On the boundedness of varieties of general type

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June, 2015

Introduction

- One of the main goals in Algebraic Geometry is to classify algebraic complex projective varieties $X \subset \mathbb{P}^N_{\mathbb{C}}$.
- Assume (for now) that X is smooth.
- Since $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$, we study X via sections of an appropriate line bundle.
- There is essentially only one choice choice, the canonical line bundle $\omega_X = \mathcal{O}(K_X)$, defined by $\omega_X = \Lambda^n T^\vee_X = \Omega^n_X$ where $n = \dim X$.
- In dimension 1, X is topologically determined by its genus $g = h^0(\omega_X) := \dim H^0(\omega_X)$.
- If $g = 0$, then $X \cong \mathbb{P}^1$ and $\omega_X = \mathcal{O}_{\mathbb{P}^1}(-2)$,
- if $g = 1$, then $X \cong \mathbb{C}/\mathit{Lattice}$ is an elliptic curve so $\omega_X \cong \mathcal{O}_X$ and there is a one parameter family of these,
- if $g \geq 2$ then $\deg \omega_X = 2g - 2 > 0$, $\omega_X^{\otimes 3}$ is very ample and there is a $3g - 3$ dimensional moduli space M_g.

Kodaira dimension

- Note that ω_X is particularly useful if $h^0(\omega_X^{\otimes m})$ has sections.
- Let $R(\omega_X) = R(K_X) = \oplus_{m\geq 0} H^0(\omega_X^{\otimes m})$ be the **canonical ring** and define **the Kodaira dimension** $\kappa(X) = tr.deg._{\mathbb{C}} R(K_X) - 1 \in \{-1, 0, 1, \ldots, \dim X\}$.
- If $\phi_{mK_X} : X \dashrightarrow \mathbb{P}H^0(\omega_X^{\otimes m})$ is the m-th canonical map then $\kappa(X) = \max\{\dim \phi_{mK_X}(X) | m > 0\}$.
- We say that X has *general type* if $\kappa(X) = \dim X$. This is the analog of curves of genus $g \geq 2$.
- Note that X is of general type iff ϕ_{mK_X} is birational for $m \gg 0$ iff $K_X \sim_{\mathbb{Q}} A + E$ with A ample and $E \geq 0$ iff $\mathrm{vol}(K_X) > 0$ where

$$\mathrm{vol}(K_X) = \lim \frac{h^0(\mathcal{O}_X(mK_X))}{m^n/n!}.$$

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- If $n = 1$ then $\mathrm{vol}(K_X) = 2g - 2$.

Stable curves

- The moduli spaces M_g are not proper, but there exists a geometrically meaningful compactification $\overline{M}_g$ whose points correspond to stable curves.
- By definition a stable curve is a (possibly reducible) curve that has only nodal singularities such that ω_X is ample.
- If $\nu : \cup X_i \to X$ is the normalization, then $\nu_i^* K_X = K_{X_i} + B_i$ where B_i is the double locus. $K_{X_i} + B_i$ is ample $\forall i$.
- Goal: generalize this picture to all dimensions.
- Define the right class of objects; find discrete invariants ν such for each value of ν we have a projective moduli space (proper and finite type).

Higher dimensions

- In higher dimensions there are many difficulties.
- Eg. if $\dim X = 2$ then by blowing up a given surface we obtain infinitely many surfaces with different invariants ($B_2(X)$) which must belong to different components of the moduli space.
- In dimension ≥ 3 it is unclear what invariants one can use to differentiate between these components.
- Another important issue is that the corresponding moduli spaces are usually not separated.
- Consider $X \to B$ a family of surfaces over a curve B with 3 sections S_1, S_2, S_3 which meet as transversely as possible in one point p lying over $O \in B$.
- Blowing up the (proper transforms) of these sections in different order, we obtain two families $X' \to B$ and $X'' \to B$ which are isomorphic over $B \setminus O$ but with non-isomorphic fibers $X'_O \ncong X''_O$.

Canonical models

- The solution to this problem is to work with varieties up to **birational isomorphism**: $X \sim X'$ iff they have isomorphic open subsets iff $\mathbb{C}(X) \cong \mathbb{C}(X')$.
- Even better, we can choose a distinguished representative of each birational equivalence class.
- $X_{\rm can} := {\rm Proj} R(K_X)$ where $R(K_X) = \oplus_{m\geq 0} H^0(mK_X)$ is finitely generated by BCHM and Siu.
- Note that $X_{\rm can}$ has mild (canonical and hence rational) singularities, $K_{X_{\rm can}}$ is $\mathbb{Q}$-Cartier and ample.
- If $f : X' \to X_{\rm can}$ is a resolution, $K_{X'} = f^*K_{X_{\rm can}} + E$ where $E \geq 0$ (and $f_*K_{X'} = K_{X_{\rm can}}$).
- Since canonical models are unique, this will give separated (but not proper) moduli spaces.

Semi-log canonical models - SLC models

- Since canonical models naturally degenerate to non-normal varieties, in order to compactify the moduli spaces, one must allow the semi-log canonical models i.e. the higher dimensional analogs of stable curves.
- X is a **semi-log canonical model**, if X is SNC in codimension 1, S_2 ($\mathrm{depth}\mathcal{O}_{X,x} \geq 2$), K_X is $\mathbb{Q}$-Cartier and ample.
- Let $f : X' \to X$ be a resolution of the normalization of X, then we can write $K_{X'} = f^*K_X + \sum a_i E_i$ where $a_i \geq -1$.
- These singularities may not be rational but they are still mild (in particular Du Bois).
- If $\nu : X^\nu \to X$ is the normalization and $\nu_i : Y_i \to X$ is the restriction of ν to the individual irreducible components, then $\nu_i^* K_X = K_{Y_i} + \Delta_i$ is a LCM (log canonical model). Note that $Y_i \cong \mathrm{Proj} R(K_{Y_i} + \Delta_i)$.
- By work of Kollár we can recover X by glueing the $\{(Y_i, \Delta_i)\}$ along the double locus.

Properness

- Once we allow SLCM, properness will follow from semistable reduction and the MMP.
- Consider $(\mathcal{X}^0, \mathcal{B}^0) \to C^0$ a family of SLCM over the curve $C^0 = C \setminus O$.
- After compactifying, resolving and base changing, we may assume that if $(\mathcal{X}', \mathcal{B}') \to C$ denotes the compactified family, then $\mathcal{B}' + \mathcal{X}'_O$ has simple normal crossings support (in particular $(\mathcal{X}', \mathcal{B}' + \mathcal{X}'_O)$ is SLC).
- Let $(\mathcal{X}, \mathcal{B}) := \mathrm{Proj}_C R(K_{\mathcal{X}'} + \mathcal{B}') \to C$ be the relative SLC model, then $(\mathcal{X}, \mathcal{B}) \to C$ is a family of SLCM over C which agrees with $(\mathcal{X}^0, \mathcal{B}^0) \to C^0$ over C^0.

Constructing the moduli space

- In order to construct the moduli space, we first fix an integer m and a Hilbert polynomial $H(t)$ and consider the SLC models X such that mK_X is very ample and $H(t) = h^0(tmK_X)$ for all $t > 0$.
- Consider the embedding $\phi : X \hookrightarrow \mathbb{P}^N = |mK_X|$, $\mathcal{O}_X(mK_X) = \phi^*\mathcal{O}_{\mathbb{P}^N}(1)$.
- And the moduli space $SLC_{H(t),m} = S_{H(t),m}/Aut(\mathbb{P}^N)$ where $S_{H(t),m} \subset Hilb_{H(t)}(\mathbb{P}^N)$ is an appropriate locally closed subset.
- $S_{H(t),m}/Aut(\mathbb{P}^N)$ is a separated algebraic space which is locally of finite type.
- To get a proper coarse moduli space of finite type $SLC_{H(t)} = \cup_{m>0} SLC_{H(t),m}$ we need a fixed integer m (depending on $H(t)$) such that mK_X is very ample.
- Projectivity is then established by Fujino, Kovács-Patakfalvi.

Boundedness of log pairs

Theorem (Hacon-McKernan-Xu)

Fix $n \in \mathbb{N}$, $c > 0$, $\mathcal{C}$ a DCC set (eg. $\mathcal{C} = \{1 - \frac{1}{r} \mid r \in \mathbb{N}\}$), then the set

$$SLC(c, n, \mathcal{C}) = \{(X, B = \sum b_i B_i) \mid SLC - models, \ \dim X = n, \\ (K_X + B)^n = c, \ b_i \in \mathcal{C}\}$$

has finitely many deformation types, i.e. there is a projective family $(\mathcal{X}, \mathcal{B}) \to S$ of finite type such that

- *for any $s \in S$ we have $(\mathcal{X}_s, \mathcal{B}_s) \in SLC(c, n, \mathcal{C})$ and*
- *for any $(X, B) \in SLC(c, n, \mathcal{C})$ there is an $s \in S$ with $(X, B) \cong (\mathcal{X}_s, \mathcal{B}_s)$.*

In dimension 2, this is due to Alexeev and Alexee-Mori.

Boundedness of log pairs

- The introduction of the coefficient set $\mathcal{C}$ is necessary because of induction on the dimension (adjunction).
- Note that instead of fixing the Hilbert polynomial, we simply fix the dimension and the volume $c = (K_X + B)^n$.
- The existence of an m such that mK_X is very ample follows by a semicontinuity argument.
- Note also, that it follows that the set of all volumes $V(\mathcal{C}, n) = \{(K_X + B)^n\}$ (where (X, B) is a SLC-model of dimension n and $b_i \in \mathcal{C}$) satisfies the DCC (any non-increasing sequence is eventually constant) and so $V(\mathcal{C}, n) \cap \mathbb{R}_{>0}$ has a minimum $v(\mathcal{C}, n) > 0$.
- This is important because if $f : \cup_{i \in I} Y_i \to X$ is the normalization and $K_{Y_i} + \Delta_i = (K_X + B)|_{Y_i}$, then $c_i := (K_{Y_i} + \Delta_i)^n \in V(\mathcal{C}, n)$ so that $|I| \leq c/v(\mathcal{C}, n)$ and there are finitely many possibilities for the c_i.

Boundedness of the moduli functor

- In fact the volumes of canonical models are discrete (but the volumes of log canonical models with 0 boundary are not discrete).
- There are several technical issues, but the above result seems to be sufficient to settle the boundedness of the corresponding moduli functor.
- This should imply that the moduli functor of SLC models with Hilbert polynomial H, $\mathcal{M}_H^{SLC}$ is bounded.
- By definition $\mathcal{M}_H^{SLC}(S)$ is the set of flat projective morphisms $\mathcal{X} \to S$ whose fibers are SLC models with Hilbert polynomial H, $\omega_{\mathcal{X}}$ is flat over S and commutes with base change.
- Disclaimer: There is a huge body of work that goes in to the construction of $\mathcal{M}_H^{SLC}(S)$ (Alexeev, Shepherd-Baron, Kollár, Kovács, Viehweg and others). I do not discuss this, but focus on the boundedness of log pairs stated above.

Curves

- The easiest case is of course the case of dimension 1: **Claim.** If $g \geq 2$, then mK_X is very ample for all $m \geq 3$.
- We must show that $|mK_X|$ separates points and tangent directions i.e. that $H^0(mK_X) \to H^0(mK_X/I_Z) \cong \mathbb{C}^2$ is surjective for any scheme $Z \subset X$ of length 2.
- There is a short exact sequence

$$0 \to \mathcal{O}(mK_X - Z) \to \mathcal{O}(mK_X) \to \mathcal{O}(mK_X)/I_Z \to 0.$$

- By the corresponding long exact sequence in cohomology, we must check that $H^1(\mathcal{O}(mK_X - Z)) = 0$.
- By Serre duality this is equivalent to $H^0(\mathcal{O}(Z - (m-1)K_X)) = 0$. This is obvious since $\deg(Z - (m-1)K_X) < 0$ for $g \geq 2$ and $m \geq 3$.

Log curves

- Consider $\mathcal{C} = \{(C, B = \sum b_i B_i\}$ where C is a curve, $b_i \in \{1 - \frac{1}{k} | k \in \mathbb{N}\}$ and $\deg(K_C + B) = 2g - 2 + \sum b_i > 0$.
- Then $\min\{\deg(K_C + B) = \frac{1}{42}\}$.
- Proof: we may assume $g = 0$ (else for $g \geq 2$ we have $2g - 2 \geq 2 > 1/42$ and for $g = 1$, $2g - 2 = 0$ so $b_1 \neq 0$ and $b_1 = 1 - 1/k \geq 1/2 > 1/42$).
- Since $1 \geq b_i \geq 1/2$, then $|I| \in \{3, 4\}$. But $b_1 + \ldots + b_4 \geq 1/2 + 1/2 + 1/2 + 2/3 = 2 + 1/6$, so $|I| = 3$.
- $2 \leq k_1 \leq k_2 \leq k_3$. It is easy to see that $b_2 > 1/2$ and $b_1 < 3/4$. An easy case by case analysis yields that the minimum is achieved by $1/2 + 2/3 + 6/7 = 2 + 1/42$.
- Corollary: If C is of general type, then $|Aut(C)| \leq 84(g - 1)$.
- Proof: $K_C = f^*(K_{\bar{C}} + B)$ where $\bar{C} = C/Aut(C)$ and B is the ramification. Then $2g - 2 = |Aut(C)|\deg(K_{\bar{C}} + B) \geq |Aut(C)|/42$.

Surfaces

- If K_X is nef and big, then $\chi(\mathcal{O}_X) > 0$ and for $m \geq 2$,
$$h^0(mK_X) = \chi(mK_X) = \frac{m(m-1)}{2}K_X^2 + \chi(\mathcal{O}_X)$$
so that $h^0(2K_X) \geq 2$.
- Let $X \dashrightarrow C$ be the Stein factorization of the map induced by a general pencil. Assume for simplicity that this is base point free and denote by F, F_i general fibers so that $F_i \in |2K_X|$.
- Let x_1, x_2 be two general points in X and consider the following short exact sequences.
- $0 \to 3K_X \to 3K_X + B \to K_B + (2K_X)|_B \to 0$ and
$0 \to 2K_X \to 2K_X + B_1 + B_2 \to \oplus(K_{B_i} + (K_X)|_{B_i} \to 0$
- Since $H^1(mK_X) = 0$ for $i > 0$ and $H^0(K_B + (mK_X)|_B)$ is non-empty and in fact very ample for $m \geq 2$, it follows that $6K_X$ defines a birational map.

Higher dimensional difficulties

- Some difficulties that occour in higher dimensions are:
- I_Z is not locally free (need theory of multiplier ideal sheaves).
- It is harder to show that $H^1(\mathcal{O}(mK_X)\otimes I_Z)=0$ (need Kawamata-Viehweg-Nadel vanishing).
- X is singular so must understand the singularities of the MMP.
- $\mathrm{vol}(K_X)$ could be very small (eg. $1/420$ for threefolds). So many arguments are non-effective and rely on Noetherian induction.
- We will need to understand how $\mathrm{vol}(K_X)$ varies in families (deformation invariance of plurigenera; Siu and others).

Singularities of the MMP

- Let X be a normal variety, U the big open subset of smooth points, then ω_U is a line bundle locally generated by $dz_1 \wedge \ldots \wedge dz_n$ where $z_1, \ldots, z_n$ are local coordinates.
- The **canonical sheaf** is defined by $\omega_X = i_*\omega_U$ where $i : U \to X$.
- A **canonical divisor** K_X is any divisor such that $\mathcal{O}_X(K_X) \cong \omega_X$ (K_X is not unique).
- A **log pair** (X, B) consists of a normal variety X and a $\mathbb{R}$-divisor $B = \sum b_i B_i$ such that $K_X + B$ is $\mathbb{R}$-Cartier.
- A **log resolution** of a log pair (X, B) is a proper birational morphism $f : Y \to X$ such that the exceptional locus $\mathrm{Ex}(f)$ is a divisor and $\mathrm{Ex}(f) + f_*^{-1}B$ has simple normal crossings support.
- Eg. Compute a log resolution of $(\mathbb{C}^2, \{y^2 - x^3 = 0\})$.

Singularities of the MMP

- We may write $K_Y = f^*(K_X + B) + A_Y(X, B)$ where $f_* K_Y = K_X$ and $f_* A_Y(X, B) = -B$.
- $A_Y(X, B)$ is the discrepancy divisor (and $\mathbf{A}(X, B)$ is the discrepancy b-divisor defined by $\mathbf{A}(X, B)_Y = A_Y(X, B)$).
- If we write $A_Y(X, B) = \sum a_P(X, B)P$ where P are the prime divisors on Y, then $a_P(X, B)$ are the **discrepancies** of (X, B) along P.
- It is convenient to write $A_Y(X, B) = E_Y(X, B) - L_Y(X, B)$ where $E_Y, L_Y \geq 0$ and $E_Y \wedge L_Y = 0$ (i.e. E_Y and L_Y have no common components).
- We have $K_Y + L_Y = f^*(K_X + B) + E_Y$.
- note that if $F \geq 0$ is f-exceptional, then $H^0(m(K_Y + L_Y + F)) = H^0(f^*(m(K_X + B)) + m(F + E_Y)) \cong H^0(m(K_X + B))$.
- The corresponding b-divisors are denoted by $\mathbf{L}$ and $\mathbf{E}$.

Singularities of the MMP

- The **total discrepancy** of (X, B) is

$$\text{tot. discrep}(X, B) = \inf\{a_P(X, B) | P \text{ ex. prime div. over } X\}.$$

- The **discrepancy** of (X, B) is

$$\text{discrep}(X, B) = \inf\{a_P(X, B) | P \text{ prime div. over } X\}.$$

- Eg, if $f : Y \to X$ is the blow up of a smooth divisor of codimension k with exceptional divisor E, then $a_E(X, B) = k - 1 - \sum_{Z \subset B_i} b_i$.
- Intuition: Smaller discrepancies = more singular varieties. Eg. X the cone over a curve of genus g, and $Y \to X$ the blow up at the vertex p with exceptional divisor E, then $a_P(X, B) \in \{> -1, 0, < -1\}$ if $g \in \{0, 1, \geq 2\}$.
- Lemma: If the total discrepancy is < -1 then it is $-\infty$ ($\dim X \geq 2$).

Singularities of the MMP

- Proof: There is $f : Y \to X$ with a divisor E of discrepancy $a_E(X,B) < -1 - e$ for some $e > 0$. Blow up a general codim 1 point on this divisor to get a divisor E' of discrepancy $a_{E'}(X,B) < - - e$. Blow up the point given by the intersection of the strict transform of E and E' to get a divisor of discrepancy $< -2e$. repeat n-times to get divisors E^n of discrepancy $< -ne$.

Singularities of the MMP

- We say that (X, B) is **log canonical / LC** and **Kawamata log terminal / klt** if the total discrepancies are ≥ -1 and > -1. We say that (X, B) is **terminal**, **canonical** if the discrepancies are > -1, ≥ 0.
- The log canonical and klt conditions can be checked on one (any) log resolution.
- Some authors work with **log discrepancies** which are the discrepancies plus one i.e. $a_P + 1$.
- KLT singularities are rational ($R^i f_* \mathcal{O}_Y = 0$ for $i > 0$) and LC singularities are Du Bois.
- In dimension 2 terminal singularities are smooth and canonical ones are du Val (rational double points).
- If $a_P(X, B) \leq 1$ (resp. < 1), then we say that P is a **NKLT place** (resp. a **NLC place**) and its image $f(P)$ is a center of NKLT singularities.

Singularities of the MMP

- If $f : Y \to X$ is the blow up at the vertex of a cone over a rational curve of degree n with exceptional curve E, then $E^2 = -n$ and by adjunction $-2 = (K_Y + E) \cdot E = (\nu^*(K_X) + (a_E + 1)E) \cdot E = -n(a_E + 1)$ so that $a_E = -1 + \frac{2}{n}$.
- The same computation shows that if the curve is elliptic, then $a_E = -1$ and if the curve has genus $g \geq 2$, then $a_E < -1$.
- If (X, B) is LC and $G \geq 0$ is an $\mathbb{R}$-Cartier divisor, then the **log canonical threshold** is

$$\mathrm{lct}(X, B; G) = \sup\{c > 0 | (X, D + cG) \text{ is LC}\}.$$

- One can compute log canonical thresholds on a single log resolution (eg, $\mathrm{lct}(\mathbb{C}^2, 0; \{y^2 = x^3\}) = 5/6$).

Canonical models

Theorem (BCHM + Siu)

Let (X, B) be a klt pair, $f : X \to Z$ a projective morphism such that $K_X + B$ is $\mathbb{Q}$-Cartier, then $R_Z(K_X + B) := \oplus_{m>0} f_*\mathcal{O}_X(m(K_X + B))$ is finitely generated (over $\mathcal{O}_Z$). Eg. if $Z = \mathrm{Spec}(k)$, then $\oplus H^0(\mathcal{O}_X(m(K_X + B)))$ is finitely generated.

- This conjecturally also holds for LC pairs (very hard but some special cases are known).
- If $K_X + B$ is big, then $\phi : X \dashrightarrow X_{can} := \mathrm{Proj}(R(K_X + B))$ has klt singularities (if $B = 0$ and X has canonical singularities, then X_{can} has canonical sings).
- $K_{X_{can}} + \phi_* B$ is ample and $R(K_{X_{can}} + \phi_* B) \cong R(K_X + B)$.
- In fact if $p : W \to X$ and $q : W \to X_{can}$ resolve ϕ, then $p^*(K_X + B) - q^*(K_{X_{can}} + \phi_* B) \geq 0$.

Canonical models

- We say that X is a **canonical model** if X has canonical singularities and K_X is ample eg. $\mathrm{Proj}(R(K_X))$ where X has canonical sings and K_X is big.
- We say that (X, B) is a **log canonical model** if it has log canonical singularities and $K_X + B$ is ample eg. $\mathrm{Proj}(R(K_X + B))$ where (X, B) has klt (conjecturally even LC) sings and $K_X + B$ is big. .
- We say that a canonical model (resp. log canonical model) X_{can} (resp. $(X_{\mathrm{can}}, B_{can})$) is a canonical model of X (resp. a log canonical model of (X, B)) if given $p : W \to X$ and $q : W \to X_{can}$, then $p^*(K_X + B) - q^*(K_{X_{can}} + \phi_* B) \geq 0$.
- It follows easily that then $R(K_X + B) \cong R(K_{X_{can}} + \phi_* B)$.

Minimal models

- If X is a smooth projective surface of general type, then by Castenuovo's criterion we may contract any -1 curve (i.e. any rational curve $E \subset X$ with $K_X \cot E = E^2 = -1$) to a smooth point say $X \to X_1$.
- Since $B_2(X) = B_2(X_1) + 1$ we can repeat this at most finitely many times $X \to X_1 \to \ldots \to X_N$. The output $X_N = X_{\min}$ is the minimal model and $K_{X_{min}}$ is nef.
- There is a morphism $K_{X_{min}} \to K_{X_{can}}$ given by contracting K-trivial curves (i.e. $E \cdot K_X = 0$).
- In higher dimensions, the minimal model has terminal sings and it can be obtained by a sequence of finitely many flips and divisorial contractions $X \dashrightarrow X_{min}$.
- $K_{X_{min}}$ is semiample and induces a morphism $X_{min} \to X_{can}$ which contracts K-trivial curves. (X_{can} has canonical sings.).

Semi log canonical models

- If X is an S_2 variety with simple normal crossing singularities in codimension 1 and B is a $\mathbb{R}$-divisor on X whose support contains no component of $\mathrm{Sing}(X)$ and $K_X + B$ is $\mathbb{R}$-Cartier then (X, B) has **SLC singularities** if given $\nu : Y \to X$ the normalization and $K_Y + B_Y = \nu^*(K_X + B)$, then (Y, B) is log canonical (i.e. its components are LC). If moreover X is projective and $K_X + B$ is ample, then we say that (X, B) is a **SLC model**.

Vanishing

- The main tool of the MMP is Kawamata-Viehweg vanishing, a far reaching generalization of the better known Kodaira vanishing:
- **Kodaira vanishing**: Let X be a smooth projective variety and A an ample divisor, then $H^i(\mathcal{O}_X(K_X + A)) = 0$ for $i > 0$.
- **Kawamata Viehweg vanishing:** Let (X, B) be a KLT pair and L be a Cartier divisor such that $L - (K_X + B)$ is big and nef, then $H^i(\mathcal{O}_X(L)) = 0$ for all $i > 0$.
- Recall, L is **nef** if $L \cdot C > 0$ for any curve $C \subset X$.

Vanishing

- The relative version of KV vanishing also holds: If $f : X \to Y$ is a projective morphism $L-(K_X+B)$ is f-nef and f-big, then $R^i f_* \mathcal{O}_X(L) = 0$ for $i > 0$.
- M is f-nef (f-big) if $M + f^*A$ is nef (big) for a sufficiently ample divisor A on Y.
- Relative KV vanishing follows easily from KV vanishing by a spectral sequence argument.
- An easy but important consequence is the Kollár-Shokurov connectedness lemma.

Theorem

$f : X \to Z$ proper of normal vars with connected fibers such that $-(K_X+B)$ is f-nef and f-big, any divisor with negative coefficient in B is f-exceptional, then $\mathrm{Nklt}(X, B) \cap f^{-1}(z)$ is connected for any $z \in Z$.

Vanishing

- Replacing (X, B) by a log resolution, we may assume that X is smooth and B has SNC. $\mathrm{Nklt}(X, B) = \mathrm{Supp}(B^{\geq 1})$.
- we have a s.e.s.

$$0 \to \mathcal{O}_X(\lceil -B \rceil) \to \mathcal{O}_X(\lceil -B^{<1} \rceil) \to \mathcal{O}_S(\lceil -B^{<1} \rceil) \to 0$$

  where $S = \lfloor B^{\geq 1} \rfloor$.
- as $\lceil -B \rceil = K_X + \lceil -(K_X + B) \rceil = K_X - (K + B) + \{K_X + B\}$, then by KV vanishing $R^i f_* \mathcal{O}_X(\lceil -B \rceil) = 0$ so that there is a surjection

$$f_* \mathcal{O}_X(\lceil -B^{<1} \rceil) \to f_* \mathcal{O}_S(\lceil -B^{<1} \rceil).$$

- $\lceil -B^{<1} \rceil \geq 0$ so

$$f_* \mathcal{O}_S \subset f_* \mathcal{O}_S(\lceil -B^{<1} \rceil)$$

  and hence $\mathcal{O}_Z \to \mathcal{O}_{f(S)} \to f_* \mathcal{O}_S$ and so $S \to f(S)$ has connected fibers.

Calculus of Nklt centers

- If (X,B) is LC and (X,B_0) is klt and W_i are Nklt centers of (X,B), then so is any irreducible component of $W_1 \cap W_2$.
- So at every point there is (locally) a unique minimal center on Nklt singularities for (X,B).
- Minimal Nklt centers are normal and satisfy sub-adjunction.
- If $(X,B+S)$ has SNC and S is prime of coefficient one in $S+B$, then $(K_X+S)|_S = K_S + B|_S$.
- If $(X,B+S)$ is LC and S is a minimal LC center, then $(K_X+S+B)|_S = K_S + \mathrm{Diff}_S(B)$, where if B is $\mathbb{R}$-Cartier in codimension 2, $\mathrm{Diff}_S(B) = \mathrm{Diff}_S(0) + B|_S$ and the coefficients of $\mathrm{Diff}_S(0)$ are standard (of the form $\{1-\frac{1}{k}|k\in\mathbb{N}\}$).

Calculus of Nklt centers

- Note that if $\tau = \mathrm{lct}(X, B)$, then $(X, \tau B)$ is log canonical.
- if W is a minimal Nklt center of a LC pair (X, B), then pick A a sufficiently ample divisor and $L \in |A \otimes I_W|$ general. If $\tau = \mathrm{lct}(X, B + \epsilon L)$, then W is the unique Nklt center for $(X, \tau(B + \epsilon L))$ and there is a unique Nklt place.

Subadjunction

- Example: Let (X, S) be the cone over a rational curve of degree n and a line through the vertex $v \in S \subset X$. If $f : Y \to X$ is the blow up of the vertex with exceptional divisor E and $S' = f_*^{-1}S \cong S$, then $f^*S = S' + \frac{1}{n}E$, so $f^*(K_X + S) = K_Y + S' + (1 - \frac{1}{n})E$, and hence $K_S + \mathrm{Diff}_S(B) = (K_X + S)|_S = (K_Y + S' + (1 - \frac{1}{n})E)|_{S'} = K_S + (1 - \frac{1}{n})v$.
- **Kawamata sub-adjunction:** (X, B) LC and W a minimal Nklt center then $(K_X + B + \epsilon H)|_W \sim_{\mathbb{Q}} K_W + B_W$ where (W, B_W) is klt (we will recall a more precise version later).

Multiplier ideals

- Let X be smooth, $B \geq 0$, $f : Y \to X$ a log resolution of (X, B) then the **multiplier ideal sheaf** of (X, D) is

$$\mathcal{J} = \mathcal{J}(X, B) = f_*\mathcal{O}_X(K_{Y/X} - \lfloor f^*B \rfloor) \subset f_*\mathcal{O}_X(K_{Y/X}) = \mathcal{O}_X.$$

- $\mathcal{J}$ is independent of the log resolution.
- $\mathcal{J} = \mathcal{O}_X$ iff (X, B) is klt.
- iF B IS SNC, then $\mathcal{J}(B) = \mathcal{O}_X(-\lfloor B \rfloor)$.
- If G Cartier, then $\mathcal{J}(G + B) = \mathcal{J}(B) \otimes \mathcal{O}_X(-G)$.
- $D_1 \leq D_2$ then $\mathcal{J}(D_2) \subset \mathcal{J}(D_1)$.
- $\mathrm{mult}_x(D) \geq n = \dim X$, then $\mathcal{J}(D) \subset m_x$ (just blow up $x \in X$).
- (Harder) If $\mathrm{mult}_x(D) < 1$, then $\mathcal{J}(D)_x = \mathcal{O}_{X,x}$.
- $\mathrm{lct}(X, B) = \sup\{t | \mathcal{J}(X, tB) = \mathcal{O}_X\}$.

Nadel vanishing

Theorem (Nadel vanishing)

X smooth, $f : X \to Z$ a projective morphism, $D \geq 0$ an $\mathbb{R}$-divisor, N a Cartier divisor such that $N - D$ is f-nef and f-big, then

$$R^i f_*(\mathcal{O}_X(K_X + N) \otimes \mathcal{J}(D)) = 0 \qquad \forall i > 0.$$

Proof: Given $g : Y \to X$ a log resolution, $g^*(N - D)$ is $(f \circ g)$-nef and $(f \circ g)$-big as well as g-nef and g-big. Thus
$R^i g_* \mathcal{O}_Y(K_Y + \lceil g^*(N - D) \rceil) = 0$ and
$R^i (f \circ g)_* \mathcal{O}_Y(K_Y + \lceil g^*(N - D) \rceil) = 0$ for $i > 0$.
But $g_* \mathcal{O}_Y(K_Y + \lceil g^*(N - D) \rceil) = \mathcal{O}_X(K_X + N) \otimes \mathcal{J}(D)$ so
$R^i f_* \mathcal{O}_X(K_X + N) \otimes \mathcal{J}(D) = R^i (f \circ g)_* \mathcal{O}_Y(K_Y + \lceil g^*(N - D) \rceil) = 0$
for $i > 0$ (by an easy Spectral sequence argument).

Restrictions

- X smooth, H smooth irreducible divisor on X, $D \geq 0$ effective $\mathbb{R}$-divisor whose support does not contain S. Then
- $\mathcal{J}(H, D|_H) \subset \mathcal{J}(X, D) \cdot \mathcal{O}_H$, and
- if $0 < s < 1$, then for all $0 < t \ll 1$ we have

$$\mathcal{J}(X, D + (1-t)H) \cdot \mathcal{O}_H \subset \mathcal{J}(H, (1-s)D|_H)$$

- This is an example of inversion of adjunction. If $\mathcal{J}(H, (1-s)D|_H) \subset m_x$ ($x \in H$ and all $0 < s < 1$), then $\mathcal{J}(X, D + (1-t)H) \subset m_x$ for all $0 < t \ll 1$.
- (analog for log pairs) $(X, S+B)$ an effective log pair, $\nu : S^\nu \to S$ the normalization of S and $K_{S^\nu} + B_{S^\nu} = \nu^*(K_X + S + B)$, then $(X, S+B)$ is plt (LC + S is the only Nklt place) iff (S^ν, B_{S^ν}) is klt and .$(X, S+B)$ is LC iff (S^ν, B_{S^ν}) is LC.
- The first is an easy consequence of the connectedness lemma, the second is a deep result of Kawakita.

Inversion of adjunction

- To see this, consider $f : X' \to X$ a log resolution and write $K_{X'} + S' + B' = f^*(K_X + S + B)$.
- Assume for simplicity that S is normal (this is true for plt pairs) and write $(K_X + S + B) = K_S + B_S$ and $(K_{X'} + S' + B')|_{S'} = K_{S'} + B_{S'}$ where as $(X', S' + B')$ is SNC, $B_{S'} = B'|_{S'}$.
- Note that $K_{S'} + B_{S'} = f^*(K_S + B_S)$.
- If $(X, S + B)$ is plt, then $B' < 1$ and so $B_{S'} < 1$ i.e. (S, B_S) is klt.
- If (S, B_S) is klt, then $B_{S'} < 1$ so that $\lfloor B' \rfloor \cap S' = \emptyset$.
- Suppose by contradiction that there is a component F of $\lfloor B' \rfloor$ and a point $x \in f(F) \cap S$, then the fiber $f^{-1}(x)$ intersects S' and $\lfloor B' \rfloor$. This is easily seen to contradict the connectedness lemma.

Restrictions

- We can now prove that if $\mathrm{mult}_x(D) < 1$, then $\mathcal{J}(D)_x = \mathcal{O}_{X,x}$.
- Fix $x \in H \subset X$ a general smooth divisor, then $\mathrm{mult}_x(D|_H) = \mathrm{mult}_x(D) < 1$.
- Thus $\mathcal{O}_{H,x} = \mathcal{J}(H, D|_H)_x \subset \mathcal{J}(X, D) \cdot \mathcal{O}_{H,x}$ and the claim follows.

More about volumes

- Using multiplier ideals and a cleaver induction, Siu proves the following beautiful result.

Theorem

Let $f : X \to S$ be a smooth projective morphism of smooth projective varieties and $m \geq 0$, then $h^0(m(K_{X_s}))$ is independent of $s \in S$. (Equivalently $f_*\mathcal{O}_X(mK_X)$ is locally free).

- There is no algebraic proof of this (except if K_X is big over S).
- Another important fact that we will need later is

Theorem (Easy addition)

Let $f : X \to S$ be a smooth projective morphism, then $\kappa(X) \leq \kappa(X_s) + \dim S$ where $s \in S$ is general. In particular if X has general type, then so does X_s.

More about volumes

- The idea is as follows. If X has general type (the other cases are similar), we may write $K_X \sim_{\mathbb{Q}} A + E$ where A is ample and E is effective.
- But then $K_{X_s} \sim_{\mathbb{Q}} A|_{X_s} + E|_{X_s}$ where $A|_{X_s}$ is ample and $E|_{X_s}$ is effective.
- We have the following important consequence.

Theorem

Let $Z \to T$ be a projective morphism and $f : Z \to X$ a dominant morphism to a projective variety. If X is of general type, then so is Z_t for general $t \in T$.

- By definition X is of general type iff so is any resolution of X.

More about volumes

- Cutting by generic hyperplanes on T, we may assume that $Z \to X$ is generically finite.
- Replacing X and $Z \to T$ by appropriate birational models we may assume that X, Z, T are smooth.
- Since $f : Z \to X$ is generically finite, we have $K_Z = f^*K_X + R$ where $R \geq 0$ is the ramification divisor.
- Thus Z is also of general type.
- By the easy addition theorem, Z_t is of general type.

Ample canonical class

Next we prove our first boundedness result (due to Anhern-Siu).

Theorem

If X is smooth, K_X is ample, then mK_X is base point free for $m \geq 2 + \binom{n+1}{2}$ (and birational for $m \geq 2 + 2\binom{n+1}{2}$).

- The (well known) idea is, for any given $x \in X$, to find a $\mathbb{Q}$-divisor $D = D_x \sim_{\mathbb{Q}} \lambda K_X$ with $\lambda < 1 + \binom{n+1}{2}$ such that $\mathcal{J}(D) = \mathfrak{m}_x$ near $x \in X$.
- Eg. $\mathrm{mult}_x D \geq n$ and $\mathrm{mult}_y D < 1$ for all $x \neq y \in U$ a neighborhood of x.
- By Nadel vanishing $H^1(\omega_X^m \otimes \mathcal{J}(D)) = 0$ and so the evaluation $H^0(\omega_X^m) \to \mathbb{C}(x)$ is surjective.

Creating LC centers

- Creating the above D is done in 2 steps.
- Step 1. Since $H^0(mK_X) = C \cdot m^n + o(m^n)$ where $C = K_X^n/n!$ and vanishing at a point to order k is $\binom{k+n}{n} = k^n/n! + o(k^n)$ conditions, it follows that there exists $D \sim_{\mathbb{Q}} \lambda K_X$ with $\mathrm{mult}_x D \geq n$ and $\lambda \leq n + \epsilon$.
- We then have that $\mathfrak{m}_x \supset \mathcal{J}(D)$ but the dimension of the co-support of $\mathcal{J}(D)$ in a neighborhood of x may be > 0.
- I.e., locally near x we have $\mathcal{J}(D) = \mathcal{I}_Z$ where $x \in Z \subset X$ and $\dim Z > 0$.
- The goal is to reduce the dimension of Z inductively, until $\dim Z = 0$.

Cutting down LC centers

- To "cut down" Z, we use (inversion of) adjunction.
- We first produce a divisor $D'_Z \sim_{\mathbb{Q}} \lambda' K_X|_Z$ (on Z) such that $\mathrm{mult}_z(D'_Z) > \dim Z$ and $\lambda' \le \dim Z + \epsilon$ where $z \in Z$ is a general point (possible since $(K_X|_Z)^{\dim Z} \ge 1$).
- Since K_X is ample, by Serre vanishing we may assume $D'_Z = D'|_Z$ where $D' \sim \lambda' K_X$.
- By inversion of adjunction $\mathfrak{m}_z \supset \mathcal{J}((1-\delta)D + D') = \mathcal{I}_{Z'} \supset \mathcal{I}_Z$ where $\dim Z' < \dim Z$.
- Repeating this at most $\dim X$ times, we obtain the required divisor.
- (In order to get nonvanishing at x we would need to degenerate z to x. This will not be used in the sequel.)

Tsuji's idea

- There are many new technical difficulties when dealing with singular varieties eg. canonical models (K_X ample $\mathbb{Q}$-Cartier).
- K_X^n can be small (eg $K_X^3 = 1/420$ for $n = 3$) and $(K_X|_Z)^{\dim Z}$ is even harder to control.
- Hard to get generation at singular points.
- Nether the less one can prove the following.

Theorem (Tsuji, Hacon-McKernan, Takayama)

Fix $n \in \mathbb{N}$ and $V > 0$, then there exists $r \in \mathbb{N}$ such that if X is a canonical model, $\dim X = n$ and $K_X^n \leq V$, then rK_X is very ample. (In particular $\{K_X^n\}$ is duscrete.)

- It will follow that canonical models with $K_X^n \leq V$ have finitely many deformation types.
- We begin by proving the following (assuming the above theorem in dimension $n-1$).

Tsuji's idea

Theorem

There exists an integer $r = r(n)$ such that if X is a canonical model of dimension n, then rK_X is birational for $r \geq r_0$.

- Tsuji's idea is to ignore these problems (at first), and begin by showing that $|mK_X|$ induces a birational map for $m \geq A(K_X^n)^{-1/n} + B$ (where A, B depend only on V, n).
- If $K_X^n \geq 1$, then let $r_0 = A + B$.
- If $K_X^n < 1$, then it follows that the degree of $\phi_{|mK_X|}(X) \leq (A(K_X^n)^{-1/n} + B)^n K_X^n \leq (A + B)^n$.
- Thus, X is birationally bounded (for K_X^n bounded from above).
- More precisely, there is a projective morphism of quasi-projective varieties $\mathcal{Z} \to S$ with a dense set of of points corresponding to birational models of canonical models X with $K_X^n < V$.

Boundedness of canonical models

- Let $\tilde{\mathcal{X}} \to \mathcal{Z}$ be a resolution.
- After decomposing S in to a finite union of locally closed subsets, we may assume that $\tilde{\mathcal{X}} \to S$ is smooth and the fibers corresponding to canonical models are dense in each component of S.
- By Siu's deformation invariance of plurigenera, all fibers are of general type and the possible volumes belong to a discrete set.
- In particular, this set has a positive minimum $v(n)$.
- Take the relative canonical model so that all fibers are canonical models.
- We may find an integer $r > 0$ such that rK_X is very ample (for any n-dimensional canonical model with $K_X^n < V$).
- The key issue is to show that rK_X is Cartier, but this is clear as being Cartier is an open condition.

Subadjunction

- The main issue is, for general $x \in X$, to produce a divisor $D_x \sim_{\mathbb{Q}} \lambda K_X$ such that locally $\mathcal{J}(D_x) = m_x$ and $\lambda = \mathcal{O}((K_X^n)^{-1/n})$ i.e. $(\lambda \leq A(K_X^n)^{-1/n} + B$ for some constants A, B).
- As before, we have $h^0(mK_X) = \frac{m^n}{n!}K_X^n + o(m^n)$ whilst vanishing at a point of order k is $k^n/n! + o(k^n)$ conditions.
- Thus we can find a divisor $D_1 \sim_{\mathbb{Q}} \lambda_1 K_X$ with $\mathrm{mult}_x(D_1) \geq n$ and $\lambda_1 = O((K_X^n)^{-1/n})$.
- Locally we have $\mathcal{J}(D_1) = \mathcal{I}_Z \subset m_x$ where we may assume that Z is reduced and irreducible.
- Our goal is to cut down Z to a point.
- To do this, we need to bound $(K_X|_Z)^{\dim Z}$ from below.
- Since $x \in X$ is general, Z is of general type and so by induction on the dimension $\mathrm{vol}(K_{Z'}) \geq v(d) > 0$ where $d = \dim Z$ and $Z' \to Z$ is a resolution.

Subadjunction

- Tsuji's idea is to use Kawamata's subadjunction to compare K_Z and K_X.
- Recall that $\mathcal{J}(D_1) = \mathcal{I}_Z$ near $x \in Z \subset X$ and $D_1 \sim_{\mathbb{Q}} \lambda_1 K_X$.
- Assume for simplicity that Z is smooth.
- Then, by Kawamata sub adjunction (as K_X is ample)

$$(1+\lambda_1+\epsilon)K_X|_Z \sim_{\mathbb{Q}} (K_X + D_1 + \epsilon K_X)_Z \geq K_Z$$

  and so $(K_X|_Z)^d \geq (\frac{1}{1+\lambda_1})^d \cdot v(d)$ where $d = \dim Z$.
- We now pick $D'_Z \sim_{\mathbb{Q}} \lambda' K_X|_Z$ such that $\mathrm{mult}_z(D'_Z) > d$ and $\lambda' \leq d(1+\lambda_1)+1$.
- Since K_X is ample, by Serre vanishing we may assume that $D'_Z = D'|_Z$ where $D' \sim_{\mathbb{Q}} \lambda' K_X$. Then
- $m_z \supset \mathcal{J}((1-\delta)D_1 + (1-\eta)D') = \mathcal{I}_{Z_2} \supset \mathcal{I}_Z$ where $\dim Z_2 < \dim Z$ (by inversion of adjunction).
- Let $D_2 = (1-\delta)D_1 + (1-\eta)D'$ so that $D_2 \sim \lambda_2 K_X$ where $\lambda_2 = O((K_X^n)^{1/n})$ and proceed by induction.

Birational boundedness of log pairs

- Not surprisingly, the case of log pairs is substantially harder.
- The first step is to show that (for fixed n, $\mathcal{C}$ and v), the set $\mathcal{LCM}$ of LC models (X, B) such that $\dim X = n$, $B \in \mathcal{C}$, $(K_X + B)^n = v$ are birationally bounded.
- This means that there exists a pair $(\mathcal{X}, \mathcal{B})$ and a projective finite type morphism $g : \mathcal{X} \to S$ such that for any $(X, B) \in \mathcal{LCM}$, there is an $s \in S$ and a birational morphism $h : X \dashrightarrow \mathcal{X}_s$ such that the support of the strict transform of B plus the $\mathcal{X}_s/X$ exceptional divisors are contained in the support of $\mathcal{B}_s$.
- To this end, it suffices to show that there is an integer $m = O(v^{-1/n})$ such that $m(K_X + B)$ is birational.
- Then X is birationally bounded (similarly to what we have seen above) but why is the pair log birationally bounded?

Birational boundedness of log pairs

- We may assume that $h : X \to \mathcal{Z}_s$ is a morphism. Let $S = \sum E_i$ where E_i are the components of the support of B.
- It then suffices to show that if $H = \mathcal{O}_{\mathcal{Z}}(1)$, then $S \cdot h^* H_s^{n-1}$ is bounded from above.
- Since $\mathcal{Z} \to S$ is a bounded family, $K_X \cdot h^* H_s^{n-1} = K_{\mathcal{Z}_s} \cdot H_s^{n-1}$ is bounded and hence so is

$$B \cdot h^* H_s^{n-1} = (K_X + B) \cdot h^* H_s^{n-1} - K_X \cdot h^* H_s^{n-1}.$$

Adjunction for log pairs

- Adjunction for log pairs is more complicated.
- We have $K_X + B$ ample, $B \in \mathcal{C}$ and $D \sim_{\mathbb{Q}} \lambda(K_X + B)$ with $\mathcal{J}(D) = \mathcal{I}_Z$ near $x \in Z \subset X$ and we would like to find (Z, B_Z) LC, $B_Z \in \mathcal{C}$ and $K_Z + B_Z$ is big such that $(K_X + D + B)|_Z \geq K_Z + B_Z$.
- We have little control over λ and the coefficients of D, but since $x \in X$ is general, we can "pretend" that Z is a fiber of a morphism $X \to T$.
- In this case $K_X|_Z = K_Z$, $B_Z = B|_Z$ and we can ignore D.
- In practice we have to do a delicate analysis of Kawamata's subadjunction.
- Let $D(\mathcal{C}) = \{a \leq 1 | a = \frac{m-1+f}{m},\ m \in \mathbb{N},\ f = \sum f_i,\ f_i \in \mathcal{C}\}$, then $D(\mathcal{C})$ is also a DCC set.
- Let $V^\nu \to V$ be the normalization and $V' \to V^\nu$ a resolution.

Adjunction for log pairs

Theorem

There exists a divisor Θ on V^ν, $\Theta \in \{1-t|t \in LCT_{n-1}(D(\mathcal{C})) \cup 1\}$ s.t. $(K_X + D + B)|_{V^\nu} - (K_{V^\nu} + \Theta)$ is PSEF.
If V is a general member of a covering family, then $K_{V'} + M_\Theta \geq (K_X + B)|_{V'}$

- Since $K_X + B$ is big (and LC), so is $K_{V'} + M_\Theta$ (where M_Θ is the strict transform of Θ plus the V'/V^ν exceptional divisors).
- Thus the pushforward $K_{V^\nu} + \Theta$ is also big.
- In order to show that the coefficients of Θ lie in a DCC, we must show that
  $LCT_{n-1}(D(\mathcal{C})) = \{LCT(X, B; M)|B \in D(\mathcal{C}),\ M \in \mathbb{N}\}$
  satisfies the ACC property (aka the ACC for LCT's).

Definition of Θ

- To define Θ we proceed as follows.
- After perturbing D, we may assume that on a neighborhood of the general point of V, $(X, B+D)$ is log canonical with a unique LC place S above V.
- Using the MMP, we may pick a model $f: Y \to X$, that extracts only NKLT places of $(X, B+D)$ including S and is $\mathbb{Q}$-factorial. Write
- $K_Y + S + \Gamma = f^*(K_X + B) + E$, $K_S + \Phi = (K_X + S + \Gamma)|_S$
  $K_Y + S + \Gamma + \Gamma' = g^*(K_X + B + D)$, $K_S + \Phi' = (K_X + S + \Gamma + \Gamma')|_S$.
- In particular $\Gamma \in \mathcal{C}$ and $\Phi \in D(\mathcal{C})$.
- For any codimension 1 point $P \in V^\nu$, let
  $t_P = LCT(S, \Phi; f|_S^* P)$ (over the generic point of P).
- Then $\Theta = \sum(1 - t_p)P$. Define Θ' similarly for (S, Φ').
- By Kawamata subadjunction $(K_X + B + D)|_{V^\nu} - (K_{V^\nu} + \Theta')$ is PSEF. Since $\Theta \leq \Theta'$ we are done (with the first claim; the second is harder and we skip it).

Good minimal models of LC families

- A second difficulty comes from the fact that once we have a bounded family $(\bar{\mathcal{X}}, \bar{\mathcal{B}}) \to S$ such that all $(X, B) \in SLC(c, n, \mathcal{C})$ are birational to a fiber $(\bar{\mathcal{X}}_s, \bar{\mathcal{B}}_s)$, in order to deduce boundedness, we must "fix the correct model and coefficients of $(\mathcal{X}, \mathcal{B})$" and take the relative log canonical model of a resolution of $(\bar{\mathcal{X}}, \bar{\mathcal{B}})$.
- This would require the LC mmp (and hence abundance!).
- Luckily, we can assume that our families are smooth and a dense set of fibers has a good minimal model. We show:

Theorem (HMX)

If $(\tilde{\mathcal{X}}, \tilde{\mathcal{B}}) \to S$ is log smooth over S and there is a dense set of points such that the fibers $(\tilde{\mathcal{X}}_s, \tilde{\mathcal{B}}_s)$ have a good minimal model, then $(\tilde{\mathcal{X}}, \tilde{\mathcal{B}})$ has a good minimal model over S.

Deformation invariance of log plurigenera

- The key ingredient is a result of Berndtson and Paun on the deformation invariance of log-plurigenera for a klt pair and a smooth morphism $(\tilde{\mathcal{X}}, \tilde{\mathcal{B}}) \to S$
- So far, the only proof of this result is analytic.

From birational boundedness to boundedness

- From this point on we may assume that our LC models (X, B) ($\dim X = n$, $B \in \mathcal{C}$, $(K_X + B)^n = C$) belong to a birationally bounded family.
- Recall that this means that there is a projective morphism of varieties of finite type $\mathcal{Z} \to S$ and a divisor $\mathcal{D}$ on $\mathcal{Z}$ such that for any (X, B) as above, there is a point $s \in S$ and a birational map $f : X \to \mathcal{Z}_s$ such that $\mathcal{D}_s$ contains the strict transform of B and the $\mathcal{Z}_s/X$ exceptional divisors.
- Blowing up $\mathcal{Z}$ and replacing $\mathcal{D}$ by its strict transform and the exceptional divisors, we may assume that each fiber $(\mathcal{Z}_s, \mathcal{D}_s)$ is SNC.
- Replacing each (X, B) by an appropriate birational model, we may assume that each (X, B) is snc and $f : X \to \mathcal{Z}_s$ is a morphism (but $K_X + B$ is not ample; $\mathrm{vol}(K_X + B) = c$).

From birational boundedness to boundedness

- We begin by considering the set of all LC SNC pairs (X, B) with $B \in \mathcal{C}$ admitting a morphism to a fixed SNC pair $(Z, D) = (\mathcal{Z}_s, \mathcal{D}_s)$ say $f : X \to Z$ such that $f_* B \leq D$.
- **Claim:** The set $\mathcal{V}(Z, D, \mathcal{C}) = \{\mathrm{vol}(K_X + B)\}$ satisfies the DCC and the set of LCM for these pairs (X, B) is finite.
- Throughout the process, we are allowed to replace (X, B) by a birational pair (X', B') such that $R(K_X + B) \cong R(K_{X'} + B')$.
- Suppose there is an infinite sequence (X_i, B_i) and define the b-divisor $\mathbf{D} = \lim \mathbf{M}_{B_i}$ as follows.
- For any divisorial valuation ν over Z, let $\mathbf{M}_{B_i}(\nu)$ be the coefficient of B_i if ν is a divisor on X_i and 1 otherwise. Thus $\mathbf{M}_{B_i}$ is the b-divisor given by the strict transform of B_i plus the reduced exceptional divisor (over X_i).
- Since the coefficients of $\mathbf{M}_{B_i}$ are in the DCC set $\mathcal{C}$, each limit $\lim \mathbf{M}_{B_i}(\nu)$ is well defined.

From birational boundedness to boundedness

- Let $\Phi = \mathbf{D}_Z$.
- Suppose that (Z, Φ) is terminal, then we claim that $R(K_{X_i} + B_i) \cong R(K_Z + f_{i,*}B_i)$ for all $i \gg 0$.
- In fact since $f_{i,*}B_i \leq \Phi$ has finitely many components which belong to a DCC, it is clear that for $i \gg 0$ we have $f_{i,*}B_i \leq \lim f_{i,*}B_i = \Phi$ so $(Z_i, f_{i,*}B_i)$ is terminal.
- But then $K_{X_i} + B_i = f_i^*(K_Z + f_{i,*}B_i) + E_i$ with $E_i \geq 0$ and f_i-exceptional.
- Thus $H^0(m(K_{X_i} + B_i)) = H^0(m(K_Z + f_{i,*}B_i))$ for all $m > 0$.
- Thus we may assume that $X_i = Z$ for all $i \gg 0$.
- Suppose that $\mathrm{vol}(K_Z + B_i) \geq \mathrm{vol}(K_Z + B_{i+1})$. Passing to a subsequence, we may assume $B_i \leq B_{i+1}$, so that $\mathrm{vol}(K_Z + B_i) \leq \mathrm{vol}(K_Z + B_{i+1})$.
- Thus $\mathrm{vol}(K_{X_i} + B_i) = \mathrm{vol}(K_{X_{i+1}} + B_{i+1})$ for all $i \gg 0$.

From birational boundedness to boundedness

- The statement about finiteness of log canonical models follows from a general result of the MMP.

Theorem

Let X be a smooth variety and $B_1 \leq B_2$ effective divisors with SNC such that $K_X + B_1$ is big and $K_X + B_2$ is klt. Then there is a finite set of birational maps $(\psi_i : X \dashrightarrow W_i)_{i \in I}$ such that for any $\mathbb{Q}$-divisor $B_1 \leq B \leq B_2$, there exists an index $i \in I$ such that ψ_i is the LCM of (X, B) and in particular $\mathrm{Proj}(R(K_X + B)) \cong W_i$.

- Next we explain how to deal with the case when (Z, Φ) is not terminal.

From birational boundedness to boundedness

- Suppose that (Z, Φ) is klt. Then it is easy to see that blowing up Z finitely many times along strata of Φ (and the exceptional divisors), we obtain a birational morphism $h : Z' \to Z$ such that $K_{Z'} + \Phi' = h^*(K_Z + \Phi)$, $\Phi' \geq 0$, and (Z', Φ') is terminal.
- One minor issue is that the coefficients of (Z', Φ') are no longer in $\mathcal{C}$ and so we must enlarge $\mathcal{C}$ slightly. The new values are determined by finitely many linear functions (in dimension 2 we consider $a_1 + a_2 - 1$ where $a_i \in \mathcal{C}$). Thus we must replace $\mathcal{C}$ by a slightly bigger DCC set.
- We must also replace (X_i, B_i) by blow ups along strata of the strict transform of Φ (and the exceptional divisors).
- Let $h_i : X_i' \to X_i$ and $f_i' : X_i' \to Z'$ be the corresponding morphisms, then we simply let $B_i' = \mathbf{M}_{X_i, B_i}$ as above.

From birational boundedness to boundedness

- The hardest case is when (Z, Φ) is log canonical but not klt. The proof proceeds by induction on the codimension of the smallest NKLT center.
- Suppose for simplicity that $\dim Z = 2$. Say that Φ consists of two components of multiplicity 1 meeting at a point $P \in Z$.
- If $\mathbf{D} \geq \mathbf{M}_{Z,\Phi}$, then we find a contradiction to $\mathrm{vol}(K_{X_i} + B_i) > \mathrm{vol}(K_{X_{i+1}} + B_{i+1})$.
- Note that then $\mathrm{vol}(K_Z + \Phi) > \mathrm{vol}(K_{X_i} + B_i)$. However $\mathrm{vol}(K_Z + \Phi) = \lim \mathrm{vol}(K_Z + (1 - \epsilon)\Phi)$ and so the contradiction follows if we show $\lim \mathrm{vol}(K_{X_i} + B_i) \geq \mathrm{vol}(K_Z + (1 - \epsilon)\Phi)$.
- But $(Z, (1 - \epsilon)\Phi)$ is klt and we can use the terminalization trick explained above.

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- So assume that there is a divisor with valuation ν over Z such that $\mathbf{D}(\nu) < \mathbf{M}_{Z,\Phi}$. In particular $\mathbf{M}_{Z,\Phi} > 0$ and so ν is a toric valuation.
- Let $\mu : Z_\nu \to Z$ be the corresponding toric blow up (eg. blow up $p \in Z$). Ste $\Phi_\nu = \mu^{-1}_* \Phi + d_\nu E_\nu$ where E_ν is the exceptional divisor and $0 \leq d_\nu = \mathbf{D}(\nu) < 1$.
- We may replace (Z, Φ) by (Z_ν, Φ_ν) and (X_i, B_i) by $X_{i,\nu} \to X_i$ (extracting the divisor corresponding to ν if necessary) and B_i by the strict transform of B_i and the exceptional divisor $E_{i,\nu}$ corresponding to ν with mutiplicity d_ν.
- Then (after possibly passing to log resolutions) the only remaining NKLT centers have codimension 1 (not 2).

From birational boundedness to boundedness

- By a similar argument, we can adress the remaining points contained in the intersection of a component of Φ coefficient 1 and a component of coefficient $a > 0$.
- Either $\mathbf{D} \geq \mathbf{L}_{Z,\phi}$ and then we can assume as above that $X_i = Z$, or there is a valuation ν such that $\mathbf{D}(\nu) < \mathbf{L}_{Z,\Phi}(\nu)$ in which case we extract the divisor ν and define Φ' by letting the coefficient along the exceptional divisor to be $a' = \mathbf{D}(\nu) < a$.
- Since the coefficients of $\mathbf{D}(\nu)$ belong to a DCC set, this procedure must terminate.

Boundedness in families

- We must now show that the analogous statements hold when (X, B) is birational to a fiber of a finite type family $(\mathcal{Z}, \mathcal{B}) \to S$.
- Decomposing S in to a finite disjoint union of locally closed subsets (and applying base change), we can assume that each strata of $(\mathcal{Z}, \mathcal{B})$ is smooth with connected fibers over S.
- By a result of Siu, Hacon-M$^{\mathrm{c}}$Kernan, Berndtson-Paun, Hacon-M$^{\mathrm{c}}$kernan-Xu, the log plurigenera $h^0(m(K_{\mathcal{Z}_s} + \mathcal{B}_s))$ are deformation invariant.
- Suppose again for simplicity that $(\mathcal{Z}, \mathcal{B})$ is terminal, then for any (X, B) we have $h^0(m(K_X + B)) = h^0(m(K_{\mathcal{Z}_s} + \mathcal{B}_s))$ and so the set of volumes $V = \{\mathrm{vol}(K_X + B)\}$ is determined by the volumes of finitely many fibers $(\mathcal{Z}_s, \mathcal{B}_s)$ (one for each component of s).

Outline of the talk

ACC for LCT's

Theorem

Fix $n \in \mathbb{N}$ and $\mathcal{C} \subset [0.1]$ a DCC set. Let $LCT_n(\mathcal{C}) = \{LCT(X, B; M)\}$ where (X, B) is LC, $B \in \mathcal{C}$, $M \geq 0$ is a $\mathbb{Z}$-Weil, $\mathbb{Q}$-Cartier divisor. Then $LCT_n(\mathcal{C})$ satisfies the DCC.

- This is Shokurov's ACC for LCT's conjecture, which was proved in the case of bounded singularities by Ein-Mustata-de Fernex.
- We will now give a sketch of the proof.
- Suppose that there is a sequence of pairs (X_i, B_i) and divisors M_i as above with $t_i = LCT(X_i, B_i; M_i)$ such that $t_i < t_{i+1}$ for all $i > 0$.
- We let $t = \lim t_i > t_i$.

ACC for LCT's

- For all i, let $\nu_i : Y_i \to X_i$ be a proper birational morphism extracting a unique divisor of discrepancy -1 with center a minimal NKLT center of $(X_i, B_i + t_i M_i)$.
- Cutting by hyperplanes on X_i, we may assume that this minimal NKLT center is a point $x_i \in X_i$.
- We may assume that $\rho(Y_i/X_i) = 1$.
- Define $K_{E_i} + \Delta_i = (K_{Y_i} + E_i + \nu_{i,*}^{-1}(B_i + t_i M_i))|_{E_i} \equiv 0$, and $K_{E_i} + \Delta_i' = (K_{Y_i} + E_i + \nu_{i,*}^{-1}(B_i + tM_i))|_{E_i}$.
- Note that the coefficients of $B_i + t_i i M_i$ and $B_i + tM_i$ are in the DCC set $\mathcal{C}' = \mathcal{C} \cup \{t_i | i \in \mathbb{N}\} \cup \{t\}$ and hence the coefficients of Δ_i and Δ_i' are in the DCC set $D(\mathcal{C}')$.
- Since $t > t_i$ and $(\nu_{i,*}^{-1} M_i)|_{E_i} \neq 0$, then $K_{E_i} + \Delta_i'$ is ample.
- Since $\lim t_i = t$, $K_{E_i} + \Delta_i'$ is LC by the ACC for LCT's in dimension $n-1$.

ACC for LCT's

- The following consequence of the results on the boundednes of LC models gives a contradiction: There exists a number $\tau < 1$ such that for all i, $K_{E_i} + \tau\Delta'_i$ is big.
- The idea is that there is an integer m (depending only on the dimension n and the DCC set $\mathcal{C}$) such that if (X, B) is a proper LC pair with $K_X + B$ ample, then $m(K_X + B)$ is birational (even for $\mathbb{R}$-divisors).
- But then, following the proof of Anhern and Siu's theorem, $K_X + (m\binom{n}{2} + 1)(K_X + B)$ is generated at a general point $x \in X$.
- In particular $K_X + (m\binom{n}{2} + 1)(K_X + B)$ is PSEF.
- Since $K_X + (m\binom{n}{2} + 1)(K_X + B) = (m\binom{n}{2} + 2)(K_X + \alpha B)$, where $\alpha = (m\binom{n}{2} + 1)/(m\binom{n}{2} + 2)$, we let $\tau = (\alpha + 1)/2$.