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SENSITIVITY TO SWITCHING RATES IN STOCHASTICALLY SWITCHED ODES*

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Abstract. We consider a stochastic process driven by a linear ordinary differential equation whose right-hand side switches at exponential times between a collection of different matrices. We construct planar examples that switch between two matrices where the individual matrices and the average of the two matrices are all Hurwitz (all eigenvalues have strictly negative real part), but nonetheless the process goes to infinity at large time for certain values of the switching rate. We further construct examples in higher dimensions where again the two individual matrices and their averages are all Hurwitz, but the process has arbitrarily many transitions between going to zero and going to infinity at large time as the switching rate varies. In order to construct these examples, we first prove in general that if each of the individual matrices is Hurwitz, then the process goes to zero at large time for sufficiently slow switching rate and if the average matrix is Hurwitz, then the process goes to zero at large time for sufficiently fast switching rate. We also give simple conditions that ensure the process goes to zero at large time for all switching rates.

Key words. Ergodicity, piecewise deterministic Markov process, switched dynamical systems, hybrid switching system, planar switched systems, linear differential equations.

AMS subject classifications. 60J75, 93E15, 37H15, 34F05, 34D23.

1. Introduction

We consider the stochastic process $(X_t)_{t\geq 0}\in\mathbb{R}^d$ where X_t solves $\dot{X}_t=A_{I_t}X_t$, with I_t a Markov process on a finite set E and $\{A_i\}_{i\in E}$ a set of $d\times d$ real matrices. The stability of this system when the switching process I_t is deterministic has been extensively studied in the past decade; see [2] and [12].

In [6], the authors study the stochastic problem in the plane with I_t a Markov process and $E=\{0,1\}$. The authors assume both A_0 and A_1 are Hurwitz (all eigenvalues have strictly negative real part) and prove the surprising result that $||X_t||$ may converge to 0 or $+\infty$ as $t\to\infty$ depending on the switching rate as long as an average matrix $\bar{A}=\lambda A_0+(1-\lambda)A_1$ has a positive eigenvalue for some $\lambda\in(0,1)$.

In this paper, we show that the assumption that the average matrix has a positive eigenvalue is not necessary to ensure a blowup. Specifically, we construct examples in the plane where A_0, A_1, and $\bar{A}=\lambda A_0+(1-\lambda)A_1$ are all Hurwitz, but $||X_t||\to+\infty$ almost surely as $t\to\infty$ for certain values of the switching rate. This is significant for the general study of switching processes because it shows that the dynamics of the switching process can be very different from both the individual dynamics (in this case, the A_i's) and the averaged dynamics (in this case, $\bar{A}$). These planar examples are also interesting because they have multiple transitions between $||X_t||$ going to 0 and going to $+\infty$ at large time as the switching rate varies. Furthermore, we construct examples in higher dimensions that have arbitrarily many such phase transitions.

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Recently researchers have devoted considerable attention to randomly switched systems and we now comment on our work in this broader context. [7, 5, 4], and [1] all study invariant measures for such processes. Our work shows that the existence of such invariant measures may depend in a complicated way on the switching rates. In [8, 9], and [3], the authors provide conditions under which their randomly switched systems behave according to the individual systems for slow switching and according to the averaged system for fast switching. We prove that our system also obeys this principle in theorems 2.3 and 2.4. However, we show in Example 3.1 that the transition between the slow and fast switching regimes can be quite complicated. Furthermore, Example 3.3 shows that it can be as complicated as we want.

As background for these surprising results, we first prove sufficient conditions to ensure stability for all switching rates in Section 2. Furthermore we also show in Section 2 that the individual matrices determine the stability for slow switching and that the average matrix determines the stability for fast switching. In Section 3 we use these theorems to construct examples that show "medium" switching can induce blowups even when the individual matrices and the average matrix are all Hurwitz.

We conclude this introduction by defining notation. Let $E=\{0,1,\dots,n-1\}$ and let $\{A_i\}_{i\in E}$ be a set of $d\times d$ real matrices. For a given switching rate $r>0$, let $(I_t)_{t\geq 0}$ be an irreducible continuous time Markov process with state space E and generator rQ. Under these assumptions, the Markov process on E with generator rQ has a unique invariant probability measure which we denote by π. Furthermore, π is the unique probability vector satisfying $\pi Q=0$.

Define $(X_t)_{t\geq 0}$ to be the solution of

$$X_t=X_0+\int_0^t A_{I_s}X_s\,ds,\quad (t\geq 0). \tag{1.1}$$

Then $(X_t,I_t)_{t\geq 0}$ is a Markov process on $\mathbb{R}^d\times E$. Unless otherwise noted, assume throughout that the distribution of the initial condition (X_0,I_0) is some given probability measure on $\mathbb{R}^d\times E$ satisfying $\mathbb{E}||X_0||<\infty$. Define the average matrix

$$\bar{A}=\sum_{i\in E}A_i\pi_i.$$

The following description of our process will be useful. Let $\xi_1,\xi_2,\dots$ denote the succession of states visited by I_t, $\tau_1,\tau_2,\dots$ the holding times in each state, $N(t)$ the number of switches before t, and $a_t=t-\sum_{k=1}^{N(t)}\tau_k$ the time since the last switch. Observe that we can write X_t as

$$X_t=\exp(A_{\xi_{N(t)+1}}a_t)\exp(A_{\xi_{N(t)}}\tau_{N(t)})\dots\exp(A_{\xi_1}\tau_1)X_0. \tag{1.2}$$

2. Basic stability theorems

THEOREM 2.1 (normal case). *If A_i is normal and Hurwitz for each $i\in E$, then $||X_t||\to 0$ monotonically as $t\to\infty$ almost surely.*

Proof. Because each A_i is normal and Hurwitz, there exists a $\gamma>0$ so that for each A_i and for every $t>0$,

$$||\exp(A_it)||\leq e^{-\gamma t}<1.$$

Therefore

$$||X_t||=||\exp(A_{\xi_{N(t)+1}}a_t)\exp(A_{\xi_{N(t)}}\tau_{N(t)})\dots\exp(A_{\xi_1}\tau_1)X_0||$$

$$\le ||\exp(A_{\xi_{N(t)+1}} a_t)|| \left(\prod_{k=1}^{N(t)} ||\exp(A_{\xi_k} \tau_k)|| \right) ||X_0||$$
$$\le e^{-\gamma t} ||X_0|| \to 0 \quad \text{as } t \to \infty.$$

To see that the convergence is monotonic, let $0 \le s \le t$ and replace X_0 by X_s in the calculation above. ∎

THEOREM 2.2 (commuting case). *Assume $\{A_i\}_{i\in E}$ is a commuting family of matrices. If $\bar{A}$ is Hurwitz, then $||X_t|| \to 0$ as $t \to \infty$ almost surely.*

Proof. Because $\bar{A}$ is Hurwitz, there exist positive β and γ so that for each $t \ge 0$

$$||\exp(\bar{A}t)|| \le \beta e^{-\gamma t}.$$

For each $t > 0$, define

$$C_t = \frac{1}{t}\left(\sum_{k=1}^{N(t)} A_{\xi_k} \tau_k + A_{\xi_{N(t)+1}} a_t \right) = \sum_{i\in E} A_i \frac{1}{t} \int_0^t 1_{I_s = i}\, ds.$$

Now because $\{A_i\}_{i\in E}$ is a commuting family of matrices, equation (1.2) becomes

$$||X_t|| = ||\exp\left(\sum_{k=1}^{N(t)} A_{\xi_k} \tau_k + A_{\xi_{N(t)+1}} a_t \right) X_0|| = ||\exp(C_t t)\, X_0||$$
$$= ||\exp\left(\bar{A}t\right) \exp\left((C_t - \bar{A})t\right) X_0|| \le \beta e^{-\gamma t} e^{||C_t - \bar{A}||t} ||X_0||.$$

Because Q is irreducible, $C_t \to \bar{A}$ almost surely as $t \to \infty$ because $\frac{1}{t}\int_0^t 1_{I_s=i} ds \to \pi_i$ almost surely as $t \to \infty$ (see [13], page 126). Thus, $||X_t|| \to 0$ almost surely as $t \to \infty$. ∎

REMARK 2.1. If $\{A_i\}_{i\in E}$ is a commuting family of matrices and each A_i is Hurwitz, then $\bar{A}$ is Hurwitz. This is an immediate consequence of the fact that eigenvalues "add"—in some order—for commuting matrices.

THEOREM 2.3 (slow switching). *Assume A_i is Hurwitz for each $i \in E$. Then there exists a constant $a > 0$ so that if $r < a$, then $||X_t|| \to 0$ as $t \to \infty$ almost surely.*

Proof. Because each A_i is Hurwitz, there exist $\beta > 1$ and $\gamma > 0$ so that for each A_i and each $t \ge 0$

$$||\exp(A_i t)|| \le \beta e^{-\gamma t}.$$

Therefore from equation (1.2), we have that

$$\begin{aligned} ||X_t|| &\le ||\exp(A_{\xi_{N(t)+1}} a_t)|| \left(\prod_{k=1}^{N(t)} ||\exp(A_{\xi_k} \tau_k)|| \right) ||X_0|| \\ &\le \beta^{N(t)+1} e^{-\gamma t} ||X_0|| = \exp\left(\Big(\frac{N(t)+1}{t} \log\beta - \gamma \Big) t \right) ||X_0||. \end{aligned} \tag{2.1}$$

Next we claim that we have the following almost sure convergence as $K \to \infty$:

$$\frac{1}{K} \sum_{k=1}^{K} \tau_k \to \left(r \sum_{i\in E} \pi_i q_i \right)^{-1}, \tag{2.2}$$

where q_i is the ith diagonal entry of Q. To see this, let s_j^i denote the duration of the jth visit of the process I_t to state $i \in E$ and let $V_i(K) := \sum_{k=1}^K 1_{\xi_k = i}$ denote the number of visits to i before the Kth jump of the process I_t. Then

$$\frac{1}{K}\sum_{k=1}^{K}\tau_k = \sum_{i\in E}\frac{1}{K}\sum_{j=1}^{V_i(K)} s_j^i = \sum_{i\in E}\frac{V_i(K)}{K}\frac{1}{V_i(K)}\sum_{j=1}^{V_i(K)} s_j^i.$$

For each $i \in E$, $\frac{V_i(K)}{K} \to q_i\pi_i/(\sum_{k\in E} q_k\pi_k)$ almost surely as $K\to\infty$ because Q is irreducible. By the strong law of large numbers, $\frac{1}{V_i(K)}\sum_{j=1}^{V_i(K)} s_j^i \to \frac{1}{rq_i}$ almost surely as $K \to \infty$. Therefore, equation (2.2) is verified.

By the definition of $N(t)$ we have that $\sum_{k=1}^{N(t)}\tau_k \le t \le \sum_{k=1}^{N(t)+1}\tau_k$. Therefore

$$\frac{\sum_{k=1}^{N(t)}\tau_k}{N(t)} \le \frac{t}{N(t)} \le \frac{\sum_{k=1}^{N(t)+1}\tau_k}{N(t)+1}\frac{N(t)+1}{N(t)}. \tag{2.3}$$

Because each τ_k is almost surely finite, $N(t)\to\infty$ almost surely as $t\to\infty$. It then follows from combining equations (2.2) and (2.3) that

$$\frac{N(t)}{t} \to r\sum_{i\in E}\pi_i q_i \quad \text{almost surely as } t\to\infty.$$

So if $r < \gamma(2\log\beta\sum_{i\in E}\pi_i q_i)^{-1}$, then $||X_t|| \to 0$ almost surely as $t\to\infty$ by equation (2.1). ∎

THEOREM 2.4 (fast switching). *Assume $\bar{A}$ is Hurwitz. Then there exists a constant $b>0$ so that if $r>b$, then $||X_t||\to 0$ as $t\to\infty$ almost surely.*

The proof relies on the following lemma. Let $\mathbb{E}_\nu$ denote the expectation with respect to the measure of the process $(I_t)_{t\ge 0}$ with I_0 distributed according to ν. Because we will consider processes with different switching rates, let us momentarily make the dependence on the switching rate explicit by letting $(I_t^{(r)})_{t\ge0}$ be the Markov process on E with generator rQ and define $(X_t^{(r)})_{t\ge0}$ with respect to $(I_t^{(r)})_{t\ge0}$ as before. Define $S_t^{(r)}$ to be the operator that maps X_0 to $X_t^{(r)}$. Observe that $S_t^{(r)}$ is a function of $(I_s^{(r)})_{0\le s\le t}$.

LEMMA 2.5. *For every probability measure ν on E and for every $t>0$,*

$$\mathbb{E}_\nu||S_t^{(r)}|| \to ||\exp(\bar{A}t)|| \quad \textit{as } r\to\infty.$$

Proof. Define $\{\xi_i^1\}_{i=1}^\infty$, $\{\tau_i^1\}_{i=1}^\infty$, $\{N^1(t)\}_{t\ge0}$, and $\{a_t^1\}_{t\ge0}$ as before but now with respect to $\{I_t^{(1)}\}_{t\ge0}$. Let the distribution of I_0 be a given probability measure ν on E and for $\lambda>0$ define

$$\tilde{S}_t^\lambda = \exp\left(A_{\xi_1^1}^T\frac{\tau_1^1}{\lambda}\right)\exp\left(A_{\xi_2^1}^T\frac{\tau_2^1}{\lambda}\right)\dots\exp\left(A_{\xi_{N^1(\lambda t)}}^T\frac{\tau_{N^1(\lambda t)}}{\lambda}\right)\exp\left(A_{\xi_{N^1(\lambda t)+1}}^T\frac{a_{\lambda t}^1}{\lambda}\right),$$

where we denote the transpose of a matrix B by B^T. Observe that if $r=\lambda$, then $\tilde{S}_t^\lambda$ has been defined so that $(\tilde{S}_t^\lambda)^T$ and $S_t^{(r)}$ are equal in distribution.

By [11], $\tilde{S}_t^\lambda \to \exp(\bar{A}^T t)$ almost surely in the strong operator topology as $\lambda\to\infty$. Because $\mathbb{R}^d$ is finite-dimensional, we actually have that the convergence holds in the uniform operator topology. Because $||B|| = ||B^T||$ for every matrix B, it follows that

$$||\tilde{S}_t^\lambda|| \to ||\exp(\bar{A}t)|| \text{ almost surely as } \lambda\to\infty.$$

Because $||\tilde{S}_t^\lambda|| \le \exp(\max_i ||A_i|| t)$ for every $\lambda > 0$, the bounded convergence theorem gives

$$\mathbb{E}||\tilde{S}_t^\lambda|| \to ||\exp(\bar{A}t)|| \text{ as } \lambda \to \infty.$$

Because $||\tilde{S}_t^\lambda||$ and $||S_t^r||$ are equal in distribution, the proof is complete. ∎

Proof. [of Theorem 2.4] Because $\bar{A}$ is Hurwitz, there exist positive numbers β and γ so that for every $t \ge 0$

$$||\exp(\bar{A}t)|| \le \beta e^{-\gamma t}.$$

Thus we can choose $T > 0$ so that $||\exp(\bar{A}T)|| < \frac{1}{4}$. By Lemma 2.5 there exists a $b > 0$ so that if $r > b$, then $\mathbb{E}_i ||S_T^{(r)}|| < \frac{1}{2}$ for each $i \in E$, where $\mathbb{E}_i$ denotes the expectation with respect to the measure of the process $(I_t)_{t\ge 0}$ with initial measure $\mathbb{P}(I_0 = i) = 1$.

Let $r > b$ and define the process $\{M_n\}_{n=0}^\infty$ and the filtration $\{\mathcal{F}_n\}_{n=0}^\infty$ by

$$M_n = ||X_{nT}|| \qquad \text{and} \qquad \mathcal{F}_n = \sigma((X_t, I_t) : 0 \le t \le nT).$$

We claim that M_n is a supermartingale with respect to $\mathcal{F}_n$. It's immediate that $M_n \in \mathcal{F}_n$ and $\mathbb{E}|M_n| \le e^{\Lambda n T} < \infty$ for $\Lambda := \max_{i\in E} ||A_i||$. For $0 \le s \le t$, define $S(s,t)$ to be the operator that maps X_s to X_t. We now check the supermartingale property.

$$\begin{aligned}
\mathbb{E}[M_{n+1}|\mathcal{F}_n] &\le \mathbb{E}[||S(nT,(n+1)T)|| \, ||X_{nT}|| |\mathcal{F}_n] \\
&= M_n \mathbb{E}[||S(nT,(n+1)T)|| |\mathcal{F}_n] \\
&= M_n \mathbb{E}_{I_{nT}} ||S(nT,(n+1)T)|| \\
&\le \frac{1}{2} M_n.
\end{aligned}$$

Taking the expectation of the above inequality and iterating yields $\mathbb{E}M_n \le \frac{1}{2^n}\mathbb{E}M_0$. Therefore M_n converges in L^1 to 0 because $M_n \ge 0$. Also because $M_n \ge 0$, the martingale convergence theorem implies that M_n must converge almost surely. Therefore M_n converges almost surely to 0.

To conclude that $||X_t|| \to 0$ almost surely, we need to control $||X_t||$ at times between multiples of T. This is easily obtained because $||X_t||$ cannot grow faster than $e^{\Lambda t}$. Let $\omega \in \Omega$ be such that $M_n(\omega) \to 0$ and let $\epsilon > 0$. There exists $N = N(\omega, \epsilon)$ so that for all $n \ge N$,

$$||M_n(\omega)|| < e^{-\Lambda T}\epsilon.$$

Thus for all $t \ge NT$,

$$||X_t(\omega)|| \le ||(S(t - T\lfloor t/T \rfloor, t) X_{T\lfloor t/T \rfloor})(\omega)|| \le e^{\Lambda T} M_{\lfloor t/T \rfloor}(\omega) < \epsilon.$$

Because this set of ω's has probability one, the proof is complete. ∎

EXAMPLE 2.2. *Assume $E = \{0,1\}$ and $Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. Define*

$$A_0 = \begin{pmatrix} 1 & 4 \\ 0 & -2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then A_0 and A_1 each have a positive eigenvalue, but $\bar{A} = \frac{1}{2}(A_0 + A_1)$ is Hurwitz. So despite the fact that each individual matrix is unstable, Theorem 2.4 guarantees that $||X_t|| \to 0$ almost surely as $t \to \infty$ for sufficiently fast switching rate.

3. Medium switching can be complicated

We will now construct a switching example with two matrices, A_0 and A_1, that is surprising for the following two reasons. First, the individual matrices A_0 and A_1 and the average $\bar{A}=\frac{1}{2}(A_0+A_1)$ are all Hurwitz, but $||X_t||$ will still blow up at large time for certain values of the switching rate. In [6], the authors show that $||X_t||$ can blow up if the two individual matrices A_0 and A_1 are Hurwitz as long as the average matrix has a positive eigenvalue. Thus our result shows that this assumption on the average matrix is not necessary.

Second, the asymptotic behavior of the following example has multiple "phase transitions" as the switching rate varies. That is, the process goes to zero at large time for both slow and fast switching, but blows up for medium switching.

We also remark that we can choose the negative real part of all the eigenvalues of A_0, A_1, and $\bar{A}$ to have arbitrarily large absolute value.

Example 3.1. *Assume $\mathbb{P}(X_0=0)=0$ and let $E=\{0,1\}$ and $Q=\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}$. We will show the existence of matrices $A_0,A_1\in\mathbb{R}^{2\times 2}$ and positive numbers $a<b$, so that*

1. *A_0, A_1 are each Hurwitz.*
2. *$\bar{A}=\frac{1}{2}(A_0+A_1)$ is Hurwitz.*
3. *If $r\notin(a,b)$, then $||X_t||\to 0$ almost surely as $t\to\infty$.*
4. *$||X_t||\to\infty$ almost surely as $t\to\infty$ for some value of $r\in(a,b)$.*

For positive α and c, we define

$$A_0=\begin{pmatrix}-\alpha & c\\ 0 & -\alpha\end{pmatrix}, \qquad A_1=\begin{pmatrix}-\alpha & 0\\ -c & -\alpha\end{pmatrix}.$$

Observe that A_0 and A_1 each have $-\alpha<0$ as their only eigenvalue. The two eigenvalues of $\bar{A}=\frac{1}{2}(A_0+A_1)$ are $-\alpha\pm ic/2$. Thus A_0, A_1, and $\bar{A}$ are each Hurwitz. By theorems 2.3 and 2.4, $||X_t||\to 0$ as $t\to\infty$ almost surely for sufficiently large r and for sufficiently small r. We will show that $||X_t||\to+\infty$ as $t\to\infty$ almost surely for some intermediate values of r.

We use polar coordinates to study the large time behavior of $||X_t||$. Our technique follows [6] in this setting and the well known utility of the polar representation when studying Lyapunov exponents (especially in two-dimensions) which dates back to at least [10]. Define the radial process $R_t:=||X_t||$ and define the angular process U_t as the point on the unit circle S^1 given by X_t/R_t. A short calculation shows that between jumps R_t and U_t satisfy

$$\dot{R}_t=R_t\langle A_{I_t}U_t,U_t\rangle, \tag{3.1}$$

$$\dot{U}_t=A_{I_t}U_t-\langle A_{I_t}U_t,U_t\rangle U_t. \tag{3.2}$$

The advantage of this decomposition is that the evolution of the angular process doesn't depend on the radial process. Therefore (U_t,I_t) is a Markov process on $S^1\times\{0,1\}$.

Lemma 3.1. *If we identify $\theta\in\mathbb{R}$ with $(\cos\theta,\sin\theta)\in S^1$, then the unique invariant measure of the angular process U_t is given by*

$$\mu(d\theta,i)=p_i(\theta;r/c)1_{[0,2\pi]}(\theta)\,d\theta,$$

where for any parameter $\lambda>0$, the functions p_0 and p_1 satisfy

$$p_i(\theta;\lambda)=p_{1-i}(\theta+\pi/2;\lambda)=p_i(\theta+\pi;\lambda)\quad \text{for } \theta\in\mathbb{R}, \tag{3.3}$$

$$\textit{and} \quad p_0(\theta;\lambda) < p_1(\theta;\lambda) \quad \textit{for } \theta \in (-\tfrac{\pi}{2},0). \tag{3.4}$$

Proof. Define the process $\Theta_t \in \mathbb{R}$ to be the lift of $U_t \in S^1$ from the circle to its covering space $\mathbb{R}$. That is to say Θ_t is the unique process so that $U_t = (\cos\Theta_t, \sin\Theta_t)$, Θ_t is continuous in t, and $\Theta_0 \in [0,2\pi)$. It follows from equation (3.2) and plugging in our values for A_0 and A_1 that between jumps Θ_t satisfies

$$\dot{\Theta}_t = -c[I_t \cos^2(\Theta_t) + (1-I_t)\sin^2(\Theta_t)] \leq 0.$$

Because $\min_{i\in\{0,1\}} -c[i\cos^2(\theta) + (1-i)\sin^2(\theta)] \leq -c/2 < 0$ for all $\theta \in \mathbb{R}$, it follows that $\Theta_t \to -\infty$ as $t \to \infty$ almost surely. Because Θ_t is continuous, we conclude that the Markov process (U_t, I_t) is recurrent and irreducible and must have a unique invariant measure.

If we identify $\theta \in \mathbb{R}$ with $(\cos\theta, \sin\theta) \in S^1$, then the adjoint of generator of the Markov process (U_t, I_t) is

$$(\mathcal{L}^* q)(\theta,i) = \partial_\theta \left(c\left[(1-i)\sin^2(\theta) + i\cos^2(\theta)\right] q(\theta,i)\right) + r(q(\theta,1-i) - q(\theta,i)).$$

For $\theta \in (-\frac{\pi}{2},0)$ and $\lambda > 0$, define

$$\begin{aligned}
H(\theta;\lambda) &= \exp(-2\lambda\cot(2\theta)) \int_\theta^0 \exp(2\lambda\cot(2y))\sec^2(y)\,dy,\\
p_0(\theta;\lambda) &= C\csc^2(\theta)\lambda H(\theta),\\
p_1(\theta;\lambda) &= C\sec^2(\theta)\left[1-\lambda H(\theta)\right],
\end{aligned}$$

where

$$C(\lambda) = \left[4\int_{-\frac{\pi}{2}}^0 \sec^2(x) + (\csc^2(x) - \sec^2(x))\lambda H(x)\,dx\right]^{-1}.$$

Define $H(0;\lambda) = 0 = p_0(0;\lambda)$ and $p_1(0;\lambda) = C(\lambda)$. Extend p_1 and p_0 to be defined on the rest of the real line by equation (3.3). It is easy to check that these three functions are well-defined.

Writing $p_i(\theta;\lambda)$ as $p(\theta,i;\lambda)$, it is easy to check that $\mathcal{L}^* p(\theta,i;\lambda) = 0$ for all $\theta \in \mathbb{R}$ and for $i = \{0,1\}$. Thus, the measure μ defined in the statement of the lemma is the unique invariant measure for (U_t, I_t).

We now check that p_0 and p_1 satisfy equation (3.4). Let $\lambda > 0$ and observe that for $\theta \in (-\frac{\pi}{2},0)$, writing $1 = \sin^2(y)\csc^2(y)$ in the integrand gives

$$\begin{aligned}
H(\theta;\lambda) &= \exp(-2\lambda\cot(2\theta)) \int_\theta^0 \exp(2\lambda\cot(2y))\sec^2(y)\sin^2(y)\csc^2(y)\,dy\\
&< \exp(-2\lambda\cot(2\theta))\sin^2(\theta) \int_\theta^0 \exp(2\lambda\cot(2y))\sec^2(y)\csc^2(y)\,dy\\
&= \frac{1}{\lambda}\sin^2(\theta),
\end{aligned} \tag{3.5}$$

Because $\sin^2(\theta)$ is strictly decreasing on $(-\frac{\pi}{2},0)$ and

$$\frac{d}{dy}\left[\exp(2\lambda\cot(2y))\right] = -\lambda\exp(2\lambda\cot(2y))\sec^2(y)\csc^2(y).$$

Observe also that for $\theta \in (-\frac{\pi}{2}, 0)$,

$$H'(\theta;\lambda) = \lambda H(\theta;\lambda)(\sec^2(\theta) + \csc^2(\theta)) - \sec^2(\theta) = \frac{1}{C}(p_0(\theta;\lambda) - p_1(\theta;\lambda)). \tag{3.6}$$

Combining equations (3.5) and (3.6), we have that for $\theta \in (-\frac{\pi}{2}, 0)$,

$$\frac{1}{C}(p_0(\theta;\lambda) - p_1(\theta;\lambda)) < 0.$$

Thus equation (3.4) holds. ∎

LEMMA 3.2. *For $\lambda > 0$, define*

$$G(\lambda) := \int_0^{2\pi} (p_0(\theta;\lambda) - p_1(\theta;\lambda)) \cos(\theta)\sin(\theta)\, d\theta.$$

Then $G(\lambda) > 0$ and

- *If $G\left(\frac{r}{c}\right) > \frac{\alpha}{c}$, then $||X_t|| \to \infty$ as $t \to \infty$ almost surely.*
- *If $G\left(\frac{r}{c}\right) < \frac{\alpha}{c}$, then $||X_t|| \to 0$ as $t \to \infty$ almost surely.*

Proof. By equations (3.3) and (3.4) in the statement of Lemma 3.1, we have that $(p_0(\theta;\lambda) - p_1(\theta;\lambda))\cos(\theta)\sin(\theta) > 0$ for all θ and thus $G(\lambda) > 0$.

Now by equation (3.1), we have that

$$\frac{1}{t}\log\left(\frac{R_t}{R_0}\right) = \frac{1}{t}\int_0^t \langle A_{I_s} U_s, U_s\rangle\, ds.$$

Identify $\theta \in \mathbb{R}$ with $e_\theta := (\cos\theta, \sin\theta) \in S^1$. It follows from Lemma 3.1 and Birkhoff's ergodic theorem that there exists a set $A \in S^1$ with $\mu(A) = 1$ so that if $U_0 \in A$, then

$$\frac{1}{t}\log\left(\frac{R_t}{R_0}\right) \to \int \langle A_i e_\theta, e_\theta\rangle\, \mu(d\theta, i) \quad \text{almost surely as } t \to \infty. \tag{3.7}$$

Define $T_A := \inf\{t \geq 0 : U_t \in A\}$ and observe that for any $U_0 \in S^1$, we have that $T_A < \infty$ almost surely because U_t is recurrent. Because T_A is a stopping time, we have that the convergence in equation (3.7) actually holds for every $U_0 \in S^1$.

Plugging in our choice of A_0 and A_1, and the definition of μ, yields

$$\begin{aligned}\int \langle A_i e_\theta, e_\theta\rangle\, \mu(d\theta, i) &= \int_0^{2\pi} \langle A_0 e_\theta, e_\theta\rangle p_0(\theta; r/c)\, d\theta + \int_0^{2\pi} \langle A_1 e_\theta, e_\theta\rangle p_1(\theta; r/c)\, d\theta \\ &= c\int_0^{2\pi} (p_0(\theta; r/c) - p_1(\theta; r/c))\cos(\theta)\sin(\theta)\, d\theta - \alpha \\ &= cG\left(\frac{r}{c}\right) - \alpha.\end{aligned}$$

Hence if $G\left(\frac{r}{c}\right) > \frac{\alpha}{c}$, then $\lim_{t\to\infty} \frac{1}{t}\log\left(\frac{R_t}{R_0}\right) > 0$ almost surely and thus $||X_t|| \to \infty$ as $t \to \infty$ almost surely. Similarly if $G\left(\frac{r}{c}\right) < \frac{\alpha}{c}$, then $||X_t|| \to 0$ as $t \to \infty$ almost surely. ∎

Because $G\left(\frac{r}{c}\right) > 0$ for every pair of positive numbers r and c, it is immediate that we can choose r, c, and α so that $||X_t|| \to \infty$ as $t \to \infty$ almost surely.

REMARK 3.2. Relating this example to the deterministic problem studied in [2], the pair A_0, A_1 defined above fall into case **S4** with $\mathcal{R} > 1$ of Theorem 1 in [2].

3.1. Many transitions between stable and unstable. The following example shows that there exist two matrices such that as the switching rate varies from zero to infinity, the asymptotic behavior of the system will switch between converging to zero and converging to infinity at least any prespecified number of times.

EXAMPLE 3.3. *Assume $\mathbb{P}(X_0=0)=0$ and let $E=\{0,1\}$ and $Q=\left(\begin{smallmatrix}-1 & 1\\ 1 & -1\end{smallmatrix}\right)$. We will show that for any positive integer k, there exist matrices $A_0,A_1\in\mathbb{R}^{2k\times 2k}$ and positive numbers $a_1<b_1<a_2<b_2<\cdots<a_k<b_k$ so that*

1. *A_0, A_1 are each Hurwitz.*
2. *$\bar{A}=\frac{1}{2}(A_0+A_1)$ is Hurwitz.*
3. *If $r\notin\bigcup_{i=1}^k(a_i,b_i)$, then $||X_t||\to 0$ almost surely as $t\to\infty$.*
4. *For every $i\in\{1,\dots,k\}$, $||X_t||\to\infty$ almost surely as $t\to\infty$ for some value of $r\in(a_i,b_i)$.*

Let k be a given positive integer and define the two block diagonal matrices $A_0,A_1\in\mathbb{R}^{2k\times 2k}$ by

$$A_0=\begin{pmatrix}A_0^1 & 0 & \cdots & 0\\ 0 & A_0^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_0^k\end{pmatrix},\qquad A_1=\begin{pmatrix}A_1^1 & 0 & \cdots & 0\\ 0 & A_1^2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & A_1^k\end{pmatrix},\tag{3.8}$$

where

$$A_0^i=\begin{pmatrix}-\alpha_i & c_i\\ 0 & -\alpha_i\end{pmatrix},\qquad A_1^i=\begin{pmatrix}-\alpha_i & 0\\ -c_i & -\alpha_i\end{pmatrix},\tag{3.9}$$

for some positive numbers $\{c_i\}_{i=1}^k$ and $\{\alpha_i\}_{i=1}^k$. It's immediate that A_0, A_1, and $\bar{A}$ are all Hurwitz.

Let X_t denote the $\mathbb{R}^{2k}$-valued process corresponding to (3.8) and $X_t^{(i)}$ the $\mathbb{R}^2$-valued process corresponding to (3.9) for each $i\in\{1,\dots,k\}$. Because the ODEs for $X^{(i)}$ and $X^{(j)}$ are not coupled for $i\neq j$, we have that $X_t=(X_t^{1)},\dots,X_t^{(k)})$ when viewed as an $(\mathbb{R}^2)^k$-valued process. In particular, one has

$$||X_t||^2=\sum_{i=1}^k||X_t^{(i)}||^2.$$

Thus $||X_t||\to 0$ if and only if $||X_t^{(i)}||\to 0$ for every $i\in\{1,\dots,k\}$. Furthermore if $||X_t^{(i)}||\to\infty$ for some $i\in\{1,\dots,k\}$, then $||X_t||\to\infty$.

The proof proceeds by choosing the parameters α_i and c_i as in Example 3.1 so that $X^{(i)}$ is unstable for switching rates r in an interval (a_i,b_i) but stable outside of the interval. By arranging so that the collection of intervals $\{(a_j,b_j):j=1,\dots,k\}$ are disjoint we will succeed at constructing the desired matrices A_0 and A_1.

More explicitly, it follows from Lemma 3.2 and theorems 2.3 and 2.4 that we can choose r_1, c_1, α_1, and $a_1<b_1$ so that

$$G\left(\frac{r_1}{c_1}\right)>\frac{\alpha_1}{c_1}\quad\text{and}\quad G\left(\frac{r}{c_1}\right)<\frac{\alpha_1}{c_1}\text{ if }r\notin(a_1,b_1).$$

Choose $N>\frac{b_1}{a_1}$ and for $i\in\{2,\dots,k\}$ define

$$a_i=\frac{a_1}{N^{i-1}},\quad b_i=\frac{b_1}{N^{i-1}},\quad \alpha_i=\frac{\alpha_1}{N^{i-1}},\quad c_i=\frac{c_1}{N^{i-1}},\quad r_i=\frac{r_1}{N^{i-1}}.$$

To see that our intervals (a_i,b_i) don't overlap, observe that $a_i < b_i$ for each i and

$$b_i = \frac{b_1}{N^{i-1}} < \frac{Na_1}{N^{i-1}} = a_{i-1}.$$

Next observe that if $r \notin (a_i,b_i)$, then $rN^{i-1} \notin (a_1,b_1)$ and therefore

$$G\left(\frac{r}{c_i}\right) = G\left(\frac{rN^{i-1}}{c_1}\right) < \frac{\alpha_1}{c_1}.$$

Thus, $||X_t^{(i)}|| \to 0$ almost surely as $t \to \infty$ if $r \notin (a_i,b_i)$.

Finally observe that $r_i \in (a_i,b_i)$ and

$$G\left(\frac{r_i}{c_i}\right) = G\left(\frac{r_1}{c_1}\right) > \frac{\alpha_1}{c_1} = \frac{\alpha_i}{c_i}.$$

Thus, $||X_t^{(i)}|| \to \infty$ almost surely as $t \to \infty$ if the switching rate is $r_i \in (a_i,b_i)$.

4. Conclusions

Stochastically switched linear ODEs are one of the simplest examples of stochastically switched systems. However despite their simplicity, we have shown that their behavior can be quite rich. First, the large time behavior can depend on the switching rate in a very delicate way. Second, this large time behavior can be very different from the large time behavior of both the individual systems and the average system.

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REFERENCES

[1] Y. Bakhtin and T. Hurth, *Invariant densities for dynamical systems with random switching*, Nonlinearity, 25, 2012.
[2] M. Balde, U. Boscain, and P. Mason, *A note on stability conditions for planar switched systems*, Int. J. Control, 82(10), 1882–1888, 2009.
[3] I. Belykh, V. Belykh, R. Jeter, and M. Hasler, *Multistable randomly switching oscillators: The odds of meeting a ghost*, Eur. Phys. J. Special Topics, Springer, 222(10), 2497–2507, 2013.
[4] M. Benaim, S. Leborgne, F. Malrieu, and P.A. Zitt, *Qualitative properties of certain piecewise deterministic Markov processes*, preprint, 2012.
[5] M. Benaim, S. Leborgne, F. Malrieu, and P.A. Zitt, *Quantitative ergodicity for some switched dynamical systems*, Elec. Commun. Prob., 17, 1–14, 2012.
[6] M. Benaim, S. Leborgne, F. Malrieu, and P.A. Zitt, *On the stability of planar randomly switched systems*, Ann. Appl. Prob., 24(1), 292–311, 2013.
[7] B. Cloez and M. Hairer, *Exponential ergodicity for Markov processes with random switching*, preprint, 2013.
[8] M. Hasler, V. Belykh, and I. Belykh, *Dynamics of stochastically blinking systems. Part i: Finite time properties*, SIAM J. Appl. Dyn. Syst., 12, 1007–1030, 2013.
[9] M. Hasler, V. Belykh, and I. Belykh, *Dynamics of stochastically blinking systems. Part ii: Asymptotic properties*, SIAM J. Appl. Dyn. Syst., 12, 1031–1084, 2013.
[10] R.Z. Khasminskii, *Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems*, Theo. Prob. Appl., 12, 144–147, 1967.
[11] T. Kurtz, *A random Trotter product formula*, Proc. Amer. Math. Soc., 35(1), 147–154, 1972.
[12] H. Lin and P.J. Antsaklis, *Stability and stabilizability of switched linear systems: A survey of recent results*, IEEE Transaction on Automatic Control, 54, 2009.
[13] J. Norris, *Markov Chains*, Cambridge University Press, 1997.