = w^d(x,0),$$

then, for each n, we have that

$$\varphi^\alpha(x) = \varphi^w(x) + O(\varepsilon^n). \tag{5.5}$$

Proof. It is straightforward to check that α and β can be decomposed into a sum of the following steady state and decay parts:

$$
\alpha(x,t) = \alpha^d(x,t) + \begin{cases} C_1 x & \text{if } 0 \le x < 0.5,\\ C_1 x + C_3 & \text{if } 0.5 < x \le 1, \end{cases}
$$

$$
\beta(x,t) = \beta^d(x,t) + \begin{cases} C_2 \sinh(\gamma x) & \text{if } 0 \le x < 0.5,\\ -C_2 \sinh(\gamma(1-x)) & \text{if } 0.5 < x \le 1, \end{cases}
$$

where $\gamma = 1/\sqrt{\varepsilon}$,

$$
C_1 = \frac{\eta}{1 + \frac{2}{\gamma}\frac{1-\rho}{\rho}\tanh(\gamma/2)}, \quad C_2 = \frac{(1-\rho)C_1}{\gamma\cosh(\gamma/2)}, \quad \text{and} \quad C_3 = \eta - C_1,
$$

and the decay parts, α^d and β^d, satisfy the boundary/interface problem in (5.4).

Further, one can quickly check that w can be decomposed into the following sum of steady state and decay parts:

$$
w(x,t) = w^d(x,t) + \begin{cases} Bx & \text{if } 0 \le x < 0.5,\\ Bx + \eta - B & \text{if } 0.5 < x \le 1, \end{cases} \quad \text{where} \quad B = \frac{\eta}{1 + \frac{2}{\gamma}\frac{1-\rho}{\rho}},
$$

and that w^d satisfies the indicated boundary/interface problem. It is easy to check that $|B - C_1| = O(\varepsilon^n)$ for each n, and thus (5.3) and (5.5) hold, even in the case that $\rho = O(\sqrt{\varepsilon})$. ■

Lemma 5.4. *Let $\mathcal{L}$ be the operator corresponding to the boundary value problem in (5.4). That is, define*

$$\mathcal{L}\begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} f'' \\ g'' - g/\varepsilon \end{pmatrix}$$

for (f, g) in the domain of $\mathcal{L}$ given by all $f, g \in L^2[0,1]$ such that their restrictions to $(0, 0.5)$ and $(0.5, 1)$ are in $H^2(0, 0.5)$ and $H^2(0.5, 1)$ and such that

$$\begin{aligned} f(0) = g(0) = f(1) = g(1) = f'_+ = f'_- = g'_+ = g'_- \\ = -\rho(f_+ - f_-) - (\beta_+ - \beta_-) = (1-\rho)f'_+ - g'_+ = 0. \end{aligned} \tag{5.6}$$

Then, $\mathcal{L}$ is self-adjoint with respect to the inner product

$$\left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} \hat{f} \\ \hat{g} \end{pmatrix} \right\rangle := \kappa \int_0^1 f(x)\hat{f}(x)\,dx + (1-\kappa)\int_0^1 g(x)\hat{g}(x)\,dx, \tag{5.7}$$

where

$$\kappa = 1 - \frac{1}{1 + \rho(1-\rho)}. \tag{5.8}$$

Furthermore, $\mathcal{L}$ is negative definite.

Proof. The proof is analogous to the proof of Lemma 4.4. ■

Lemma 5.5. *Let $w^d(x, t)$ be as in Lemma 5.3. Then, for each $t > 0$, it is given by the uniformly convergent series*

$$w^d(x,t) = \begin{cases} \sum_{n\geq 1} b_n e^{-\bar{\mu}_n^2 t}\sin(\bar{\mu}_n x) & \text{if } 0 \leq x < 0.5, \\ -\sum_{n\geq 1} b_n e^{-\bar{\mu}_n^2 t}\sin(\bar{\mu}_n(1-x)) & \text{if } 0.5 < x \leq 1, \end{cases} \tag{5.9}$$

where $0 < \bar{\mu}_1 < \bar{\mu}_2 < \cdots$ are the solutions to

$$\tan(\bar{\mu}/2) = -\frac{1-\rho}{\gamma\rho}\bar{\mu}, \tag{5.10}$$

where $\gamma = 1/\sqrt{\varepsilon}$, and

$$b_n = 2\big(1 - \sin(\bar{\mu}_n)/\bar{\mu}_n\big)^{-1}\left(\int_0^{0.5} \sin(\bar{\mu}_n y)\varphi^w(y)\,dy - \int_{0.5}^1 \sin(\bar{\mu}_n(1-y))\varphi^w(y)\,dy\right). \tag{5.11}$$

Further, we have that

$$|b_n| \leq 4\|\varphi^w\|_{L^2}. \tag{5.12}$$

Finally, if $n \neq k$, then

$$|\bar{\mu}_n - \bar{\mu}_k| \geq \pi. \tag{5.13}$$

Proof. Deriving the formulas in (5.10) and (5.11) is routine.

The proofs of (5.12) and (5.13) are analogous to the proofs of (4.18) and (4.19). The uniform convergence of (5.9) for $t > 0$ follows immediately from (4.18) and (4.19). ■

Lemma 5.6. *Let $\alpha^d(x,t)$ be as in Lemma 5.3. Then, for each $t > 0$, it is given by the uniformly convergent series*

$$\alpha^d(x,t) = \begin{cases} \sum_{n\geq 1} c_n e^{-\mu_n^2 t} \sin(\mu_n x) & \text{if } 0 \leq x < 0.5, \\ -\sum_{n\geq 1} c_n e^{-\mu_n^2 t} \sin(\mu_n(1-x)) & \text{if } 0.5 < x \leq 1, \end{cases}$$

where $0 < \mu_1 < \mu_2 < \cdots$ are the solutions to either

$$\tan(\mu/2) = -\frac{1-\rho}{\sqrt{\gamma^2-\mu^2}\rho}\tanh(0.5\sqrt{\gamma^2-\mu^2})\mu \tag{5.14}$$

or

$$\tan(\mu/2) = -\frac{1-\rho}{\sqrt{\mu^2-\gamma^2}\rho}\tan(0.5\sqrt{\mu^2-\gamma^2})\mu, \tag{5.15}$$

depending on whether $\mu_n^2 < \gamma^2$ or $\mu_n^2 > \gamma^2$, where $\gamma = 1/\sqrt{\varepsilon}$ and $\lambda = \sqrt{\gamma^2-\mu^2}$. If ε is such that $\tan(\gamma/2) = -((1-\rho)/\rho)\gamma/2$, then there is one value $\mu_n = \gamma$. Further,

$$c_n = d_n\left(\int_0^{0.5} \sin(\mu_n y)\varphi^\alpha(y)\,dy - \int_{0.5}^1 \sin(\mu_n(1-y))\varphi^\alpha(y)\,dy\right), \tag{5.16}$$

where

$$d_n = \left(\frac{\mu_n \rho \sin^2(\mu_n/2)(\sinh(\lambda_n)-\lambda_n)}{\lambda_n(\rho-1)\sinh^2(\lambda_n/2)(\sin(\mu_n)-\mu_n)} + 1\right)^{-1}, \tag{5.17}$$

if $\mu_n^2 < \gamma^2$, where $\lambda_n = \sqrt{\gamma^2-\mu_n^2}$. And for all $n \geq 1$, we have that

$$|c_n| \leq 4\|\varphi^\alpha\|_{L^2}. \tag{5.18}$$

Further, if $n \neq k$ are such that $\max\{\mu_n^2, \mu_k^2\} < \gamma^2$, then

$$|\mu_n - \mu_k| \geq \pi. \tag{5.19}$$

Finally, if n is a positive integer, then there are at most two solutions to (4.21) on the interval $(2n\pi + \pi, 2(n+1)\pi + \pi)$.

Proof. Since $\mathcal{L}$ is self-adjoint and negative definite, we seek eigenvalues $-\mu^2 < 0$ and eigenfunctions $\bar{\alpha}$, $\bar{\beta}$ satisfying

$$\begin{aligned} \bar{\alpha}'' + \mu^2\bar{\alpha} &= 0, \\ \bar{\beta}'' + (\mu^2 - 1/\varepsilon)\bar{\beta} &= 0. \end{aligned} \tag{5.20}$$

If $\mu^2 - 1/\varepsilon < 0$, then the homogeneous boundary conditions and the equality of the derivatives at the interface imply that the eigenfunctions are of the form

$$\bar{\alpha}(x) = \begin{cases} A\sin(\mu x) & \text{if } 0 \le x < 0.5, \\ -A\sin(\mu(1-x)) & \text{if } 0.5 < x \le 1, \end{cases}$$

$$\bar{\beta}(x) = \begin{cases} B\sinh(\lambda x) & \text{if } 0 \le x < 0.5, \\ -B\sinh(\lambda(1-x)) & \text{if } 0.5 < x \le 1, \end{cases}$$

where

$$\lambda = \sqrt{\gamma^2 - \mu^2}. \tag{5.21}$$

In order for the eigenfunctions to satisfy the interface conditions $-\rho(\alpha_+ - \alpha_-) = \beta_+ - \beta_-$ and $(1-\rho)\partial_x\alpha_+ = \partial_x\beta_+$ at the interface $x = 0.5$, we need the following system of linear equations to be satisfied:

$$\rho A\sin(\mu/2) + B\sinh(\lambda/2) = 0, \tag{5.22}$$

$$(1-\rho)\mu A\cos(\mu/2) - \lambda B\cosh(\lambda/2) = 0. \tag{5.23}$$

Since we seek nonzero A and B, we need the determinant of the matrix in (5.22) and (5.23) to be zero. Equivalently, we need

$$\mu(\rho-1)\sinh(\lambda)\cos^2(\mu/2) - \lambda\rho\cosh^2(\lambda/2)\sin(\mu) = 0,$$

which can be written as

$$\tan(\mu/2) = -\frac{1-\rho}{\lambda\rho}\tanh(\lambda/2)\mu. \tag{5.24}$$

For each $\varepsilon > 0$, this equation has a finite number of solutions μ satisfying $\mu^2 - 1/\varepsilon < 0$. Let $\mu_n^2 < \gamma^2$ denote the nth positive solution to (5.24) and λ_n be defined as in (5.21) with respect to μ_n.

Hence, the eigenfunctions are of the form

$$\alpha_n(x) = \begin{cases} A_n\sin(\mu_n x) & \text{if } 0 \le x < 0.5, \\ -A_n\sin(\mu_n(1-x)) & \text{if } 0.5 < x \le 1, \end{cases} \tag{5.25}$$

$$\beta_n(x) = \begin{cases} B_n\sinh(\lambda_n x) & \text{if } 0 \le x < 0.5, \\ -B_n\sinh(\lambda_n(1-x)) & \text{if } 0.5 < x \le 1. \end{cases}$$

Now we want the eigenfunctions (α_n, β_n) to have norm equal to one with respect to the inner product defined in (5.7) for each n. That is, we want

$$\kappa\int_0^{0.5} A_n^2\sin^2(\mu_n x)\,dx + (1-\kappa)\int_0^{0.5} B_n^2\sinh^2(\lambda_n x)\,dx = 0.5, \tag{5.26}$$

where κ is defined in (5.8). Since (5.22) implies that $A_n = -B_n \sinh(\lambda/2)/(\rho \sin(\mu/2))$, it follows from (5.26) that

$$B_n = \frac{1}{K_n} \quad \text{and} \quad A_n = \frac{-\sinh(\lambda_n/2)}{K_n \rho \sin(\mu_n/2)}, \tag{5.27}$$

where

$$K_n = \frac{1}{\sqrt{2}}\sqrt{\kappa\left(\frac{\sinh(\lambda_n/2)}{\rho\sin(\mu_n/2)}\right)^2\left[1-\frac{\sin(\mu_n)}{\mu_n}\right]+(1-\kappa)\left[\frac{\sinh(\lambda_n)}{\lambda_n}-1\right]}, \tag{5.28}$$

which comes from explicitly evaluating the integrals in (5.26).

It is easy to see from (5.20) that all the normalized eigenfunctions for α^d are of the form given in (5.25), even for $\mu_n \geq \gamma^2$. So, putting this together, we have that

$$\alpha^d(x,t) = \begin{cases} \sum_{n\geq 1} a_n e^{-\mu_n^2 t} A_n \sin(\mu_n x) & \text{if } 0 \leq x < 0.5, \\ -\sum_{n\geq 1} a_n e^{-\mu_n^2 t} A_n \sin(\mu_n(1-x)) & \text{if } 0.5 < x \leq 1, \end{cases} \tag{5.29}$$

where the a_n's are chosen to meet the initial conditions. The remainder of the proof is analogous to the proof of Lemma 4.6. ■

Lemma 5.7. *Let $p > 0$ and n be such that $\max\{\mu_n, \bar{\mu}_n\} < \varepsilon^{-p/2}$, where μ_n and $\bar{\mu}_n$ are as in Lemma 5.6. If ρ is independent of ε, as is the case in Theorem 5.1, then*

$$|\bar{\mu}_n - \mu_n| = O(\varepsilon^{3/2-3p/2}). \tag{5.30}$$

If $\rho = O(\sqrt{\varepsilon})$, as is the case in Theorem 5.2, then

$$|\bar{\mu}_n - \mu_n| = O(\varepsilon^{1-3p/2}). \tag{5.31}$$

Proof. The proof is analogous to the proof of Lemma 4.7. ■

Lemma 5.8. *Let $t \geq 0$ and $p > 0$. If ρ is independent of ε, as is the case in Theorem 5.1, then there exists an M and an ε_0 such that the functions $\alpha^d(x,t)$ and $w^d(x,t)$ defined in Lemma 5.3 satisfy*

$$|\alpha^d(x,t) - w^d(x,t)| \leq M\varepsilon^{3/2-2p}$$

for all $0 < \varepsilon < \varepsilon_0$ and $x \in [0,1]$. If $\rho = O(\sqrt{\varepsilon})$, as is the case in Theorem 5.2, then the same statement holds but with $\varepsilon^{3/2-2p}$ replaced by ε^{1-2p}.

Proof. The proof is analogous to the proof of Lemma 5.8. ■

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