**Math 6310, Assignment 2** **Due in class: Friday, September 25**

1. Let $x, y \neq e$ be elements of a group such that $xyx^{-1} = y^5$, the element $x$ has order 3, and $y$ has odd order. Find the order of $y$.

2. Let $V$ be the set of $r \times s$ matrices over $\mathbb{R}$. Consider the group $G = \mathrm{GL}_r(\mathbb{R}) \times \mathrm{GL}_s(\mathbb{R})$ acting on $V$, where

$$(A,B)\colon M \longmapsto AMB^{-1} \qquad \text{for } M \in V \text{ and } (A,B) \in G.$$

   How many orbits are there? Describe them.

3. Let $G$ be a group, and $Z$ its center.
   (a) If $G/Z$ is cyclic, prove that $G$ is abelian.
   (b) If $\mathrm{Aut}(G)$ is cyclic, prove that $G$ is abelian.
   (c) If $|G| = p^3$ for $p$ a prime, show that either $G$ is abelian or $|Z| = p$.

4. Let $G$ be a $p$-group. If $H$ is a normal subgroup of order $p$, prove that $H$ is contained in the center of $G$.

5. Let $G$ be a finite group with an automorphism $\varphi$ that fixes only the group identity.
   (a) Show that each element of $G$ can be written as $x^{-1}\varphi(x)$.
   (b) If $\varphi^2$ is the identity, prove that $\varphi(x) = x^{-1}$ for each $x$, and conclude that $G$ is abelian.

6. Let $G$ be a finite group such that $\mathrm{Aut}(G)$ acts transitively on the set $G \smallsetminus \{e\}$. Show that $G$ is a $p$-group, and that $G$ is abelian.

7. Let $G$ be an infinite group, and $H$ a subgroup of finite index. Show that $G$ has a normal subgroup $K$ of finite index, with $K < H$.

8. Let $G$ be an infinite group with an element $x \neq e$ that has only finitely many conjugates. Prove that $G$ is not simple, i.e., that it has a proper normal subgroup.

9. Let $H$ and $K$ be subgroups of a group $G$. For $x \in G$, the set $HxK = \{hxk \mid h \in H, k \in K\}$ is a *double coset*.
   (a) Prove that $G$ is a disjoint union of double cosets $HxK$, and that $|HxK| = \dfrac{|H||K|}{|H \cap xKx^{-1}|}$.
   (b) Prove that $|HK| = \dfrac{|H||K|}{|H \cap K|}$.
   (c) If all double cosets $HxH$ for $x \in G$ have the same number of elements, show that $H \lhd G$.

10. Let $G$ be a group of odd order acting transitively on a set $S$. Fix $s \in S$. Show that the orbits of the action of $G_s$ on $S \setminus \{s\}$ have lengths that are equal in pairs.