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The Ising model and critical behavior of transport in binary composite media

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We present a general theory for critical behavior of transport in binary composite media. The theory holds for lattice and continuum percolation models in both the static case with real parameters and the quasi–static case (frequency dependent) with complex parameters. Through a direct, analytic correspondence between the magnetization of the Ising model and the effective parameter problem of two phase random media, we show that the critical exponents of the transport coefficients satisfy the standard scaling relations for phase transitions in statistical mechanics. Our work also shows that delta components form in the underlying spectral measures at the spectral endpoints precisely at the percolation threshold p_c and at $1 - p_c$. This is analogous to the Lee-Yang-Ruelle characterization of the Ising model phase transition, and identifies these transport transitions with the collapse of spectral gaps in these measures.
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I. INTRODUCTION

Lattice and continuum percolation models have been used to study a broad range of disordered composite materials including semiconductors,[52] radar absorbing coatings,[36] bone ,[28,50] rocks,[10,11] glacial ice,[20] polycrystalline metals,[13] carbon nanotube composites,[37] and sea ice.[26,27] A key feature of these materials is the critical dependence of the effective transport properties on the connectedness, or percolation characteristics, of a particular component. The behavior of such composite media is particularly challenging to describe physically, and to predict mathematically.

Here we construct a mathematical framework which unifies the critical theory of transport in two phase random media. By adapting techniques developed by Baker for the Ising model,[2] we provide a detailed description of percolation-driven critical transitions in transport exhibited by such media. The most natural formulation is in terms of the conduction problem in the continuum $\mathbb{R}^d$, which includes the lattice $\mathbb{Z}^d$ as a special case.[23,29] Although, symmetries in Maxwell's equations[40] immediately extend our results to the effective parameter problem of electrical permittivity.

An original motivation for this work was to gain a better understanding of critical transitions in the transport properties of sea ice. In particular, fluid flow through sea ice mediates a broad range of processes that are important to studying its role in the climate system, and the impact of climate change on polar ecosystems.[25] In fact, the brine microstructure of sea ice displays a percolation threshold at a critical brine volume fraction ϕ of about 5% in columnar sea ice.[26,27,42] This leads to critical behavior of fluid flow, where sea ice is effectively impermeable to fluid transport for ϕ below 5%, and is increasingly permeable for ϕ above 5%, which is known as the *rule of fives*.[26] Percolation theory can then be used to capture the behavior of the fluid permeability of sea ice.[27] There has also been evidence[31,41] that this critical behavior in the microstructure also induces similar behavior in the effective electromagnetic properties of sea ice, such as its effective complex permittivity ϵ^*. In Refs. 31 and 41, for example, microstructural properties of the brine phase were recovered from measurements of the complex permittivity of sea ice. The current paper helps lay the groundwork for the analysis of sea ice permittivity data collected in the polar regions, and how it can be used

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to monitor changes in the microstructure, the fluid transport properties, and the geophysical and biological processes that are controlled by fluid flow.

II. BACKGROUND AND SUMMARY OF THE RESULTS

The partition function Z of the Ising model is a polynomial in the activity variable.[2,38,46,48] In 1952 Lee and Yang[38] showed that the roots of Z lie on the unit circle, which is known as the Lee–Yang theorem.[38,46] They also demonstrated that the distribution of the roots determines the associated equation of state,[55] and that the properties of the system, in relation to phase transitions, are governed by the behavior of these roots near the positive real axis.

In 1968, Baker[1] used the Lee–Yang Theorem to represent the Gibbs free energy per spin $f = -(N\beta)^{-1}\ln Z$ as a logarithmic potential,[49] where N is the number of spins, $\beta = (kT)^{-1}$, k is Boltzmann's constant, and T is the absolute temperature. He used this special analytic structure to prove that the magnetization per spin $M(T, H) = -\partial f/\partial H$ (Ref. 44) may be represented in terms of a Stieltjes function G in the variable $\tau = \tanh \beta m H$,

$$\frac{M}{m} = \tau(1 + (1-\tau^2)G(\tau^2)), \quad G(\tau^2) = \int_0^\infty \frac{d\psi(y)}{1+\tau^2 y}, \tag{1}$$

where H is the applied magnetic field strength, m is the (constant) magnetic dipole moment of each spin,[30] and ψ is a non-negative definite measure.[1,2] The integral representation in (1) immediately leads to the inequalities

$$G \geq 0, \qquad \frac{\partial G}{\partial u} \leq 0, \qquad \frac{\partial^2 G}{\partial u^2} \geq 0, \tag{2}$$

where $u = \tau^2$. The last formula in Eq. (2) is the Griffiths–Hurst–Sherman (GHS) inequality, which is an important tool in the study of the Ising model.[2,23]

In 1970, Ruelle[47] extended the Lee–Yang theorem and proved that there exists a *gap* $\theta_0(T) > 0$ in the roots of Z about the positive real axis for high temperatures. Moreover, he proved that the gap collapses, $\theta_0(T) \to 0$, as T decreases to a critical temperature $T_c > 0$. Consequently, the temperature-driven phase transition (spontaneous magnetization) is unique, and is characterized by the pinching of the real axis by the roots of Z.[46]

Baker[2,3] then exploited the Lee–Yang–Ruelle theorem to provide a detailed description of the critical behavior of the parameters characterizing the phase transition exhibited by the Ising model.[14] He defined a critical exponent Δ for the gap in the distribution of the Lee–Yang–Ruelle zeros, $\theta_0(T) \sim (T - T_c)^\Delta$, as $T \to T_c^+$, and proved that the measure ψ is supported on the compact interval $[0, S(T)]$ for $T > T_c$, with $S(T) \sim (T - T_c)^{-2\Delta}$ as $T \to T_c^+$. He demonstrated that the moments $\psi_n = \int_0^\infty y^n \, d\psi(y)$ of ψ diverge as $T \to T_c^+$ according to the power law $\psi_n \sim (T - T_c)^{-\gamma_n}$, $n \geq 0$, by proving that the sequence γ_n satisfies Baker's inequalities $\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0$. They imply that this sequence increases at least linearly with n. He later proved that this sequence is actually linear in n, $\gamma_n = \gamma + 2\Delta n$, with constant gap $\gamma_i - \gamma_{i-1} = 2\Delta$.[2] The critical exponent γ is defined via the magnetic susceptibility per spin $\chi = \partial M/\partial H = -\partial^2 f/\partial H^2 \sim (T - T_c)^{-\gamma}$, as $T \to T_c^+$.

The phase transition may be concisely described with two other critical exponents. When $H = 0$, $M(T, 0) \sim (T - T_c)^\beta$, as $T \to T_c^-$, where the critical exponent β is not to be confused with $(kT)^{-1}$, and along the critical isotherm $T = T_c$, $M(T_c, H) \sim H^{1/\delta}$, as $H \to 0$.[2,14] Using the integral representation in (1), Baker obtained (two–parameter) scaling relations for these critical exponents[2]

$$\beta = \Delta - \gamma, \qquad \delta = \Delta/(\Delta - \gamma), \qquad \gamma_n = \gamma + 2\Delta n. \tag{3}$$

The critical exponent γ, for example, is defined in terms of the following limit, and γ exists when this limit exists,[2]

$$\gamma = \limsup_{T \to T_c^+,\ H=0} \left(\frac{-\ln \chi(T, H)}{\ln(T - T_c)} \right). \tag{4}$$

In 1997, Golden[24] demonstrated that Baker's critical theory may be adapted to provide a precise description of percolation-driven critical transitions in transport, exhibited by two phase random media in the static regime. This result puts these two classes of seemingly unrelated problems on an equal mathematical footing. He did so by considering percolation models of classical conductive two phase composite media, where the connectedness of the system is determined, for example, by the volume fraction p of inclusions with conductance σ_2 in an otherwise homogeneous medium of conductivity σ_1, with $h = \sigma_1/\sigma_2 \in [0, 1]$. He demonstrated that the function $m(p, h) = \sigma^*(p, h)/\sigma_2$ plays the role of the magnetization $M(T, H)$, where σ^* is the effective conductivity of the medium.[4,29,39] Moreover, he showed that the volume fraction p mimics the temperature T while the contrast ratio h is analogous to the applied magnetic field strength H. More specifically, critical behavior of transport arises when $h = 0$ ($\sigma_1 = 0, 0 < \sigma_2 < \infty$), as $p \to p_c^+$,[24] and critical behavior of the magnetization in the Ising model arises when $H = 0$, as $T \to T_c^+$.[14] Using these mathematical parallels, it was shown that the critical exponents of transport satisfy an analogue of Baker's scaling relations (3).

Here, using a novel unified approach, we reproduce Golden's static results ($h \in \mathbb{R}$) and obtain the analogous static results associated with a conductive–superconductive medium in terms of $w(p, z) = \sigma^*(p, z)/\sigma_1$, where $z = 1/h$. Using Stieltjes function integral representations of $m(p, h; \mu)$ and $w(p, z; \alpha)$, where μ and α are each spectral measures of a random self-adjoint operator, we determine the (two–parameter) critical exponent scaling relations of each system. We then extend these results to the frequency dependent quasi-static regime ($h \in \mathbb{C}$). We also link these two sets of critical exponents and, assuming a symmetry in the properties of μ and α, the resultant scaling relations linking the two sets of critical exponents are in agreement with the seminal paper by Efros and Shklovskii.[19] We remark that there are similar critical exponents involving ϵ^* for two phase binary dielectric media,[7,15] and there are direct analogs of our results regarding such media.

In arbitrary finite lattice systems we explicitly show that there are *gaps* in the supports of the measures $\alpha(d\lambda)$ and $\mu(d\lambda)$ about the spectral endpoints $\lambda = 0, 1$ for $p \ll 1$ and $1 - p \ll 1$, respectively. Moreover, in infinite lattice or continuum composite systems, we demonstrate that critical transitions in transport are due to the formation of delta components in μ and α located at $\lambda = 0, 1$. We do so by constructing a measure ϱ which is supported on the set $\{0, 1\}$ that links μ and α. This general result demonstrates that, for percolation models, the onset of criticality (the formation of these delta components) occurs *precisely* at the percolation threshold p_c and at $1 - p_c$.

III. THE ANALYTIC CONTINUATION METHOD

We now formulate the effective parameter problem for two component conductive media. Let (Ω, P) be a probability space, and let $\boldsymbol{\sigma}(\vec{x}, \omega)$ and $\boldsymbol{\rho}(\vec{x}, \omega)$ be the local conductivity and resistivity tensors, respectively, which are (spatially) stationary random fields in $\vec{x} \in \mathbb{R}^d$ and $\omega \in \Omega$. Here, Ω is the set of all geometric realizations of our random medium, $P(d\omega)$ is the underlying probability measure, which is compatible with stationarity, and $\boldsymbol{\rho} = \boldsymbol{\sigma}^{-1}$.[29] Define the Hilbert space of stationary random fields $\mathscr{H}_s \subset L^2(\Omega, P)$, and the underlying Hilbert spaces of stationary curl free $\mathscr{H}_\times \subset \mathscr{H}_s$ and divergence free $\mathscr{H}_\bullet \subset \mathscr{H}_s$ random fields

$$\mathscr{H}_\times = \{\vec{Y}(\omega) \in \mathscr{H}_s \mid \vec{\nabla} \times \vec{Y} = 0 \text{ weakly and } \langle \vec{Y} \rangle = 0\}, \tag{5}$$
$$\mathscr{H}_\bullet = \{\vec{Y}(\omega) \in \mathscr{H}_s \mid \vec{\nabla} \cdot \vec{Y} = 0 \text{ weakly and } \langle \vec{Y} \rangle = 0\},$$

where $\vec{Y} : \Omega \mapsto \mathbb{R}^d$ and $\langle \cdot \rangle$ means ensemble average over Ω, or by an ergodic theorem spatial average over all of $\mathbb{R}^d$.[29]

Consider the following variational problems: find $\vec{E}_f \in \mathscr{H}_\times$ and $\vec{J}_f \in \mathscr{H}_\bullet$ such that[29]

$$\langle \boldsymbol{\sigma}(\vec{E}_0 + \vec{E}_f) \cdot \vec{Y} \rangle = 0 \quad \forall\, \vec{Y} \in \mathscr{H}_\times \qquad \text{and} \qquad \langle \boldsymbol{\rho}(\vec{J}_0 + \vec{J}_f) \cdot \vec{Y} \rangle = 0 \quad \forall\, \vec{Y} \in \mathscr{H}_\bullet\,, \tag{6}$$

respectively. When the bilinear forms $a(\vec{u}, \vec{v}) = \vec{u}^{\,T} \boldsymbol{\sigma}\, \vec{v}$ and $\tilde{a}(\vec{u}, \vec{v}) = \vec{u}^{\,T} \boldsymbol{\rho}\, \vec{v}$ are bounded and coercive, these problems have unique solutions satisfying[29]

$$\begin{aligned} &\vec{\nabla} \times \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{J} = 0, \qquad \vec{J} = \boldsymbol{\sigma} \vec{E}, \qquad \vec{E} = \vec{E}_0 + \vec{E}_f, \qquad \langle \vec{E}\, \rangle = \vec{E}_0, \\ &\vec{\nabla} \times \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{J} = 0, \qquad \vec{E} = \boldsymbol{\rho} \vec{J}, \qquad \vec{J} = \vec{J}_0 + \vec{J}_f, \qquad \langle \vec{J}\, \rangle = \vec{J}_0, \end{aligned} \tag{7}$$

respectively. Here, $\vec{E}_f$ and $\vec{J}_f$ are the fluctuating electric field and current density of mean zero, respectively, about the (constant) averages $\vec{E}_0$ and $\vec{J}_0$, respectively.

We assume that the local conductivity $\sigma(\vec{x}, \omega)$ of the medium takes the *complex* values σ_1 and σ_2 and write $\sigma(\vec{x}, \omega) = \sigma_1 \chi_1(\vec{x}, \omega) + \sigma_2 \chi_2(\vec{x}, \omega)$, where χ_j is the characteristic function of medium $j = 1, 2$, which equals one for all $\omega \in \Omega$ having medium j at $\vec{x}$, and zero otherwise, with $\chi_1 = 1 - \chi_2$.[29] Similarly, we assume that the local resistivity $\rho(\vec{x}, \omega)$ takes the values $1/\sigma_1$ and $1/\sigma_2$ and write $\rho(\vec{x}, \omega) = \chi_1(\vec{x}, \omega)/\sigma_1 + \chi_2(\vec{x}, \omega)/\sigma_2$.

As $\vec{E}_f \in \mathscr{H}_\times$ and $\vec{J}_f \in \mathscr{H}_\bullet$, Eq. (6) yields the energy (power density) constraints $\langle \vec{J} \cdot \vec{E}_f \rangle = \langle \vec{E} \cdot \vec{J}_f \rangle = 0$, which lead to the reduced energy representations

$$\langle \vec{J} \cdot \vec{E} \rangle = \langle \vec{J} \rangle \cdot \vec{E}_0 \quad \text{and} \quad \langle \vec{E} \cdot \vec{J} \rangle = \langle \vec{E} \rangle \cdot \vec{J}_0. \tag{8}$$

The effective complex conductivity and resistivity tensors, $\boldsymbol{\sigma}^*$ and $\boldsymbol{\rho}^*$, are defined by

$$\langle \vec{J}\, \rangle = \boldsymbol{\sigma}^* \vec{E}_0 \quad \text{and} \quad \langle \vec{E}\, \rangle = \boldsymbol{\rho}^* \vec{J}_0, \tag{9}$$

respectively, yielding $\langle \vec{J} \cdot \vec{E} \rangle = \boldsymbol{\sigma}^* \vec{E}_0 \cdot \vec{E}_0 = \boldsymbol{\rho}^* \vec{J}_0 \cdot \vec{J}_0$. For simplicity, we focus on one diagonal component of these tensors, $\sigma^* = \sigma^*_{kk}$ and $\rho^* = \rho^*_{kk}$, for some $k = 1, \ldots, d$. Assuming that $0 < |\sigma_1| < |\sigma_2| < \infty$, these functions have the following bounds[40,54]

$$|\sigma_1| \leq |\sigma^*| \leq |\sigma_2|, \qquad |\sigma_2|^{-1} \leq |\rho^*| \leq |\sigma_1|^{-1}. \tag{10}$$

Due to the homogeneity of these functions, e.g., $\sigma^*(a\sigma_1, a\sigma_2) = a\sigma^*(\sigma_1, \sigma_2)$ for any complex number a, they depend only on the ratio $h = \sigma_1/\sigma_2$, and we define the functions

$$m(h) = \sigma^*/\sigma_2, \quad w(z) = \sigma^*/\sigma_1, \quad \tilde{m}(h) = \sigma_1 \rho^*, \quad \tilde{w}(z) = \sigma_2 \rho^*, \tag{11}$$

where $z = 1/h$. The dimensionless functions $m(h)$ and $\tilde{m}(h)$ are analytic off the negative real axis in the h-plane, while $w(z)$ and $\tilde{w}(z)$ are analytic off the negative real axis in the z-plane.[29] Each take the corresponding upper half plane to the upper half plane, so that they are examples of Herglotz functions.[29] As a function of h, z: $(-\infty, 0) \mapsto (-\infty, 0)$. Therefore, the functions $w(z(h))$ and $\tilde{w}(z(h))$ are also analytic off the negative real axis in the h-plane. We, henceforth, restrict h in the complex plane to the set

$$\mathcal{U}_\varepsilon = \{h \in \mathbb{C} : |h| < 1 \text{ and } |h - h_0| > \varepsilon \text{ for all } h_0 \in (-1, 0]\}, \tag{12}$$

which is parameterized by $0 < \varepsilon \ll 1$. When $\varepsilon = 0$ in Eq. (12) we write $\mathcal{U}_0$.

A key step in the method is obtaining integral representations for σ^* and ρ^* in terms of Herglotz functions $\mathcal{A}_{i,j}$ and $\mathcal{S}_{i,j}$, $i, j = 0, 1, 2, \ldots$, of the form[33]

$$\mathcal{A}_{i,j}(\xi; \nu) = \int_0^1 \frac{\lambda^i d\nu(\lambda)}{(\xi - \lambda)^j}, \quad \mathcal{S}_{i,j}(\xi; \nu) = \int_0^\infty \frac{y^i d\nu(y)}{(1 + \xi y)^j}, \tag{13}$$

which follow from resolvent representations of the electric field $\vec{E}$ and the current density $\vec{J}$,

$$\vec{E} = s(s - \Gamma\chi_1)^{-1}\vec{E}_0 = t(t - \Gamma\chi_2)^{-1}\vec{E}_0 \quad \text{and} \quad \vec{J} = s(s - \Upsilon\chi_2)^{-1}\vec{J}_0 = t(t - \Upsilon\chi_1)^{-1}\vec{J}_0, \tag{14}$$

respectively. Here we have defined $s = 1/(1 - h)$, $t = 1/(1 - z) = 1 - s$, $\Gamma = \vec{\nabla}\Delta^{-1}\vec{\nabla}\cdot$, and $\Upsilon = -\vec{\nabla} \times \Delta^{-1}\vec{\nabla}\times$. These formulas follow from manipulations of Eq. (7). The operator Γ is a projection onto curl-free fields, based on convolution with the free-space Green's function for the Laplacian $\Delta = \nabla^2$.[29] More specifically $\Gamma : \mathscr{H}_s \mapsto \mathscr{H}_\times$, and for every $\vec{\zeta} \in \mathscr{H}_\times$ we have $\Gamma\vec{\zeta} = \vec{\zeta}$. For the convenience of the reader we recall a few vector calculus facts. For every $\vec{\zeta} \in \mathscr{H}_\bullet$ we have $\vec{\zeta} = \vec{\nabla} \times (\vec{A} + \vec{C})$ weakly, where $\vec{\nabla} \times \vec{C} = 0$ weakly.[21,34] The arbitrary vector $\vec{C}$ can be chosen so that the vector potential $\vec{A}$ satisfies $\vec{\nabla} \cdot \vec{A} = 0$ weakly.[34] Hence,

$\vec{\nabla}\times\vec{\zeta}=\vec{\nabla}\times\vec{\nabla}\times\vec{A}=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\Delta\vec{A}=-\Delta\vec{A}$ weakly. The vector $\vec{C}$ chosen in this manner gives the Coulomb (or transverse) *gauge* of $\vec{\zeta}$.[34] Choosing the members of the Hilbert space $\mathscr{H}_\bullet$ to have Coulomb gauge, one can similarly show that the operator Υ is a projection onto divergence-free fields. More specifically $\Upsilon:\mathscr{H}_s\mapsto\mathscr{H}_\bullet$, and for every $\vec{\zeta}\in\mathscr{H}_\bullet$ we have $\Upsilon\vec{\zeta}=\vec{\zeta}$.

It is more convenient to consider the functions $F(s)=1-m(h)$ and $E(s)=1-\tilde{m}(h)$, which are analytic off [0, 1] in the s-plane, and $G(t)=1-w(z)$ and $H(t)=1-\tilde{w}(z)$, which are analytic off [0, 1] in the t-plane.[4,29] By Eq. (10) they satisfy

$$0<|F(s)|,|E(s)|<1,\quad 0<|G(t)|,|H(t)|<\infty,\quad h\in\mathcal{U}_0. \tag{15}$$

Here $G(t)$ and $H(t)$ are not to be confused with the Stieltjes function in (1) and the magnetic field strength in the Ising model, respectively. We write $\vec{E}_0=E_0\,\vec{e}_k$ and $\vec{J}_0=J_0\,\vec{j}_k$, where $\vec{e}_k$ and $\vec{j}_k$ are standard basis vectors, for some $k=1,\ldots,d$. Using Eqs. (7), (9), and (14), and the spectral theorem,[43] we obtain the following Herglotz integral representations of $F(s)$, $E(s)$, $G(t)$, and $H(t)$[4,6,29]

$$F(s)=\langle\chi_1(s-\Gamma\chi_1)^{-1}\vec{e}_k\cdot\vec{e}_k\rangle=\int_{\lambda_0}^{\lambda_1}\frac{d\mu(\lambda)}{s-\lambda},\quad E(s)=\langle\chi_2(s-\Upsilon\chi_2)^{-1}\vec{j}_k\cdot\vec{j}_k\rangle=\int_{\tilde{\lambda}_0}^{\tilde{\lambda}_1}\frac{d\eta(\lambda)}{s-\lambda}, \tag{16}$$

$$G(t)=\langle\chi_2(t-\Gamma\chi_2)^{-1}\vec{e}_k\cdot\vec{e}_k\rangle=\int_{\hat{\lambda}_0}^{\hat{\lambda}_1}\frac{d\alpha(\lambda)}{t-\lambda},\quad H(t)=\langle\chi_1(t-\Upsilon\chi_1)^{-1}\vec{j}_k\cdot\vec{j}_k\rangle=\int_{\check{\lambda}_0}^{\check{\lambda}_1}\frac{d\kappa(\lambda)}{t-\lambda},$$

or in the compact notation of (13) $F(s)=\mathcal{A}_{0,1}(s;\mu)$, $E(s)=\mathcal{A}_{0,1}(s;\eta)$, $G(t)=\mathcal{A}_{0,1}(t;\alpha)$, and $H(t)=\mathcal{A}_{0,1}(t;\kappa)$. Equation (16) displays Stieltjes transforms of the bounded positive measures μ, η, α, and κ which are supported on Σ_μ, Σ_η, Σ_α, $\Sigma_\kappa\subseteq[0,1]$, respectively, and depend only on the geometry of the medium.[6,29] The supremum and infimum of these sets are defined to be the upper and lower limits of integration displayed in Eq. (16).

The integro-differential operators $M_j=\chi_j\Gamma\chi_j$ and $K_j=\chi_j\Upsilon\chi_j$, $j=1,2$, are compositions of projection operators on the associated Hilbert spaces $\mathscr{H}_\times$ and $\mathscr{H}_\bullet$, respectively, and are consequently positive definite and bounded by 1 in the underlying operator norm.[45] They are self-adjoint on $L^2(\Omega,P)$.[29] Consequently, in the Hilbert space $L^2(\Omega,P)$ with weight χ_2 in the inner product, for example, $\Gamma\chi_2$ is a bounded self-adjoint operator.[29] Equation (16) is based upon spectral representations of resolvents involving these self-adjoint operators. The measures μ, η, α, and κ are spectral measures of the family of projections of these operators in the respective $\langle\vec{e}_k,\vec{e}_k\rangle$ or $\langle\vec{j}_k,\vec{j}_k\rangle$ state.[29,43]

A key feature of Eqs. (8), (9), and (16) is that the parameter information in s and E_0 is *separated* from the geometry of the composite, which is encapsulated in the measures μ, η, α, and κ through their moments μ_n, η_n, α_n, and κ_n, $n\geq 0$, respectively, which depend on the correlation functions of the medium.[29] For example, $\alpha_0=\eta_0=p$ and $\mu_0=\kappa_0=1-p$. A principal application of the analytic continuation method is to derive *forward bounds* on σ^* and ρ^*, given partial information on the microgeometry.[5,6,29,39] One can also use the representations in (16) to obtain *inverse bounds*, allowing one to use data about the electromagnetic response of a sample to bound its structural parameters such as p.[17,28]

IV. STIELTJES FUNCTION REPRESENTATIONS OF σ^* AND ρ^*

In Sec. III we formulated the effective parameter problem for two-component conductive media and obtained integral representations of the effective complex conductivity σ^* and resistivity ρ^*. In this section we derive Stieltjes function representations of σ^* and ρ^*. These alternate representations will be used in Secs. V and VI to provide spectral characterizations of critical behavior exhibited by σ^* and ρ^*.

In order to illuminate the many symmetries of this mathematical framework, we will henceforth focus on the complex variable $h=h_r+\mathrm{i}h_i$, where $h_r=\mathrm{Re}\,h$ and $h_i=\mathrm{Im}\,h$. Moreover, in the last two formulas of Eq. (16), we will make the change of variables $t(s)=1-s$ and $\lambda\mapsto 1-\lambda$, so that $G(t(s))=-\int_{1-\hat{\lambda}_1}^{1-\hat{\lambda}_0}[-d\alpha(1-\lambda)]/(s-\lambda)$, for example. The change of variables $s(h)=1/(1-h)$ and $\lambda(y)=y/(1+y)\Leftrightarrow y(\lambda)=\lambda/(1-\lambda)$ yield Stieltjes function representations[2] of the formulas in (16). For example,

$$F(s) = (1-h)\int_{S_0}^{S} \frac{(1+y)d\mu(\lambda(y))}{1+hy}\,, \quad G(t(s)) = (h-1)\int_{\hat{S}_0}^{\hat{S}} \frac{(1+y)[-d\alpha(1-\lambda(y))]}{1+hy}\,, \tag{17}$$

where $S_0 = \lambda_0/(1 - \lambda_0)$, $S = \lambda_1/(1 - \lambda_1)$, $\hat{S}_0 = (1-\hat{\lambda}_1)/\hat{\lambda}_1$, $\hat{S} = (1-\hat{\lambda}_0)/\hat{\lambda}_0$, and the supports $\Sigma_\mu = [\lambda_0, \lambda_1]$ and $\Sigma_\alpha = [\hat{\lambda}_0, \hat{\lambda}_1]$ are defined in (16). Therefore, $\lim_{\lambda_0\to 0} S_0 = \lim_{\hat{\lambda}_1\to 1}\hat{S}_0 = 0$ and $\lim_{\lambda_1\to 1} S = \lim_{\hat{\lambda}_0\to 0}\hat{S} = \infty$. Moreover, $d\mu(\lambda(y))$ is the measure $d\mu(\lambda)$ under the variable change $\lambda \mapsto \lambda(y) = y/(1 + y)$ and $[-d\alpha(1 - \lambda(y))]$ is the measure $d\alpha(\lambda)$ under the variable change $\lambda \mapsto 1 - \lambda(y)$, where the negative sign accounts for the switch of integration limits in the second formula of (17). By Eqs. (16) and (17), the Stieltjes function representations of $m(h)$ and $w(z(h))$ are given by

$$m(h) = 1 + (h-1)g(h), \quad g(h) = \int_0^\infty \frac{d\phi(y)}{1+hy}\,, \quad d\phi(y) = (1+y)d\mu(\lambda(y)), \tag{18}$$

$$w(z(h)) = 1 - (h-1)\hat{g}(h), \quad \hat{g}(h) = \int_0^\infty \frac{d\hat{\phi}(y)}{1+hy}\,, \quad d\hat{\phi}(y) = (1+y)[-d\alpha(1-\lambda(y))],$$

with analogous formulas for $\tilde{m}(h)$ and $\tilde{w}(z(h))$ involving Stieltjes functions $\tilde{g}(h) = \mathcal{S}_{0,1}(h;\tilde{\phi})$ and $\check{g}(h) = \mathcal{S}_{0,1}(h;\check{\phi})$, respectively. Equation (18) should be compared to Eq. (1) regarding the Ising model. The Stieltjes functions $g(h)$, $\tilde{g}(h)$, $\hat{g}(h)$, and $\check{g}(h)$ are analytic for all $h \in \mathcal{U}_0$.[29] As μ, η, α, and κ are positive measures on [0, 1], ϕ, $\tilde{\phi}$, $\hat{\phi}$, and $\check{\phi}$ are positive measures on $[0, \infty]$. Consequently, the following inequalities hold (see Lemma 4.1 below)

$$\frac{\partial^{2n}\zeta}{\partial h^{2n}} > 0, \quad \frac{\partial^{2n+1}\zeta}{\partial h^{2n+1}} < 0, \quad \left|\frac{\partial^{n}\zeta}{\partial h^{n}}\right| > 0, \quad \zeta = g(h), \tilde{g}(h), \hat{g}(h), \check{g}(h), \quad h \in \mathcal{U}_0, \tag{19}$$

for $n \geq 0$, which are analogs of Eq. (2) for the Ising model.[23] The first two inequalities in (19) hold for $h \in \mathcal{U}_0 \cap \mathbb{R}$, and the last inequality holds for $h \in \mathcal{U}_0$ such that $h_i \neq 0$.

By Eq. (18), the moments ϕ_n of ϕ satisfy

$$\phi_n = \int_0^\infty y^n d\phi(y) = \int_0^\infty y^n(1+y)d\mu(\lambda) = \int_0^1 \frac{\lambda^n d\mu(\lambda)}{(1-\lambda)^{n+1}} = \mathcal{A}_{n,n+1}(1;\mu). \tag{20}$$

A partial fraction expansion of $\lambda^n/(1 - \lambda)^{n+1}$ then shows that (see Lemma 4.1 below)

$$\frac{(-1)^n}{n!}\lim_{s\to 1}\frac{\partial^n F(s)}{\partial s^n} = \int_0^1 \frac{d\mu(\lambda)}{(1-\lambda)^{n+1}} = \sum_{j=0}^{n}\binom{n}{j}\phi_j. \tag{21}$$

Equation (21) demonstrates that ϕ_n depends on $\int_0^1 d\mu(\lambda)/(1-\lambda)^{n+1}$ *and* all the lower moments ϕ_j, $j = 0, 1, \ldots, n - 1$, of ϕ. Equations (15) and (20) imply that ϕ_0 is bounded. In Lemma 5.1 below, we prove that the higher moments ϕ_n, $n \geq 1$, diverge as $\sup\{\Sigma_\mu\} \to 1$.

We now show that the moments ϕ_j have physical significance. The energy constraints $\langle \vec{J}\cdot\vec{E}_f\rangle = \langle \vec{E}\cdot\vec{J}_f\rangle = 0$ lead to detailed decompositions of the system energy in terms of Herglotz functions involving μ, η, α, and κ. For example, $\langle \vec{J}\cdot\vec{E}_f\rangle = 0$, $\vec{E} = \vec{E}_0 + \vec{E}_f$, $\langle \vec{E}_f\rangle = 0$, and $\sigma = \sigma_2(1 - \chi_1/s)$ imply that $0 = \langle\sigma\vec{E}\cdot\vec{E}_f\rangle = \langle\sigma_2(1-\chi_1/s)(\vec{E}_f\cdot\vec{E}_0 + E_f^2)\rangle = \sigma_2\left[\langle E_f^2\rangle - (\langle\chi_1\vec{E}_f\cdot\vec{E}_0\rangle + \langle\chi_1 E_f^2\rangle)/s\right]$. The spectral theorem[43] and (13) then yield

$$\langle E_f^2\rangle/E_0^2 = \mathcal{A}_{1,2}(s;\mu) = \mathcal{A}_{1,2}(t;\alpha), \qquad \langle J_f^2\rangle/J_0^2 = \mathcal{A}_{1,2}(s;\eta) = \mathcal{A}_{1,2}(t;\kappa). \tag{22}$$

Equation (22) leads to Herglotz representations of all such energy components involving μ, η, α, and κ, e.g., $\langle\chi_1\vec{E}_f\cdot\vec{E}_0\rangle/E_0^2 = \mathcal{A}_{1,1}(s;\mu) = \mathcal{A}_{1,1}(t;\alpha)$. Equations (16), (20), and (22) show that the first two moments, ϕ_0 and ϕ_1, of ϕ are identified with energy components,

$$\phi_0 = \lim_{s\to 1} \frac{\langle \chi_1 \vec{E} \cdot \vec{E}_0 \rangle}{E_0^2}, \quad \phi_1 = \lim_{s\to 1} \frac{\langle E_f^2 \rangle}{E_0^2}. \tag{23}$$

By Eq. (21), *all* of the higher moments ϕ_j, $j \geq 2$, depend on these energy components.

Similarly, the moments $\hat{\phi}_n$ of $\hat{\phi}$ satisfy (see Lemma 4.1 below)

$$\hat{\phi}_n = \int_0^1 \frac{(1-\lambda)^n d\alpha(\lambda)}{\lambda^{n+1}}, \qquad \frac{(-1)^{n+1}}{n!} \lim_{s\to 1} \frac{\partial^n G(t(s))}{\partial^n t} = \int_0^1 \frac{d\alpha(\lambda)}{\lambda^{n+1}} = \sum_{j=0}^{n} \binom{n}{j} \hat{\phi}_j. \tag{24}$$

Equations (16), (22), and (24) also identify the first two moments, $\hat{\phi}_0$ and $\hat{\phi}_1$, of $\hat{\phi}$ with energy components. Equation (24) then implies that all of the higher moments $\hat{\phi}_j$, $j \geq 2$, depend on these energy components. We prove in Lemma 5.1 below that *all* the moments $\hat{\phi}_n$, $n \geq 0$, diverge as $\inf\{\Sigma_\alpha\} \to 0$. By the symmetries in Eqs. (16) and (18), Eqs. (20) and (21) hold for $\tilde{\phi}$ with $E(s)$ and η in lieu of $F(s)$ and μ, respectively, and Eq. (24) holds for $\check{\phi}$ with $H(t(s))$ and κ in lieu of $G(t(s))$ and α, respectively.

We now give some key formulas which will be used extensively. Equations (8) and (9) yield the energy representations $\langle \vec{J} \cdot \vec{E} \rangle = \sigma_2 m(h) E_0^2 = \sigma_1 w(z(h)) E_0^2$ and $\langle \vec{E} \cdot \vec{J} \rangle = \tilde{m}(h) J_0^2/\sigma_1 = \tilde{w}(z(h)) J_0^2/\sigma_2$ involving σ^* and ρ^*, which imply that

$$m(h) = h w(z(h)) \iff 1 - F(s) = (1 - 1/s)(1 - G(t(s))), \quad h \in \mathcal{U}_0 \tag{25}$$

and an analogous formula linking $\tilde{m}(h)$ and $\tilde{w}(z(h))$. Equations (18) and (25) then yield

$$g(h) + h\hat{g}(h) = 1, \qquad \tilde{g}(h) + h\check{g}(h) = 1, \quad h \in \mathcal{U}_0. \tag{26}$$

For $h \in \mathcal{U}_0$, the functions $g(h)$, $\hat{g}(h)$, $\tilde{g}(h)$, and $\check{g}(h)$ are analytic[29] and have bounded h derivatives of all orders.[45] An inductive argument applied to Eq (26) yields

$$\frac{\partial^n g}{\partial h^n} + n\frac{\partial^{n-1}\hat{g}}{\partial h^{n-1}} + h\frac{\partial^n \hat{g}}{\partial h^n} = 0, \qquad \frac{\partial^n \tilde{g}}{\partial h^n} + n\frac{\partial^{n-1}\check{g}}{\partial h^{n-1}} + h\frac{\partial^n \check{g}}{\partial h^n} = 0, \quad n \geq 1. \tag{27}$$

When $h \in \mathcal{U}_0$ such that $h_i \neq 0$, the complex representation of Eq (27) is, for example,

$$\begin{gathered}\frac{\partial^n g_r}{\partial h^n} + n\frac{\partial^{n-1}\hat{g}_r}{\partial h^{n-1}} + h_r\frac{\partial^n \hat{g}_r}{\partial h^n} - h_i\frac{\partial^n \hat{g}_i}{\partial h^n} = 0, \qquad \frac{\partial^n g_i}{\partial h^n} + n\frac{\partial^{n-1}\hat{g}_i}{\partial h^{n-1}} + h_r\frac{\partial^n \hat{g}_i}{\partial h^n} + h_i\frac{\partial^n \hat{g}_r}{\partial h^n} = 0, \\ \frac{\partial^n g_r}{\partial h^n} = \mathrm{Re}\frac{\partial^n g}{\partial h^n}, \quad \frac{\partial^n g_i}{\partial h^n} = \mathrm{Im}\frac{\partial^n g}{\partial h^n}, \qquad \frac{\partial^n \hat{g}_r}{\partial h^n} = \mathrm{Re}\frac{\partial^n \hat{g}}{\partial h^n}, \quad \frac{\partial^n \hat{g}_i}{\partial h^n} = \mathrm{Im}\frac{\partial^n \hat{g}}{\partial h^n},\end{gathered} \tag{28}$$

and analogous equations involving $\tilde{g}$ and $\check{g}$.

The integral representations of (27) and (28) follow from Eq (29) of Lemma 4.1 below, involving the functions $\mathcal{S}_{i,j}$ defined in (13). In the remainder of this section, we focus on the measures ϕ and $\hat{\phi}$, as the analogous results involving $\tilde{\phi}$ and $\check{\phi}$ follow by symmetry.

Lemma 4.1: For all $h \in \mathcal{U}_0$ and $i, j \in \mathbb{Z}$ satisfying $0 \leq i \leq j$, we have $|\mathcal{S}_{i,j}(h;\phi)| < \infty$, and for $0 \leq i \leq j - 1$, $|\mathcal{S}_{i,j}(h;\hat{\phi})| < \infty$. Consequently (Theorem 2.27 of Ref. 22), the Stieltjes functions $g(h)$ and $\hat{g}(h)$ may be repeatedly differentiated under the integral sign:

$$\frac{\partial^n g(h)}{\partial h^n} = (-1)^n n! \int_0^\infty \frac{y^n d\phi(y)}{(1+hy)^{n+1}}, \qquad \frac{\partial^n \hat{g}(h)}{\partial h^n} = (-1)^n n! \int_0^\infty \frac{y^n d\hat{\phi}(y)}{(1+hy)^{n+1}}, \quad n \geq 0. \tag{29}$$

Before we prove Lemma 4.1, we note that Eqs. (27) and (29) imply that

$$\int_0^\infty \frac{y^n d\phi(y)}{(1+hy)^{n+1}} = \int_0^\infty \frac{y^{n-1} d\hat{\phi}(y)}{(1+hy)^n} - h\int_0^\infty \frac{y^n d\hat{\phi}(y)}{(1+hy)^{n+1}}, \quad n \geq 1, \; h \in \mathcal{U}_0. \tag{30}$$

Moreover, Eq. (29) also yields the integral representations of (28) using

$$\frac{(-1)^n}{n!}\frac{\partial^n g(h)}{\partial h^n}=\int_0^\infty \frac{y^n d\phi(y)}{|1+hy|^{2(n+1)}}(1+\bar{h}y)^{n+1}=\sum_{j=0}^{n+1}\binom{n+1}{j}\bar{h}^j\int_0^\infty \frac{y^{n+j}d\phi(y)}{|1+hy|^{2(n+1)}}\,, \tag{31}$$

for example, where $\bar{h}$ denotes complex conjugation of the complex variable h.

Proof of Lemma 4.1: Let $\mathcal{S}_{i,j}(\xi;\nu)$ be defined as in Eq. (13). The supports of the measures ϕ and $\hat{\phi}$ are $\Sigma_\phi=[S_0,S]$ and $\Sigma_{\hat{\phi}}=[\hat{S}_0,\hat{S}]$, respectively, which are defined in terms of $\Sigma_\mu=[\lambda_0,\lambda_1]$ and $\Sigma_\alpha=[\hat{\lambda}_0,\hat{\lambda}_1]$, respectively, directly below Eq. (17). Recalling that $\lambda(y)=y/(1+y)\Leftrightarrow y(\lambda)=\lambda/(1-\lambda)$ and $s=1/(1-h)$, Eq. (18) implies that

$$\mathcal{S}_{i,j}(h;\phi)=s^j\int_{\lambda_0}^{\lambda_1}\frac{\lambda^i(1-\lambda)^{j-i-1}d\mu(\lambda)}{(s-\lambda)^j}\,,\quad \mathcal{S}_{i,j}(h;\hat{\phi})=s^j\int_{\hat{\lambda}_0}^{\hat{\lambda}_1}\frac{(1-\lambda)^i\lambda^{j-i-1}d\alpha(\lambda)}{(s-(1-\lambda))^j}\,. \tag{32}$$

We now show that $|\mathcal{S}_{i,j}(h;\phi)|$ and $|\mathcal{S}_{i,j}(h;\hat{\phi})|$ in (32) are uniformly bounded for all $h\in\mathcal{U}_\varepsilon$.

Set $0<\varepsilon\ll 1$ and let $h\in\mathcal{U}_\varepsilon$, so that $|h|<1$ and $|h-h_0|>\varepsilon$ for all $h_0\in(-1,0]$. As a complex variable, $s=1/(1-h)=|s|^2(1-h_r+\mathrm{i}h_i)$. Therefore, by the lower bound

$$|s|^2=\frac{1}{|1-h|^2}=\frac{1}{1-2h_r+|h|^2}>\frac{1}{2(1-h_r)}>\frac{1}{2(1-\varepsilon)}>\frac{1}{2}\,, \tag{33}$$

when $h_i\neq 0$, we have $|s_i|=|s|^2|h_i|>\varepsilon/2>0$, and when $h_i=0$, $s>1/(1-\varepsilon)=1+\varepsilon/(1-\varepsilon)>1+\varepsilon$. Moreover, $t=1-s$ then implies that, when $h_i\neq 0$, $|t_i|>\varepsilon/2>0$, and when $h_i=0$, $t<-\varepsilon<0$. By Eq. (10) we also have $F(1)=\int_0^1 d\mu(\lambda)/(1-\lambda)\in[0,1]$. Thus, for $i,j\in\mathbb{Z}$ such that $0\le i\le j$ and $0\le i\le j-1$, Eq (32) now implies that[45]

$$|\mathcal{S}_{i,j}(h;\phi)|\le|s|^j\,\lambda_1^i\,(1-\lambda_0)^{j-i}\,F(1)\sup_{\lambda\in[\lambda_0,\lambda_1]}|s-\lambda|^{-j}<\infty\quad\text{and} \tag{34}$$
$$|\mathcal{S}_{i,j}(h;\hat{\phi})|\le|s|^j\,(1-\hat{\lambda}_0)^i\,\hat{\lambda}_1^{j-i-1}\,\alpha_0\sup_{\lambda\in[\hat{\lambda}_0,\hat{\lambda}_1]}|t-\lambda|^{-j}<\infty,$$

respectively, where α_0 is the mass of the measure α. We stress that these bounds, hence (29) hold even if $\lambda_0=\hat{\lambda}_0=0$ and $\lambda_1=\hat{\lambda}_1=1$, when $h\in\mathcal{U}_\varepsilon$. As $g(h)$ and $\hat{g}(h)$ are analytic on $\mathcal{U}_0$,[29] hence have bounded derivatives of all orders,[45] we may now let $\varepsilon\to 0$. □

All the equations given in this section display general formulas holding for two-component stationary random media in lattice and continuum settings.[24] In Sec. VI we will investigate a class of composites for which the transport properties of σ^* exhibit critical behavior in the limit $|h|\to 0$ ($|s|\to 1$, $|t|\to 0$). When $h=0$, the bounds in (34) are violated when $\lambda_1=1$ and $\hat{\lambda}_0=0$, respectively. However, there is a class of composites for which there are *gaps* in the support of the measures μ and α about the spectral endpoints $\lambda=0,1$, including: matrix/particle composites[12] and the composite underlying effective medium theory (EMT) (Ref. 40) (see Sec. VI A below). For such composite media, the bounds (34) and Eq. (29) are valid for $h\in\mathcal{U}_0\cup\{0\}$. However, we will show in Sec. V below that the limit of Eq. (30) as $h\to 0$ is more subtle than simply setting $h=0$.

While in general the spectra of μ and α extend all the way to $\lambda=0,1$, it has been argued that there are composites for which the spectrum close to $\lambda=0,1$ give exponentially small contributions to the transport properties of σ^* as $h\to 0$ (Lifshitz phenomenon).[15,35] However, in this case, $|\partial^n g(h)/\partial h^n|$ and $|\partial^n\hat{g}(h)/\partial h^n|$ may diverge as $h\to 0$ for some $n\ge 1$.

Definition 4.1: Define $\mathcal{B}_{n_\nu}$ to be the class of composites such that the functions $\mathcal{S}_{i,j}(h;\nu)$ in (13) *satisfy* $\lim_{|h|\to 0}|\mathcal{S}_{n,n+1}(h;\nu)|<\infty$, *where $\nu=\phi,\hat{\phi}$, for all $0\le n\le n_\nu$.*

By Eq (30), we have $n_\phi\ge n_{\hat{\phi}}$. The class $\mathcal{B}_{n_\nu}$, $\nu=\phi,\hat{\phi}$, will be used extensively in the Proof of Theorem 6.1 below.

V. SPECTRAL CHARACTERIZATION OF CRITICALITY IN TRANSPORT

In this section we construct measures ϱ and $\tilde{\varrho}$ that are supported on $\{0, 1\}$ which link the measures μ and α, and η and κ, respectively. The properties of ϱ and $\tilde{\varrho}$ imply that critical transitions in the transport properties of σ^* and ρ^* are due to the formation of delta function components in the underlying spectral measures at $\lambda = 0, 1$. In Sec. VI, this identifies these transport transitions with the collapse of spectral gaps in these measures and leads to a precise spectral characterization of critical transport behavior in binary composite media.

The Stieltjes transform of the spectral measures μ, α, η, and κ completely determines the effective transport properties of the medium. Conversely, given the Stieltjes transform of a measure, the Stieltjes-Perron inversion theorem[33] recovers the underlying measure,

$$\mu(\upsilon) = -\frac{1}{\pi}\lim_{\epsilon\downarrow 0} \operatorname{Im} F(\upsilon + \mathrm{i}\epsilon), \quad \upsilon \in \Sigma_\mu, \tag{35}$$

for example. To evoke this theorem directly, in Eq. (16) we define the measures $d\tilde{\alpha}(\lambda) = [-d\alpha(1-\lambda)]$ and $d\tilde{\kappa}(\lambda) = [-d\kappa(1-\lambda)]$, and write $G(t(s)) = -\int_0^1 d\tilde{\alpha}(\lambda)/(s-\lambda)$ and $H(t(s)) = -\int_0^1 d\tilde{\kappa}(\lambda)/(s-\lambda)$. Setting $s = \upsilon + \mathrm{i}\epsilon$, Eqs. (25) and (35) imply that

$$\upsilon\mu(\upsilon) = (1-\upsilon)[-\alpha(1-\upsilon)] - \upsilon\varrho(\upsilon), \qquad \varrho(\upsilon) = \lim_{\epsilon\downarrow 0}\frac{-\epsilon/\pi}{\upsilon^2+\epsilon^2}\int_0^1 \frac{(\upsilon+\lambda-1)\,d\alpha(\lambda)}{(\upsilon+\lambda-1)^2+\epsilon^2}, \tag{36}$$

and an analogous formula involving a measure $\tilde{\varrho}$ which links η and κ. We now demonstrate that Eqs. (25), (26), and (36) explicitly determine the measures ϱ and $\tilde{\varrho}$.

The integral representations of Eq. (26) follow from Eq. (18), and are given by

$$\int_0^\infty \frac{d\phi(y)}{1+hy} + h\int_0^\infty \frac{d\hat{\phi}(y)}{1+hy} = 1, \qquad \int_0^\infty \frac{d\tilde{\phi}(y)}{1+hy} + h\int_0^\infty \frac{d\check{\phi}(y)}{1+hy} = 1. \tag{37}$$

Due to the underlying symmetries of this framework, without loss of generality, we henceforth focus on $F(s(h);\mu)$, $G(t(h);\alpha)$, $g(h;\phi)$, and $\hat{g}(h;\hat{\phi})$. We wish to re-express the first formula in Eq. (37) in a more suggestive form by adding and subtracting the quantity $h\int_0^\infty y\,d\phi(y)/(1+hy)$. This is permissible if the modulus of this quantity is finite for all $h \in \mathcal{U}_0$.[22,45] The affirmation of this fact is given by Lemma 4.1 and we may therefore add and subtract it in Eq. (37), yielding

$$h\int_0^\infty \frac{d\Phi_0(y)}{1+hy} \equiv 1 - \phi_0 = m(0), \quad d\Phi_0(y) = d\hat{\phi}(y) - y\,d\phi(y), \quad h \in \mathcal{U}_0, \tag{38}$$

as $1 - \phi_0 = 1 - F(s)|_{s=1} = m(h)|_{h=0}$ (see Eq. (20)). We stress that $\sigma_1 \neq \sigma_2$ when $h = 0$ so that in (10) $0 \le m(0) < 1$. Equation (38) provides another representation for the quantity $m(0)$ and shows that the transform $h\int_0^\infty d\Phi_0(y)/(1+hy)$ of Φ_0, a signed measure,[45] is independent of h for all $h \in \mathcal{U}_0$. Equation (18) and the identity $y = \lambda/(1-\lambda) \Leftrightarrow \lambda = y/(1+y)$ relates this representation of $m(0)$ to the measure ϱ defined in Eq. (36),

$$d\Phi_0(y) = \frac{1}{(1-\lambda)^2}((1-\lambda)\,[-d\alpha(1-\lambda)] - \lambda\,d\mu(\lambda)) = \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2} = y(1+y)\,d\varrho\,(\lambda(y))\,.$$

We may now express Eq. (38) in terms of $\varrho(d\lambda)$ as follows:

$$m(0) = h\int_0^\infty \frac{d\Phi_0(y)}{1+hy} = h\int_0^\infty \frac{y(1+y)d\varrho(\lambda(y))}{1+hy} = \int_0^1 \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2/h + \lambda(1-\lambda)}. \tag{39}$$

Remark 5.1: Define the transform $\mathcal{D}(h;\varrho)$ of the measure ϱ by

$$\mathcal{D}(h;\varrho) = \int_0^1 \frac{\lambda\,d\varrho(\lambda)}{(1-\lambda)^2/h + \lambda(1-\lambda)}. \tag{40}$$

Equations (10) and (39) show that $\mathcal{D}(h;\varrho)$ has the following properties for all $h \in \mathcal{U}_0$: (1) $\mathcal{D}(h;\varrho)$ is independent of h, (2) $0 \le |\mathcal{D}(h;\varrho)| < 1$, and (3) $\mathcal{D}(h;\varrho) = m(0) \not\equiv 0$.

Lemma 5.1: Let the quantities $m(0) = m(h)|_{h=0} = 1 - F(s)|_{s=1}$ and $w(0) = w(z)|_{z=0} = 1 - G(t)|_{t=1}$ be defined as in Eq. (16), which satisfy $0 \le m(0), w(0) < 1$. If $\mathcal{D}(h;\varrho)$, defined in Eq. (40), satisfies the properties of Remark 5.1 for all $h \in \mathcal{U}_0$, then

$$\varrho(d\lambda) = -w(0)\delta_0(d\lambda) + m(0)(1-\lambda)\delta_1(d\lambda), \tag{41}$$

where $\delta_{\lambda_0}(d\lambda)$ is the Dirac measure concentrated at λ_0.

Proof: Let $\mathcal{D}(h;\varrho)$, defined in Eq. (40), satisfy properties (1)–(3) of Remark 5.1. The measure ϱ is independent of h.[29] If the support Σ_ϱ of the measure ϱ is over continuous spectrum[43] then $\mathcal{D}(h;\varrho)$ depends on h, contradicting property (1). Therefore, the measure ϱ is defined over pure point spectrum.[43] The most general pure point set Σ_ϱ which satisfies properties (1)–(3) is given by $\Sigma_\varrho = \{0, 1\}$. This implies that the measure ϱ is of the form

$$\varrho(d\lambda) = W_0(\lambda)\delta_0(d\lambda) + W_1(\lambda)\delta_1(d\lambda),$$

where the $W_j(\lambda)$, $j = 0, 1$, are bounded functions of $\lambda \in [0, 1]$ which are to be determined. In view of the numerator of the integrand in Eq. (40), we may assume that the function $W_0(\lambda) \equiv W_0(0) = W_0 \not\equiv 0$ is independent of λ. In order for properties (2) and (3) to be satisfied we must have $W_1(\lambda) \sim (1-\lambda)^1$ as $\lambda \to 1$ (any other power of $1 - \lambda$ would contradict one of these two properties). Therefore, without loss of generality, we may set $W_1(\lambda) = \mathrm{w}_1\,(1-\lambda)$, where w_1 is independent of λ. Property (3) now yields $\mathrm{w}_1 = m(0)$.

We have shown that $\varrho(d\lambda) = W_0\,\delta_0(d\lambda) + m(0)(1-\lambda)\delta_1(d\lambda)$, $W_0 \not\equiv 0$. By plugging this formula into Eq. (36) ($\lambda d\mu(\lambda) = (1-\lambda)[-d\alpha(1-\lambda)] - \lambda d\varrho(\lambda)$), we are able determine W_0. Indeed, using Eq. (25) ($F(s) - (1 - 1/s)G(t(s)) = 1/s$), the definition of $F(s)$ in Eq. (16), and $(1-\lambda)/(\lambda(s-\lambda)) = -(1-1/s)/(s-\lambda) + 1/(s\lambda)$, we find that

$$\begin{aligned} F(s) &= -\left(1 - \frac{1}{s}\right)\int_0^1 \frac{[-d\alpha(1-\lambda)]}{s-\lambda} + \frac{1}{s}\int_0^1 \frac{[-d\alpha(1-\lambda)]}{\lambda} - \int_0^1 \frac{d\varrho(\lambda)}{s-\lambda} \\ &= \left(1 - \frac{1}{s}\right)G(t(s)) + \frac{1}{s}\int_0^1 \frac{d\alpha(\lambda)}{1-\lambda} - \frac{W_0}{s} - m(0)\lim_{\lambda\to 1}\frac{1-\lambda}{s-\lambda}, \quad \forall\, h \in \mathcal{U}_0 \end{aligned} \tag{42}$$

which implies that $-W_0 = 1 - \int_0^1 d\alpha(\lambda)/(1-\lambda) = w(0)$. □

Corollary 5.1: If we instead focus on the contrast variables z and t in lieu of h and s, respectively, Eqs. (36) and (41) become

$$\upsilon\alpha(\upsilon) = (1-\upsilon)[-\mu(1-\upsilon)] - \upsilon\varrho(\upsilon), \qquad \varrho(d\lambda) = -m(0)\delta_0(d\lambda) + w(0)(1-\lambda)\delta_1(d\lambda). \tag{43}$$

It is worth mentioning that Eq. (30) can be written as $\int_0^\infty d\Phi_{n-1}(y)/(1+hy)^{n+1} \equiv 0$, for all $n \ge 1$, in terms of the signed measure $d\Phi_{n-1}(y) = y^{n-1}d\Phi_0(y)$. Furthermore, in Eq. (28) for $n = 1$, Eq. (31) implies that $\int_0^\infty d\Phi_1(y)/|1+hy|^4 \equiv 0$. By Lemma 4.1, these integral involving $\Phi_{n-1}(dy)$ are defined for all $h \in \mathcal{U}_0$. These formulas are consistent with Eq. (41) of Lemma 5.1.

Lemma 5.1 and Corollary 5.1 are the key results of this section. They provide a rigorous justification, and a generalization of an analogous result found in Ref. 16 by heuristic means. They demonstrate that $\lambda = 1$ is a removable *simple* singularity under μ, α, η, and κ, and illustrate how the relations in (15), $0 < |F(s)|, |E(s)| < 1$, can hold even when $s = 1$ ($h = 0$) and the spectra extend all the way to $\lambda = 1$. In Sec. VI, we discuss how these general features relate to percolation models of binary composite media.

VI. SCALING LAWS FOR CRITICAL EXPONENTS OF TRANSPORT IN LATTICE AND CONTINUUM PERCOLATION MODELS

We now formulate the problem of percolation-driven critical transitions in transport exhibited by two-component conductive media. In modeling transport in such materials, one often considers a two component random medium with component conductivities σ_1 and σ_2, in the volume fractions $1 - p$ and p, respectively. The medium may be continuous, like the random checkerboard[8,51] and Swiss cheese models,[7,32,53] or discrete, like the random resistor network (RRN).[7,15,53] In the simplest case of the 2–d square RRN,[53,54] the average cluster size of the σ_2 inclusions grows as p increases, and there is a critical volume fraction p_c, $0 < p_c < 1$, called the *percolation threshold*, where an infinite cluster of σ_2 bonds first appears. In the limit $h = \sigma_1/\sigma_2 \to 0$, the system exhibits two types of critical behavior. First, as $h \to 0$ ($\sigma_1 \to 0$ and $0 < |\sigma_2| < \infty$), the effective complex conductivity $\sigma^*(p, h) = \sigma_2\, m(p, h)$ and the effective complex resistivity $\rho^*(p, z(h)) = \tilde{w}(p, z(h))/\sigma_2$ undergo a conductor–insulator critical transition,[7]

$$|\sigma^*(p,0)| = 0, \ \text{ for } \ p < p_c, \ \text{ and } \ 0 = |\sigma_1| < |\sigma^*(p,0)| < |\sigma_2| < \infty, \ \text{ for } \ p > p_c, \tag{44}$$
$$\lim_{p\to p_c^+} |\rho^*(p, z(0))| = \infty, \ \text{ and } \ 0 < |\sigma_2|^{-1} < \left|\rho^*(p, z(0))\right| < |\sigma_1|^{-1} = \infty, \ \text{ for } \ p > p_c.$$

Second, as $h \to 0$ ($|\sigma_2| \to \infty$ and $0 < |\sigma_1| < \infty$), the effective complex conductivity $\sigma^*(p, z(h)) = \sigma_1 w(p, z(h))$ and the effective complex resistivity $\rho^*(p, h) = \tilde{m}(p, h)/\sigma_1$ undergo a conductor–superconductor critical transition,[7]

$$0 < |\sigma_1| < |\sigma^*(p, z(0))| < |\sigma_2| = \infty, \ \text{ for } \ p < p_c, \ \text{ and } \ \lim_{p\to p_c^-} |\sigma^*(p, z(0))| = \infty, \tag{45}$$
$$0 = |\sigma_2|^{-1} < \left|\rho^*(p,0)\right| < |\sigma_1|^{-1} < \infty, \ \text{ for } \ p < p_c, \ \text{ and } \ \left|\rho^*(p,0)\right| = 0, \ \text{ for } \ p > p_c.$$

We will focus on the conductor–insulator critical transition of the effective complex conductivity $\sigma^*(p, h) = \sigma_2\, m(p, h)$ and the conductor–superconductor critical transition of the effective complex conductivity $\sigma^*(p, z(h)) = \sigma_1 w(p, z(h))$. It is clear from Eqs. (18), (44), and (45) that our results immediately generalize to $\rho^*(p, h) = \tilde{m}(p, h)/\sigma_1$ and $\rho^*(p, z(h)) = \tilde{w}(p, z(h))/\sigma_2$, respectively, with $p \mapsto 1 - p$.

This critical behavior in transport is made more precise through the definition of critical exponents. Recall that the existence of a critical exponent is determined by the existence of a limit like that given in (4). In the static limit, $h \in \mathcal{U}_0 \cap \mathbb{R}$, as $h \to 0$ the effective conductivity $\sigma^*(p, h) = \sigma_2\, m(p, h)$ exhibits critical behavior near the percolation threshold $\sigma^*(p, 0) \sim (p - p_c)^t$, as $p \to p_c^+$. Here, the critical exponent t, not to be confused with the contrast parameter, is believed to be *universal* for lattices, depending only on dimension.[24] At $p = p_c$, $\sigma^*(p_c, h) \sim h^{1/\delta}$ as $h \to 0$. We assume the existence of the critical exponents t and δ, as well as γ, defined via a conductive susceptibility $\chi(p, 0) = \partial m(p, 0)/\partial h \sim (p - p_c)^{-\gamma}$, as $p \to p_c^+$. For $p > p_c$, we assume that there is a gap $\theta_\mu \sim (p - p_c)^{\Delta}$ in the support of μ around $h = 0$ or $s = 1$ which collapses as $p \to p_c^+$, or that any spectrum in this region does not affect power law behavior.[24] Consequently, for $p > p_c$ we think of the support of ϕ as being contained in the interval $[0, S(p)]$, with $S(p) \sim (p - p_c)^{-\Delta}$ as $p \to p_c^+$. We demonstrated in (20) that the moments ϕ_j of ϕ become singular as $\theta_\mu \to 0$. We therefore assume the existence of critical exponents γ_n such that $\phi_n(p) \sim (p - p_c)^{-\gamma_n}$ as $p \to p_c^+$, $n \geq 0$. When $h \in \mathcal{U}_0$ such that $h_i \neq 0$, we also assume the existence of critical exponents t_r, δ_r, t_i and δ_i corresponding to $m_r(p, h) = \mathrm{Re}\, m(p, h)$ and $m_i(p, h) = \mathrm{Im}\, m(p, h)$. In summary,

$$\begin{aligned}
&m(p,0) \sim (p-p_c)^t, && m_r(p,0) \sim (p-p_c)^{t_r}, && m_i(p,0) \sim (p-p_c)^{t_i}, && \text{as } \ p \to p_c^+, \\
&m(p_c,h) \sim h^{1/\delta}, && m_r(p_c,h) \sim |h|^{1/\delta_r}, && m_i(p_c,h) \sim |h|^{1/\delta_i}, && \text{as } \ |h| \to 0, \\
&\chi(p,0) \sim (p-p_c)^{-\gamma}, && \phi_n \sim (p-p_c)^{-\gamma_n}, && S(p) \sim (p-p_c)^{-\Delta}, && \text{as } \ p \to p_c^+.
\end{aligned} \tag{46}$$

In a similar way, we define critical exponents for the conductor–superconductor system,

$$\begin{aligned} &w(p,z(0)) \sim (p_c-p)^{-s}, && w_r(p,z(0)) \sim (p_c-p)^{-s_r}, && w_i(p,z(0)) \sim (p_c-p)^{-s_i}, && \text{as } p \to p_c^-, \\ &w(p_c,z(h)) \sim h^{-1/\hat{\delta}}, && w_r(p_c,z(h)) \sim |h|^{-1/\hat{\delta}_r}, && w_i(p_c,z(h)) \sim |h|^{-1/\hat{\delta}_i}, && \text{as } |h| \to 0, \\ &\hat{\chi}(p,z(0)) \sim (p_c-p)^{-\hat{\gamma}'}, && \hat{\phi}_n \sim (p_c-p)^{-\hat{\gamma}_n'}, && \hat{S}(p) \sim (p_c-p)^{-\hat{\Delta}'}, && \text{as } p \to p_c^-, \end{aligned} \tag{47}$$

where s is the superconductor critical exponent, not to be confused with the contrast parameter. We also assume the existence of critical exponents γ', γ_n', and Δ', associated with $m(p, h; \phi)$ for the left hand limit $p \to p_c^-$, and critical exponents $\hat{\gamma}$, $\hat{\gamma}_n$, and $\hat{\Delta}$, associated with $w(p, z(h); \hat{\phi})$ for the right hand limit $p \to p_c^+$. The critical exponents γ, δ, Δ, and γ_n for transport are different from those defined in Sec. II for the Ising model in (3).

The key result of this section is the two–parameter scaling relations between the critical exponents of the conductor–insulator system, defined in Eq. (46), and that of the conductor–superconductor system, defined in Eq. (47). Moreover, Lemma 5.1 shows that measures μ and α, hence ϕ and $\hat{\phi}$ are related, and we therefore anticipate that these two sets of critical exponents are also related. This is indeed the case and, assuming a symmetry in the properties of μ and α, the resultant relationship between the critical exponents t and s is in agreement with the seminal paper by Efros and Shklovskii.[19]

These results are summarized in Theorem 6.1 below. In this theorem, we assume that the percolation model under consideration is of class $\mathcal{B}_{n_\phi}$ and $\mathcal{B}_{n_{\hat{\phi}}}$ for $p > p_c$ and $p < p_c$, respectively (see Definition 4.1). By Eqs. (30), (44), and (45), the $2-d$ square RRN is of class $\mathcal{B}_{n_{\hat{\phi}}}$ and $\mathcal{B}_{n_\phi}$ for $p > p_c$ and $p < p_c$ with $0 \le n_{\hat{\phi}} \le n_\phi$. We assume that this holds for the percolation model under consideration, and we further assume that $1 \le n_{\hat{\phi}} \le n_\phi$ so that $\chi(p, 0) = \partial m(p, 0)/\partial h$ and $\hat{\chi}(p, z(0)) = \partial w(p, z(0))/\partial h$ in (46) and (47) exist.

Theorem 6.1: *Consider a percolation model of a binary conductive medium of class* $\mathcal{B}_{n_\phi}$ and $\mathcal{B}_{n_{\hat{\phi}}}$ *for* $p > p_c$ and $p < p_c$, *respectively, and* $1 \le n_{\hat{\phi}} \le n_\phi$. *Let the critical exponents associated with the model:* $t, t_r, t_i, \delta, \delta_r, \delta_i, \gamma, \gamma_n, \Delta, \gamma', \gamma_n'$, *and* Δ', *and* $s, s_r, s_i, \hat{\delta}, \hat{\delta}_r, \hat{\delta}_i, \hat{\gamma}', \hat{\gamma}_n', \hat{\Delta}', \hat{\gamma}, \hat{\gamma}_n$, *and* $\hat{\Delta}$, *be defined as in Eqs.* (46) *and* (47), *respectively, and in the paragraph following Eq.* (47). *Then the following scaling relations hold:*

(1) $\gamma_1 = \gamma,\ \gamma_1' = \gamma',\ \hat{\gamma}_1 = \hat{\gamma}$, *and* $\hat{\gamma}_1' = \hat{\gamma}'$. **(2)** $\gamma_0' = 0,\ \gamma_0 < 0,\ \gamma_n' > 0$ *and* $\gamma_n > 0,\ n \ge 1$.
(3) $\hat{\gamma}_n' > 0$ *for* $n \ge 0$. **(4)** $\gamma = \hat{\gamma}_0$ *and* $\Delta = \hat{\Delta}$. **(5)** $\gamma' = \hat{\gamma}_0'$ *and* $\Delta' = \hat{\Delta}'$.
(6) $\gamma_n = \gamma + \Delta(n-1)$ *for* $1 \le n \le n_\phi$. **(7)** $\hat{\gamma}_n' = \hat{\gamma}_0' + \hat{\Delta}' n = \hat{\gamma}' + \hat{\Delta}'(n-1)$ *for* $1 \le n \le n_{\hat{\phi}}$.
(8) $t = \Delta - \gamma$. **(9)** $s = \hat{\gamma}_0' = \hat{\gamma}' - \hat{\Delta}'$. **(10)** $\delta = \frac{\Delta}{\Delta - \gamma}$. **(11)** $\hat{\delta} = \frac{\hat{\Delta}'}{\hat{\gamma}_0'} = \frac{\hat{\Delta}'}{\hat{\gamma}' - \hat{\Delta}'}$.
(12) $t_r = t_i = t$. **(13)** $s_r = s_i = s$. **(14)** $\delta_r = \delta_i = \delta$. **(15)** $\hat{\delta}_r = \hat{\delta}_i = \hat{\delta}$.
(16) *If* $\Delta = \Delta'$ *and* $\gamma = \gamma'$, *then* $t + s = \Delta$ *and* $1/\delta + 1/\hat{\delta} = 1$.
(17) *In general* $1/\delta + 1/\hat{\delta} = 1$. *If* $1 \le n_{\hat{\phi}} \le n_\phi$, *then* $t/\Delta + s/\hat{\Delta}' = 1,\ \Delta = \hat{\Delta}' \iff \gamma = \hat{\gamma}_0'$.

It is important to note that the scaling relations $t_r = t_i = t$ and $s_r = s_i = s$ are a fundamental identity, as these sets of critical exponents are defined in terms of $m(p, 0)$ and $w(p, z(0))$, where $h = 0 \in \mathbb{R}$. The relation $1/\delta + 1/\hat{\delta} = 1$ is also a fundamental identity which follows from Eq. (25) and the definition of these critical exponents. The calculation of these scaling relations will serve as a consistency check of this mathematical framework.

Before we present the Proof of Theorem 6.1, which is given in Sec. VI B below, we first demonstrate that the critical exponents of EMT satisfy the critical exponent scaling relations therein. This verification is essential, as there exists a binary composite medium which realizes the effective parameter of EMT.[40] Through our exploration of EMT, we will uncover aspects which illuminate general features of critical transport transitions exhibited by two phase random media. These features will be discussed in detail in Sec. VI C.

A. Effective medium theory

An EMT for the effective parameter problem may be constructed from dilute limits.[16] The EMT approximation for σ^* with percolation threshold p_c is given by[16]

$$p\,\frac{\sigma_2-\sigma^*}{1+p_c\,(\sigma_2/\sigma^*-1)}+(1-p)\,\frac{\sigma_1-\sigma^*}{1+p_c\,(\sigma_1/\sigma^*-1)}=0. \tag{48}$$

Equation (48) leads to quadratic formulas involving $m(p,h)=\sigma^*/\sigma_2$ and $w(p,z(h))=\sigma^*/\sigma_1$. The quadratic equation demonstrates that the relation $m(p,h)=h\,w(p,z(h))$ in (25) is exactly satisfied and that

$$m(p,h(s))=\frac{-b\,(s,p,p_c)+\sqrt{-\zeta(s,p)}}{2s(1-p_c)},\quad \zeta(\lambda,p)=-\lambda^2+2(1-\varphi)\lambda+\upsilon^2-(1-\varphi)^2, \tag{49}$$
$$w(p,z(t))=\frac{-b\,(s,1-p,p_c)+\sqrt{-\zeta(t,1-p)}}{2t(1-p_c)},\quad \zeta(\lambda,1-p)=-\lambda^2+2\varphi\lambda+\upsilon^2-\varphi^2,$$

where $b(\lambda,p,p_c)=(2p_c-1)\lambda+(1-p-p_c)$, $\varphi=\varphi(p,p_c)=p\,(1-p_c)+p_c(1-p)$, and $\upsilon=\upsilon(p,p_c)=2\sqrt{p\,(1-p)\,p_c(1-p_c)}$.

The spectral measures μ and α in (16) may be extracted from Eq. (49) using the Stieltjes–Perron inversion theorem in (35). These measures are absolutely continuous, i.e., there exist density functions such that $\mu(d\lambda)=\mu(\lambda)d\lambda$ and $\alpha(d\lambda)=\alpha(\lambda)d\lambda$. Direct calculation shows that, for $p\neq p_c$, $1-p_c$, these measures have gaps in the spectra about $\lambda=0, 1$: $\mu(\lambda)=0\Leftrightarrow\zeta(\lambda,p)\le 0\Leftrightarrow|\lambda-(1-\varphi)|\ge\upsilon$ and $\alpha(\lambda)=0\Leftrightarrow\zeta(\lambda,1-p)\le 0\Leftrightarrow|\lambda-\varphi|\ge\upsilon$. Therefore, the composite medium underlying the percolation model of EMT (Ref. 40) is of class $\mathcal{B}_{n_\phi}$ and $\mathcal{B}_{n_{\hat\phi}}$ for $p\neq p_c$, $1-p_c$ with n_ϕ, $n_{\hat\phi}=\infty$ (see Definition 4.1). The Stieltjes transformations of μ and α are given by

$$F(p,s)=\int_{\lambda_0}^{1-\theta}\frac{\sqrt{\zeta(\lambda,p)}\,d\lambda}{2\pi(1-p_c)\,\lambda(s-\lambda)},\qquad G(p,t)=\int_{\theta}^{\hat\lambda_1}\frac{\sqrt{\zeta(\lambda,1-p)}\,d\lambda}{2\pi(1-p_c)\,\lambda(t-\lambda)}, \tag{50}$$

where $\theta=\theta(p,p_c)=\varphi-\upsilon$ and $\hat\lambda_1=1-\lambda_0=\varphi+\upsilon$ define *spectral gaps*, which satisfy $\lim_{p\to1-p_c}\lambda_0=0$, $lim_{p\to p_c}\theta=0$, and $\lim_{p\to1-p_c}\hat\lambda_1=1$.

Define a critical exponent Δ for the spectral gap $\theta(p)\sim|p-p_c|^\Delta$, as $p\to p_c$, in $\mu(d\lambda)$ about $\lambda=1$ and $\alpha(d\lambda)$ about $\lambda=0$. Using the definition of a critical exponent in (4) and L'Hôpital's rule we have shown that $\Delta=2$. Moreover, $\lambda_0=1-\hat\lambda_1\sim|p-(1-p_c)|^\Delta$, as $p\to 1-p_c$, with the same critical exponent. The absolutely continuous nature of the measures μ and α in EMT implies that critical indices are the same for $p\to p_c^+$ and $p\to p_c^-$. Therefore, the spectral symmetry properties in the hypothesis of Lemma 6.11 hold for EMT.

We have explicitly calculated the integrals in Eq. (50) for real and complex h using the symbolic mathematics software Maple 15. Using the exact representation in (50) of $G(p,t(h))$, as a function of $0\le\theta\ll1$ and $0\le|h|\ll1$, we have calculated the critical exponents s, $\hat\delta$, $\hat\delta_r$, $\hat\delta_i$, and $\hat\gamma_n$, for $n=0,1,2,\ldots$ These results are in agreement with our general theory. With $h=0$ and $0<\theta\ll1$, we found that $w(p,z(0))\sim\theta^{-1/2}$ which yields $s=\Delta/2=1$. When $\theta=0$ and $0<h\ll1$, one must split up the integration domain, $\Sigma_\alpha\supset(0,h-\epsilon)\cup(h+\epsilon,\hat\lambda_1)$, and take the principal value of the integral as $\epsilon\to0$. Doing so yields $\hat\delta=\hat\delta_r=\hat\delta_i=2$. As in our general theory, the values of the exponents are independent of the path of h to zero. More specifically, these relations hold for $0<|h_r|=|ah_i|\ll1$ with arbitrary $a\in\mathbb{R}$, and for independent h_r and h_i satisfying $0<|h_r|$, $|h_i|\ll1$. The critical exponents $\hat\gamma_n$ associated with the moments $\hat\phi_n$ of the measure $\hat\phi$ satisfy our general relation $\hat\gamma_n=\hat\gamma_0+\Delta n$ with $\hat\gamma_0=\Delta=2$ so that $\hat\gamma_n=\Delta(n+1)$.

Similarly, using the exact representation of $F(p,s(h))$ in (50), as a function of $0\le\theta\ll1$ and $0\le|h|\ll1$, we have calculated the critical exponents t, δ, δ_r, δ_i, and γ_n, for $n=0,1,2,\ldots$ These results are also in agreement with our general theory. In accordance with Ref. 16, we obtain $t=\Delta/2=1$, so that the relation $s+t=\Delta=2$ is satisfied. By direct calculation we have obtained $\delta=\delta_r=\delta_i=2$. We have also obtained these values using $m(p,h)=hw(p,z(h))$ and the associated relations for complex h, $m_r=h_rw_r-h_iw_i$ and $m_i=h_rw_i+h_iw_r$, with $\hat\delta=\hat\delta_r=\hat\delta_i$ and $1/\delta+1/\hat\delta=1$. The mass $\phi_0(p)=F(p,1)$ of the measure ϕ behaves logarithmically as $\theta\to0$, yielding

$\gamma_0 = 0$. The exponents of the higher moments satisfy our general relation $\gamma_n = \gamma_0 + \Delta n = \gamma + \Delta(n-1)$, or $\gamma_n = \Delta n$, $n \geq 0$.

In summary, we have extended EMT to the complex quasi-static regime and shown that the critical exponents of EMT exactly satisfy our scaling relations displayed in Theorem 6.1. Moreover, we have shown that, in EMT, the percolation threshold p_c and $1 - p_c$ coincide with the collapse of gaps in the spectral measures about the spectral endpoints $\lambda = 0, 1$. We will discuss this link between spectral gaps and the percolation threshold in more detail in Sec. VI C.

B. Proof of Theorem 6.1

Baker's critical theory characterizes phase transitions of a given system via the asymptotic behavior of the underlying Stieltjes functions near a critical point. This powerful method has been very successful for the Ising model, precisely characterizing the phase transition (spontaneous magnetization).[2] We will now show how this method may be adapted to provide a detailed description of phase transitions in transport, exhibited by binary composite media. Theorem 6.1 will be proven via a sequence of lemmas as we collect some important properties of $m(p, h)$, $g(p, h)$, $w(p, z(h))$, and $\hat{g}(p, h)$, and how they are related. We stress that the only assumption needed for Theorem 6.1 is that our percolation model is of class $\mathcal{B}_{n_\phi}$ and $\mathcal{B}_{n_{\hat{\phi}}}$ for $p > p_c$ and $p < p_c$, respectively, and $1 \leq n_{\hat{\phi}} \leq n_\phi$ (see Definition 4.1). The following theorem characterizes Stieltjes functions (series of Stieltjes).[2]

Theorem 6.2: *Let $D(i, j)$ denote the determinant*

$$D(i, j) = \begin{vmatrix} \xi_i & \xi_{i+1} & \cdots & \xi_{i+j} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{i+j} & \xi_{i+j+1} & \cdots & \xi_{i+2j} \end{vmatrix}. \tag{51}$$

The ξ_n form a series of Stieltjes if and only if $D(i, j) \geq 0$ for all $i, j = 0, 1, 2, \ldots$

Baker's inequalities for the sequences γ_n and $\hat{\gamma}_n$ of transport follow directly from Theorem 6.2. For example, $\phi_n \sim (p - p_c)^{-\gamma_n}$ and Theorem 6.2 with $\phi_i = \xi_i$, $i = n$, and $j = 1$, imply that, for $0 < p - p_c \ll 1$,

$$\begin{aligned} &(p - p_c)^{-\gamma_n - \gamma_{n+2}} - (p - p_c)^{-2\gamma_{n+1}} \geq 0 \iff (p - p_c)^{-\gamma_n - \gamma_{n+2} + 2\gamma_{n+1}} \geq 1 \\ &\iff -\gamma_n - \gamma_{n+2} + 2\gamma_{n+1} \leq 0 \iff \boxed{\gamma_{n+1} - 2\gamma_n + \gamma_{n-1} \geq 0}. \end{aligned} \tag{52}$$

The sequence of inequalities in (52) are *Baker's inequalities* for transport, corresponding to $m(p, h)$, and they imply that the sequence γ_n increases at least linearly with n. The symmetries in Eqs. (18), (46), and (47) imply that Baker's inequalities also hold for the sequences $\hat{\gamma}'_n$, γ'_n, and $\hat{\gamma}_n$.

The following lemma provides the asymptotic behavior of the h derivatives of $g(p, h)$ and $\hat{g}(p, h)$, which will be used extensively in this section.

Lemma 6.1: Let $h \in \mathcal{U}_0$, $0 < |h| \ll 1$, and $|p - p_c| \ll 1$. Then the integrals in Eq. (29) have the following asymptotics for $n \geq 0$

$$\left| \frac{\partial^n g(p, h)}{\partial h^n} \right| \sim \phi_n, \qquad \left| \frac{\partial^n \hat{g}(p, h)}{\partial h^n} \right| \sim \hat{\phi}_n. \tag{53}$$

Proof: The asymptotic behavior in Eq. (53) follows from Eqs. (20), (21), and (24), Baker's inequalities (52), and equation (18) ($g(p, h) = sF(p, s)$ and $\hat{g}(p, h) = -s\, G(p, t(s))$). These equations imply that, for $c_j, b_j \in \mathbb{Z}$ and $|p - p_c| \ll 1$,

$$\lim_{h \to 0} \frac{\partial^n g(p, h)}{\partial h^n} = \sum_{j=0}^{n} c_j \lim_{s \to 1} \frac{\partial^j F(p, s)}{\partial s^j} \sim \phi_n, \quad \lim_{h \to 0} \frac{\partial^n \hat{g}(p, h)}{\partial h^n} = \sum_{j=0}^{n} b_j \lim_{s \to 1} \frac{\partial^j G(p, t(s))}{\partial t^j} \sim \hat{\phi}_n.$$

□

Lemma 6.1 demonstrates that the numbers n_ϕ and $n_{\hat{\phi}}$ introduced in Definition 4.1 are also related to the number of *finite* moments of the measures ϕ and $\hat{\phi}$, respectively.

Lemma 6.2: $\gamma_1 = \gamma$, $\gamma_1' = \gamma'$, $\hat{\gamma}_1 = \hat{\gamma}$, and $\hat{\gamma}_1' = \hat{\gamma}'$.

Proof: Set $0 < p - p_c \ll 1$. By Eqs. (18) ($g(p, h) = sF(p, s)$), (21), (46), and (52)

$$(p - p_c)^{-\gamma} \sim \chi(p, 0) = \frac{\partial m(p, 0)}{\partial h} = \lim_{s\to 1}\left[-\frac{\partial F(p, s)}{\partial s}\right] = \phi_0 + \phi_1 \sim \phi_1 \sim (p - p_c)^{-\gamma_1}, \quad (54)$$

hence $\gamma_1 = \gamma$. Similarly, for $0 < p_c - p \ll 1$, we have $\gamma_1' = \gamma'$. By Eq. (54), the symmetries between $m(p, h)$ and $w(p, z(h))$ given in (18), and the critical exponent definitions given in (46) and (47), we also have $\hat{\gamma}_1 = \hat{\gamma}$ and $\hat{\gamma}_1' = \hat{\gamma}'$. □

Equation (25) is consistent with, and provides a link between Eqs. (44) and (45). We will see that the fundamental asymmetry between $m(p, h)$ and $w(p, z(h))$ ($\gamma_0' = 0$ and $\hat{\gamma}_0' > 0$), given in Theorem 6.1.(2) and 6.1.(3), is a direct and essential consequence of Eq. (25), and has deep and far reaching implications.

Lemma 6.3: Let the sequences γ_n and γ_n', $n \geq 0$, be defined as in Eq. (46). Then

$$(\mathbf{1}) \quad \gamma_0' = 0,\ \ \gamma_0 < 0,\ \ \gamma_n' > 0,\ \text{ and }\ \gamma_n > 0,\ \text{ for }\ n \geq 1.$$

$$(\mathbf{2}) \quad 0 < \lim_{h\to 0}\langle \chi_1 \vec{E}\cdot\vec{E}_0\rangle / E_0^2 < 1 \ \text{ for all }\ p \in [0, 1].$$

Proof: By Eq. (45) $|w(p, z(0))|$ is bounded for all $p < p_c$. Thus, for all $p < p_c$, Eqs. (21), (25), and (46) imply that

$$0 = \lim_{h\to 0} hw(p, z(h)) = \lim_{h\to 0} m(p, h) = \lim_{s\to 1}(1 - F(p, s)) = 1 - \phi_0(p) \sim 1 - (p_c - p)^{-\gamma_0'},$$

where the rightmost relation holds for $0 < p_c - p \ll 1$ and the leftmost relation is consistent with Eq. (44). Therefore, $\gamma_0' = 0$ and ϕ is a probability measure for all $p < p_c$. The strict positivity of the γ_n', for $n \geq 1$, follows from Baker's inequalities in (52). Thus, from the analogy of Eq. (54) for $p < p_c$, we have

$$\infty = \lim_{p\to p_c^-} \phi_1(p) = -\lim_{p\to p_c^-} \frac{\partial m(p, 0)}{\partial h}. \quad (55)$$

For $p > p_c$, Eqs. (21) and (44) imply that $0 < \lim_{h\to 0}|m(p, h)| = 1 - \phi_0 < 1$. Therefore, $(p - p_c)^{-\gamma_0} \sim \phi_0 < 1$ for all $0 < p - p_c \ll 1$, hence $\gamma_0 < 0$. The strict positivity of γ_1 follows from Eq. (55), and the strict positivity of the γ_n for $n \geq 2$ follows from Baker's inequalities (52). Equation (23) and the inequality $0 < \lim_{h\to 0}|m(p, h)| = 1 - \phi_0 < 1$ imply that $0 < \lim_{h\to 0}\langle \chi_1 \vec{E}\cdot\vec{E}_0\rangle / E_0^2 < 1$ for all $p \in [0, 1]$.

Lemma 6.4: Let the sequence $\hat{\gamma}_n'$, $n \geq 0$, be defined as in Eq. (47). Then

$$(\mathbf{1})\ \ \hat{\gamma}_n' > 0 \ \textit{ for all }\ n \geq 0. \qquad\qquad (\mathbf{2})\ \ \lim_{p\to p_c, h\to 0}\langle E_f^2\rangle = \infty.$$

Proof: By Eq. (44) we have $0 < \lim_{h\to 0}|m(p, h)| < 1$, for all $p > p_c$. Therefore, Eq. (25) implies that $\lim_{h\to 0} w(p, z(h)) = \lim_{h\to 0} m(p, h)/h = \infty$, for all $p > p_c$, which is consistent with Eq. (45). More specifically, for all $p > p_c$, Eqs. (25) and (44) imply that $0 < \lim_{h\to 0}|m(p, h)| = \lim_{h\to 0}|hw(p, z(h))| = L(p) < 1$, where $L(p) = 0$ for all $p < p_c$. Therefore, by Eq. (18), we have

$$\lim_{h\to 0}|h\, w(p, z(h))| = \lim_{h\to 0}|h\, \hat{g}(p, h)| \in (0, 1),\ \text{ for all }\ p > p_c, \quad (56)$$

$$\lim_{h\to 0}|h\, w(p, z(h))| = \lim_{h\to 0}|h\, \hat{g}(p, h)| = 0,\ \text{ for all }\ p < p_c.$$

By Eqs. (24), (45), and (47) we have, for all $p > p_c$,

$$\infty = \lim_{p\to p_c^-}\lim_{h\to 0} w(p,z(h)) = \lim_{p\to p_c^-}\lim_{s\to 1}(1-G(p,t(s))) = 1 + \lim_{p\to p_c^-}\hat{\phi}_0(p) \sim 1 + \lim_{p\to p_c^-}(p_c-p)^{-\hat{\gamma}_0'},$$

hence $\hat{\gamma}_0' > 0$. Baker's inequalities then imply that $\hat{\gamma}_n' > 0$ for all $n \geq 0$. Equation (23) and the analogy thereof involving $\hat{\phi}_1$, and the strict positivity of $\hat{\gamma}_1'$, γ_1, and γ_1', shown here and in Lemma 6.3, imply that $\lim_{p\to p_c, h\to 0}\langle E_f^2\rangle = \infty$. □

The asymptotic behavior of $|\hat{g}(p,h)|$ in Eq. (53), and Lemma 6.4 motivates the following fundamental homogenization assumption of this section:[2]

Remark 6.1: Near the critical point $(p, h) = (p_c, 0)$, *the asymptotic behavior of the Stieltjes function* $\hat{g}(p,h)$ *is determined primarily by the mass* $\hat{\phi}_0(p)$ of the measure $\hat{\phi}$ *and the rate of collapse of the spectral gap* θ_α.

By remark 6.1, and in light of Lemmas 6.2–6.4, we make the following variable changes:

$$\begin{aligned} \hat{q} &= y(p_c-p)^{\hat{\Delta}'}, \qquad \hat{Q}(p) = \hat{S}(p)(p_c-p)^{\hat{\Delta}'}, \qquad d\hat{\pi}(\hat{q}) = (p_c-p)^{\hat{\gamma}_0'}\, d\hat{\phi}(y), \\ q &= y(p-p_c)^{\Delta}, \qquad Q(p) = S(p)(p-p_c)^{\Delta}, \qquad d\pi(q) = (p-p_c)^{\gamma}\; y\, d\phi(y), \end{aligned} \tag{57}$$

so that, by Eqs. (46) and (47), $\hat{Q}(p)$, $Q(p) \sim 1$ and the masses $\hat{\pi}_0$ and π_0 of the measures $\hat{\pi}$ and π, respectively, satisfy $\hat{\pi}_0, \pi_0 \sim 1$ as $p \to p_c$.

Equation (57) defines the following scaling functions $G_{n-1}(x)$, $\hat{G}_n(\hat{x})$, $\mathcal{G}_{n-1,j}(x)$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$ as follows. For $h \in \mathcal{U}_0 \cap \mathbb{R}$, Eqs. (29) and (57) imply, for all $n \geq 1$, that

$$\begin{aligned} &\frac{\partial^n g}{\partial h^n} \propto (p-p_c)^{-(\gamma+\Delta(n-1))} G_{n-1}(x), \qquad \frac{\partial^n \hat{g}}{\partial h^n} \propto (p_c-p)^{-(\hat{\gamma}_0'+\hat{\Delta}' n)} \hat{G}_n(\hat{x}), \\ &G_{n-1}(x) = \int_0^{Q(p)} \frac{q^{n-1} d\pi(q)}{(1+xq)^{n+1}}, \qquad \hat{G}_n(\hat{x}) = \int_0^{\hat{Q}(p)} \frac{\hat{q}^{\,n} d\hat{\pi}(\hat{q})}{(1+\hat{x}\hat{q})^{n+1}}, \\ &x = h(p-p_c)^{-\Delta}, \quad 0 < p-p_c \ll 1, \qquad \hat{x} = h(p_c-p)^{-\hat{\Delta}'}, \quad 0 < p_c - p \ll 1, \end{aligned} \tag{58}$$

respectively, and an analogue of (58) for the opposite limits involving $\hat{\Delta}$, $\hat{\gamma}_0$, Δ', and γ'. For $h \in \mathcal{U}_0$ such that $h_i \neq 0$, we define the scaling functions $\mathcal{R}_{n-1}(x)$, $\mathcal{I}_{n-1}(x)$, $\hat{\mathcal{R}}_n(\hat{x})$, and $\hat{\mathcal{I}}_n(\hat{x})$ as follows. Using Eqs. (31) and (57) we have, for $0 < p - p_c \ll 1$,

$$\begin{aligned} \frac{\partial^n g}{\partial h^n} &= (-1)^n n! \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{h}^j \int_0^{S(p)} \frac{y^{n+j} d\phi(y)}{|1+hy|^{2(n+1)}} \\ &= (-1)^n n! \sum_{j=0}^{n+1} \binom{n+1}{j} [\bar{x}(p-p_c)^{\Delta}]^j (p-p_c)^{-(\gamma+\Delta(n-1+j))} \mathcal{G}_{n-1,j}(x) \\ &= (-1)^n n! (p-p_c)^{-(\gamma+\Delta(n-1))} \mathcal{K}_{n-1}(x), \quad \mathcal{K}_{n-1}(x) = \mathcal{R}_{n-1}(x) + \mathrm{i}\, \mathcal{I}_{n-1}(x), \\ \frac{\partial^n \hat{g}}{\partial h^n} &= (-1)^n n! (p-p_c)^{-(\hat{\gamma}_0+\hat{\Delta} n)} \hat{\mathcal{K}}_n(\hat{x}), \quad \hat{\mathcal{K}}_n(\hat{x}) = \hat{\mathcal{R}}_n(\hat{x}) + \mathrm{i}\, \hat{\mathcal{I}}_n(\hat{x}). \end{aligned} \tag{59}$$

Here, x and $\hat{x}$ are defined in Eq. (58) and

$$\begin{aligned} &\mathcal{G}_{n-1,j}(x) = \int_0^{Q(p)} \frac{q^{n-1+j} d\pi(q)}{|1+xq|^{2(n+1)}}, \qquad \hat{\mathcal{G}}_{n,j}(\hat{x}) = \int_0^{\hat{Q}(p)} \frac{\hat{q}^{\,n+j} d\hat{\pi}(\hat{q})}{|1+\hat{x}\hat{q}|^{2(n+1)}}, \\ &\mathcal{K}_{n-1}(x) = \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{x}^{\,j} \mathcal{G}_{n-1,j}(x), \qquad \hat{\mathcal{K}}_n(\hat{x}) = \sum_{j=0}^{n+1} \binom{n+1}{j} \bar{\hat{x}}^{j} \hat{\mathcal{G}}_{n,j}(\hat{x}), \end{aligned} \tag{60}$$

where we have made the definitions $\mathcal{R}_{n-1}(x) = \mathrm{Re}\,\mathcal{K}_{n-1}(x)$, $\mathcal{I}_{n-1}(x) = \mathrm{Im}\,\mathcal{K}_{n-1}(x)$, $\hat{\mathcal{R}}_n(\hat{x}) = \mathrm{Re}\,\hat{\mathcal{K}}_n(\hat{x})$, and $\hat{\mathcal{I}}_n(\hat{x}) = \mathrm{Im}\,\hat{\mathcal{K}}_n(\hat{x})$. Analogous formulas are defined for the opposite limit,

$0 < p_c - p \ll 1$, involving $\hat{\Delta}'$, $\hat{\gamma}_0'$, Δ', and γ'. By (19) we have, for $p \in [0, 1]$ and $n \geq 0$,

$$G_{n-1}(x) > 0, \quad \mathcal{G}_{n-1,j}(x) > 0, \qquad \hat{G}_n(\hat{x}) > 0, \quad \hat{\mathcal{G}}_{n,j}(\hat{x}) > 0. \tag{61}$$

By hypothesis, our percolation model is of class $\mathcal{B}_{n_\phi}$ and $\mathcal{B}_{n_{\hat{\phi}}}$ for every $p > p_c$ and $p < p_c$, respectively, and $1 \leq n_\phi \leq n_{\hat{\phi}}$ (see Definition 4.1). We therefore have

$$\begin{aligned} &\lim_{h\to 0} G_{n-1}(x) < \infty, \qquad \lim_{h\to 0} \mathcal{G}_{n-1,j}(x) < \infty, \qquad \text{for all } \ p > p_c, \ \ 0 \leq n \leq n_\phi, \\ &\lim_{h\to 0} \hat{G}_n(\hat{x}) < \infty, \qquad \lim_{h\to 0} \hat{\mathcal{G}}_{n,j}(\hat{x}) < \infty, \qquad \text{for all } \ p < p_c, \ \ 0 \leq n \leq n_{\hat{\phi}}. \end{aligned} \tag{62}$$

Lemma 6.5: Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in Eq. (58)*, for $p > p_c$. Then, if $n_\phi \geq 1$,*

(1) $G_{n-1}(x) \sim 1$ *as* $x \to 0\ (h \to 0$ *and* $0 < p - p_c \ll 1)$ *for all* $1 \leq n \leq n_\phi$.
(2) $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ *as* $\hat{x} \to 0\ (h \to 0$ *and* $0 < p - p_c \ll 1)$ *for all* $1 \leq n \leq n_{\hat{\phi}}$.
(3) $\gamma = \hat{\gamma}_0$.
(4) $\Delta = \hat{\Delta}$.

Proof: Let $h \in \mathcal{U}_0 \cap \mathbb{R}$ and $p > p_c$. Equations (30), (58), (61), and (62) imply that we have, for all $1 \leq n \leq n_\phi$, $0 < p - p_c \ll 1$, and $0 < h \ll 1$,

$$(0, \infty) \ni (p - p_c)^{-(\gamma+\Delta(n-1))} G_{n-1}(x) = (p - p_c)^{-(\hat{\gamma}_0+\hat{\Delta}(n-1))}[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})]. \tag{63}$$

Equations (61) and (62) imply that $G_{n-1}(x) \sim 1$ as $x \to 0$, for all $1 \leq n \leq n_\phi$. Equation (63) then implies that $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ as $\hat{x} \to 0$, for all $1 \leq n \leq n_\phi$ (a possible competition in sign between two diverging terms). Or equivalently, generalizing Eq. (56), $[\hat{G}_0(\hat{x}) - \hat{x}^n\hat{G}_n(\hat{x})] \sim 1$. Therefore, Eq. (63) implies that

$$\gamma + \Delta(n-1) = \hat{\gamma}_0 + \hat{\Delta}(n-1), \quad 1 \leq n \leq n_\phi, \tag{64}$$

which in turn, implies that $\gamma = \hat{\gamma}_0$ and $\Delta = \hat{\Delta}$. □

Lemma 6.6: Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in Eq. (58)*, for $p < p_c$. Then, if $n_{\hat{\phi}} \geq 1$,*

(1) $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ *as* $\hat{x} \to 0\ (h \to 0$ *and* $0 < p_c - p \ll 1)$, *for all* $1 \leq n \leq n_{\hat{\phi}}$.
(2) $G_{n-1}(x) \sim 1$ *as* $x \to 0\ (h \to 0$ *and* $0 < p_c - p \ll 1$, *for all* $1 \leq n \leq n_{\hat{\phi}}$.
(3) $\gamma' = \hat{\gamma}_0'$.
(4) $\Delta' = \hat{\Delta}'$.

Proof: Let $h \in \mathcal{U}_0 \cap \mathbb{R}$ and $p < p_c$. Equations (30), (58), (61), and (62) imply that, for all $1 \leq n \leq n_{\hat{\phi}}$, $0 < p_c - p \ll 1$, and $0 < h \ll 1$,

$$(0, \infty) \ni (p_c - p)^{-(\hat{\gamma}_0'+\hat{\Delta}'(n-1))}[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] = (p_c - p)^{-(\gamma'+\Delta'(n-1))} G_{n-1}(x). \tag{65}$$

Equations (61) and (62) imply that $[\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim 1$ as $\hat{x} \to 0$ for all $1 \leq n \leq n_{\hat{\phi}}$. Equation (65) then implies that $G_{n-1}(x) \sim 1$ as $x \to 0$ for all $1 \leq n \leq n_{\hat{\phi}}$. Therefore,

$$\gamma' + \Delta'(n-1) = \hat{\gamma}_0' + \hat{\Delta}'(n-1), \quad 1 \leq n \leq n_{\hat{\phi}}.$$

Which in turn, implies that $\gamma' = \hat{\gamma}_0'$ and $\Delta' = \hat{\Delta}'$. □

Lemma 6.7: Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in Eq. (58)*. Then, if $1 \leq n_{\hat{\phi}} \leq n_\phi$,*

(1) $\gamma_n = \gamma + \Delta(n-1)$, *for all* $1 \leq n \leq n_\phi$.
(2) $\hat{\gamma}_n' = \hat{\gamma}_0' + \hat{\Delta}'n = \hat{\gamma}' + \hat{\Delta}'(n-1)$, *for all* $1 \leq n \leq n_{\hat{\phi}}$.

(3) $t = \hat{\Delta} - \hat{\gamma}_0 = \Delta - \gamma$.
(4) $s = \hat{\gamma}_0' = \hat{\gamma}' - \hat{\Delta}'$.

Proof: Let $0 < p - p_c \ll 1$. By Eqs. (46), (53), and (58), and Lemma 6.5 we have, for all $1 \le n \le n_\phi$,

$$(p - p_c)^{-\gamma_n} \sim \phi_n \sim \lim_{h\to 0} \frac{\partial^n g(p,h)}{\partial h^n} \sim (p-p_c)^{-(\gamma+\Delta(n-1))} \lim_{x\to 0} G_{n-1}(x) \sim (p-p_c)^{-(\gamma+\Delta(n-1))}.$$

Therefore, $\gamma_n = \gamma + \Delta(n-1)$ for all $1 \le n \le n_\phi$, with constant gap $\gamma_i - \gamma_{i-1} = \Delta$. When this is true for all $n \ge 0$, as in EMT where $n_\phi = \infty$, this is consistent with the absence of multifractal behavior for the bulk conductivity $\sigma^*(p,h) = \sigma_2 m(p,h)$.[53]

Now let $0 < p_c - p \ll 1$. By Eqs. (47), (53), (58), (61), and (62) we have, for all $1 \le n \le n_{\hat\phi}$,

$$(p_c - p)^{-\hat{\gamma}_n} \sim \hat{\phi}_n \sim \lim_{h\to 0} \frac{\partial^n \hat{g}(p,h)}{\partial h^n} \propto (p_c - p)^{-(\hat{\gamma}_0'+\hat{\Delta}'n)} \lim_{\hat{x}\to 0} \hat{G}_n(\hat{x}) \sim (p_c-p)^{-(\hat{\gamma}_0'+\hat{\Delta}'n)}.$$

Therefore, by Lemma 6.2, we have $\hat{\gamma}_n = \hat{\gamma}_0' + \hat{\Delta}'n = \hat{\gamma}' + \hat{\Delta}'(n-1)$ for all $1 \le n \le n_{\hat\phi}$, with constant gap $\hat{\gamma}_i' - \hat{\gamma}_{i-1}' = \hat{\Delta}$. When this is true for all $n \ge 0$, as in EMT where $n_{\hat\phi} = \infty$, this is consistent with the absence of multifractal behavior for the bulk conductivity $\sigma^*(p, z(h)) = \sigma_1 w(p, z(h))$.[53]

Again let $0 < p - p_c \ll 1$. Equations (18), (26), (46), (56), and (58) yield

$$\begin{aligned}(p-p_c)^t &\sim \lim_{h\to 0} m(p,h) = 1 - \lim_{h\to 0} g(p,h) = \lim_{h\to 0} h\hat{g}(p,h) = (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0} \lim_{\hat{x}\to 0} \hat{x}\hat{G}_0(\hat{x}) \\ &\sim (p-p_c)^{\hat{\Delta}-\hat{\gamma}_0}. \end{aligned} \tag{66}$$

Therefore, by Lemma 6.5 we have, for $n_\phi \ge 1$, $t = \hat{\Delta} - \hat{\gamma}_0 = \Delta - \gamma$.

Finally, let $0 < p_c - p \ll 1$. By Eqs. (18), (47), (58), (61), and (62), and Lemma 6.4, we have

$$(p_c - p)^{-s} \sim \lim_{h\to 0} w(p, z(h)) \sim 1 + \lim_{h\to 0} \hat{g}(p,h) = 1 + (p_c-p)^{-\hat{\gamma}_0'} \lim_{\hat{x}\to 0} \hat{G}_0(\hat{x}) \sim (p_c-p)^{-\hat{\gamma}_0'}.$$

Therefore, by Lemma 6.7.2 we have, for $n_{\hat\phi} \ge 1$, $s = \hat{\gamma}_0' = \hat{\gamma}' - \hat{\Delta}'$. □

Lemma 6.8: Let $\hat{G}_n(\hat{x})$, $G_{n-1}(x)$, and the associated critical exponents be defined as in Eq. (58)*, for $p > p_c$ and $p < p_c$. Then for all $1 \le n \le n_{\hat\phi} \le n_\phi$*

(1) $G_{n-1}(x) \sim [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim x^{-(\gamma+\Delta(n-1))/\Delta}$, *as* $\hat{x} \to \infty$ ($p \to p_c^+$ *and* $0 < h \ll 1$).
(2) $G_{n-1}(x) \sim [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim x^{-(\gamma'+\Delta'(n-1))/\Delta'}$, *as* $x \to \infty$ ($p \to p_c^-$ *and* $0 < h \ll 1$).
(3) $\delta = \hat{\Delta}/(\hat{\Delta} - \hat{\gamma}_0) = \Delta/(\Delta - \gamma)$.
(4) $\hat{\delta} = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\Delta}'/(\hat{\gamma}' - \hat{\Delta}')$.

Proof: Let $0 < h \ll 1$, so that $g(p,h)$ and $\hat{g}(p,h)$ are analytic for all $p \in [0, 1]$.[29] The analyticity of $g(p,h)$ and $\hat{g}(p,h)$ implies that all orders of h derivatives of these functions are bounded as $p \to p_c$, from the left or the right. Therefore, Eq. (63) holds for $0 < p - p_c \ll 1$, and Eq. (65) holds for $0 < p_c - p \ll 1$. Moreover, in order to cancel the diverging p dependent prefactors in (63) and (65) we must have, *in general*, for all $n \ge 1$,

$$\begin{aligned} &G_{n-1}(x) \sim x^{-(\gamma+\Delta(n-1))/\Delta}, \quad [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim \hat{x}^{-(\hat{\gamma}_0+\hat{\Delta}(n-1))/\hat{\Delta}}, \quad \text{as } p \to p_c^+, \\ &G_{n-1}(x) \sim x^{-(\gamma'+\Delta'(n-1))/\Delta'}, \quad [\hat{G}_{n-1}(\hat{x}) - \hat{x}\hat{G}_n(\hat{x})] \sim \hat{x}^{-(\hat{\gamma}_0'+\hat{\Delta}'(n-1))/\hat{\Delta}'}, \quad \text{as } p \to p_c^-. \end{aligned} \tag{67}$$

For $1 \le n \le n_{\hat\phi} \le n_\phi$, Lemmas 6.8.(1) and 6.8.(2) follow from (67) and Lemmas 6.5 and 6.6.

Now by Eqs. (18), (25), (46), (58), and (67) for $n = 1$, we have

$$\begin{aligned} h^{1/\delta} &\sim \lim_{p\to p_c^+} m(p,h) = \lim_{p\to p_c^+} h w(p, z(h)) \sim \lim_{p\to p_c^+} h\hat{g}(p,h) = h \lim_{p\to p_c^+} (p-p_c)^{-\hat{\gamma}_0} \hat{G}_0(\hat{x}) \\ &\sim h(p-p_c)^{-\hat{\gamma}_0} h^{-\hat{\gamma}_0/\hat{\Delta}} (p-p_c)^{-\hat{\Delta}(-\hat{\gamma}_0/\hat{\Delta})} = h^{(\hat{\Delta}-\hat{\gamma}_0)/\hat{\Delta}}. \end{aligned} \tag{68}$$

Therefore, $\delta = \hat{\Delta}/(\hat{\Delta} - \hat{\gamma}_0)$, and for $1 \leq n_{\hat{\phi}} \leq n_\phi$, Lemma 6.5 implies $\delta = \Delta/(\Delta - \gamma)$. Similarly by Eqs. (18), (47), (58), and (67) for $n = 1$, and Lemma 6.4, we have

$$h^{-1/\hat{\delta}} \sim \lim_{p \to p_c^-} w(p, z(h)) \sim 1 + \lim_{p \to p_c^-} \hat{g}(p, h) = 1 + \lim_{p \to p_c^-} (p - p_c)^{-\hat{\gamma}_0'} \hat{G}_0(\hat{x}) = h^{-\hat{\gamma}_0'/\hat{\Delta}'}, \tag{69}$$

in general. By Lemma 6.7, for $1 \leq n_{\hat{\phi}} \leq n_\phi$, we have $\hat{\delta} = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\Delta}'/(\hat{\gamma}' - \hat{\Delta}')$. □

Lemma 6.9: Let $h \in \mathcal{U}_0$ such that $h_i \neq 0$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$, $\hat{\mathcal{R}}_n(\hat{x})$, $\hat{\mathcal{I}}_n(\hat{x})$, and the associated critical exponents be defined as in Eqs. (59) and (60) for $p > p_c$ and $p < p_c$. Furthermore, let s_r, s_i, t_r, and t_i be defined as in Eqs. (46) and (47). Then,

(1) $[\hat{\mathcal{G}}_{0,0}(\hat{x}) + \hat{x}_r \hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \hat{x}_i \hat{\mathcal{G}}_{0,1}(\hat{x}) \sim 1$ *as* $\hat{x} \to 0$ ($h \to 0$ *and* $0 < p_c - p \ll 1$).
(2) $\lim_{\hat{x} \to 0} [\hat{x}_r \hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}|^2 \hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \lim_{\hat{x} \to 0} [\hat{x}_i \hat{\mathcal{G}}_{0,0}(\hat{x})] \sim 1$ *for* $0 < p - p_c \ll 1$.
(3) $s_r = s_i = \hat{\gamma}_0' = s$.
(4) $t_r = t_i = \Delta - \gamma = t$.

Proof: Let $0 < p_c - p \ll 1$, $h \in \mathcal{U}_0$ such that $h_i \neq 0$, and $0 < |h| \ll 1$. By Eqs. (59) and (60), for $n = 0$, we have

$$\hat{g}(p, h) = \int_0^{\hat{S}(p)} \frac{d\hat{\phi}(y)}{|1 + hy|^2} + \bar{h} \int_0^{\hat{S}(p)} \frac{y\, d\hat{\phi}(y)}{|1 + hy|^2} = (p_c - p)^{-\hat{\gamma}_0'} [\hat{\mathcal{G}}_{0,0}(\hat{x}) + \bar{\hat{x}} \hat{\mathcal{G}}_{0,1}(\hat{x})], \tag{70}$$

so that

$$\begin{aligned} \hat{g}_r(p, h) &= (p_c - p)^{-\hat{\gamma}_0'} \hat{\mathcal{R}}_0(\hat{x}) = (p_c - p)^{-\hat{\gamma}_0'} [\hat{\mathcal{G}}_{0,0}(\hat{x}) + \hat{x}_r \hat{\mathcal{G}}_{0,1}(\hat{x})], \\ \hat{g}_i(p, h) &= (p_c - p)^{-\hat{\gamma}_0'} \hat{\mathcal{I}}_0(\hat{x}) = -(p_c - p)^{-\hat{\gamma}_0'} \hat{x}_i \hat{\mathcal{G}}_{0,1}(\hat{x}). \end{aligned} \tag{71}$$

Equations (56) and (61) imply that $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim 1$ as $\hat{x} \to 0$ ($h \to 0$ and $0 < p_c - p \ll 1$). Therefore, Eqs. (18), (47), and (71) and Lemma 6.4 imply that

$$\begin{aligned} (p_c - p)^{-s_r} &\sim w_r(p, z(0)) \sim 1 + \hat{g}_r(p, 0) \sim 1 + (p_c - p)^{-\hat{\gamma}_0'} \lim_{\hat{x} \to 0} \hat{\mathcal{R}}_0(\hat{x}) \sim (p_c - p)^{-\hat{\gamma}_0'}, \\ (p_c - p)^{-s_i} &\sim w_i(p, z(0)) \sim \hat{g}_i(p, 0) \sim (p_c - p)^{-\hat{\gamma}_0'} \lim_{\hat{x} \to 0} \hat{\mathcal{I}}_0(\hat{x}) \sim (p_c - p)^{-\hat{\gamma}_0'}. \end{aligned} \tag{72}$$

Equation (72) and Lemma 6.7 imply that $s_r = s_i = \hat{\gamma}_0' = s$.

Now let $0 < p - p_c \ll 1$ with h as before. In Eq (66) we demonstrated that $m(p, 0) = \lim_{h \to 0} h \hat{g}(p, h)$. Therefore, Eq. (71), for $p > p_c$, implies that

$$\begin{aligned} m_r(p, 0) &\sim \lim_{h \to 0} [h_r \hat{g}_r(p, h) - h_i \hat{g}_i(p, h)] = (p - p_c)^{\hat{\Delta} - \hat{\gamma}_0} \lim_{\hat{x} \to 0} [\hat{x}_r \hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}_r|^2 \hat{\mathcal{G}}_{0,1}(\hat{x})], \\ m_i(p, 0) &\sim \lim_{h \to 0} [h_i \hat{g}_r(p, h) + h_r \hat{g}_i(p, h)] = (p - p_c)^{\hat{\Delta} - \hat{\gamma}_0} \lim_{\hat{x} \to 0} [\hat{x}_i \hat{\mathcal{G}}_{0,0}(\hat{x})]. \end{aligned} \tag{73}$$

By Eq. (56) we have $\lim_{\hat{x} \to 0} [\hat{x}_r \hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}|^2 \hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \lim_{\hat{x} \to 0} [\hat{x}_i \hat{\mathcal{G}}_{0,0}(\hat{x})] \sim 1$ for all $0 < p - p_c \ll 1$. Therefore, Eqs. (46) and (73) imply that

$$(p - p_c)^{t_r} \sim m_r(p, 0) \sim (p - p_c)^{\hat{\Delta} - \hat{\gamma}_0}, \qquad (p - p_c)^{t_i} \sim m_i(p, 0) \sim (p - p_c)^{\hat{\Delta} - \hat{\gamma}_0}. \tag{74}$$

Equation (74) and Lemmas 6.5 and 6.7 imply that, for $1 \leq n_{\hat{\phi}} \leq n_\phi$, $t_r = t_i = \hat{\Delta} - \hat{\gamma}_0 = \Delta - \gamma = t$. □

Lemma 6.10: Let $h \in \mathcal{U}_0$ such that $h_i \neq 0$, and $\hat{\mathcal{G}}_{n,j}(\hat{x})$, $\hat{\mathcal{R}}_n(\hat{x})$, $\hat{\mathcal{I}}_n(\hat{x})$, and the associated critical exponents be defined as in Eqs. (59) and (60) for $p > p_c$ and $p < p_c$. Furthermore, let $\hat{\delta}_r$, $\hat{\delta}_i$, δ_r, and δ_i be defined as in Eqs. (46) and (47). Then,

(1) $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim |\hat{x}|^{-\hat{\gamma}_0'/\hat{\Delta}'}$, *as* $\hat{x} \to \infty$ ($p \to p_c^-$ *and* $0 < |h| \ll 1$).
(2) $[\hat{x}_r \hat{\mathcal{R}}_0(\hat{x}) - \hat{x}_i \hat{\mathcal{I}}_0(\hat{x})] \sim [\hat{x}_r \hat{\mathcal{I}}_0(\hat{x}) + \hat{x}_i \hat{\mathcal{R}}_0(\hat{x})] \sim |\hat{x}|^{(\hat{\Delta} - \hat{\gamma}_0)/\hat{\Delta}}$, *as* $\hat{x} \to \infty$.
(3) $\hat{\delta}_r = \hat{\delta}_i = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\delta}$.
(4) $\delta_r = \delta_i = \Delta/(\Delta - \gamma) = \delta$.

Proof: Let $h \in \mathcal{U}_0$ such that $h_i \neq 0$ and $0 < |h| \ll 1$, so that $g(p, h)$ and $\hat{g}(p, h)$ are analytic for all $p \in [0, 1]$.[29] Equations (18), (47), and (71) and Lemma 6.4 imply that

$$|h|^{-1/\hat{\delta}_r} \sim w_r(p_c, z(h)) \sim 1 + \hat{g}_r(p_c, h) \sim \lim_{p \to p_c^-} (p_c - p)^{-\hat{\gamma}_0'} \hat{\mathcal{R}}_0(\hat{x}), \tag{75}$$

$$|h|^{-1/\hat{\delta}_i} \sim w_i(p_c, z(h)) \sim \hat{g}_i(p_c, h) \sim \lim_{p \to p_c^-} (p_c - p)^{-\hat{\gamma}_0'} \hat{\mathcal{I}}_0(\hat{x}).$$

The analyticity of $g(p, h)$ and $\hat{g}(p, h)$ implies that they are bounded for all $p \in [0, 1]$. Therefore, in order to cancel the diverging p dependent prefactors in Eq. (75), we must have $\hat{\mathcal{R}}_0(\hat{x}) \sim \hat{\mathcal{I}}_0(\hat{x}) \sim |x|^{-\hat{\gamma}_0'/\hat{\Delta}'}$ as $\hat{x} \to \infty$ ($p \to p_c^-$ and $0 < h \ll 1$). Equation (75) then implies

$$|h|^{-1/\hat{\delta}_r} \sim (p_c - p)^{-\hat{\gamma}_0'} |h|^{-\hat{\gamma}_0'/\hat{\Delta}'} (p_c - p)^{-\hat{\Delta}'(-\hat{\gamma}_0'/\hat{\Delta}')} = |h|^{-\hat{\gamma}_0'/\hat{\Delta}'}, \qquad |h|^{-1/\hat{\delta}_i} \sim |h|^{-\hat{\gamma}_0'/\hat{\Delta}'}. \tag{76}$$

Therefore, by Lemma 6.8, $\hat{\delta}_r = \hat{\delta}_i = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\delta}$. It's worth mentioning that these scaling relations are independent of the path of the limit $h \to 0$.

Equations (18) and (25) imply that $m(p_c, h) \sim \lim_{p \to p_c^+} h\hat{g}(p, h)$, for $0 < |h| \ll 1$. Therefore, Eqs. (46) and (73) imply that

$$|h|^{1/\delta_r} \sim m_r(p_c, h) = (p - p_c)^{\hat{\Delta} - \hat{\gamma}_0} \lim_{p \to p_c^+} [\hat{x}_r \hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}_r|^2 \hat{\mathcal{G}}_{0,1}(\hat{x})], \tag{77}$$

$$|h|^{1/\delta_i} \sim m_i(p_c, h) = (p - p_c)^{\hat{\Delta} - \hat{\gamma}_0} \lim_{p \to p_c^+} [\hat{x}_i \hat{\mathcal{G}}_{0,0}(\hat{x})].$$

The analyticity of $g(p, h)$ and $\hat{g}(p, h)$ implies that they are bounded for all $p \in [0, 1]$. Therefore, in order to cancel the diverging p dependent prefactors in Eq. (77), we must have $[\hat{x}_r \hat{\mathcal{G}}_{0,0}(\hat{x}) + |\hat{x}_r|^2 \hat{\mathcal{G}}_{0,1}(\hat{x})] \sim \hat{x}_i \hat{\mathcal{G}}_{0,0}(\hat{x}) \sim |x|^{(\hat{\Delta} - \hat{\gamma}_0)/\hat{\Delta}}$ as $\hat{x} \to \infty$ ($p \to p_c^+$ and $0 < h \ll 1$). Therefore, Eq. (77) implies that $\delta_r = \delta_i = \hat{\Delta}/(\hat{\Delta} - \hat{\gamma}_0)$ *in general*, and for $1 \leq n_{\hat{\phi}} \leq n_\phi$, Lemmas 6.5 and 6.8 imply that $\delta_r = \delta_i = \hat{\Delta}/(\hat{\Delta} - \hat{\gamma}_0) = \Delta/(\Delta - \gamma) = \delta$. As before, these scaling relations are independent of the path of $h \to 0$. □

Lemma 6.11: For $1 \leq n_{\hat{\phi}} \leq n_\phi$, the measure $y\, d\phi(y)$ has the symmetry property ($\Delta = \Delta'$ and $\gamma = \gamma'$) if and only if the measure $d\hat{\phi}(y)$ has the symmetry property ($\hat{\Delta} = \hat{\Delta}'$ and $\hat{\gamma}_0 = \hat{\gamma}_0'$). If either measure has this symmetry, then

$$\textbf{(1)}\ \ s + t = \Delta. \quad \textbf{(2)}\ \ 1/\delta + 1/\hat{\delta} = 1. \quad \textbf{(3)}\ \ \Delta = \hat{\Delta} = \Delta' = \hat{\Delta}'. \quad \textbf{(4)}\ \ \gamma = \gamma' = \hat{\gamma}_0 = \hat{\gamma}_0'.$$

Proof: We have shown in Lemmas 6.5 and 6.6 that, for $1 \leq n_{\hat{\phi}} \leq n_\phi$, $\gamma = \hat{\gamma}_0$, $\Delta = \hat{\Delta}$, $\gamma' = \hat{\gamma}_0'$, and $\Delta' = \hat{\Delta}'$. Therefore, it is clear that, ($\Delta = \Delta'$ and $\gamma = \gamma'$) $\Leftrightarrow$ ($\hat{\Delta} = \hat{\Delta}'$ and $\hat{\gamma}_0 = \hat{\gamma}_0'$). Assume that either of the measures, $d\hat{\phi}(y)$ or $y\, d\phi(y)$, has this symmetry. Thus, $\Delta = \hat{\Delta} = \hat{\Delta}' = \Delta'$ and $\gamma = \hat{\gamma}_0 = \hat{\gamma}_0' = \gamma'$. By Lemma 6.7 we have $t = \Delta - \gamma$ and $s = \hat{\gamma}_0'$, and by Lemma 6.8 we have $\delta = \Delta/(\Delta - \gamma)$ and $\hat{\delta} = \hat{\Delta}'/\hat{\gamma}_0'$. Therefore,

$$s + t = \hat{\gamma}_0' + \Delta - \gamma = \hat{\gamma}_0 + \Delta - \gamma = \Delta.$$

$$\delta = \Delta/(\Delta - \gamma) = 1/(1 - \gamma/\Delta) = 1/(1 - \hat{\gamma}_0/\hat{\Delta}) = 1/(1 - \hat{\gamma}_0'/\hat{\Delta}') = 1/(1 - 1/\hat{\delta}).$$ □

As mentioned above, the scaling relations $t_r = t_i = t$ and $s_r = s_i = s$ that we proved in Lemma 6.9 are fundamental identities, and serve as a consistency check of this mathematical framework. Another consistency check was given in Lemma 6.11, where we proved that $1/\delta + 1/\hat{\delta} = 1$. This is also a fundamental identity which follows from the relation in (25), $m(p, h) = h\, w(p, z(h))$, and the definition of these critical exponents in (46) and (47): $h^{1/\delta} \sim m(p_c, h) = h\, w(p_c, z(h)) \sim h\, h^{-1/\hat{\delta}} \sim h^{1-1/\hat{\delta}}$, for $0 < |h| \ll 1$. It follows that the relation in (25) provides a partial converse to the assumption underlying Lemma 6.11. Indeed, as $1/\delta + 1/\hat{\delta} = 1$ in general, and for $1 \leq n_{\hat{\phi}} \leq n_\phi$ we have $\delta = \Delta/(\Delta - \gamma) = \Delta/t$ and $\hat{\delta} = \hat{\Delta}'/\hat{\gamma}_0' = \hat{\Delta}'/s$, then $1 - \gamma/\Delta = 1/\delta = 1 - 1/\hat{\delta} = 1 - \hat{\gamma}_0'/\hat{\Delta}'$ implies that $t/\Delta + s/\hat{\Delta}' = 1$, and $\Delta = \hat{\Delta}' \iff \gamma = \hat{\gamma}_0'$. This concludes the proof of Theorem 6.1. □

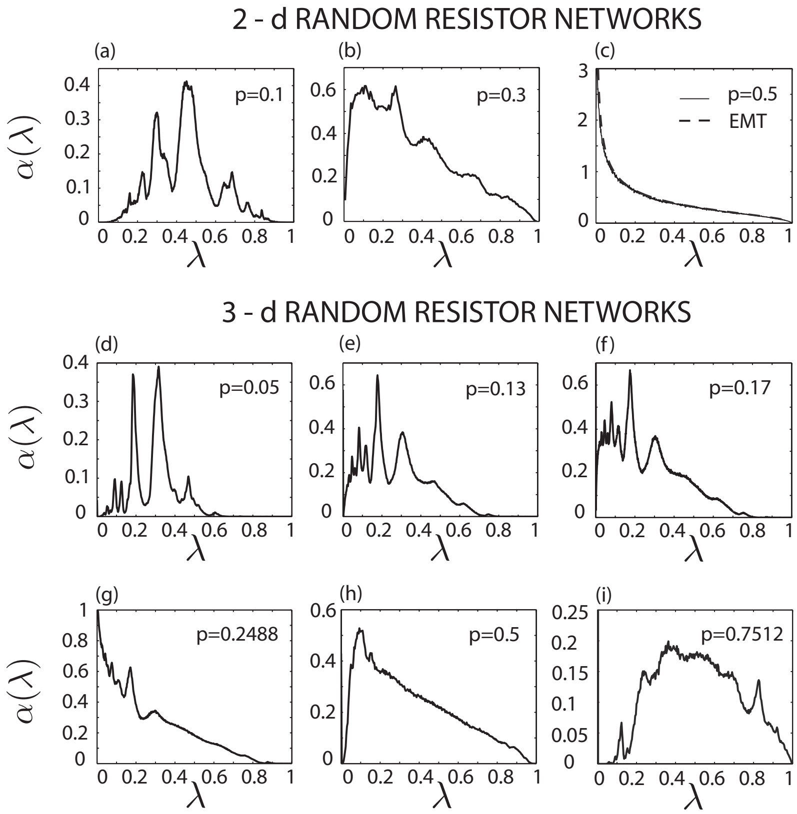

FIG. 1. *The spectral function for the 2-d and 3-d square random resistor networks (RRN).* In the 2-d RRN (a)–(c), as the volume fraction p increases from left to right, the width of the gaps in the spectrum near $\lambda = 0, 1$ shrink to 0 *symmetrically* with increasing connectedness as $p \to p_c = 1 - p_c = 0.5$. In (c) the effective medium theory (EMT) prediction of the spectral measure, which coincides with the exact duality prediction, is also displayed. In the 3-d RRN (d)–(i), as $p \to p_c \approx 0.2488$ the width of the gap near $\lambda = 0$ shrinks to 0, and as $p \to 1 - p_c \approx 0.7512$ the width of the gap near $\lambda = 1$ shrinks to 0.

C. Spectral gaps and critical behavior of transport

We now discuss the gaps θ_μ and θ_α in the spectral measures μ and α, respectively. As the operators Γ and Υ are projectors on the associated Hilbert spaces $\mathscr{H}_\times$ and $\mathscr{H}_\bullet$, respectively, their eigenvalues are confined to the set $\{0, 1\}$.[43] The associated operators M_j and K_j, $j = 1, 2$ are positive definite compositions of projection operators, thus their eigenvalues are confined to the set $[0, 1]$.[29,45] While in general, the spectrum actually extends all the way to the spectral endpoints $\lambda = 0, 1$, it has been argued that the part close to $\lambda = 0, 1$ corresponds to very large, but very rare connected regions (Lifshitz phenomenon). It is believed that this phenomenon gives exponentially small contributions to the effective complex conductivity (resistivity), and does not affect power law behavior.[24]

In Ref. 12 Bruno has proven the existence of spectral gaps in matrix/particle systems with polygonal inclusions, and studied how the gaps vanish as the inclusions touch (like $p \to p_c$). In Figure 1, we give a graphical representation of the spectral measure α for finite, square 2–d and 3–d RRN (Ref. 28) (explained in more detail below). In the 2–d RRN, as $p \to p_c = 0.5$ the gaps in the spectrum near $\lambda = 0, 1$ shrink to 0 *symmetrically*. In the 3–d RRN, as $p \to p_c \approx 0.2488$ the spectral gap near $\lambda = 0$ shrinks to 0, and as $p \to 1 - p_c \approx 0.7512$ the spectral gap near $\lambda = 1$ shrinks to 0. As p surpasses p_c and $1 - p_c$ the spectrum piles up at $\lambda = 0$ and $\lambda = 1$, respectively, forming delta function-like components in the measure. In Sec. VI A we showed that, for EMT, there are gaps in the spectral measures μ and α for $p \neq p_c$, $1 - p_c$. The gaps in μ and α about $\lambda = 1$ and $\lambda = 0$, respectively, collapse as $p \to p_c$, and the gaps in μ and α about $\lambda = 0$ and $\lambda = 1$, respectively, collapse as $p \to 1 - p_c$.

This is the behavior displayed in Lemma 5.1 and Corollary 5.1, which hold for general percolation models of stationary two phase random media with $m(0) = m(p, 0)$ and $w(0) = w(p, 0)$. In this way, the spectral measures μ and α truly are independent of the material contrast ratio, and are independent of how we define it. For example, we have focused on the contrast ratio $h = \sigma_1/\sigma_2$ and defined an insulator–conductor system by letting $\sigma_1 \to 0$, resulting in critical behavior (the formation of a delta component in μ at $\lambda = 1$ with weight $m(p, 0)$) as p surpasses p_c, where $p = \langle \chi_2 \rangle$ (see Lemma 5.1). We could have instead focused on $z = \sigma_2/\sigma_1$ and defined an insulator–conductor system by letting $\sigma_2 \to 0$, resulting in critical behavior (the formation of a delta component in α at $\lambda = 1$ with weight $w(p, 0)$) as p surpasses $1 - p_c$ (see Corollary 5.1). Lemma 5.1 and Corollary 5.1 demonstrate, through spectral means, the equivalence of these two systems. Moreover, these lemmas rigorously prove, for general percolation models of two phase stationary random media in lattice and continuum settings, that the onset of critical behavior in transport is identified with the formation of delta components in μ and α at $\lambda = 0, 1$ *precisely* at $p = p_c$ and $p = 1 - p_c$.

For bond lattice systems with a finite number n of bonds, the differential equations in (7) become difference equations (Kirchoff's laws).[23] Consequently, the operators M_j, $j = 1, 2$, are given by $N \times N$ matrices,[23,28] and the spectral measure $\alpha_{ik}(d\lambda)$ of the matrix M_2 is given by a sum of "Dirac δ functions,"

$$\alpha_{ik}(d\lambda) = \left[\sum_{j=1}^{N} m_j \delta_{\lambda_j}(d\lambda)\right] d\lambda = \alpha_{ik}(\lambda) d\lambda, \tag{78}$$

where $\delta_{\lambda_j}(d\lambda)$ is the Dirac delta measure concentrated at λ_j, $m_j = \langle \vec{e}_i^{\,T} [\vec{v}_j \vec{v}_j^{\,T}] \vec{e}_k \rangle$, $\vec{e}_k$ is a standard basis vector on the lattice, for some $k = 1, \ldots, d$, and λ_j and $\vec{v}_j$ are the eigenvalues and eigenvectors of M_2, respectively.[28] The associated Stieltjes transformation of the measure in (16) is given by the sum $G(t) = \sum_{j=1}^{n} m_j/(t - \lambda_j)$, and $\alpha_{ik}(\lambda)$ in Eq. (78) is called "*the spectral function*," which is defined only pointwise on the set of eigenvalues $\{\lambda_j\}$.

In Figure 1 we give a graphical representation of the spectral measure for finite 2–d and 3–d RRN. It displays linearly connected peaks of histograms with bin sizes on the order of 10^{-2}. The apparent smoothness of the spectral function graphs in this figure is due to the large number ($\sim 10^6$) of eigenvalues and eigenvectors calculated, and ensemble averaged. Consistent with the isotropy of the RRN, the diagonal components α_{kk} are virtually identical, positive measures of equal mass $1/d$, while the α_{ik}, $i \neq k$, are signed measures of zero mass, up to numerical accuracy and finite size effects. The α_{kk} do not have mass p, as the eigenvectors are normalized in the l_2 inner product, not that weighted by the characteristic function χ_2, like in the general theory. In Figure 1 the mass of the measures has been scaled to the volume fraction p.

We now provide an analytical proof for the existence of spectral gaps in α_{ik} about the spectral endpoints $\lambda = 0, 1$ for arbitrary, finite lattice systems. More specifically, for $p \ll 1$, $\inf\{\Sigma_\alpha\} > 0$ and $\sup\{\Sigma_\alpha\} < 1$. We focus on $M_2 = \chi_2 \Gamma \chi_2$ and α_{ik}, as our results extend to $M_1 = \chi_1 \Gamma \chi_1$ and μ_{ik} by symmetry. In this lattice setting, Γ is a real symmetric projection matrix which can be diagonalized: $\Gamma = QDQ^T$, where $D = \mathrm{diag}(1,\ldots,1,0,\ldots,0)$ is a diagonal matrix of L ones and $N - L$ zeros, $0 < L < N$ when $N \gg 1$, and Q is a real orthogonal matrix with columns $\vec{q}_i$, $i = 1, \ldots, N$, which are the eigenvectors of Γ. More specifically,

$$\Gamma_{ij} = (\vec{q}_i \cdot \vec{q}_j)_L,$$

where $(\vec{q}_i \cdot \vec{q}_j)_L = \sum_{l=1}^{L} (\vec{q}_i)_l (\vec{q}_j)_l$, and $(\vec{q}_i)_l$ is the lth component of the vector $\vec{q}_i \in \mathbb{R}^N$. Here, we consider the non-degenerate case $L < N$.

In the matrix case, the action of χ_2 is given by that of a square diagonal matrix of zeros and ones.[28] The action of χ_2 in the matrix $\chi_2 \Gamma \chi_2$ introduces a row and column of zeros in the matrix Γ, corresponding to every diagonal entry of χ_2 with value 0. When there is only one σ_2 inclusion ($p = 1/n$) located at the jth bond, χ_2 has all zero entries except at the jth diagonal: $\chi_2 = \mathrm{diag}(0, \cdots, 0, 1, 0, \cdots, 0) = \mathrm{diag}(\vec{\mathrm{v}}_j)$. Therefore, the only non-trivial eigenvalue is given by $\hat{\lambda}_0 = (\vec{q}_j \cdot \vec{q}_j)_L = \sum_{l=1}^{L} (\vec{q}_j)_l^2 = 1 - \sum_{l=L+1}^{N} (\vec{q}_j)_l^2$, with eigenvector $\vec{\mathrm{v}}_j$ and weight $m_0 = \vec{e}_i^{\,T} \vec{\mathrm{v}}_j \vec{\mathrm{v}}_j^T \vec{e}_k$. This implies that there is a gap at $\lambda = 0$, $\theta_0 = \sum_{l=1}^{L} (\vec{q}_j)_l^2 > 0$, and a gap at $\lambda = 1$, $\theta_1 = \sum_{l=L+1}^{N} (\vec{q}_j)_l^2$

> 0. It is clear that these bounds hold for all $\omega \in \Omega$ such that $p = 1/n$ when $L < N$. We have already mentioned that the eigenvalues of M_2 are restricted to the set $\{0, 1\}$ when $p = 1(\chi_2 \equiv I_N)$. Therefore, there exists $0 < p_0 < 1$ such that, for all $p \geq p_0$, there exists a $\omega \in \Omega$ such that $\theta_0(\omega) = 0$ and/or $\theta_1(\omega) = 0$. This concludes our proof.

VII. CONCLUDING REMARKS

We have constructed a mathematical framework which unifies the critical theory of transport for binary composite media, in continuum and lattice settings. We have focused on critical transitions exhibited by the effective complex conductivity $\sigma^* = \sigma_2 m(h) = \sigma_1 w(z)$, as the symmetries underlying this framework extend our results to that regarding the effective complex resistivity $\rho^* = \tilde{m}(h)/\sigma_1 = \tilde{w}(z)/\sigma_2$. We have shown in Sec. V that critical transitions in transport properties are, in general, characterized by the formation of delta components in the underlying spectral measures at the spectral endpoints. Moreover, for percolation models, we have shown that the onset of the critical transition (the formation of these delta components) occurs *precisely* at the percolation threshold p_c and $1 - p_c$.

The mathematical transport properties of such systems, displayed in Secs. III and IV, hold for general two-component stationary random media in lattice and continuum settings.[29] While the critical exponent scaling relations and the various transport properties, displayed in Lemmas 6.2–6.11, hold for percolation models of the composites class $\mathcal{B}_{n_\phi}$ and $\mathcal{B}_{n_{\hat{\phi}}}$ for $p > p_c$ and $p < p_c$, respectively (see Definition 4.1). Under the condition that $n_\phi, n_{\hat{\phi}} \geq 1$, i.e., that $m(p, 0)$ and $\chi(p, 0)$ in (46) exist for $p > p_c$, and $w(p, z(0))$ and $\hat{\chi}(p, z(0))$ in (47) exist for $p < p_c$, we linked the two sets of critical exponents in (46) and (47). We showed that, for percolation models like EMT where $n_\phi, n_{\hat{\phi}} = \infty$, they are all determined by only three critical exponents, and are determined by only two critical exponents under the symmetry condition of Lemma 6.11. This type of critical behavior has been studied before for the lattice,[7,15,19] and alternate methods have shown that $\Delta = s + t$, $\delta = (s + t)/t$, and $\gamma = s$. These are precisely the relations that we have shown to hold for lattice and continuum percolation models of this class, under these conditions. The EMT percolation model satisfies these conditions, however, there is no apparent mathematical necessity for these conditions to hold, in general. Although they lead to the well known two dimensional duality relation $s = t$ for the lattice.[7,15,19]

Numerical and analytical work on the sequence of critical exponents $\tilde{\psi}(q)$ for the moments of the current distribution in RRN, e.g., Refs. 9, 18, and 53 has shown that this sequence exhibits nonlinear dependence in q, or multifractal behavior. We proved in Lemma 6.7 that the exponents γ_n have linear dependence in n for all $1 \leq n \leq n_{\hat{\phi}} \leq n_\phi$. This is consistent with the absence of multifractal behavior for percolation models of class $\mathcal{B}_\infty$, such as EMT. However, for percolation models with $n_{\hat{\phi}}, n_\phi < \infty$, multifractal behavior is not ruled out by Lemma 6.7. It is interesting, though, that the $\tilde{\psi}(q)$ satisfy Baker's inequalities (52).[9]

As in EMT, our general scaling relations involving $|h|$ are independent of the limiting path as $h \to 0$. This represents an alternative to the results of other workers[7,15,19] who have used heuristic scaling forms as a starting point. For our critical theory the starting point is Eq. (16), which displays *exact* formulas for infinite systems.[24]

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[1] G. A. Baker, "Some rigorous inequalities satisfied by the ferromagnetic Ising model in a magnetic field," Phys. Rev. Lett. **20**, 990–992 (1968).
[2] G. A. Baker, *Quantitative Theory of Critical Phenomena* (Academic, New York, 1990).
[3] G. A. Baker and D. S. Gaunt, "Low-temperature critical exponents from high-temperature series: The Ising model," Phys. Rev. B **1**(3), 1184–1210 (1970).
[4] D. J. Bergman, "The dielectric constant of a composite material – A problem in classical physics," Phys. Rep. C **43**(9), 377–407 (1978).

[5] D. J. Bergman, "Exactly solvable microscopic geometries and rigorous bounds for the complex dielectric constant of a two-component composite material," Phys. Rev. Lett. **44**, 1285–1287 (1980).
[6] D. J. Bergman, "Rigorous bounds for the complex dielectric constant of a two–component composite," Ann. Phys. **138**, 78–114 (1982).
[7] D. J. Bergman and D. Stroud, "Physical properties of macroscopically inhomogeneous media," Solid State Phys. **46**, 147–269 (1992).
[8] L. Berlyand and K. Golden, "Exact result for the effective conductivity of a continuum percolation model," Phys. Rev. B **50**, 2114–2117 (1994).
[9] R. Blumenfeld, Y. Meir, A. Aharony, and A. B. Harris, "Resistance fluctuations in randomly diluted networks," Phys. Rev. B **35**, 3524–3535 (1987).
[10] T. Bourbie and B. Zinszner, "Hydraulic and acoustic properties as a function of porosity in Fontainebleau sandstone," J. Geophys. Res. **90**(B13), 11524–11532, doi:10.1029/JB090iB13p11524 (1985).
[11] S. R. Broadbent and J. M. Hammersley, "Percolation processes I. Crystals and mazes," Proc. Cambridge Philos. Soc. **53**, 629–641 (1957).
[12] O. Bruno, "The effective conductivity of strongly heterogeneous composites," Proc. R. Soc. London, Ser. A **433**, 353–381 (1991).
[13] Y. Chen and C. A. Schuh, "Percolation of diffusional creep: A new universality class," Phys. Rev. Lett. **98**, 035701–1–4 (2007).
[14] K. Christensen and N. R. Moloney, *Complexity and Criticality* (Imperial College, London, 2005).
[15] J. P. Clerc, G. Giraud, J. M. Laugier, and J. M. Luck, "The electrical conductivity of binary disordered systems, percolation clusters, fractals, and related models," Adv. Phys. **39**(3), 191–309 (1990).
[16] A. R. Day and M. F. Thorpe, "The spectral function of random resistor networks," J. Phys.: Condens. Matter **8**, 4389–4409 (1996).
[17] A. R. Day and M. F. Thorpe, "The spectral function of composites: the inverse problem," J. Phys.: Condens. Matter **11**, 2551–2568 (1999).
[18] E. Deuring, R. Blumenfeld, D. J. Bergman, A. Aharony, and M. Murat, "Current distributions in a two-dimensional random-resistor network," J. Stat. Phys. **3267**, 113–121 (1992).
[19] A. L. Efros and B. I. Shklovskii, "Critical behavior of conductivity and dielectric constant near the metal–non–metal transition threshold," Phys. Status Solidi B **76**(2), 475–485 (1976).
[20] I. G. Enting, "A lattice statistics model for the age distribution of air bubbles in polar ice," Nature (London) **315**, 654–655 (1985).
[21] G. B. Folland, *Introduction to Partial Differential Equations* (Princeton University Press, Princeton, NJ, 1995).
[22] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, (Wiley, New York, 1999).
[23] K. M. Golden, "Statistical mechanics of conducting phase transitions," J. Math. Phys. **36**(10), 5627–5642 (1995).
[24] K. M. Golden, "Critical behavior of transport in lattice and continuum percolation models," Phys. Rev. Lett. **78**(20), 3935–3938 (1997).
[25] K. M. Golden, "Climate change and the mathematics of transport in sea ice," Not. Am. Math. Soc. **56**(5), 562–584 (2009).
[26] K. M. Golden, S. F. Ackley, and V. I. Lytle, "The percolation phase transition in sea ice," Science **282**, 2238–2241 (1998).
[27] K. M. Golden, H. Eicken, A. L. Heaton, J. Miner, D. Pringle, and J. Zhu, "Thermal evolution of permeability and microstructure in sea ice," Geophys. Res. Lett. **34**, L16501, doi:10.1029/2007GL030447 (2007).
[28] K. M. Golden, N. B. Murphy, and E. Cherkaev, "Spectral analysis and connectivity of porous microstructures in bone," J. Biomech. **44**(2), 337–344 (2011).
[29] K. M. Golden and G. Papanicolaou, "Bounds for effective parameters of heterogeneous media by analytic continuation," Commun. Math. Phys. **90**, 473–491 (1983).
[30] D. J. Griffiths, *Introduction to Electrodynamics* (Prentice Hall, Upper Saddle River, NJ, 1999).
[31] A. Gully, L. G. E. Backstrom, H. Eicken, and K. M. Golden, "Complex bounds and microstructural recovery from measurements of sea ice permittivity," Physica B **394**(2), 357–362 (2007).
[32] B. I. Halperin, S. Feng, and P. N. Sen, "Differences between lattice and continuum percolation transport exponents," Phys. Rev. Lett. **54**(22), 2391–2394 (1985).
[33] P. Henrici, *Applied and Computational Complex Analysis* Vol. 3 (Wiley, New York, 1974).
[34] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999).
[35] T. Jonckheere and J. M. Luck, "Dielectric resonances of binary random networks," J. Phys. A **31**, 3687–3717 (1998).
[36] R. P. Kusy and D. T. Turner, "Electrical resistivity of a polymeric insulator containing segregated metallic particles," Nature (London) **229**, 58–59 (1971).
[37] A. V. Kyrylyuk and P. van der Schoot, "Continuum percolation of carbon nanotubes in polymeric and colloidal media," Proc. Natl. Acad. Sci. U.S.A. **105**, 8221–8226 (2008).
[38] T. D. Lee and C. N. Yang, "Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model," Phys. Rev. **87**, 410–419 (1952).
[39] G. W. Milton, "Bounds on the complex dielectric constant of a composite material," Appl. Phys. Lett. **37**, 300–302 (1980).
[40] G. W. Milton, *Theory of Composites* (Cambridge University Press, Cambridge, 2002).
[41] C. Orum, E. Cherkaev, and K. M. Golden, "Recovery of inclusion separations in strongly heterogeneous composites from effective property measurements," Proc. R. Soc. London, Ser. A **468**(2139), 784–809 (2012).
[42] D. J. Pringle, J. E. Miner, H. Eicken, and K. M. Golden, "Pore-space percolation in sea ice single crystals," J. Geophys. Res., [Oceans] **114**, C12017, doi:10.1029/2008JC005145 (2009).
[43] M. C. Reed and B. Simon, *Functional Analysis* (Academic, San Diego, CA, 1980).

[44] H. S. Robertson, *Statistical Thermophysics* (Prentice Hall, Englewood Cliffs, NJ, 1993).
[45] W. Rudin, *Real and Complex Analysis* (McGraw-Hill, New York, 1987).
[46] D. Ruelle, *Statistical Mechanics: Rigorous Results* (Benjamin, New York, 1969).
[47] D. Ruelle, "Extension of the Lee-Yang circle theorem," Phys. Rev. Lett. **26**(6), 303–304 (1971).
[48] D. Ruelle, "Characterization of Lee–Yang polynomials," Ann. Math. **171**, 589–603 (2010).
[49] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields* (Springer, New York, 1997).
[50] N. Sasaki, H. Yamamura, and N. Matsushima, "Is there a relation between bone strength and percolation?," J. Theor. Biol. **122**(1), 25–31 (1986).
[51] P. Sheng and R. V. Kohn, "Geometric effects in continuous-media percolation," Phys. Rev. B **26**, 1331–1335 (1982).
[52] B. I. Shklovskii and A. L. Efros, *Electronic Properties of Doped Semiconductors* (Springer-Verlag, New York, 1984).
[53] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor & Francis, London, 1992).
[54] S. Torquato, *Random Heterogeneous Materials: Microstructure and Macroscopic Properties* (Springer-Verlag, New York, 2002).
[55] C. N. Yang and T. D. Lee, "Statistical theory of equations of state and phase transitions. I. Theory of condensation," Phys. Rev. **87**, 404–409 (1952).