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Uniqueness theorems for displacement fields with locally finite energy in linear elastostatics*

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ABSTRACT

Uniqueness theorems are proved for the fundamental boundary value problems of linear elastostatics in bodies of arbitrary shape. The displacement fields are required to have finite strain energy in bounded portions of the bodies and satisfy the principle of virtual work. For bounded bodies, the total strain energy is finite and uniqueness is proved without additional hypotheses. In particular, no restrictions other than the energy condition are placed on the field singularities that may occur at sharp edges and corners. For unbounded bodies, uniqueness can be proved as in the bounded case if the total strain energy is finite. Sufficient conditions for this are shown to be the finiteness of the strain energy in bounded portions of the body together with the growth restriction

$$\int_{\Omega_{r,\delta}} u_i(x)u_i(x)\,dx = 0(r),\ r\to\infty$$

on the displacement field u_i, where $\Omega_{r,\delta}$ is the portion of the body that lies between concentric spheres with radii r and $r+\delta$ and $\delta>0$.

Introduction

The classical theory of linear elastostatics. The fundamental problem of linear elastostatics is to determine the equilibrium displacement field that is produced in an elastic body of known shape and composition by the action of known body forces and surfaces tractions or displacements. In the classical formulation of the theory the displacements and stresses are required to be differentiable and satisfy the differential equations of equilibrium in the interior of the body and to be continuous and satisfy the prescribed surface traction or displacement conditions on the boundary. This boundary value problem has a history that begins with A. L. Cauchy's discovery of the equilibrium equations in 1822; see reference [18, p. 8]. The uniqueness of classical solutions for bounded bodies with smooth surfaces was proved by G.

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Kirchhoff in 1859 [12]. General existence theorems for classical solutions were first proved during the period 1906–1908 by integral equation methods. The principal contributors were I. Fredholm [6], G. Lauricella [17], R. Marcolongo [19], A. Korn [15, 16] and T. Boggio [2, 3]. More recently G. Fichera has proved the existence of classical solutions in bounded bodies with smooth boundaries by the methods of modern functional analysis [4, 5]. Thus the theory of the classical boundary value problems of linear elastostatics is essentially complete.

The need for a more general theory. Unfortunately the classical theory described above provides an inadequate foundation for the analysis of most of the problems studied by applied scientists in their applications of linear elastostatics. Examination of any of the numerous books on theoretical elasticity, beginning with the classical treatise of A. E. H. Love [18], reveals that most of the problems treated in them involve unbounded bodies, such as infinite plate or bars, and/or bodies having sharp edges or corners. Moreover, the stress fields are known to have singularities at re-entrant edges and corners. Examples of these difficulties can be found in the theory of cracks; see I. N. Sneddon and M. Lowengrub [22]. It is sometimes argued that the classical theory is a sufficient foundation for applications because real bodies are always bounded and boundaries with sharp edges and corners can be approximated by smooth ones. However, although this procedure simplifies the problems from the viewpoint of the classical theory, it makes them inaccessible to techniques such as separation of variables and integral transform methods that are used by applied scientists. Thus the real issue is whether a mathematical theory can be devised that is sufficiently general to provide a foundation for the analysis of the singular problems that are actually studied by applied scientists. The purpose of this paper is to provide the beginnings of such a theory comprising a formulation of the elastostatic boundary value problems that is applicable to bodies of arbitrary shape and corresponding uniqueness theorems.

Remarks on the formulation of boundary value problems. A "formulation" of a boundary value problem is a definition of the class of functions in which solutions are to be sought. The classical formulation of the elastostatic boundary value problem was described above. Many other formulations are possible. For example, the continuity conditions may be replaced at some or all boundary points by boundedness or integrability conditions, the equilibrium equations may be required to hold in a weak sense, etc. In principle, any formulation is acceptable if there is an existence theorem, stating that there is at least one solution in the class, and a uniqueness theorem, stating that there is at most one solution in the class. In practice the choice of a solution class turns on technical considerations. The proof of an existence theorem is facilitated by choosing a large solution class but uniqueness is lost if the class is too large. The proof of a uniqueness theorem is facilitated by choosing a small solution class but existence is lost if the class is too small. For example, Kirchhoff's theorem on the uniqueness of classical solutions of the elastostatic boundary value problem can be proved for bodies having re-entrant sharp edges but in this case no classical solution exists.

The role of existence and uniqueness theorems. A pure existence theorem for a boundary value problem demonstrates that the properties chosen to define the solution class are not contradictory; i.e., there are functions with these properties. In the presence of an existence theorem a uniqueness theorem shows that the defining properties of the solution class characterize the solution completely. However, a uniqueness theorem can be even more valuable when no general existence theorem is known. In such cases it may still be possible in certain instances, corresponding to special choices of the boundary or data, to construct a solution in the chosen solution class. A uniqueness theorem then shows that the solution is the correct one. An interesting example of this occurred in the theory of the diffraction of electromagnetic waves by a perfectly conducting circular disk. In 1948 J. Meixner [20] proved a uniqueness theorem for this problem and used it to show that a solution that had been published in 1927 was incorrect. Of course, in the absence of a general existence theorem it is desirable to prove uniqueness in as large a solution class as possible since this facilitates application of the uniqueness theorem in specific instances.

The boundedness question for the displacement fields. Linear elastostatics is an approximation that is valid for small displacements. If the displacements are bounded then by suitable scaling they may be made arbitrarily small. Hence it is natural to make boundedness of the displacements a defining property of the solution class. Indeed, this property has often been employed in constructing solutions of particular problems. It has also been used by J. K. Knowles and T. A. Pucik [14] in the formulation and proof of a general uniqueness theorem for plane crack problems. However, it is shown in this paper that uniqueness holds in the larger class of solutions with locally finite energy, without boundedness conditions. This result shows that the boundedness hypothesis is redundant and the boundedness property, in instances where it holds, must be derivable from the other hypotheses.

Displacement fields with locally finite energy. In this paper it is taken as a fundamental principle that equilibrium displacement fields in elastic bodies must have finite strain energy in bounded portions of the bodies. Such displacement fields will be called displacement fields with locally finite energy (or, for brevity, fields wLFE). The equilibrium displacement field corresponding to prescribed body forces will be characterized among all fields wLFE, by the principle of virtual work. The class of displacement fields that obey these two principles will be called the solutions with locally finite energy (for brevity, solutions wLFE) of the elastostatic boundary value problems. The principal results of this paper are uniqueness theorems for this class of solutions. In particular, the uniqueness of solutions wLFE in bounded bodies is proved without additional hypotheses concerning the boundary or the displacement field. The uniqueness of solutions wLFE in unbounded bodies is proved under a growth restriction on the behavior of the stress or displacement fields at infinity. Moreover, it is shown by examples that a growth restriction is necessary for uniqueness.

The remainder of the paper is organized as follows. The class of displacement fields wLFE is defined in § 1. § 2 contains the definition of the class of solutions wLFE in homogeneous elastic bodies of arbitrary shape, subject to prescribed surface tractions, prescribed body forces and prescribed displacements or stresses at infinity. The regularity properties of solutions wLFE are also discussed in this section. § 3 presents the uniqueness theorems for solutions wLFE of problems with prescribed surface tractions. In § 4 the methods and results of § 3 are extended to the other classical boundary value problems of linear elastostatics including problems with prescribed surface displacements, problems with mixed boundary conditions, problems for inhomogeneous elastic bodies and n-dimensional generalizations. § 5 contains a discussion of related literature.

1. Displacement fields with locally finite energy

A fixed system of Cartesian coordinates is used throughout the paper and points of Euclidean space are identified with their coordinate triples $(x_1, x_2, x_3) = x \in R^3$. With this convention each elastic body in space is associated with a domain (open connected set) $\Omega \subset R^3$ that describes the set of interior points of the body. The closure and boundary of Ω are denoted by $\bar{\Omega}$ and $\partial\Omega = \bar{\Omega} - \Omega$, respectively. The notation of Cartesian tensor analysis [11] is used to describe the physical variables associated with elastic bodies. In particular, tensors of various orders are denoted by subscripts and the summation convention is used.

The fundamental unknown of elastostatic boundary value problems is the displacement field. It is denoted below by $u_i = u_i(x)$. The notation $u_{i,j} = \partial u_i/\partial x_j$ is used for the covariant derivative of u_i. The strain tensor field $e_{ij}(u)$ associated with u_i is defined by the differential operator

$$e_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.1}$$

It is assumed, following G. Green [7 and 18, pp. 11–12 and 95–99], that for quasi-static isothermal small deformations of an elastic body there is a positive definite quadratic function of e_{ij},

$$w = \tfrac{1}{2} c_{ijkl} e_{ij} e_{kl}, \tag{1.2}$$

such that for all $K \subset \Omega$

$$W_K = \frac{1}{2}\int_K c_{ijkl} e_{ij}(u) e_{kl}(u)\, dx \tag{1.3}$$

is the strain energy of the displacement field u_i in the set K. The positivity assumption means that

$$c_{ijkl} e_{ij} e_{kl} > 0 \quad \text{for all} \quad e_{ij} = e_{ji} \neq 0 \tag{1.4}$$

The stress-strain tensor c_{ijkl} is uniquely determined by w if the natural symmetries

$$c_{ijkl} = c_{jikl} = c_{klji} \tag{1.5}$$

are assumed. The stress tensor field $\sigma_{ij}(u)$ associated with u_i is given by the differential operator

$$\sigma_{ij}(u) = c_{ijkl}e_{kl}(u) \tag{1.6}$$

The positive definiteness of w implies that $\sigma_{ij} = c_{ijkl}e_{kl}$ has a unique solution $e_{ij} = \gamma_{ijkl}\sigma_{kl}$ and $w = \frac{1}{2}\sigma_{ij}e_{ij} = \frac{1}{2}\gamma_{ijkl}\sigma_{ij}\sigma_{kl}$. In particular,

$$W_K = \frac{1}{2}\int_K \sigma_{ij}(u)e_{ij}(u)\,dx = \frac{1}{2}\int_K \gamma_{ijkl}\sigma_{ij}(u)\sigma_{kl}(u)\,dx \tag{1.7}$$

is a functional of $\sigma_{ij}(u)$ alone. A body is homogeneous if and only if c_{ijkl} is constant in Ω. It is isotropic if and only if [11, 18]

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{1.8}$$

where λ and μ are scalars such that $\mu > 0$, $3\lambda + 2\mu > 0$. The results in §2 and §3 are formulated for the case of homogeneous anisotropic bodies. In §4 it is shown that the uniqueness theorems hold for the more general case of inhomogeneous anisotropic media with bounded uniformly positive definite stress-strain tensor. This means that the components $c_{ijkl}(x)$ are Lebesgue measurable and there exist positive constants c_0 and $c_1 \geq c_0$ such that

$$c_0 e_{ij}e_{ij} \leq c_{ijkl}(x)e_{ij}e_{kl} \leq c_1 e_{ij}e_{ij} \quad \text{for all} \quad x \in \Omega \tag{1.9}$$

and all $e_{ij} = e_{ji}$.

The most general uniqueness theorems for solutions wLFE will be obtained by making the class of displacement fields wLFE as large as possible subject to the LFE condition. Hence it is natural to define the energy integrals $W_K(u)$ to be Lebesgue integrals and to interpret the differential operators e_{ij} in the distribution-theoretic sense. It can be shown that this choice has the additional advantage that the set of displacement fields wLFE is a complete space in the sense of convergence in energy on bounded sets. It was by using such complete function spaces that Fichera proved the existence of solutions of the elastostatic boundary value problems in bounded domains.

In the remainder of this section several function spaces are defined that are needed for the formulation and proof of the uniqueness theorems. In the definitions $\Omega \subset R^3$ denotes an arbitrary domain.

The definitions are based on the Lebesgue space

$$L_2(\Omega) = \left\{ u : \Omega \to R \,|\, u(x) \text{ is } L\text{-measurable}, \int_\Omega u(x)^2\,dx < \infty \right\} \tag{1.10}$$

and the associated spaces

$$L_2^{loc}(\Omega) = \{u : \Omega \to R \,|\, u \in L_2(K) \text{ for every bounded measurable } K \subset \Omega\} \tag{1.11}$$

$$L_2^{int}(\Omega) = \{u : \Omega \to R \mid u \in L_2(C) \text{ for every compact } C \subset \Omega\} \tag{1.12}$$

and

$$L_2^{com}(\Omega) = L_2(\Omega) \cap \{u \mid u(x) \text{ is equivalent to 0 outside a bounded set}\} \tag{1.13}$$

It is clear that $L_2^{com}(\Omega) \subset L_2(\Omega) \subset L_2^{loc}(\Omega) \subset L_2^{int}(\Omega)$. Moreover, $L_2^{com}(\Omega) = L_2(\Omega) = L_2^{loc}(\Omega)$ if and only if Ω is bounded. Note that the condition $u \in L_2^{loc}(\Omega)$ restricts the behavior of u near $\partial\Omega$ because the sets K in (1.11) can be any bounded open subsets of Ω. The condition $u \in L_2^{int}(\Omega)$ is weaker because it does not restrict the behavior of u near $\partial\Omega$. All of the function spaces used below are spaces of tensor fields on Ω whose components lie in certain linear subspaces of $L_2^{int}(\Omega)$.

The space $L_2^{int}(\Omega)$ may be interpreted as a linear subspace of L. Schwartz's space $\mathscr{D}'(\Omega)$ of all distributions on Ω [21]. Thus functions $u \in L_2^{int}(\Omega)$ have derivatives of all orders in $\mathscr{D}'(\Omega)$ and if

$$A = \sum_{0 \le |\alpha| \le m} A_\alpha \partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3} \tag{1.14}$$

(where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$) is a partial differential operator with constant coefficients then $Au \in \mathscr{D}'(\Omega)$. The notation $Au \in L_2^{int}(\Omega)$ (resp. $L_2^{loc}(\Omega)$, $L_2(\Omega)$, $L_2^{com}(\Omega)$, etc.) will be interpreted to mean that the distribution Au is in the subspace $L_2^{int}(\Omega)$ (resp. $L_2^{loc}(\Omega)$, $L_2(\Omega)$, $L_2^{com}(\Omega)$, etc.). If $A_1, A_2, \ldots, A_n$ is a set of partial differential operators with constant coefficients the following notation will be used.

$$L_2(A_1, A_2, \ldots, A_n; \Omega) = L_2(\Omega) \cap \{u \mid A_j u \in L_2(\Omega), j = 1, 2, \ldots, n\} \tag{1.15}$$

$$L_2^{loc}(A_1, A_2, \ldots, A_n; \Omega) = L_2^{loc}(\Omega) \cap \{u \mid A_j u \in L_2^{loc}(\Omega), j = 1, 2, \ldots, n\} \tag{1.16}$$

$$L_2^{int}(A_1, A_2, \ldots, A_n; \Omega) = L_2^{int}(\Omega) \cap \{u \mid A_j u \in L_2^{int}(\Omega), j = 1, 2, \ldots, n\} \tag{1.17}$$

$$L_2^{com}(A_1, A_2, \ldots, A_n; \Omega) = L_2^{com}(\Omega) \cap L_2(A_1, A_2, \ldots, A_n; \Omega) \tag{1.18}$$

In particular, if $\{A_1, A_2, \ldots, A_n\} = \{\partial^{|\alpha|}/\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3} \mid 0 \le |\alpha| \le m\}$ the following notation will be used.

$$L_2^m(\Omega) = L_2(A_1, A_2, \ldots, A_n; \Omega) \tag{1.19}$$

$$L_2^{m, loc}(\Omega) = L_2^{loc}(A_1, A_2, \ldots, A_n; \Omega) \tag{1.20}$$

$$L_2^{m, int}(\Omega) = L_2^{int}(A_1, A_2, \ldots, A_n; \Omega) \tag{1.21}$$

$$L_2^{m, com}(\Omega) = L_2^{com}(A_1, A_2, \ldots, A_n; \Omega) \tag{1.22}$$

Notations such as $u_i \in L_2^{loc}(\Omega)$, $e_{ij} \in L_2(\Omega)$, etc. will be interpreted to mean that each component of the tensor field is in the indicated space. With this convention the classes of displacement fields wFE (with finite energy) and wLFE may be defined as follows.

DEFINITION. A vector field u_i on Ω is said to be a displacement field wFE if and only if it is in the function space

$$E(\Omega) = \{u \mid u_i \in L_2^{loc}(\Omega), e_{ij}(u) \in L_2(\Omega)\} \tag{1.23}$$

Similarly, u_i is said to be a displacement field wLFE if and only if it is in the function space

$$E^{loc}(\Omega)=\{u \mid u_i \in L_2^{loc}(\Omega),\ e_{ij}(u)\in L_2^{loc}(\Omega)\} \tag{1.24}$$

Note that $E^{loc}(\Omega)=E(\Omega)$ if and only if Ω is bounded.

The terminology used in the definition is justified by the observation that if the stress-strain tensor satisfies (1.9) then $e_{ij}(u)\in L_2(\Omega)$ implies $\sigma_{ij}(u)\in L_2(\Omega)$ and hence $u\in E(\Omega)$ implies

$$W_\Omega=\frac{1}{2}\int_\Omega \sigma_{ij}(u)e_{ij}(u)\,dx<\infty \tag{1.25}$$

Similarly, if (1.9) holds then $e_{ij}(u)\in L_2^{loc}(\Omega)$ implies $\sigma_{ij}(u)\in L_2^{loc}(\Omega)$ and hence $u\in E^{loc}(\Omega)$ implies

$$W_K=\frac{1}{2}\int_K \sigma_{ij}(u)e_{ij}(u)\,dx<\infty \tag{1.26}$$

for all bounded measurable sets $K\subset\Omega$.

Each of the function spaces defined above is a complete space with respect to a suitable topology. Several examples of this will be indicated. It is well known that $L_2(\Omega)$ is a Hilbert space with scalar product

$$(u,v)=\int_\Omega u(x)v(x)\,dx \tag{1.27}$$

Similarly, $E(\Omega)$ and $E^{loc}(\Omega)$ are Fréchet spaces [28] with respect to the families of semi-norms defined by

$$\rho_{K,E}(u)=\left(\int_K u_i(x)u_i(x)\,dx+\int_\Omega \sigma_{ij}(u)\sigma_{ij}(u)\,dx\right)^{1/2} \tag{1.28}$$

and

$$\rho_{K,E}^{loc}(u)=\left(\int_K \{u_i(x)u_i(x)+\sigma_{ij}(u)\sigma_{ij}(u)\}\,dx\right)^{1/2} \tag{1.29}$$

respectively, where K is any bounded measurable subset of Ω. In particular, if Ω is bounded then $E^{loc}(\Omega)=E(\Omega)$ is a Hilbert space. These completeness results play no role in the uniqueness theorems given below. However, they are essential for the validity of existence theorems for solutions wLFE. This is evident from the proofs of Fichera's existence theorems for bounded bodies.

In the definition of $E^{loc}(\Omega)$ the operators $e_{ij}(u)$ defined by (1.1) are interpreted in the distribution-theoretic sense. Hence the condition $u\in E^{loc}(\Omega)$ does not necessarily imply that the individual derivatives $u_{i,j}\in L_2^{loc}(\Omega)$. However, it is known that if $u\in E^{loc}(\Omega)$ then $u_{i,j}\in L_2(C)$ for every compact set $C\subset\Omega$. This is a consequence of Korn's inequality in the form

$$\|u_{i,j}\|^2_{L_2(C)}\le\gamma\left(\sum_{i=1}^{3}\|u_i\|^2_{L_2(K)}+\sum_{i,j=1}^{3}\|e_{ij}(u)\|^2_{L_2(K)}\right) \tag{1.30}$$

which is valid for all $u \in E^{loc}(\Omega)$, all bounded open sets $K \subset \Omega$ and all compact sets $C \subset K$ with a constant $\gamma = \gamma(C, K)$. This result can be derived from the version of Korn's inequality due to J. Gobert [8]. Moreover if Ω has the cone property [1, 9] then one may take $C = K$ in (1.30). Hence in this case

$$u \in E^{loc}(\Omega) \Rightarrow u_i \in L_2^{1,loc}(\Omega) \tag{1.31}$$

In particular, for domains that are bounded and have the cone property

$$u \in E(\Omega) \Rightarrow u_i \in L_2^1(\Omega) \tag{1.32}$$

2. Equilibrium problems with prescribed surface tractions

In this section elastostatic equilibrium problems are formulated, and regularity properties of the solutions are discussed, for homogeneous anisotropic elastic bodies of arbitrary shape that are subject to prescribed body forces, prescribed surface tractions and, in the case of unbounded bodies, prescribed displacements or stresses at infinity. The cases of prescribed body forces F_i, zero surface tractions and zero displacements or stresses at infinity are discussed first.

The principle of virtual work. Let $\Omega \subset R^3$ be an arbitrary domain and let $u \in E^{loc}(\Omega)$ be the equilibrium displacement field wLFE corresponding to body forces $F_i \in L_2^{com}(\Omega)$ and zero surface tractions. Imagine that the equilibrium is disturbed slightly by changing u_i to $u_i + v_i$ where v_i is a field wFE from the set

$$E^{com}(\Omega) = E(\Omega) \cap \{v \mid e_{ij}(v) \in L_2^{com}(\Omega)\} \tag{2.1}$$

Let $K \subset \Omega$ be a bounded measurable set such that $e_{ij}(v)$ is equivalent to zero in $\Omega - K$. Then $W_K(\sigma(u))$ and $W_K(\sigma(u+v))$ are the strain energies in K before and after the disturbance. Hence the work done against internal forces during the disturbance is $W_K(\sigma(u+v)) - W_K(\sigma(u))$. The energy norm of v can be made arbitrarily small. If this is done and terms quadratic in v are dropped, in keeping with the linear theory, the difference becomes

$$\int_\Omega \sigma_{ij}(u) e_{ij}(v)\, dx = \text{Work done against internal forces} \tag{2.2}$$

Moreover, if the body forces are constant during the displacement then

$$-\int_\Omega F_i v_i \, dx = \text{Work done against body forces} \tag{2.3}$$

No further work is done during the disturbance if the surface tractions are zero. The principle of virtual work states that the true equilibrium field $u_i(x)$ is characterized by the property that the total work done against the internal and external forces in any (small) disturbance of u_i consistent with the constraints is zero [23]. Thus in the present case

$$\int_{\Omega} \sigma_{ij}(u)e_{ij}(v)\,dx - \int_{\Omega} F_i v_i\,dx = 0 \tag{2.4}$$

for all $v \in E^{com}(\Omega)$. This motivates the following

DEFINITION. A displacement field u_i is said to be a *solution wLFE* of the equilibrium problem for the domain Ω with body forces $F_i \in L_2^{com}(\Omega)$ and zero surface tractions if and only if $u \in E^{loc}(\Omega)$ and (2.4) holds for all $v \in E^{com}(\Omega)$.

Necessary conditions for the solvability of problems with zero surface tractions. The fields

$$v_i(x) = a_i + \varepsilon_{ijk}b_j x_k, \quad x \in R^3 \tag{2.5}$$

where a_i and b_i are constant vectors and ε_{ijk} is the alternating tensor [11] satisfy $e_{ij}(v) = 0$ in R^3 and hence $v \in E^{com}(R^3)$. In particular, $v \in E^{com}(\Omega)$. It follows from (2.4) with this choice of v that necessary conditions for the existence of a solution wLFE are

$$\int_{\Omega} F_i\,dx = 0 \tag{2.6}$$

$$\int_{\Omega} (F_i x_j - F_j x_i)\,dx = 0 \tag{2.7}$$

Physically, these conditions mean that the body forces F_i exert no net resultant or moment on the body. They are assumed to be satisfied in the remainder of the discussion of problems with zero surface tractions.

Non-uniqueness of the displacements for problems with zero surface tractions. Equations (2.5) define a displacement field that describes a rigid body displacement [11]. Moreover, since $e_{ij}(v) = 0$ in R^3 the fields (2.5) may be added to any solution u of (2.4). Physically, this means that the equilibrium displacement fields are determined only up to rigid body displacements. Hence, the natural uniqueness theorem for problems with zero surface tractions asserts that the stress and strain fields are unique while the displacement fields are unique modulo fields of the form (2.5).

Bounded bodies and displacement fields wFE. If Ω is bounded then $E^{loc}(\Omega) = E(\Omega)$ and every solution wLFE actually has finite total strain energy in Ω. More generally, if u is a solution wLFE for an arbitrary domain Ω and if $u \in E(\Omega)$ then u is said to be a *solution wFE* provided (2.4) holds for all $v \in E(\Omega)$. The uniqueness of solutions wFE is proved in §3 without additional hypotheses concerning Ω or the displacement field.

Unbounded bodies and equilibrium states with prescribed stresses or displacements at infinity. If Ω is unbounded then, in general, solutions wLFE in Ω are not unique. Simple examples of non-uniqueness are available for the case $\Omega = R^3$. The field $u_i(x) = b_{ij}x_j$ with constant $b_{ij} = b_{ji} \neq 0$ is a solution wLFE in R^3 with $F_i(x) \equiv 0$ and

$\sigma_{ij}(u)=c_{ijkl}b_{kl}\neq 0$ since $e_{ij}(u)=b_{ij}$ and $\sigma_{ij}(u)e_{ij}(u)=c_{ijkl}b_{ij}b_{kl}>0$. A second example is provided by the homogeneous isotropic plate with domain $\Omega=\{x \mid x_1, x_2 \in R, |x_3|<h\}$ rand stress-strain tensor (1.8). In this case $u_1=(\lambda+2\mu)x_1$, $u_2=(\lambda+2\mu)x_2$, $u_3=-2\lambda x_3$ defines a displacement field in Ω with $F_i(x)\equiv 0$, zero surface tractions and constant non-zero stress field $\sigma_{11}=\sigma_{22}=6\lambda\mu+4\mu^2$, all other $\sigma_{ij}=0$. These examples show that uniqueness theorems for solutions wLFE in unbounded domains cannot hold without some growth restrictions at infinity on u_i or σ_{ij}.

The problem of finding suitable growth restrictions on u_i or σ_{ij} that guarantee the uniqueness of solutions wLFE is a special case of the classical problem of elastostatics of finding equilibrium displacement fields that have prescribed stresses or displacements at infinity. Many problems of this type are discussed in the treatise of Love [18]. To formulate the problem with prescribed stresses at infinity let

$$\Omega_{R,\infty}=\Omega\cap\{x \mid |x|>R\} \tag{2.8}$$

and let $u_i^\infty(x)$ be a solution wLFE in $\Omega_{R,\infty}$, for some R, whose stress field $\sigma_{ij}^\infty=\sigma_{ij}(u^\infty)$ has the desired behavior at infinity. A solution wLFE in Ω is sought such that $\sigma_{ij}(u)(x)$ is close to $\sigma_{ij}^\infty(x)$ at infinity, in a suitable sense. One possibility is to extend u_i^∞ to Ω and require

$$W_\Omega(\sigma(u)-\sigma^\infty)<\infty \tag{2.9}$$

This suggests the

DEFINITION. A solution wLFE of the equilibrium problem for an unbounded domain Ω is said to have prescribed stresses σ_{ij}^∞ at infinity if and only if $u_i-u_i^\infty$ is a solution wFE for Ω.

Solutions wLFE with stresses $\sigma_{ij}^\infty=0$ at infinity are just the solutions wFE defined above. Condition (2.9) is correct in this case, at least for exterior domains where the stresses generated by body forces $F_i\in L_2^{com}(\Omega)$ are known to satisfy $\sigma_{ij}(u)(x)=O(|x|^{-2})$, $|x|\to\infty$ [13].

To formulate the problem with prescribed displacements at infinity let $u_i^\infty(x)$ be a displacement field that is defined in $\Omega_{R,\infty}$ for some R, and has the desired behavior at infinity. A solution wLFE in Ω is sought such that $u_i(x)$ is close to $u_i^\infty(x)$ at infinity, in a suitable sense. One might try the condition $u_i-u_i^\infty\in L_2(\Omega_{R,\infty})$, in analogy with (2.9). However, this condition is too strong. In fact, it is known that if $u_i^\infty=0$ and Ω is an exterior domain then the displacements generated by body forces $F_i\in L_2^{com}(\Omega)$ have the exact order $u_i(x)=O(|x|^{-1})$, $|x|\to\infty$ [10]. Thus a weaker condition consistent with this estimate is needed. In what follows the condition

$$\|u-u^\infty\|_{r,\delta}^2=O(r), \qquad r\to\infty \tag{2.10}$$

is used where

$$\|u\|_{r,\delta}^2=\int_{\Omega_{r,\delta}} u_i(x)u_i(x)\,dx \tag{2.11}$$

$$\Omega_{r,\delta}=\Omega\cap\{x \mid r\le|x|\le r+\delta\} \tag{2.12}$$

and $\delta>0$ is a constant.

DEFINITION. A solution wLFE of the equilibrium problem for an unbounded domain Ω is said to have prescribed displacements u_i^∞ at infinity if and only if (2.10) holds for some $\delta > 0$.

A sufficient condition for (2.10) to hold with $u_i^\infty = 0$ is

$$u_i(x) = O(|x|^{-1/2}), \qquad |x| \to \infty \tag{2.13}$$

Of course, the precise order condition on u_i that is sufficient to guarantee (2.10) in particular cases will depend on the geometry of Ω near infinity. For example, if $\Omega = \{x \mid |x_3| < h\}$ then $\int_{\Omega_{r,\delta}} dx = O(r)$, $r \to \infty$, and $u_i(x) = O(1)$ is a sufficient condition for (2.10) with $u_i^\infty = 0$. If $\Omega = \{x \mid (x_1, x_2) \in G, x_3 \in R\}$ where $G \subset R^2$ is bounded then $\int_{\Omega_{r,\delta}} dx = O(1)$ and $u_i(x) = (|x|^{1/2})$, $|x| \to \infty$, is sufficient.

Ellipticity of the Cauchy-Green operator. The principle of virtual work (2.4) with $v_i \in C_0^\infty(\Omega) \subset E^{com}(\Omega)$ implies that the equilibrium fields u_i are weak solutions of the system of partial differential equations $\sigma_{ij,j}(u) + F_i = 0$ in Ω. If the body is homogeneous, as is assumed in this section, then the system may be written

$$A_{ik} u_k + F_i = 0 \tag{2.14}$$

where

$$A_{ik} = c_{ijkl} \partial^2 / \partial x_j \partial x_l \tag{2.15}$$

The matrix differential operator (A_{ik}), with coefficients that satisfy the positivity and symmetry conditions (1.4), (1.5), will be called the Cauchy-Green operator. Conditions (1.4), (1.5) imply that (A_{ik}) is strongly elliptic ($c_{ijkl}\eta_i\eta_k\xi_j\xi_l \neq 0$ for all non-zero η_i, ξ_i) and hence elliptic ($\det(c_{ijkl}\xi_j\xi_l) \neq 0$ for all non-zero ξ_i) [13, p. 20]. G. Fichera [5] has used the theory of elliptic boundary value problems to prove both interior and boundary regularity theorems for weak solutions of (2.14). The interior and boundary regularity properties of solutions wLFE that are implied by Fichera's results and methods are described here briefly.

Interior regularity of solutions wLFE. Fichera's interior regularity theorem [5, p. 36] implies the following results.

THEOREM 2.1. *Let $\Omega \subset R^3$ be an arbitrary domain. Let $u_i \in L_2^{int}(\Omega)$, $e_{ij}(u) \in L_2^{int}(\Omega)$ and $F_i \in L_2^{m,int}(\Omega)$ where $m \geq 0$ is an integer. Assume that* (2.4) *holds for all $v_i \in C_0^\infty(\Omega)$. Then $u_i \in L_2^{m+2,int}(\Omega)$.*

COROLLARY 2.2. *Let $\Omega \subset R^3$ be an arbitrary domain and let u be a solution wLFE of the equilibrium problem for Ω with $F_i \in L_2^{m,com}(\Omega)$. Then $u_i \in L_2^{m+2,int}(\Omega)$.*

COROLLARY 2.3. *If the hypotheses of Theorem* 2.1 *or Corollary* 2.2 *hold then $u_i \in C^m(\Omega)$.*

COROLLARY 2.4. *Let $\Omega \subset R^3$ be an arbitrary domain and let $u \in E^{loc}$ satisfy $e_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}) = 0$ in $L_2^{loc}(\Omega)$. Then there exist constants a_i, b_i such that $u_i(x) = a_i + \varepsilon_{ijk} b_j x_k$ in Ω.*

Fichera proved Theorem 2.1 in [5] under the hypotheses $f \in L_2^m(\Omega)$, $u \in L_2(\Omega)$. However, the theorem as stated above is an immediate consequence of his theorem.

Corollary 2.2 is a special case of Theorem 2.1. Corollary 2.3 follows from Theorem 2.1 and Sobolev's imbedding theorem [5, p. 26]. Corollary 2.4 may be verified by noting that u is a solution wLFE in Ω with body forces $F_i = 0$ in Ω. Thus $u_i \in C^\infty(\Omega)$, by Corollary 2.3, and $u_{i,j} + u_{j,i} = 0$ *in* Ω. The proof that every such u_i has the form $u_i = a_i + \varepsilon_{ijk} b_j x_k$ is classical [11, p. 71].

Boundary regularity of solutions wLFE. Fichera's theorems on regularity at the boundary imply the following results (see [5, Chapters 10 and 12]).

THEOREM 2.5. *Let* $\Omega \subset R^3$ *be a domain with boundary* $\partial\Omega \in C^\infty$. *Let* u *be a solution wLFE of the equilibrium problem for* Ω *with* $F_i \in L_2^{com}(\Omega) \cap C^\infty(\bar\Omega)$. *Then* $u_i \in C^\infty(\bar\Omega)$ *and*

$$\sigma_{ij}(u) n_j = 0 \text{ on } \partial\Omega \tag{2.16}$$

where n_i *is the unit exterior normal field on* $\partial\Omega$.

COROLLARY 2.6. *Let* $x_0 \in \partial\Omega$ *and assume that there is a neighborhood* $N_\delta(x_0) = \{x \mid |x - x_0| < \delta\}$ *such that* $\partial\Omega \cap N_\delta(x_0) \in C^\infty$. *Moreover, let* $F_i \in L_2^{com}(\Omega) \cap C^\infty(\bar\Omega \cap N_\delta(x_0))$. *Then* $u_i \in C^\infty(\bar\Omega \cap N_\delta(x_0))$ *and* $\sigma_{ij}(u) n_j = 0$ *on* $\partial\Omega \cap N_\delta(x_0)$.

Corollary 2.6 is an immediate consequence of Theorem 2.5 since boundary regularity is a local property. Boundary regularity results can also be proved when $\partial\Omega$ and F_i have a finite number of derivatives. The following results can be proved by the methods of [5]; see also [1].

THEOREM 2.7. *Let* $\Omega \subset R^3$ *have a boundary point* x_0 *such that* $\partial\Omega \cap N_\delta(x_0) \in C^{k+2}$ *for some* $\delta > 0$ *where* $k \geq 0$ *is an integer. Let* u *be a solution wLFE of the equilibrium problem for* Ω *with* $F_i \in L_2^{com}(\Omega) \cap L_2^k(\Omega \cap N_\delta(x_0))$. *Then* $u_i \in L_2^{k+2}(\Omega \cap N_\delta(x_0))$.

COROLLARY 2.8. *Under the hypotheses of Theorem* 2.7, $u_i \in C^k(\bar\Omega \cap N_\delta(x_0))$. *Moreover, if* $k \geq 1$ *then* $\sigma_{ij}(u) n_j = 0$ *on* $\partial\Omega \cap N_\delta(x_0)$.

COROLLARY 2.9. *Let* $\Omega \subset R^3$ *be a domain with boundary* $\partial\Omega \in C^{k+2}$, $k \geq 0$. *Let* u *be a solution wLFE of the equilibrium problem for* Ω *with* $F_i \in L_2^{k,com}(\Omega)$. *Then* $u_i \in C^k(\bar\Omega)$. *Moreover, if* $k \geq 2$ *then* u_i *is a classical solution of the equilibrium boundary value problem with body forces* $F_i \in C_0^{k-2}(\bar\Omega) \subset L_2^{k,com}(\Omega)$ *and zero surface tractions; i.e.,* u_i *satisfies* (2.10) *and*

$$c_{ijkl} u_{k,jl} + F_i = 0 \text{ in } \Omega. \tag{2.17}$$

Bodies whose boundary $\partial\Omega$ is a piece-wise smooth surface with piece-wise smooth edges with corners are of great interest for applications. A class of bodies of this type are the C-domains, defined and studied by N. Weck [24]. Solutions wLFE in such domains are regular and satisfy the boundary condition (2.16) near smooth points of $\partial\Omega$, by Corollary 2.8. At edge and corner points of $\partial\Omega$ condition (2.16) is meaningless, because n_i is undefined, and the only regularity property that remains is the LFE condition. For this reason the LFE condition is sometimes called the "edge condition" [20].

Equilibrium problems with non-zero surface tractions. The formulation (2.4) of the principle of virtual work is appropriate for the case of zero surface tractions. The surface traction at a point $x_0 \in \partial\Omega$ is by definition the vector $\sigma_{ij}(u(x_0))n_j(x_0)$ and hence is defined only at boundary points where the boundary values $\sigma_{ij}(u(x_0))$ and the normal vector $n_j(x_0)$ exist. If a portion $S \subset \partial\Omega$ is sufficiently smooth for n_j and boundary values of $\sigma_{ij}(u)$ to exist on it then the principle of virtual work can be extended to include the boundary condition

$$\sigma_{ij}(u)n_j = \begin{cases} t_i & \text{on } S \\ 0 & \text{on } \partial\Omega - S \end{cases} \tag{2.18}$$

To do this the term

$$-\int_S t_i v_i \, dS = \text{Work done against surface tractions} \tag{2.19}$$

must be added to (2.4), so that the extended principle becomes

$$\int_\Omega \sigma_{ij}(u)e_{ij}(v)\, dx - \int_\Omega F_i v_i \, dx - \int_S t_i v_i \, dS = 0 \tag{2.20}$$

for all $v \in E^{com}(\Omega)$. Moreover, it is known from Sobolev's imbedding theorem that every $v \in E^{com}(\Omega)$ has boundary values $v \in L_2(S)$ on smooth portions $S \subset \partial\Omega$ [1, p. 38]. In the important special case where $\partial\Omega$ is piece-wise smooth then $\sigma_{ij}(u)n_j$ exists almost everywhere on $\partial\Omega$ and S may be replaced by $\partial\Omega$ in (2.18), (2.19) and (2.20).

3. Uniqueness theorems for problems with prescribed surface tractions

The strain energy theorem for classical solutions of the elastostatic equilibrium problem with body forces F_i and zero surface tractions states that [18, p. 173]

$$W_\Omega = \frac{1}{2}\int_\Omega \sigma_{ij}(u)e_{ij}(u)\, dx = \frac{1}{2}\int_\Omega F_i u_i \, dx \tag{3.1}$$

The uniqueness of classical solutions is a corollary. In this section the strain energy theorem is extended to arbitrary domains Ω and all solutions wFE (= solutions wLFE and zero stresses at infinity if Ω is unbounded) and solutions wLFE and zero displacements at infinity. The uniqueness of solutions wLFE with prescribed stresses or displacements at infinity follow as corollaries. The simple case of solutions wFE is treated first.

THEOREM 3.1. *Let u be a solution wFE of the equilibrium problem with body forces* $F_i \in L_2^{com}(\Omega)$ *and zero surface tractions in a domain* $\Omega \subset R^3$. *Then the strain energy equation* (3.1) *holds.*

The proof is immediate from the representation (1.7) for W_Ω and the definition of solution wFE, since one may take $v_i = u_i \in E(\Omega)$ in (2.4).

COROLLARY 3.2. *Uniqueness of Solutions wFE. Let* $u_i^{(1)}$, $u_i^{(2)}$ *be two solutions wFE of the equilibrium problem with the same body forces* $F_i \in C^{\mathrm{com}}(\Omega)$ *and zero surface tractions. Then*

$$\sigma_{ij}(u^{(1)}) = \sigma_{ij}(u^{(2)}) \text{ in } \Omega \tag{3.2}$$

and there exist constant vectors a_i, b_i *such that*

$$u_i^{(1)}(x) - u_i^{(2)}(x) = a_i + \varepsilon_{ijk} b_j x_k \text{ in } \Omega \tag{3.3}$$

Proof. $u_i = u_i^{(1)} - u_i^{(2)}$ is a solution wFE with body forces $F_i \equiv 0$ in Ω and zero surface tractions. Thus (3.1) holds with $F_i = 0$ and $\sigma_{ij}(u) = 0$ in $L_2(\Omega)$ by the positive-definiteness of the energy. Moreover, $\sigma_{ij}(u) \in C^\infty(\Omega)$ by Corollary 2.3 and hence $\sigma_{ij}(u)(x) \equiv 0$ in Ω which implies (3.2). Finally, Corollary 3.4 implies $u_i(x) = a_i + \varepsilon_{ijk} b_j x_k$ which implies (3.3).

COROLLARY 3.3. *Uniqueness of solutions wLFE with prescribed stresses at infinity. Let* $\Omega \subset R^3$ *be unbounded and let* $u_i^{(1)}$, $u_i^{(2)}$ *be two solutions wLFE of the equilibrium problem with the same body forces* F_i, *zero surface tractions and the same stresses* σ_{ij}^∞ *at infinity. Then* (3.2) *and* (3.3) *hold.*

Proof. By hypothesis, both $u_i^{(1)} - u_i^\infty$ and $u_i^{(2)} - u_i^\infty$ are solutions wFE for Ω. It follows that the difference field $u_i = u_i^{(1)} - u_i^{(2)}$ is a solution wFE with body forces $F_i \equiv 0$ in Ω and zero surface tractions. Equations (3.2), (3.3) follow as in the proof of Corollary 3.2.

The uniqueness theorem for solutions wLFE with prescribed displacements at infinity will be based on the following generalization of Theorem 3.1.

THEOREM 3.4. *Let u be a solution wLFE of the equilibrium problem with body forces* $F_i \in L_2^{\mathrm{com}}(\Omega)$ *and zero surface tractions in an unbounded domain* $\Omega \subset R^3$. *Moreover, let u satisfy*

$$\int_R^\infty \|u\|_{r,\delta}^{-2}\, dr = +\infty \tag{3.4}$$

for some $R > 0$ *and* $\delta > 0$. *Then the strain energy equation* (3.1) *holds.*

A proof of Theorem 3.4 is given at the end of the section, following the statement and discussion of the remaining uniqueness theorems.

COROLLARY 3.5. *Uniqueness of solutions wLFE with prescribed displacements at infinity. Let* $\Omega \subset R^3$ *be unbounded and let* $u_i^{(1)}$, $u_i^{(2)}$ *be two solutions wLFE of the equilibrium problem with the same body forces* F_i, *zero surface tractions and the same displacements* u_i^∞ *at infinity. Then* (3.2) *and* (3.3) *hold.*

Proof. By hypothesis $\|u^{(k)} - u^\infty\|_{r,\delta} = O(r^{1/2})$, $r \to \infty$, $k = 1, 2$. It follows by the triangle inequality that the differences field $u_i = u_i^{(1)} - u_i^{(2)}$ satisfies $\|u\|_{r,\delta} = O(r^{1/2})$, $r \to \infty$, or equivalently

$$\|u\|_{r,\delta}^2 = O(r),\ r \to \infty \tag{3 5}$$

which implies condition (3.4). Moreover, u is a solution wLFE with $F_i = 0$ and zero surface tractions. Hence (3.1) holds with $F_i = 0$, by Theorem 3.4, and the conclusions (3.2), (3.3) follow as before.

Uniqueness theorems for problems with non-zero surface tractions. The uniqueness theorems proved above are valid for arbitrary bounded and unbounded domains $\Omega \subset R^3$. No local or global restrictions are imposed on Ω or $\partial\Omega$. If a portion $S \subset \partial\Omega$ is smooth enough for the surface tractions $\sigma_{ij}(u)n_j$ and surface integrals (2.19) to be defined then solutions wLFE with non-zero surface tractions t_i on S are defined by the principle of virtual work. The uniqueness theorems for solutions with zero surface tractions extend immediately to this case because the difference of two solutions with the same surface tractions t_i is a solution with zero surface tractions.

Other growth conditions at infinity. It is clear from condition (3.4) of Theorem 3.4 that condition (3.5) is only one sufficient condition for uniqueness. Generalizations are obtained by replacing (3.5) by

$$\|u\|^2_{r,\delta} = O(p(r)), \; r \to \infty \tag{3.6}$$

where $p(r)$ is a function such that

$$\int_R^\infty p(r)^{-1}\, dr = +\infty \tag{3.7}$$

If Ω is an exterior domain ($\{x \mid |x| > R\} \subset \Omega$ for $R \geq R_0$) and if the body is isotropic as well as homogeneous; i.e., (1.8) holds, then the uniqueness theorem can be proved under weaker growth restrictions than (3.4). Indeed, under these conditions Fichera [4] has proved that

$$u_i(x) = o(1) \Rightarrow u_i(x) = O(|x|^{-1}) \quad \text{and} \quad \sigma_{ij}(x) = O(|x|^{-2}) \tag{3.8}$$

M. E. Gurtin and E. Sternberg [10] have rederived this result and proved the complementary result that

$$\sigma_{ij}(x) = o(1) \Rightarrow u_i(x) = O(|x|^{-1}) \quad \text{and} \quad \sigma_{ij}(x) = O(|x|^{-2}) \tag{3.9}$$

Moreover, these results are based on an expansion theorem for biharmonic functions in a neighborhood of infinity and are independent of $\partial\Omega$. Thus the uniqueness theorems for solutions wLFE with prescribed displacements or stresses at infinity in homogeneous isotropic solids are valid for arbitrary exterior domains Ω under the conditions

$$u_i(x) - u_i^\infty(x) = o(1), \qquad |x| \to \infty \tag{3.10}$$

and

$$\sigma_{ij}(u)(x) - \sigma_{ij}^\infty(x) = o(1), \qquad |x| \to \infty \tag{3.11}$$

respectively.

Proof of Theorem 3.4. The idea of the proof is to put $v_i = u_i$ in the principle of virtual work identity (2.4), as in the proof of Theorem 3.1. However, this cannot be done directly when u is a solution wLFE because $v \in E^{com}(\Omega)$ must have compact support. Instead, let $v_i(x) = \phi(x)u_i(x)$ where

$$\phi(x) = \psi((|x| - R)/\delta), \quad R > 0, \qquad \delta > 0, \qquad x \in R^3 \tag{3.12}$$

and $\psi \in C^\infty(R)$ is a function such that $\psi'(\tau) \leq 0$, $0 \leq \psi(\tau) \leq 1$ and

$$\psi(\tau) = \begin{cases} 1, & \tau \leq 0 \\ 0, & \tau \geq 1 \end{cases} \tag{3.13}$$

These properties imply that $\phi \in C_0^\infty(R^3)$, $0 \leq \phi(x) \leq 1$ and

$$\phi(x) = \begin{cases} 1, & |x| \leq R \\ 0, & |x| \geq R + \delta \end{cases} \tag{3.14}$$

It follows that for all $u \in E^{loc}(\Omega)$, $v = \phi u \in E^{com}(\Omega)$ and

$$v_{i,j} = \phi u_{i,j} + \phi_{,j} u_i \tag{3.15}$$

Moreover,

$$\phi_{,j}(x) = \psi'((|x| - R)/\delta) x_j / \delta |x| \tag{3.16}$$

and

$$\operatorname{supp} \phi_{,j} \subset \Omega_{R,\infty} \tag{3.17}$$

With this choice of v_i

$$e_{ij}(v) = \phi e_{ij}(u) + \tfrac{1}{2}(\phi_{,i} u_j + \phi_{,j} u_i) \tag{3.18}$$

and hence

$$\begin{aligned} \sigma_{ij}(u) e_{ij}(v) &= \phi \sigma_{ij}(u) e_{ij}(u) + \sigma_{ij}(u) \phi_{,i} u_j \\ &= \phi \sigma_{ij}(u) e_{ij}(u) + \delta^{-1} \psi'(|x| - R)/\delta) \sigma_{ij}(u) \hat{x}_i u_j \end{aligned}$$

where $\hat{x}_j = x_j/|x|$. By assumption $F_i \in L_2^{com}(\Omega)$. Choose R_0 so large that $\operatorname{supp} F_i \subset \{x \mid |x| \leq R_0\}$ and substitute $v_i = \phi u_i$ and (3.19) in (2.4) with $R \geq R_0$. The result can be written

$$\int_\Omega \phi \sigma_{ij}(u) e_{ij}(u)\, dx + \delta^{-1} \int_{\Omega_{R,\delta}} \psi' \sigma_{ij}(u) \hat{x}_i u_j \, dx - \int_\Omega F_i u_i \, dx = 0 \tag{3.20}$$

The goal of the remainder of the proof is to calculate the limit of equation (3.20) for $R \to \infty$ and to show that the limiting form is the energy equation (3.1). To this end define

$$f(R) = \int_\Omega \psi(\delta^{-1}(|x| - R)) \sigma_{ij}(u) e_{ij}(u)\, dx - \int_\Omega F_i u_i \, dx, \qquad R \geq R_0 \tag{3.21}$$

By equation (3.20) an alternative representation is

$$f(R) = -\delta^{-1} \int_{\Omega_{R,\delta}} \psi'(\delta^{-1}(|x| - R)) \sigma_{ij}(u) \hat{x}_i u_j \, dx \tag{3.22}$$

The properties of $f(R)$ that are needed to complete the proof of Theorem 3.4 are described by

LEMMA 3.6. $f \in C^1[R_0, \infty)$ *and has derivative*

$$f'(R) = -\delta^{-1}\int_{\Omega_{R,\delta}} \psi'(\delta^{-1}|x|-R))\sigma_{ij}(u)e_{ij}(u)\,dx \geq 0 \tag{3.23}$$

In particular, $f(R)$ *is monotone non-decreasing on* $[R_0, \infty)$. *Moreover,*

$$f^2(R) \leq M^2\|u\|^2_{R,\delta} f'(R), \qquad R \geq R_0 \tag{3.24}$$

where $M^2 = (\delta^{-1}c_1) \max\limits_{0\leq\tau\leq1} \psi'(\tau)|$.

Proof of Lemma 3.6. Form the difference quotient

$$h^{-1}\{f(R+h)-f(R)\} = \int_{\Omega_{R,R+h+\delta}} h^{-1}\{\psi(\delta^{-1}(|x|-R-h))-\psi(\delta^{-1}(|x|-R))\} \times\sigma_{ij}(u)e_{ij}(u)\,dx \tag{3.25}$$

The quotient

$$h^{-1}\{\psi(\delta^{-1}(|x|-R-h))-\psi(\delta^{-1}(|x|-R))\} \to -\delta^{-1}\psi'(\delta^{-1}(|x|-R)),\ h\to 0 \tag{3.26}$$

uniformly for x in bounded sets in R^3. Moreover, $\sigma_{ij}(u)e_{ij}(u)$ is Lebesgue integrable on bounded subsets of Ω. Thus passage to the limit $h \to 0$ in (3.25) is permissible by Lebesgue's dominated convergence theorem. Hence $f'(R)$ exists for all $R \geq R_0$ and is given by (3.23). It is easy to show that the integral in (3.23) defines a continuous function of R which is non-negative. The monotonicity of $f(R)$ follows.

To prove the inequality (3.24) note that (3.22) implies the estimate

$$|f(R)| \leq \delta^{-1}\int_{\Omega_{R,\delta}} |\psi'(\delta^{-1}(|x|-R))|\,|\sigma_{ij}(u)\hat{x}_i u_j|\,dx, \qquad R \geq R_0 \tag{3.27}$$

Moreover, by repeated application of Schwarz's inequality

$$|\sigma_{ij}(u)\hat{x}_i u_j| \leq (\sigma_{ij}(u)\hat{x}_i\sigma_{kj}(u)\hat{x}_k)^{1/2}(u_j u_j)^{1/2} \tag{3.28}$$

$$|\sigma_{ij}(u)\hat{x}_j| \leq \left(\sum_{i=1}^{3} \sigma_{ij}^2(u)\right)^{1/2} \tag{3.29}$$

$$\begin{aligned}\sigma_{ij}(u)\hat{x}_i\sigma_{kj}(u)\hat{x}_k &\leq \left(\sum_{j=1}^{3} (\sigma_{ij}(u)\hat{x}_i)^2\right)^{1/2}\left(\sum_{k=1}^{3} (\sigma_{kj}(u)\hat{x}_k)^2\right)^{1/2} \\ &= \sum_{j=1}^{3} (\sigma_{ij}(u)\hat{x}_j)^2 \leq \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}^2(u) = \sigma_{ij}(u)\sigma_{ij}(u)\end{aligned} \tag{3.30}$$

Now $e_{ij} = \gamma_{ijkl}\sigma_{kl}$ together with (1.9) imply

$$c_1^{-1}\sigma_{ij}\sigma_{ij} \leq \sigma_{ij}e_{ij} = \gamma_{ijkl}\sigma_{ij}\sigma_{kl} \leq c_0^{-1}\sigma_{ij}\sigma_{ij} \tag{3.31}$$

for all $\sigma_{ij} = \sigma_{ji}$. Combining these inequalities gives

$$|\sigma_{ij}(u)\hat{x}_i u_j| \leq c_1^{1/2}(\sigma_{ij}(u)e_{ij}(u))^{1/2}(u_j u_j)^{1/2} \tag{3.32}$$

Substituting in (3.27) and using Schwarz's inequality again and equation (3.23) gives

$$
\begin{aligned}
|f(R)| &\leq \delta^{-1} c_1^{1/2} \int_{\Omega_{R,\delta}} |\psi'(\delta^{-1}(|x|-R))| (\sigma_{ij}(u) e_{ij}(u))^{1/2} (u_j u_j)^{1/2} \, dx \\
&\leq \delta^{-1} c_1^{1/2} \left(\int_{\Omega_{R,\delta}} |\psi'| \sigma_{ij}(u) e_{ij}(u) \, dx \right)^{1/2} \left(\int_{\Omega_{R,\delta}} |\psi'| u_j u_j \, dx \right)^{1/2} \\
&\leq \delta^{-1} c_1^{1/2} \mu^{1/2} (\delta f'(R))^{1/2} \|u\|_{R,\delta} \qquad (3.33)
\end{aligned}
$$

where $\mu = \operatorname{Max} |\psi'(x)|$. Squaring (3.33) gives (3.24).

Proof of Theorem 3.4 *Concluded.* Lemma 3.6 implies that $f(+\infty)$ exists as a finite number or $+\infty$. It will be shown that $f(+\infty)=0$. There are three cases to consider.

Case 1. $0 < f(+\infty) \leq +\infty$. In this case there exists $R_1 \geq R_0$ such that $f(R) \geq f(R_1) > 0$ for $R \geq R_1$. Hence (3.24) can be written

$$
-\frac{d}{dR}\left(\frac{1}{f(R)}\right) = \frac{f'(R)}{f^2(R)} \geq M^{-2} \|u\|_{R,\delta}^{-2}, \qquad R \geq R_1 \tag{3.34}
$$

and integration gives

$$
\frac{1}{f(R_1)} - \frac{1}{f(R)} \geq M^{-2} \int_{R_1}^{R} \|u\|_{r,\delta}^{-2} \, dr, \qquad R \geq R_1 \tag{3.35}
$$

In particular, since $f(R) > 0$ for $R \geq R_1$,

$$
\frac{M^2}{f(R_1)} \geq \int_{R_1}^{R} \|u\|_{r,\delta}^{-2} \, dr \quad \text{for} \quad R \geq R_1 \tag{3.36}
$$

But this contradicts hypothesis (3.4) of the theorem. Hence Case 1 cannot occur.

Case 2. $f(+\infty) \leq 0$ and $f(R_1) = 0$ for some $R_1 \geq R_0$. In this case $0 \leq f(R_1) \leq f(+\infty) \leq 0$; i.e. $f(+\infty) = 0$.

Case 3. $f(+\infty) \leq 0$ and $f(R) < 0$ for all $R \geq R_0$. In this case (3.34) and (3.35) hold and the latter can be written, since $|f(R)| = -f(R)$,

$$
\frac{1}{|f(R)|} \geq \frac{1}{|f(R_1)|} + M^{-2} \int_{R_1}^{R} \|u\|_{r,\delta}^{-2} \, dr, \qquad R \geq R_1 \tag{3.37}
$$

Hence condition (3.4) implies that $f(+\infty) = 0$.

It has been shown that (3.4) implies $f(+\infty) = 0$; that is,

$$
\lim_{R \to +\infty} \int_{\Omega} \psi(\delta^{-1}(|x|-R)) \sigma_{ij}(u) e_{ij}(u) \, dx = \int_{\Omega} F_i u_i \, dx \tag{3.38}
$$

Since $\psi(\delta^{-1}(|x|-R))$ is a monotone increasing function of R for each fixed $x \in R^3$ and tends to 1 everywhere when $R \to \infty$, (3.38) implies equation (3.1). In particular $W_\Omega < \infty$ because $\int_\Omega F_i u_i \, dx$ is finite. This completes the proof.

4. Uniqueness theorems for other equilibrium problems

The purpose of this section is to show how the methods and results developed above can be extended to the most general equilibrium problems of linear elastostatics. Equilibria subject to other boundary conditions, equilibria in inhomogeneous anisotropic bodies and n-dimensional generalizations are discussed. In each case the boundary conditions for displacement fields wFE and wLFE are defined by appropriate subspaces of $E(\Omega)$ and $E^{loc}(\Omega)$, respectively, and a corresponding form of the principle of virtual work is given. Regularity and uniqueness results for the new problems are indicated without proofs. In fact, the proofs of sections 2 and 3 are valid with minor modifications.

Equilibrium problems with prescribed surface displacements. The case of zero surface displacements is discussed first. Suitable subspaces of displacements fields are

$$E_0(\Omega) = \text{Closure in } E(\Omega) \text{ of } E^{com}(\Omega) \cap \{u \mid \operatorname{supp} u \subset \Omega\} \tag{4.1}$$

$$E_0^{loc}(\Omega) = \text{Closure in } E^{loc}(\Omega) \text{ of } E^{com}(\Omega) \cap \{u \mid \operatorname{supp} u \subset \Omega\} \tag{4.2}$$

The topologies in $E(\Omega)$ and $E^{loc}(\Omega)$ are those defined by (1.28) and (1.29), respectively. The notation

$$E_0^{com}(\Omega) = E^{com}(\Omega) \cap E_0(\Omega) \tag{4.3}$$

is also used. A solution wFE of the equilibrium problem with body forces $F_i \in L_2^{com}(\Omega)$ and zero surface displacements in a field $u \in E_0(\Omega)$ that satisfies (2.4) for all $v \in E_0(\Omega)$. Similarly, a solution wLFE of the same problem is a field $u \in E_0^{loc}(\Omega)$ that satisfies (2.4) for all $v \in E_0^{com}(\Omega)$. Problems with non-zero surface displacements

$$u_i(x) = f_i(x), \qquad x \in \partial\Omega \tag{4.4}$$

may be reduced to the preceding problem if there exists a field $u_i^0 \in E^{loc}(\Omega) \cap \{u \mid \sigma_{ij}(u^0) \in L_2^{com}(\Omega)\}$. Then $u_i' = u_i - u_i^0$ is a solution wLFE with zero boundary displacements.

The remaining boundary conditions can be formulated only when $\partial\Omega$ is piecewise smooth. It will be assumed that $\partial\Omega$ is a C-domain in the sense of [24]. For such domains the unit exterior normal field $n_i(x)$ is defined and continuous at all points of $\partial\Omega$ except edges and corners and one can define the normal and tangential components of vector field on $\partial\Omega$ by

$$u_i = u_i^\nu + u_i^\tau,\ u_i^\nu = (u_j \nu_j)\nu_i \tag{4.5}$$

moreover, $u_i^\nu v_i^\tau = 0$ for all u_i, v_i and hence

$$u_i v_i = u_i^\nu v_i^\nu + u_i^\tau v_i^\tau \tag{4.6}$$

Equilibrium problems with prescribed tangential surface tractions and normal surface displacements. Suitable subspaces of displacement fields are defined by

$$E_\nu(\Omega) = E(\Omega) \cap \{u \mid u^\nu = 0 \text{ on } \partial\Omega\} \tag{4.7}$$

$$E_\nu^{loc}(\Omega) = E^{loc}(\Omega) \cap \{u \mid u^\nu = 0 \text{ on } \partial\Omega\} \tag{4.8}$$

The existence of u^ν and u^τ on $\partial\Omega$ for all $u \in E^{loc}(\Omega)$ follows from Korn's inequality and Sobolev's imbedding theorem. A solution wFE of the equilibrium problem with body forces $F_i \in L_2^{com}(\Omega)$, zero tangential surface tractions and zero normal surface displacements is a field $u \in E_\nu(\Omega)$ that satisfies (2.4) for all $v \in E_\nu(\Omega)$. Similarly, a solution wLFE of the same problem is a field $u \in E_\nu^{loc}(\Omega)$ such that (2.4) holds for all $v \in E_\nu^{com}(\Omega) = E_\nu(\Omega) \cap E^{com}(\Omega)$. Problems with non-zero surface tractions and displacements are treated by reducing them to the preceding case through subtraction of a suitable field.

Equilibrium problems with prescribed normal surface tractions and tangential surface displacements. This problem is dual to the preceding one. Appropriate classes of displacements are

$$E_\tau(\Omega) = E(\Omega) \cap \{u \mid u^\tau = 0 \text{ on } \partial\Omega\} \tag{4.9}$$

$$E_\tau^{loc}(\Omega) = E^{loc}(\Omega) \cap \{u \mid u^\tau = 0 \text{ on } \partial\Omega\} \tag{4.10}$$

Equilibrium problems with elastically supported surface. Physically, this corresponds to the case where surface displacements produce surface tractions that satisfy Hooke's law:

$$\sigma_{ij}(u)n_j + \beta u_i = 0 \text{ on } \partial\Omega \tag{4.11}$$

where $\beta > 0$ is defined on $\partial\Omega$. A solution wLFE is a field $u \in E^{loc}(\Omega)$ such that

$$\int_\Omega \sigma_{ij}(u)e_{ij}(v)\,dx - \int_\Omega F_i v_i\,dx + \int_{\partial\Omega} \beta u_i v_i\,dS = 0 \tag{4.12}$$

for all $v \in E^{com}(\Omega)$. Identity (4.12) is the principle of virtual work for this problem, the last term being the virtual work done against the induced surface tractions by the virtual displacement v. If follows from (4.12) that (4.11) holds at smooth points of $\partial\Omega$.

Equilibrium problems with mixed boundary conditions. A mixed problem that includes the preceding problems as special cases can be formulated by decomposing $\partial\Omega$ into five portions and imposing one of the boundary conditions defined above on each portion. Thus, if

$$\partial\Omega = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5 \text{ (disjoint union)} \tag{4.13}$$

and

$$E_m^{loc}(\Omega) = E^{loc}(\Omega) \cap \{u \mid u = 0 \text{ on } S_1,\ u^\nu = 0 \text{ on } S_2,\ u^\tau = 0 \text{ on } S_3\} \tag{4.14}$$

then the principle of virtual work

$$\int_\Omega \sigma_{ij}(u)e_{ij}(v)\,dx - \int_\Omega F_i v_i\,dx + \int_{S_5} \beta u_i v_i\,dS = 0 \tag{4.15}$$

for all $v \in E_m^{com}(\Omega) = E_m^{loc}(\Omega) \cap E^{com}(\Omega)$ characterizes the solutions of the equilibrium problem that satisfy $u=0$ on S_1, $u^\nu=0$ and $(\sigma_{ij}(u)n_j)^\tau=0$ on S_2, $u^\tau=0$ and $(\sigma_{ij}(u)n_j)^\nu=0$ on S_3, $\sigma_{ij}(u)n_j=0$ on S_4 and $\sigma_{ij}(u)n_j+\beta u_i=0$ on S_5.

Regularity and uniqueness theorems will be discussed for this mixed problem since it includes the others as special cases.

Regularity theorems. The interior regularity properties of solutions wLFE of the mixed problem follow from Theorem 2.1 and are exactly the same as for the case discussed in Section 2. Concerning boundary regularity, it can be shown by the methods of Fichera's monograph [5] that if Ω is a C-domain of class C^∞ such that S_k^0 = interior of S_k in $\partial\Omega$ is a C^∞ manifold for $k=1,\ldots,5$, and if $F_i \in C^\infty(\bar{\Omega})\cap L_2^{com}(\Omega)$ then solutions wLFE of the mixed problem satisfy

$$u_i \in C^\infty(\Omega \cup S_1^0 \cup S_2^0 \cup S_3^0 \cup S_4^0 \cup S_5^0)\cap L_2^{1,loc}(\Omega).$$

The condition $u_i \in L_2^{1,loc}(\Omega)$, which follows from Korn's inequality and Sobolev's theorem, is the "edge condition" that is needed for uniqueness. The boundary conditions on S_2, S_3 and S_5 are not discussed by Fichera in [5] but can be treated by his methods.

Uniqueness theorems. Solutions wFE of the mixed problem lie in

$$E_m(\Omega)=E(\Omega)\cap\{u \mid u=0 \text{ on } S_1,\ u^\nu=0 \text{ on } S_2,\ u^\tau=0 \text{ on } S_3\} \tag{4.16}$$

and satisfy (4.15) for all $v\in E_m(\Omega)$. The strain energy theorem for the problem is

$$W_\Omega=\frac{1}{2}\int_\Omega \sigma_{ij}(u)e_{ij}(u)\,dx+\frac{1}{2}\int_{S_5}\beta u_i u_i\,dS=\frac{1}{2}\int_\Omega F_i u_i\,dx \tag{4.17}$$

where the first equation defines the strain energy for the mixed problem. The uniqueness of solutions wFE is an immediate corollary. Solutions with prescribed stresses or displacements at infinity will be defined by (2.9) and (2.10), respectively, as in the surface tractions problem. Moreover, the strain energy theorem, Theorem 3.4, extends to solutions wLFE of the mixed problem. In fact, the same proof is valid because if $u\in E_m^{loc}(\Omega)$ and $\phi\in C_0^\infty(R^3)$ then $v=\phi u\in E_m^{com}(\Omega)=E_m(\Omega)\cap E^{com}(\Omega)$. The uniqueness of solutions wLFE of the mixed problem with prescribed displacements at infinity is an immediate corollary. It can also be shown that the displacement fields for the mixed problem are unique except in the special case of pure surface tractions boundary condition ($S_4=\partial\Omega$).

Inhomogeneous bodies. The uniqueness and energy theorems given above remain valid if the constant stress strain tensor c_{ijkl} is replaced by a field $c_{ijkl}(x)$ that is Lebesgue measurable in Ω and satisfies (1.9). The interior and boundary regularity theorems of Section 2 are valid when $c_{ijkl}(x)$ has sufficient differentiability in Ω and $\bar{\Omega}$, respectively; cf. [1, p. 132].

n-Dimensional problems. Fichera [5] has developed his theory for an n-dimensional generalization of the equations of elastostatics. All of the theorems given above extend to this n-dimensional problem with only notational changes. The cases $n=1$

and $n = 2$ are applicable to elastostatic fields that are functions of only one or two of the Cartesian coordinates.

5. A discussion of related literature

Fichera's paper [4] of 1950 provided the first significant extension of Kirchhoff's uniqueness theorem to unbounded domains. His result (3.8) implies that equilibrium fields in homogeneous isotropic bodies in exterior domains have finite energy if the displacements vanish at infinity. The uniqueness of equilibrium fields in such bodies is an immediate corollary. Corresponding results for fields whose stresses vanish at infinity follow from the 1961 result (3.9) of Gurtin and Sternberg [10]. The author knows of no general uniqueness results for anisotropic bodies in exterior domains or for bodies whose boundary is unbounded.

In Fichera's monograph [5] of 1965 the existence and uniqueness of classical solutions to elastostatic equilibrium problems in bounded domains with smooth boundaries is proved by the methods of functional analysis. This provides an alternative to the classical integral equation methods cited in the introduction. However, the formulation and techniques employed by Fichera can provide more general results. Fichera's semi-weak solutions (Lecture 7) are essentially the solutions wFE of this paper. Hence, Fichera's results (Lectures 7 and 12) imply the uniqueness of solutions wFE for bounded domains and boundary conditions for which Korn's inequality is valid. For the zero surface displacements problem the inequality holds for every bounded domain. For the zero surface tractions problem it holds for domains with the cone property.

The literature on uniqueness theorems in linear elastostatics up to 1970 was surveyed in a monograph by R. J. Knops and L. E. Payne [13] published in 1971. This work also contains uniqueness theorems for a class of weak solutions. However, the hypothesis that the displacement fields are continuous in $\bar{\Omega}$ restricts the scope of these results.

Uniqueness theorems for plane crack problems were proved by J. K. Knowles and T. A. Pucik in 1973 [14] under the assumption that the displacements are bounded, but not necessarily continuous, at the crack tips. The elegant differential inequality method used in this work provided the inspiration for the proof of Theorem 3.4.

The methods employed in this paper to prove uniqueness theorems for solutions wLFE in arbitrary domains were introduced by the author during the period 1962–64 in a series of papers on boundary value problems of the theory of wave propagation [25, 26, 27]. The article [27] contains as a special case uniqueness theorems for elastodynamic problems in arbitrary domains.

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