

FINAL REPORT
ON THE INTERACTION BETWEEN STRESS WAVES
AND CRACKS IN LAMINATED COMPOSITE PLATES

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ABSTRACT

This thesis concerns wave propagation in composite materials, and the effects of these waves on interlaminar cracking. In addition to general results, there are numerical examples for a specific material.

First, several types of waves in a layered composite material are examined. In order to obtain explicit, if approximate, solutions, a perturbation method is used. The apparently novel approach views the nonhomogeneous layering as a perturbation on a homogeneous but anisotropic solid. Two waves are examined that travel normal to the laminae. The longitudinal wave is a general travelling wave, while the transverse wave has, for short wavelenghts, a surprising inverse quarter power amplitude modulation by the perturbation function. Two waves that travel parallel to the laminae are examined. These waves have a tumbling structure. All these waves are also examined numerically, and there is excellent agreement between the perturbation expansions and the numerical result. Expressions are then arrived at for stresses along the interface between two layers.

Second, an interlaminar crack is "placed" in the material. The above interlaminar stresses, restricted to a short interlaminar segment, are taken to be the boundary conditions for a new linear elasticity problem. The idea is to use this solution to "cancel" the stresses from the previous problem along a short line segment. As this line segment is then free of stress it behaves as if it were a crack in the material. The approach to this problem is the assumption of a Fourier form for the solution, which leads to a dual integral equation, which leads to a Cauchy singular integral equation for the displacements along the crack face. An explicit solution for the homogeneous anisotropic crack problem is obtained in this fashion,

and the nonhomogeneous case is discussed.

It is thought that this information could be of use in choosing quality control parameters for the manufacture of these composite materials.

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CHAPTER 1

INTRODUCTION

This thesis studies wave propagation in layered solids, and the effect of some of these waves on a pre-existing crack. This introduction describes what will be discussed, and also looks at some of the literature.

1.1 Wave Propagation

There has been some work done on wave propagation in layered solids. Some early work was done with a view towards understanding earthquakes, as the earth can be viewed as a layered solid and the earthquakes as a wave passing through it. In [Ivakin 1960], the author studies wave propagation in a periodically layered material by an analogy with electric circuits. It involves matching impedances at each boundary and is rather tedious. In fact, the author concludes his book with "... the determination of the velocity of propagation and of the amplitudes of sinusoidal waves in fine-scale nonhomogeneous media constitutes a laborious problem ..." (page 108).

More recently, some work has been done particularly with reference to artificial materials such as carbon composites. In [SAH 1968], Sun, Achenbach and Herrmann present a continuum theory and display dispersion curves for various waves in a layered material. They linearly expand the displacements about the midplanes of the layers, and then require that some continuity conditions be satisfied at the layer boundaries. An expression for the energy is derived, and Hamilton's principle is used to obtain (approximate) equations of motion.

In two papers, [HN 1973] [HB 1974], Hegemier and Nayfeh and Hegemier and Bache develop a continuum theory for layered media with a similar approach. The latter paper, in fact, attacked the problem of waves travelling at an arbitrary angle in a layered solid and included phase velocity versus angle of propagation curves.

In a series of two papers in 1975, Ben-Amoz examined wave propagation in a direction parallel to the laminae and normal to the laminae ([BA 1975]). This was done by assuming relative orders of magnitude for the solution, and then discarding the pieces of the solid equation which would then be negligible. He concludes that the behavior of a laminated composite material is "predominantly that of a macroscopically homogeneous medium" (page 43).

The above shows that the problem in this monograph has been examined. The solution methods are tedious, and always use intermediate notions rather than attacking the partial differential equations of linear elasticity directly. It is hoped that this monograph will present a straightforward approach to the propagation of waves in a layered solid, and that the behavior of such waves will be apparent.

The solution technique is to reduce a nonconstant coefficient linear partial differential equation which describes the material to an ordinary differential equation, and then apply perturbation theory to the latter. Linear elasticity is reviewed with the aim of making sure the equations are set up correctly. Where, exactly, do the coefficients of Hooke's law lie in the equation? It is necessary to know this as nonconstant coefficients cannot be pulled out of differential expressions. Also, numerical solutions are found for the ordinary differential equation by Galerkin's method. The numerical results are compared with the perturbation results, and good agreement is viewed as a sign of accuracy of both. The existence of a nonlinear ordinary differential equation is discussed.

1.2 The Crack

As failure of materials usually occurs at pre-existing cracks, the way waves affect the crack will be examined. The displacements on a pre-existing crack face will be calculated. For more general information on fracture the reader is referred to some excellent texts, such as [Broek 1978] and [Knott 1973]. Here, since the material is brittle, crack opening displacement is taken as a measure of "badness" for a crack situation.

The approach to the crack is rather interesting in that by assuming a Fourier form for the solution, one is lead first to a dual integral equation and then directly to a Cauchy singular integral equation. This approach gives the displacements for the complementary crack problem directly for a homogeneous anisotropic material. The complementary crack problem is one in which the tractions are specified on the faces of the crack.

The homogeneous anisotropic crack is considered in detail, and an explicit expression for the displacements inside the crack and along the line containing the crack is obtained. It is viewed as the lowest order perturbation term to the complete problem, that of an interlaminar crack for the layered solid. This latter problem has proved to be quite difficult, but is discussed in a qualitative fashion (an approximate solution is given). There has been work done on the problem of a crack at the junction of two isotropic half planes, and the reader is referred to [England 1965] and [Rice and Sih 1965] for details.

CHAPTER 2

LINEAR ELASTICITY

First, the mathematical model to be used will be discussed. In linear elasticity, one takes a Newtonian momentum balance and then, through a general Hooke's law, arrives at equations of motion. The general Hooke's law, which is based on experimental observation, relates the forces in the momentum balance to the metric tensor of the map from the initial unstrained (undeformed) material configuration to a strained (deformed) material configuration, and in linear elasticity all the nonlinear terms of this metric tensor are discarded. The justification for this is that if there is little change in the shape of the body, these nonlinear terms should be negligible. This chapter breezes through these arguments and begins to set up the equations which will be studied in this monograph. Throughout this chapter the summation convention will be used.

2.1 The Momentum Balance

The section is mostly taken from Malvern's book (see [Malvern 1969]). To apply Newton's third law, one looks at a mass and sets the forces upon it equal to the change in momentum. Considered here is a continuous medium, so the mass will be a small volume of material.

The material is described by

$$X_i(x_1^0, x_2^0, x_3^0, t), \quad i = 1, 2, 3$$

where (x_1^0, x_2^0, x_3^0) is some original configuration which is time independent, and t is time. Let the volume $V(X)$ of material be enclosed by a surface $S(X)$. This volume in the medium is arbitrary. It seems prudent to point out that this is a fixed amount of material and it will be treated as if it were a "particle" when applying Newton's law. This is *not* a fixed volume in space. On the surface $S(X)$ are certain surface "tractions" t_i , which are a force per unit area. These can be described by introducing a stress tensor T_{ji} , where $t_i = T_{ji}n_j$ with n_j being the unit normal to the surface. With ρ being the density of the material, and writing external forces such as gravity as f_i , the momentum balance looks like

$$\int_{S(X)} t_i dS(X) + \int_{V(X)} \rho f_i dV(X) = \frac{d}{dt} \int_{V(X)} \rho \frac{dX_i}{dt} dV(X).$$

The $\frac{dX_i}{dt}$ refers to the velocity of the material, and one can see that this equation is just force equals the time derivative of momentum for a volume of material. In the first term, replace t_i by $T_{ji}n_j$, and use the divergence theorem:

$$\int_{V(X)} \left(\frac{\partial T_{ji}}{\partial X_j} \right) dV(X) + \int_{V(X)} \rho f_i dV(X) = \frac{d}{dt} \int_{V(X)} \rho \frac{dX_i}{dt} dV(X).$$

To deal with the rightmost term, needed is the following version of

REYNOLD'S TRANSPORT THEOREM: *If A is a function of X and ρ is the density of the material, then*

$$\frac{d}{dt} \int_{V(X)} \rho A dV(X) = \int_{V(X)} \rho \frac{dA}{dt} dV(X).$$

PROOF: First one transforms the integration to one which occurs over the original configuration (x_i^0) . Let J be the Jacobian determinant

$$J = \det\left(\frac{\partial(X_i)}{\partial(x_j^0)}\right).$$

Carrying out the change of variables gives

$$\frac{d}{dt} \int_{V(X)} \rho A dV(X) = \frac{d}{dt} \int_{V(x^0)} \rho A J dV(x^0).$$

As a fixed piece of material is being looked at, $V(x^0)$ is time independent and the time differentiation can be taken inside the integration. So

$$\frac{d}{dt} \int_{V(x^0)} \rho A J dV(x^0) = \int_{V(x^0)} \frac{d}{dt} (\rho A J) dV(x^0) = \int_{V(x^0)} \left(\frac{d(\rho J)}{dt} A + \rho \frac{dA}{dt} J \right) dV(x^0).$$

Consider a fixed subpiece $V'(X)$ of $V(X)$. If M' is the mass of the material, one has

$$M' = \int_{V'(X)} \rho(X) dV'(X) = \int_{V'(x^0)} \rho(X(x^0)) J dV'(x^0).$$

Since it is a fixed piece of material, the mass does not change:

$$\frac{dM'}{dt} = 0 = \frac{d}{dt} \int_{V'(x^0)} \rho(X(x^0)) J dV'(x^0) = \int_{V'(x^0)} \frac{d(\rho J)}{dt} dV'(x^0).$$

As the subpiece of material was arbitrary, this yields

$$\frac{d}{dt} (\rho J) = 0.$$

As an aside, this says that the density is inversely proportional to the Jacobian determinant.

Using this and transforming back to the original variables X_i gives

$$\int_{V(x^0)} \left(\frac{d(\rho J)}{dt} A + \rho \frac{dA}{dt} J \right) dV(x^0) = \int_{V(x^0)} \rho \frac{dA}{dt} J dV(x^0) = \int_{V(X)} \rho \frac{dA}{dt} dV(X)$$

which is the desired result. \square

Using this gives for the momentum balance

$$\int_{V(X)} \left(\frac{\partial T_{ji}}{\partial X_j} \right) dV(X) + \int_{V(X)} \rho f_i dV(X) = \int_{V(X)} \rho \frac{d^2 X_i}{dt^2} dV(X).$$

As the original volume was an arbitrary one in the material, this yields Cauchy's equations of motion:

$$\frac{\partial T_{ji}}{\partial X_j} + \rho f_i = \rho \frac{d^2 X_i}{dt^2}. \quad (2.1)$$

This will lead to the equation of motion which will be used.

The next step is to find a relation (by defining it) between the stress tensor and the strains (deformations) in the material.

2.2 Linearizing the Metric Tensor

If the deformation of the material is viewed as a map (which can be time dependent) from some initial configuration x_i to some final configuration X_i , then one can view the resulting metric tensor as a measure of the deformation of the body, or a measure of the “strains” as they are called. So let g_{ij} be the metric tensor of

$$x_i \longrightarrow X_i.$$

Now with solids the interest is in how much the body has deformed, or changed from its original shape, so X_i is written as $x_i + u_i$, where u_i is the amount of deformation. Writing g_{ij} in terms of u_i gives

$$g_{ij} = \frac{\partial X_k}{\partial x_i} \frac{\partial X_k}{\partial x_j} = \left(\frac{\partial(x_k + u_k)}{\partial x_i} \right) \left(\frac{\partial(x_k + u_k)}{\partial x_j} \right).$$

Carrying out the differentiation yields

$$\begin{aligned} g_{ij} &= \left(\delta_{ik} + \frac{\partial u_k}{\partial x_i} \right) \left(\delta_{jk} + \frac{\partial u_k}{\partial x_j} \right) \\ &= \delta_{ik} \delta_{jk} + \delta_{ik} \frac{\partial u_k}{\partial x_j} + \delta_{jk} \frac{\partial u_k}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \\ &= \delta_{ij} + \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j}. \end{aligned}$$

In linear elasticity, the nonlinear rightmost term is dropped, and the strains ϵ_{ij} are defined to be

$$\epsilon_{ij} \equiv \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

to arrive at

$$g_{ij} = \delta_{ij} + 2\epsilon_{ij}. \quad (2.2)$$

The strains, the ϵ_{ij} , are the linear or first order deviations of the deformation mapping as compared with the identity map.

2.3 A General Hooke's Law

It seems reasonable to suppose that the material will respond with a force to try and return to its original configuration. Writing these forces in terms of the ε_{ij} is a general form of Hooke's law for springs. Recall that Hooke's law specifies a restoring force by multiplying the distance of the spring from its equilibrium position by a spring constant:

$$F = -kx.$$

The same thing occurs here. One specifies a restoring stress (a force per area of the surface) as a linear function of the ε_{ij} .

The way this is done is as follows. Let ε be the matrix of the ε_{ij} . In a deformed configuration, the eigenvectors of the ε are called the principal directions. If the material is one such that when it is stressed along any one of the coordinate axes the resulting principal directions are the coordinate axes, then one can write the way in which the material will respond as follows (see [Malvern 1969]):

$$\begin{pmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{21} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{31} & c_{32} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{pmatrix} \quad (2.3)$$

where the c_{ij} 's are material constants. The six terms are sufficient as the stress tensor is assumed symmetric. For those familiar with the equations of homogeneous isotropic linear elasticity, in that case the off diagonal terms are equal to $E\nu/(1 + 2\nu)(1 - 2\nu)$ where E is Young's modulus and ν is Poisson's ratio, c_{44}, c_{55}, c_{66} are equal to G , the shear modulus, and c_{11}, c_{22}, c_{33} are equal to $c_{12} + c_{44}$.

To obtain the equations of motion, replace the stress tensor terms T_{ij} in Cauchy's equations of motion with the above linear combinations of ε 's. This will be done in a subsequent section, but first is briefly considered a surprising result which occurs because of approximating the metric tensor as above.

2.4 Some Geometry

For those who enjoy differential geometry, the first job in a specific situation is evaluating the Christoffel symbols. Recall (see [Kreyszig 1968])

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k} \right)$$

are the Christoffel symbols of the first kind. Evaluating them using Equation 2.2, where the δ_{ij} 's are constants and have derivative zero, gives

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial^2 u_j}{\partial x_i \partial x_k} + \frac{\partial^2 u_k}{\partial x_i \partial x_j} + \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{\partial^2 u_k}{\partial x_i \partial x_j} - \frac{\partial^2 u_i}{\partial x_j \partial x_k} - \frac{\partial^2 u_j}{\partial x_i \partial x_k} \right).$$

On the right side the first and third terms cancel with the last two, and the remaining terms are the same, yielding the simple

$$\Gamma_{ijk} = \frac{\partial^2 u_k}{\partial x_i \partial x_j}.$$

This is, of course, the nice result. If one wants to find the curvature tensor, one will need the Christoffel symbols of the second kind also. In terms of the first Christoffel symbol and the metric tensor these are

$$\Gamma_{ij}^k = g^{kl} \Gamma_{ijl}.$$

Needed is g^{kl} which is the inverse of the metric tensor.

If g_{ij} is written as the square matrix g , δ_{ij} as the square matrix I , and ε_{ij} as the square matrix ε , then

$$g = I + 2\varepsilon$$

is the equivalent of Equation 2.2. Since each of the ε_{ij} is assumed small, this is easily inverted by a power series expansion:

$$g^{-1} = I - 2\varepsilon + (2\varepsilon)^2 - (2\varepsilon)^3 + \dots$$

To be consistent the nonlinear terms should be discarded (besides, they are supposed to be small), which gives a simple expression for the inverse of g :

$$g^{ij} = \delta_{ij} - 2\varepsilon_{ij} = \delta_{ij} - \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

This is all one needs to calculate the curvature tensor,

$$R_{jhkl} = \frac{\partial \Gamma_{jlh}}{\partial x_k} - \frac{\partial \Gamma_{jkh}}{\partial x_l} + \Gamma_{jk}^i \Gamma_{hli} - \Gamma_{jl}^i \Gamma_{hki}.$$

If the u_i are smooth enough to exchange the differentiation for third derivatives, the first two terms on the right cancel, leaving

$$R_{jhkl} = \Gamma_{jk}^i \Gamma_{hli} - \Gamma_{jl}^i \Gamma_{hki} = g^{im} (\Gamma_{jkm} \Gamma_{hli} - \Gamma_{jlm} \Gamma_{hki}).$$

However, this does not turn out to be simple and will not be pursued.

2.5 The Solid Equations

The background has been developed to write down the equations describing the behavior of a solid, according to the theory of linear elasticity. As will be seen, many terms are yet to be discarded, and at times it might appear that one is knee deep in approximations, but solutions to these equations agree well with experimental observations. Two types of approximations are used: first, all the nonlinear terms will be discarded; and second, it will be assumed that $\frac{\partial u_i}{\partial X_i}$ is small compared to 1, that is

$$1 + \frac{\partial u_i}{\partial X_i} \approx 1. \quad (2.4)$$

Recall Equation 2.1:

$$\frac{\partial T_{ji}}{\partial X_j} + \rho f_i = \rho \frac{d^2 X_i}{dt^2}.$$

When the T_{ij} are replaced by the expressions in Equation 2.3, the first difficulty occurs. This is because the u_i are in terms of the undeformed configuration x_i , and not the deformed configuration X_i . Let A be some function, and one has

$$\frac{\partial A}{\partial X_i} = \frac{\partial A}{\partial x_j} \frac{\partial x_j}{\partial X_i} = \frac{\partial A}{\partial x_j} \frac{\partial (X_j - u_j)}{\partial X_i} = \frac{\partial A}{\partial x_j} \left(\delta_{ij} - \frac{\partial u_j}{\partial X_i} \right).$$

So if $A = T_{jk}$ where T_{jk} is related to ε_{jk} by Hooke's law (Equation 2.3), and then all nonlinear terms are discarded, one has

$$\frac{\partial T_{jk}}{\partial X_i} = \frac{\partial T_{jk}}{\partial x_i}.$$

This is the first approximation.

Second, the density ρ depends on the deformed configuration. Let J be the Jacobian determinant

$$J = \det\left(\frac{\partial(x_i)}{\partial(X_j)}\right).$$

This is the opposite of the Jacobian used in Section 1, since now the map is from x_i to X_i , while there it was from X_i to x_i . As the density is inversely proportional to the Jacobian determinant,

$$\rho = \frac{\rho_0}{J}$$

where ρ_0 is the density in the original state. Thus

$$\rho = \frac{\rho_0}{J} = \rho_0 \det\left(\frac{\partial(X_i)}{\partial(x_j)}\right) = \rho_0 \det\left(I + \frac{\partial(u_i)}{\partial(x_j)}\right) = \rho_0\left(1 + \frac{\partial u_i}{\partial x_i} + \dots\right)$$

where the \dots are nonlinear terms. After discarding these nonlinear terms one is left with

$$\rho = \rho_0\left(1 + \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\right)$$

and based on the above assumption in Equation 2.4, the equation will reduce to $\rho = \rho_0$.

Finally, the X_i in the last term depend on the x_i , which are assumed time independent, and t . So the derivative becomes a partial derivative with respect to time, or

$$\frac{d^2 X_i}{dt^2} = \frac{\partial^2 X_i}{\partial t^2} = \frac{\partial^2(x_i + u_i)}{\partial t^2} = \frac{\partial^2(u_i)}{\partial t^2}.$$

Putting all these together gives

$$\frac{\partial T_{ji}}{\partial x_i} + \rho_0 f_i = \rho_0 \frac{\partial^2 u_i}{\partial t^2}. \quad (2.5)$$

Now the equations are in terms of the fixed coordinate system and time.

Before using Hooke's law to put the equations entirely in terms of the u_i as unknowns, some definitions will be made. It is customary to have

$$u \equiv u_1, \quad v \equiv u_2, \quad w \equiv u_3.$$

Also, to simplify the appearance of the equations, let

$$\partial_i \equiv \frac{\partial}{\partial x_i}.$$

Substitution into Equation 2.5 of Hooke's law, Equation 2.3, now yields

$$\begin{aligned} \partial_1(c_{11}\epsilon_{11} + c_{12}\epsilon_{22} + c_{13}\epsilon_{33}) + 2\partial_2(c_{66}\epsilon_{12}) + 2\partial_3(c_{55}\epsilon_{13}) + \rho f_1 &= \rho u_{tt} \\ 2\partial_1(c_{66}\epsilon_{12}) + \partial_2(c_{21}\epsilon_{11} + c_{22}\epsilon_{22} + c_{23}\epsilon_{33}) + 2\partial_3(c_{44}\epsilon_{23}) + \rho f_2 &= \rho v_{tt} \quad (2.6) \\ 2\partial_1(c_{55}\epsilon_{13}) + 2\partial_2(c_{44}\epsilon_{23}) + \partial_3(c_{31}\epsilon_{11} + c_{32}\epsilon_{22} + c_{33}\epsilon_{33}) + \rho f_3 &= \rho w_{tt} \end{aligned}$$

where use has been made of the symmetry of the index in ϵ_{ij} , u_{tt} means $\frac{\partial^2 u}{\partial t^2}$, and ρ has been written for ρ_0 .

These are the equations of motion. The c_{ij} may depend on x_i . Also, recall the definition of the strains

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

to see that these are three equations in terms of three unknowns.

In the next chapter these equations will be written out completely in terms of the u_i for the case being considered.

CHAPTER 3

COMPOSITE MATERIALS

In this chapter composite materials are discussed and the equations are set up which will be used to examine them.

3.1 A Specific Material

The authors were aware of experiments being performed on a carbon composite manufactured by Hercules, Inc., namely Hercules A.S. 4/3501-6. The mathematical analysis in this monograph applies to many composite materials, but in the numerical work this is the specific material which will be considered. For other materials the material parameters ("constants") differ.

The material in question is made of thin carbon fibers (very thin – they are about 10 microns, or .001 centimeters, in diameter) which are placed in an epoxy prepreg and pressed into thin sheets, where the fibers all run in one direction. These sheets are then stacked so that in each layer the fibers run perpendicularly to the fibers in the previous layer. At this point the sheets are heated and pressed together, so that the epoxy is like a continuous matrix supporting fibers which run in two different directions. Thus the material is of the form $0^\circ/90^\circ/0^\circ/90^\circ \dots$. Each layer is quite thin, as a plate of this material one quarter inch thick has about 25 layers.

To give the material constants for this composite, take a homogeneous block of material in which the fibers run in only one direction, say x_1 . Experimentally the

coefficient matrix for one batch of the material, corresponding to Equation 2.3, is

$$\begin{pmatrix} 20.83 & 0.6233 & 0.6702 & 0 & 0 & 0 \\ 0.6233 & 1.557 & 0.5850 & 0 & 0 & 0 \\ 0.6702 & 0.5850 & 1.718 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.502 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.71 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.71 \end{pmatrix} \quad (3.1)$$

with the units being million pounds (force) per square inch. As the fibers only run in one direction, there is nearly the expected symmetry for x_2 and x_3 (as the manufacturing process treats x_2 differently from x_3 , one direction being in the sheet and the other normal to it, some differences are to be expected). The density is 0.0571 pounds (mass) per cubic inch. Also, one should keep in mind that these constants are only representative, as each batch of the material has different material constants.

3.2 The Model Used

The material described above is considered as follows. Let x_1 and x_3 be the directions in which the fibers lie, and let the sheets be stacked in the x_2 direction. The sheets each have a thickness $h/2$ (so that two sheets have a thickness h).

In order to make the problem tractable, the choice is to look at a slice through the plate in the x_1, x_2 plane. So assume that none of the displacements (the u_i or u, v, w) depends on the x_3 direction, and that $w = u_3 = 0$. These assumptions imply that $\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = 0$, since, for example,

$$\epsilon_{13} = \frac{1}{2} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \frac{1}{2} \left(0 + \frac{\partial 0}{\partial x_1} \right) = 0.$$

This situation is called "plane strain" and effectively reduces the three dimensional problem to a two dimensional one. Also, external forces (like gravity) will not be

dealt with, so $f_i = 0$. Equations 2.6 become

$$\partial_1(c_{11}\epsilon_{11} + c_{12}\epsilon_{22}) + 2\partial_2c_{66}\epsilon_{12} = \rho u_{tt}$$

$$2\partial_1c_{66}\epsilon_{12} + \partial_2(c_{21}\epsilon_{11} + c_{22}\epsilon_{22}) = \rho v_{tt}$$

$$0 = 0.$$

As can be seen, all the terms in the last equation drop out, and left are two equations in terms of the two unknowns u and v . There are five material constants left. The general Hooke's law can be written

$$\begin{pmatrix} T_{11} \\ T_{22} \\ T_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{pmatrix}. \quad (3.2)$$

With these assumptions, the problem has been reduced to this. There are two different layers alternating with spacing $h/2$, which stack in the x_2 direction (see Figure 3.1). Let layer 1 have fibers running in the x_1 direction, and layer 2 have fibers running in the x_2 direction. Then their material parameters are as follows:
layer 1,

$$\begin{pmatrix} T_{11} \\ T_{22} \\ T_{12} \end{pmatrix} = \begin{pmatrix} 20.83 & .6702 & 0 \\ .6702 & 1.718 & 0 \\ 0 & 0 & 0.710 \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{pmatrix}; \quad (3.3)$$

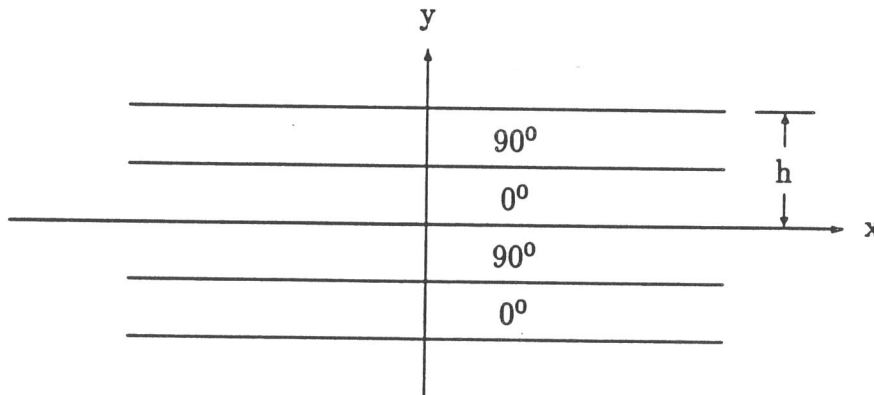


Figure 3.1. The material set up.

and layer 2,

$$\begin{pmatrix} T_{11} \\ T_{22} \\ T_{12} \end{pmatrix} = \begin{pmatrix} 1.557 & 0.5850 & 0 \\ 0.5850 & 1.718 & 0 \\ 0 & 0 & 0.502 \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{pmatrix}. \quad (3.4)$$

In both cases the units are million pounds (force) per square inch.

Suppose one is trying to find waves travelling in the above medium. One way of viewing the problem is as separate layers where the above sets of constants hold, and trying to match the solutions in each layer at the layer boundaries. This is very difficult, but it is the way the problem is usually attacked. Another approach is to choose to write the parameters as functions which are position dependent. Then there are no boundaries, but the partial differential equations no longer have constant coefficients.

This latter approach is the method used here. The purpose of writing down the above constants for a specific case was to show that they can be approximated as follows:

$$\begin{pmatrix} T_{11} \\ T_{22} \\ T_{12} \end{pmatrix} = \begin{pmatrix} c_1 + d_1 p(\alpha y) & c_3 + d_3 p(\alpha y) & 0 \\ c_3 + d_3 p(\alpha y) & c_2 + d_2 p(\alpha y) & 0 \\ 0 & 0 & c_6 + d_6 p(\alpha y) \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{pmatrix}, \quad \alpha = \frac{2\pi}{h}. \quad (3.5)$$

The position of each expression in the matrix is a definition of its value. The c_i are constants and are thought of as the average values of their corresponding terms in layers 1 and 2. The $p(\alpha y)$ is a periodic function of $y \equiv x_2$ which contains the structure of the layers. For the two-layered case considered here,

$$p(\alpha y) = \begin{cases} 1, & 0 \leq \alpha y < \pi \\ -1, & \pi \leq \alpha y < 2\pi. \end{cases}$$

The d_i are seen to be the deviations from the average of the coefficients in each layer. For the material constants given at the beginning of the chapter, one would

have

$$\begin{aligned} c_1 &= 11.1900 & d_1 &= 9.6350 \\ c_2 &= 1.7180 & d_2 &= 0.0000 \\ c_3 &= 0.6276 & d_3 &= 0.0426 \\ c_6 &= 0.6060 & d_6 &= 0.1040 \end{aligned}$$

with the units being million pounds (force) per square inch. In the next chapter a continuous p will be used for a perturbation expansion, so a Fourier sine expansion is suggested

$$p_{\text{sugg}}(\alpha y) = \sum_{n=0}^{\infty} p_n \sin(n\alpha y). \quad (3.6)$$

Now it is seen why h is the spacing between *two* layers. So if the $\{p_n\}$ correspond to the expansion of the periodic square function of amplitude one, as the above p , then

$$p_n = \begin{cases} 4/\pi n, & n \text{ odd} \\ 0, & n \text{ even.} \end{cases}$$

For an approximation to the periodic square function, one truncates this series.

So writing the coefficients as in Equation 3.5 the solid equations become

$$\partial_1((c_1 + d_1 p)\varepsilon_{11} + (c_3 + d_3 p)\varepsilon_{22}) + 2\partial_2((c_6 + d_6 p)\varepsilon_{12}) = \rho u_{tt}$$

$$2\partial_1((c_6 + d_6 p)\varepsilon_{12}) + \partial_2((c_3 + d_3 p)\varepsilon_{11} + (c_2 + d_2 p)\varepsilon_{22}) = \rho v_{tt}.$$

Noting that p only depends on y this becomes

$$(c_1 + d_1 p)\partial_1 \varepsilon_{11} + (c_3 + d_3 p)\partial_1 \varepsilon_{22} + 2\partial_2((c_6 + d_6 p)\varepsilon_{12}) = \rho u_{tt}$$

$$2(c_6 + d_6 p)\partial_1 \varepsilon_{12} + \partial_2((c_3 + d_3 p)\varepsilon_{11} + (c_2 + d_2 p)\varepsilon_{22}) = \rho v_{tt}.$$

Finally the ε_{ij} terms are replaced with their corresponding partials of u and v to yield

$$(c_1 + d_1 p)\partial_1^2 u + (c_3 + d_3 p)\partial_1 \partial_2 v + \partial_2((c_6 + d_6 p)(\partial_2 u + \partial_1 v)) = \rho u_{tt}$$

$$(c_6 + d_6 p)\partial_1(\partial_2 u + \partial_1 v) + \partial_2((c_3 + d_3 p)\partial_1 u + (c_2 + d_2 p)\partial_2 v) = \rho v_{tt}.$$

These are the equations to be used. However, in both layer 1 and layer 2 the c_{22} are the same, so it will also be taken that $d_2 = 0$.

3.3 Possibilities

In this monograph the periodic square will be the “understood” underlying $p(\alpha y)$ function. However, an approach using such a function to describe the inhomogeneity of the material makes it possible to model many situations. By picking appropriate p , one could describe layers of different thickness, layers with glue between the layers, or layers of completely different material. In the following chapters, periodicity of p is assumed, namely a period of 2π .

One can deal with a lot of structure with this simple device. The only caution would be that the equations derived here assumed that the principal stresses aligned themselves with the principal strains (see the sentences before Equation 2.3). So for really complex structure, it may be that many of the constants assumed to be zero in the Hooke’s law given here are not really zero, and the equations become much more complicated. This occurs, for example, if a layer is placed at 45° to a previous layer.

CHAPTER 4

SOME WAVES IN THE LAMINAE

In this chapter special cases of waves in the composite material are examined: in particular, two waves that propagate parallel to the laminae and two waves that propagate normal (perpendicular) to the laminae.

Recall from the previous chapter the solid equations: let u and v be the displacements in the x and y directions respectively, and

$$\begin{aligned}(c_1 + d_1 p)\partial_1^2 u + \partial_2((c_6 + d_6 p)\partial_2 u) + (c_3 + d_3 p)\partial_1 \partial_2 v + \partial_2((c_6 + d_6 p)\partial_1 v) &= \rho u_{tt} \\(c_6 + d_6 p)\partial_1 \partial_2 u + \partial_2((c_3 + d_3 p)\partial_1 u) + (c_6 + d_6 p)\partial_1^2 v + c_2 \partial_2^2 v &= \rho v_{tt}\end{aligned}\tag{4.1}$$

where the c_i 's, d_i 's, and ρ (the density) are constants. p is a function of αy and $\alpha = 2\pi/h$, where h is the spacing between two layers in the material. Thus the function $p(\alpha y)$ represents the inhomogeneity in the solid and can be anything, but recall the $p_{sugg}(\alpha y)$ which was a Fourier sine series. In general, $u(x, y, t)$ and $v(x, y, t)$ are functions of x , y , and t , but each of the cases discussed below makes some assumptions on the form u and v can take.

Finally some notes on waves. The form of the solution will generally be

$$f = \sin(m(\varphi(y) - t)).$$

where in this case the wave is propagating in the y direction. The frequency ν of the wave, or number of oscillations per unit time for a fixed point in space is then

$2\pi\nu = m \cdot 1$ or

$$\text{frequency} = \nu = \frac{m}{2\pi}.$$

The wave speed is found by following a constant angle or phase in the sine term, and is often called the phase speed. So with

$$m(\varphi(y) - t) = \text{constant}$$

one has

$$\frac{d\varphi}{dy} \frac{dy}{dt} = 1$$

or

$$\text{wave (phase) speed} = \frac{dy}{dt} = \frac{1}{\varphi'}$$

where the prime denotes differentiation with respect to y .

4.1 Normal Longitudinal Waves

The simplest case is a longitudinal wave travelling normal to the laminae. Here, assume $u = 0$ to reduce Equation 4.1 to

$$\begin{aligned} (c_3 + d_3p)\partial_1\partial_2v + \partial_2((c_6 + d_6p)\partial_1v) &= 0 \\ (c_6 + d_6p)\partial_1^2v + c_2\partial_2^2v &= \rho v_{tt}. \end{aligned}$$

If it is assumed that there is no x dependence in v , this equation reduces to

$$c_2\partial_2^2v = \rho v_{tt}$$

which has the simple solution

$$v = f\left(\sqrt{\frac{\rho}{c_2}}y - t\right), \quad (4.2)$$

f being a twice differentiable function.

So longitudinal waves travelling normal to the laminae are general travelling waves.

4.2 Normal Transverse Waves

The next case is a transverse wave travelling normal to the laminae. Assuming a solution of the form

$$\begin{aligned} u &= \eta(y)f(\varphi(y) - t) \\ v &= 0. \end{aligned} \tag{4.3}$$

Equations 4.1 yield

$$\partial_2((c_6 + d_6p)\partial_2 u) = \rho u_{tt}.$$

Carrying out the substitution $u = \eta f$ gives

$$\partial_2((c_6 + d_6p)\{\eta' f + \eta f' \varphi'\}) = \rho \eta f''$$

where the prime denotes differentiation. Expanding the left hand side gives

$$d_6 p' \{\eta' f + \eta f' \varphi'\} + (c_6 + d_6 p) \{\eta'' f + 2\eta' f' \varphi' + \eta (f'' (\varphi')^2 + f' \varphi'')\} = \rho \eta f''. \tag{4.4}$$

The next step is to remove the f' term. To do this requires that the coefficient of the f' term be zero, or that

$$d_6 p' \eta \varphi' + (c_6 + d_6 p) \{2\eta' \varphi' + \eta \varphi''\} = 0.$$

If this is divided by $(c_6 + d_6 p)\eta\varphi'$ one obtains

$$\frac{d_6 p'}{c_6 + d_6 p} + 2\frac{\eta'}{\eta} + \frac{\varphi''}{\varphi'} = 0.$$

This can be integrated to give

$$\varphi' = \frac{A}{(c_6 + d_6 p)\eta^2}$$

where A is a constant of integration (A is nonzero as it is the exponential of the actual integration constant). Placing this rather nice result back in Equation 4.4 gives

$$d_6 p' \eta' f + (c_6 + d_6 p) \left\{ \eta'' f + \eta f'' \left(\frac{A}{(c_6 + d_6 p)\eta^2} \right)^2 \right\} = \rho \eta f''$$

and with some gathering of terms and cancellation gives

$$((c_6 + d_6 p)\eta')' f + \frac{A^2}{(c_6 + d_6 p)\eta^3} f'' = \rho \eta f''.$$

This equation has periodic boundary conditions with period h . To proceed further, some assumptions on f must be made. Suppose that $f'' = -m^2 f$ as it would if it were complex exponential. This yields

$$((c_6 + d_6 p(\alpha y))\eta')' - \frac{A^2 m^2}{(c_6 + d_6 p(\alpha y))\eta^3} + m^2 \rho \eta = 0 \quad (4.5)$$

as an equation for $\eta(y)$. Now fix m , where m can be any real number.

In this case the wave (phase) speed is

$$\text{wave speed} = \frac{1}{\varphi'} = \frac{\eta^2 (c_6 + d_6 p)}{A}.$$

This leads one to wonder about the effect of the constant A in the η equation. Let η be a solution with A_η as the constant, and let μ be a solution with A_μ as the constant. Clearly $A_\mu = c A_\eta$ for some c (recall that A is nonzero), and making the substitution in the μ equation gives

$$((c_6 + d_6 p)\mu')' - \frac{c^2 A_\eta^2 m^2}{(c_6 + d_6 p)\mu^3} + m^2 \rho \mu = 0.$$

If this is divided by \sqrt{c} it is concluded that $\mu/\sqrt{c} = \eta$ and that

$$\frac{A_\mu}{\mu^2} = \frac{c A_\eta}{(\sqrt{c} \eta)^2} = \frac{A_\eta}{\eta^2}$$

so A is quite arbitrary as far as physical meaning goes. In the following, therefore, let $A = 1$.

As nonlinear equations are hard to solve, a perturbation approach is used to linearize the equation and glean some information on η . ε will be used as an expansion parameter to obtain equations in various powers of ε , after which ε will be set equal to 1. Suppose

$$\eta \approx \eta^0 + \varepsilon \eta^1.$$

This leads to

$$\frac{1}{\eta^3} = \frac{1}{(\eta^0 + \varepsilon\eta^1)^3} \approx \frac{1}{(\eta^0)^3} (1 - 3\varepsilon \frac{\eta^1}{\eta^0}).$$

Using a similar expansion for the coefficient involving p (let p be written εp), one has

$$\frac{1}{c_6 + \varepsilon d_6 p(\alpha y)} \approx \frac{1}{c_6} (1 - \varepsilon \frac{d_6}{c_6} p(\alpha y)).$$

Putting these in Equation 4.5 gives

$$((c_6 + \varepsilon d_6 p)(\eta^0 + \varepsilon\eta^1)')' - m^2 \left\{ \frac{1}{c_6} (1 - \varepsilon \frac{d_6}{c_6} p) \right\} \left\{ \frac{1}{(\eta^0)^3} (1 - 3\varepsilon \frac{\eta^1}{\eta^0}) \right\} + m^2 \rho (\eta^0 + \varepsilon\eta^1) = 0.$$

Separating and equating powers of ε yields

$$\begin{aligned} \varepsilon^0: \quad & c_6(\eta^0)'' - \frac{m^2}{c_6(\eta^0)^3} + m^2 \rho \eta^0 = 0 \\ \varepsilon^1: \quad & c_6(\eta^1)'' + (d_6 p(\eta^0)')' + \frac{m^2}{c_6(\eta^0)^3} \left(\frac{d_6}{c_6} p + \frac{3}{\eta^0} \eta^1 \right) + m^2 \rho \eta^1 = 0. \end{aligned}$$

There is a simple solution to the ε^0 equation, namely

$$\eta^0 = (\rho c_6)^{-1/4},$$

a constant. This of course simplifies the ε^1 equation giving

$$c_6(\eta^1)'' + 4m^2 \rho \eta^1 = -\rho^{3/4} \frac{m^2 d_6}{c_6^{5/4}} p.$$

This equation has periodic boundary conditions of period h .

This equation, then, is of the form $u'' + \lambda u = f$ and a Green's function may be built for it. $\lambda_0 = 0$ is an eigenvalue with $u_0 = 1/\sqrt{h}$ an eigenfunction. Also

$$\lambda_n = \frac{4\pi^2 n^2}{h^2}, \quad u_n = \sqrt{\frac{2}{h}} \cos\left(\frac{2n\pi}{h} y\right), \quad u_n = \sqrt{\frac{2}{h}} \sin\left(\frac{2n\pi}{h} y\right)$$

are eigenvalues and eigenfunctions. So the Green's function is

$$G(y, \zeta, \lambda) = \sum_{n=0}^{\infty} \frac{u_n(y) \overline{u_n(\zeta)}}{\lambda - \lambda_n} = \frac{1}{h\lambda} + \frac{2}{h} \sum_{n=1}^{\infty} \frac{\cos(n\alpha y) \cos(n\alpha \zeta) + \sin(n\alpha y) \sin(n\alpha \zeta)}{\lambda - n^2 \alpha^2}$$

recalling $\alpha = 2\pi/h$. With $\lambda = 4\frac{m^2\rho}{c_6}$ this can be used for any p , in particular for the one suggested in Equation 3.6 to obtain

$$\begin{aligned}\eta^1(y) &= -\rho^{3/4}\frac{m^2d_6}{c_6^{9/4}}\int_0^h G(y,\zeta,4\frac{m^2\rho}{c_6})p_{sugg}(\alpha\zeta)d\zeta \\ &= -\rho^{3/4}\frac{m^2d_6}{c_6^{9/4}}\sum_{n=1}^{\infty}p_n\frac{1}{4\frac{m^2\rho}{c_6}-n^2\alpha^2}\sin(n\alpha y).\end{aligned}$$

Of course there is an easier way to arrive at this last result if one is using the p_{sugg} Fourier sine series. Assume a form of solution for η^1 , namely

$$\eta^1(y) = \sum_{n=1}^{\infty} a_n \sin(n\alpha y).$$

A substitution and use of orthogonality of the $\sin(n\alpha y)$ gives

$$-c_6n^2\alpha^2a_n + 4m^2\rho a_n = -\rho^{3/4}\frac{m^2d_6}{c_6^{5/4}}p_n$$

or that

$$a_n = -\rho^{3/4}\frac{d_6}{c_6^{5/4}}(4\rho - c_6n^2\frac{\alpha^2}{m^2})^{-1}p_n. \quad (4.6)$$

As will also occur for the parallel waves, $n\frac{\alpha}{m}$ always occur together. α/m is a measure of the frequency of the wave to the "frequency" of the material. α is large for thin layers and m is large for high frequency. For this case the perturbation is expected to be good for both large and small $n\frac{\alpha}{m}$. Equation 4.6 implies that a_n is approximately a constant times p_n for small $n\frac{\alpha}{m}$, and that a_n is roughly inversely proportional to $n\frac{\alpha}{m}$ for large $n\frac{\alpha}{m}$.

The only problem is when $n\frac{\alpha}{m}$ is around $2\sqrt{\rho/c_6}$. To come up with a physical interpretation for this, consider for the ε^0 perturbation:

$$m(\varphi(y) - t) \approx m(\varphi^0 y - t) = m\left(\frac{1}{(\eta^0)^2 c_6} y - t\right) = m\left(\sqrt{\frac{\rho}{c_6}} y - t\right).$$

The peak to peak distance ("wavelength") of this wave is given by

$$m\sqrt{\frac{\rho}{c_6}} y_{peak\ to\ peak} = 2\pi.$$

So

$$y_{\text{peak to peak}} = \sqrt{\frac{c_6}{\rho}} \frac{2\pi}{m}.$$

Now at the zero in the denominator of a_n one has

$$n \frac{\alpha}{m} = 2 \sqrt{\frac{\rho}{c_6}} = 2 \frac{2\pi}{m \cdot y_{\text{peak to peak}}},$$

or that, since $\alpha = 2\pi/h$,

$$y_{\text{peak to peak}} = \frac{2}{n} h.$$

Thus the perturbation fails when the wavelength is about $2h/n$, which is admittedly rather odd, since $2h$ is *four* layers. This will be examined, in a mathematical way, in a later chapter.

This at least gives a better feel for the solution. It is interesting to note that for high frequencies (large m), Equation 4.5 gives

$$\eta \approx (\rho(c_6 + d_6 p(\alpha y)))^{-1/4} \quad (4.7)$$

and for low frequencies one has $\eta \approx (\rho c_6)^{-1/4}$. An interesting sidelight from Equation 4.6 is that for large m and small n ,

$$a_n \approx -\rho^{-1/4} \frac{d_6}{4c_6^{5/4}} p_n,$$

which gives

$$\eta \approx \eta^0 + \eta^1 \approx (\rho c_6)^{-1/4} - \sum_{\text{truncated}} \rho^{-1/4} \frac{d_6}{4c_6^{5/4}} p_n \sin(n\alpha y) \approx (\rho c_6)^{-1/4} - \frac{d_6}{4\rho^{1/4} c_6^{5/4}} p(\alpha y).$$

This is the first term of the Taylor's expansion of Equation 4.7. Thus the perturbation solution agrees with the large m limit solutions.

In summary, it is of some practical interest to note that at high frequencies there is a surprising inverse quarter power amplitude modulation of the wave, while at low frequencies the wave hardly notices the inhomogeneity of the material at all.

4.3 Waves Propagating Parallel to the Laminae

Here there do not exist purely longitudinal or purely transverse waves, but the same idea applies. A perturbation method is used to obtain some approximate solutions.

First, the following form of the solution is assumed:

$$\begin{aligned} u &= \eta(y) \cos(m(\lambda x - t)) \\ v &= \mu(y) \sin(m(\lambda x - t)) \end{aligned} \quad (4.8)$$

where λ is a constant. This form is chosen since a travelling wave is being looked for, and the solution is expected to have a y dependence in the amplitude.

It seems wise to elucidate some assumptions in the above equations. It has been assumed that the actual solid “particles” follow an elliptical path. To see this, fix y and note that

$$\frac{u^2}{\eta(y)^2} + \frac{v^2}{\mu(y)^2} = 1.$$

As will become apparent, the “longitudinal” wave will have a major axis in the direction the wave travels, while the “transverse” wave will have a major axis normal to the direction of wave propagation. At the end of the chapter this assumption will be loosened somewhat. On a physical note, it has been assumed that the frequency is not so high that the layers are acting as waveguides. Waveguiding will occur at high frequencies, meaning the wave will separate in each of the layers, the wave in one layer travelling faster than the wave in the other layer. Though the assumed form of solution leads to solutions of the solid equations, in an experimental situation these type high frequency waves would be very difficult to produce, as waveguiding would more naturally occur.

The assumed u and v are placed in the original solid equations, Equation 4.1,

to give

$$\begin{aligned}
& (c_1 + d_1 p)(-m^2 \lambda^2) \eta \cos(\cdot) + \partial_2(c_6 + d_6 p) \eta' \cos(\cdot) \\
& + (c_3 + d_3 p) m \lambda \mu' \cos(\cdot) + \partial_2(c_6 + d_6 p) m \lambda \mu \cos(\cdot) = -m^2 \rho \eta \cos(\cdot) \\
& (c_6 + d_6 p)(-m \lambda) \eta' \sin(\cdot) + \partial_2(c_3 + d_3 p)(-m \lambda) \eta \sin(\cdot) \\
& + (c_6 + d_6 p)(-m^2 \lambda^2) \mu \sin(\cdot) + c_2 \mu'' \sin(\cdot) = -m^2 \rho \mu \cos(\cdot)
\end{aligned}$$

where \cdot corresponds to $m(\lambda x - t)$ and the prime denotes differentiation with respect to y . The $\cos(\cdot)$ can be factored out of the first equation and the $\sin(\cdot)$ can be factored out of the second equation, since they do not depend on y .

This is what is left:

$$\begin{aligned}
& (c_1 + d_1 p)(-m^2 \lambda^2) \eta + ((c_6 + d_6 p) \eta')' \\
& + (c_3 + d_3 p) m \lambda \mu' + m \lambda ((c_6 + d_6 p) \mu)' = -m^2 \rho \eta \\
& (c_6 + d_6 p)(-m \lambda) \eta' - m \lambda ((c_3 + d_3 p) \eta)' \\
& + (c_6 + d_6 p)(-m^2 \lambda^2) \mu + c_2 \mu'' = -m^2 \rho \mu.
\end{aligned} \tag{4.9}$$

These are coupled ordinary differential equations, with periodic boundary conditions.

Once again the system only seems amenable to a perturbation approach, so this is what is done. Think of inserting an ε in front of the p term, and use an expansion of the form

$$\begin{aligned}
\eta & \approx \eta^0 + \varepsilon \eta^1 \\
\mu & \approx \mu^0 + \varepsilon \mu^1.
\end{aligned}$$

The zero order (ε^0) equation is

$$\begin{aligned}
& -c_1 m^2 \lambda^2 \eta^0 + c_6 (\eta^0)'' + (c_3 + c_6) m \lambda (\mu^0)' = -m^2 \rho \eta^0 \\
& -(c_6 + c_3) m \lambda (\eta^0)' - c_6 m^2 \lambda^2 \mu^0 + c_2 (\mu^0)'' = -m^2 \rho \mu^0.
\end{aligned} \tag{4.10}$$

Let $a = c_3 + c_6$ as this term will occur again. The first order (ε^1) equation is

$$\begin{aligned}
 -c_1 m^2 \lambda^2 \eta^1 + c_6 (\eta^1)'' + a m \lambda (\mu^1)' + m^2 \rho \eta^1 & \\
 = m^2 \lambda^2 d_1 p \eta^0 - (d_6 p (\eta^0)')' - m \lambda d_3 p (\mu^0)' - m \lambda (d_6 p \mu^0)' & \\
 -a m \lambda (\eta^1)' - c_6 m^2 \lambda^2 \mu^1 + c_2 (\mu^1)'' + m^2 \rho \mu^1 & \\
 = m \lambda d_6 p (\eta^0)' + m \lambda (d_3 p \eta^0)' + m^2 \lambda^2 d_6 p \mu^0. &
 \end{aligned} \tag{4.11}$$

There are two simple solutions to the equations, first

$$\eta_1^0 = 1, \quad \mu_1^0 = 0, \quad \lambda_1 = \sqrt{\frac{\rho}{c_1}} \tag{4.12}$$

where the $\eta_1^0 = 1$ is an arbitrary selection as the equations are linear, and

$$\eta_2^0 = 0, \quad \mu_2^0 = 1, \quad \lambda_2 = \sqrt{\frac{\rho}{c_6}}. \tag{4.13}$$

Similar to the normal wave, λ is inversely proportional to the wave speed, and it is immediately seen that the above two solutions are two different types of waves, as they travel at different speeds. This is, of course, not unexpected: transverse and longitudinal waves in a homogeneous, isotropic solid also travel at different speeds.

4.4 Parallel Longitudinal Waves

This section examines the longitudinal waves, or those arising from Equation 4.12. The first order perturbation equation (Equation 4.11) becomes

$$\begin{aligned}
 c_6 (\eta_1^1)'' + a m \sqrt{\frac{\rho}{c_1}} (\mu_1^1)' &= m^2 \rho \frac{d_1}{c_1} p(\alpha y) \\
 -a m \sqrt{\frac{\rho}{c_1}} (\eta_1^1)' + c_2 (\mu_1^1)'' + (1 - \frac{c_6}{c_1}) m^2 \rho \mu_1^1 &= m \sqrt{\frac{\rho}{c_1}} d_3 p'(\alpha y).
 \end{aligned} \tag{4.14}$$

For the suggested $p_{sugg}(\alpha y)$ one has a solution by assuming the following forms:

$$\begin{aligned}
 \eta_1^1(y) &= \sum_{n=1}^{\infty} a_n \sin(n\alpha y) \\
 \mu_1^1(y) &= \sum_{n=1}^{\infty} b_n \cos(n\alpha y).
 \end{aligned}$$

Placement of these in the first order Equations 4.14 gives a system for the a_n and b_n , namely

$$\begin{aligned} -c_6 n^2 \alpha^2 a_n - am \sqrt{\frac{\rho}{c_1}} n \alpha b_n &= m^2 \rho \frac{d_1}{c_1} p_n \\ -am \sqrt{\frac{\rho}{c_1}} n \alpha a_n + \{-c_2 n^2 \alpha^2 + (1 - \frac{c_6}{c_1}) m^2 \rho\} b_n &= m \sqrt{\frac{\rho}{c_1}} d_3 n \alpha p_n. \end{aligned}$$

Solution of these gives the first order perturbation for the longitudinal case. Dividing both equations by m^2 leaves the equations in a form where m and α do not exist independently, but only the ratio $\frac{\alpha}{m}$ occurs,

$$\begin{pmatrix} -c_6 n^2 \frac{\alpha^2}{m^2} & -a \sqrt{\frac{\rho}{c_1}} n \frac{\alpha}{m} \\ -a \sqrt{\frac{\rho}{c_1}} n \frac{\alpha}{m} & -c_2 n^2 \frac{\alpha^2}{m^2} + (1 - \frac{c_6}{c_1}) \rho \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} \rho \frac{d_1}{c_1} p_n \\ \sqrt{\frac{\rho}{c_1}} d_3 n \frac{\alpha}{m} p_n \end{pmatrix}. \quad (4.15)$$

Solution of this system gives the a_n and the b_n .

Thus the coefficients a_n and b_n only depend on the ratio α/m , which measures the wavelength of the wave in comparison to the spacing of the layers of material. For large α/m , meaning low frequencies, the a_n and b_n are given approximately by

$$\begin{aligned} a_n &\approx \frac{\rho}{c_1} \frac{ad_3}{c_2 c_6} \frac{m^2}{n^2 \alpha^2} p_n \\ b_n &\approx -\sqrt{\frac{\rho}{c_1}} \frac{d_3}{c_2} \frac{m}{n \alpha} p_n. \end{aligned} \quad (4.16)$$

The perturbation terms a_n and b_n are small when α/m is large, or when the wavelength of the wave is large compared to the spacing of layers.

The wave travels at the root mean square average speed in the material. To see this, recall c_1 is the average of the c_{11} in the layers, and

$$\left(\frac{dy}{dt}\right)^2 = \left(\frac{1}{\lambda}\right)^2 = \frac{c_1}{\rho} = \frac{c_{11,layer 1} + c_{11,layer 2}}{2\rho}.$$

The wave has a tumbling structure. It travels in the high speed layer and then tumbles into the low speed layer, which results in the average wave speed observed.

To justify the statement about the longitudinal waves corresponding to ellipses with the major axis parallel to the axis of propagation, η_1^1 and μ_1^1 are small, and

$$\begin{aligned}\eta_1(y) &\approx 1 + \eta_1^1(y) \\ \mu_1(y) &\approx \mu_1^1(y).\end{aligned}$$

Recalling that

$$\frac{u^2}{(\eta(y))^2} + \frac{v^2}{(\mu(y))^2} = 1$$

it is seen that u is the major axis, and u corresponds to material displacement in the x direction.

4.5 Parallel Transverse Waves

For the transverse wave, from Equations 4.11 and 4.13, the first order perturbation is

$$\begin{aligned}c_6(\eta_2^1)'' + m^2\rho\left(1 - \frac{c_1}{c_6}\right)\eta_2^1 + am\sqrt{\frac{\rho}{c_6}}(\mu_2^1)' &= -m\sqrt{\frac{\rho}{c_6}}d_6p'(\alpha y) \\ -am\sqrt{\frac{\rho}{c_6}}(\eta_2^1)' + c_2(\mu_2^1)'' &= \rho\frac{d_6}{c_6}m^2p(\alpha y)\end{aligned}$$

and, proceeding as above save that the roles of η^1 and μ^1 are reversed, let

$$\begin{aligned}\eta_2^1(y) &= \sum_{n=1}^{\infty} a_n \cos(n\alpha y) \\ \mu_2^1(y) &= \sum_{n=1}^{\infty} b_n \sin(n\alpha y)\end{aligned}$$

and the equation solved by the coefficients is (already dividing through by m^2)

$$\begin{pmatrix} -c_6n^2\frac{\alpha^2}{m^2} + \rho\left(1 - \frac{c_1}{c_6}\right) & a\sqrt{\frac{\rho}{c_6}}n\frac{\alpha}{m} \\ a\sqrt{\frac{\rho}{c_6}}n\frac{\alpha}{m} & -c_2n^2\frac{\alpha^2}{m^2}b_n \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{\rho}{c_6}}n\frac{\alpha}{m}d_6p_n \\ \rho\frac{d_6}{c_6}p_n \end{pmatrix}. \quad (4.17)$$

As in the longitudinal case, the wave speed is the root mean squared average of the wave speeds of transverse waves in the two layers.

Here, as

$$\eta_2(y) \approx \eta_2^1(y)$$

$$\mu_2(y) \approx 1 + \mu_2^1(y)$$

the major axis of displacement is v , or perpendicular to the direction of wave propagation, which is why these waves are called transverse waves.

4.6 The Other Cases

This section records the results supposing one had started with

$$u = \eta(y) \sin(m(\lambda x - t))$$

$$v = \mu(y) \cos(m(\lambda x - t)).$$

It is seen that the above results go through if one replaces

$$\sqrt{\frac{\rho}{c_1}} \text{ by } -\sqrt{\frac{\rho}{c_1}}.$$

So the longitudinal wave is

$$\eta_3(y) \approx 1 + \sum a_n \sin(n\alpha y)$$

$$\mu_3(y) \approx \sum b_n \cos(n\alpha y)$$

where the a_n and b_n are given by Equation 4.15 with the stated replacement. The transverse wave is

$$\eta_4(y) \approx \sum a_n \cos(n\alpha y)$$

$$\mu_4(y) \approx 1 + \sum b_n \sin(n\alpha y)$$

where the a_n and b_n are given by Equation 4.17, this time with

$$\sqrt{\frac{\rho}{c_6}} \mapsto -\sqrt{\frac{\rho}{c_6}}.$$

4.7 The Parallel Wave

To summarize the preceding sections, let

$$\begin{aligned}
 H_1(y) &= \begin{pmatrix} \eta_1(y) & 0 \\ 0 & \mu_1(y) \end{pmatrix}, \\
 H_2(y) &= \begin{pmatrix} \eta_2(y) & 0 \\ 0 & \mu_2(y) \end{pmatrix}, \\
 H_3(y) &= \begin{pmatrix} 0 & \eta_3(y) \\ \mu_3(y) & 0 \end{pmatrix}, \\
 H_4(y) &= \begin{pmatrix} 0 & \eta_4(y) \\ \mu_4(y) & 0 \end{pmatrix}
 \end{aligned}$$

where H is a capital η . Then a more general wave travelling parallel to the laminae can be written

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{i=1}^4 a_i H_i(y) \begin{pmatrix} \cos(m(\lambda_i x - t)) \\ \sin(m(\lambda_i x - t)) \end{pmatrix}$$

where the a_i are constants.

This completes analytical analysis of the waves in the material.

CHAPTER 5

A NUMERICAL SOLUTION FOR THE NORMAL WAVE

This chapter involves the numerical solution of Equation 4.5, in order to verify the perturbation solution obtained previously and for interest in its own right. The solution will be based on the Galerkin method.

As a first step, the change of variable $z = \alpha y$ gives

$$\frac{\alpha^2}{m^2}((c_6 + d_6 p(z))\eta')' - \frac{1}{(c_6 + d_6 p(z))\eta^3} + \rho\eta = 0 \quad (5.1)$$

with periodic boundary conditions of period 2π , and where the prime denotes differentiation with respect to z .

A note on the choice of the numerical method. There are three basic approaches to solving a nonlinear boundary value problem. One is shooting, another relaxation, and then the Galerkin approach which gives rise to nonlinear algebraic equations. All these methods involve iteration to obtain a solution. The authors prefer the latter approach, as it separates the iteration process from the solution process of the differential equation.

5.1 In a Weak Sense

Let Ω be a region, and φ be a member of $F(\Omega)$, which is some set of functions on Ω . Then one says $f = 0$ in a weak sense (with respect to F) if

$$\int_{\Omega} \varphi f \, d\Omega = 0, \quad \text{all } \varphi \in F(\Omega).$$

Below f will be a differential operator $L(u)$. $f = 0$ in a weak sense does not imply that $f \equiv 0$, but if one chooses a suitable $F(\Omega)$, f should be "close" to zero in some sense.

5.2 The Galerkin Method

Galerkin's idea was to take a set of functions and produce a weak solution of the differential equation in the linear space spanned by those functions. So if $\{\varphi_i\}$ are the basis functions, $\sum a_i \varphi_i$ is a function in the linear space spanned by the basis functions. With $L(f) = 0$ the differential operator, Galerkin's method then requires

$$\int_{\Omega} \sum a_i \varphi_j L(\sum c_i \varphi_i) d\Omega = 0.$$

As this must hold for all $\{a_i\}$, this gives

$$\int_{\Omega} \varphi_j L(\sum c_i \varphi_i) d\Omega = 0, \quad \text{all } j.$$

For a finite basis set with N elements, this leads to a system of N equations for the N c_i 's. If L is a linear operator, then the equations are linear. If L is a nonlinear operator, then the equations are nonlinear.

5.3 Setting Up the Equations

In the case studied here, the operator L is nonlinear and is

$$L(\eta) \equiv \frac{\alpha^2}{m^2} ((c_6 + d_6 p(z)) \eta')' - \frac{1}{(c_6 + d_6 p(z)) \eta^3} + \rho \eta.$$

Assuming u and v are periodic with period 2π , an integration by parts gives

$$\int_0^{2\pi} v L(u) d\mu = -\frac{\alpha^2}{m^2} \int_0^{2\pi} v' (c_6 + d_6 p(z)) u' dz - \int_0^{2\pi} \left(\frac{v}{(c_6 + d_6 p(z)) u^3} - \rho v u \right) dz.$$

To use this, assume the φ_i are periodic. The weak solution in the subspace is written as

$$\eta(z) = \sum_{i=1}^N c_i \varphi_i(z).$$

It is best at an early stage to put this into a matrix notation, so let

$$\Phi = (\varphi_1 \cdots \varphi_N), \quad \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix}.$$

With the usual definition of matrix multiplication, this gives

$$\eta(z) = \Phi(z)\vec{c}$$

as the approximate solution. With Φ^T meaning the transpose of Φ , Galerkin's equations can be written

$$\int_0^{2\pi} \Phi^T L(\Phi\vec{c}) dz = \vec{0},$$

or

$$-\frac{\alpha^2}{m^2} \int_0^{2\pi} (c_6 + d_6 p(z)) (\Phi')^T (\Phi\vec{c})' dz - \int_0^{2\pi} \Phi^T \left(\frac{1}{(c_6 + d_6 p(z)) (\Phi\vec{c})^3} - \rho \Phi\vec{c} \right) dz = \vec{0}.$$

These are N equations for N unknowns.

The equations will be broken into linear and nonlinear terms to give

$$\int_0^{2\pi} \left\{ -\frac{\alpha^2}{m^2} (c_6 + d_6 p(z)) (\Phi')^T \Phi' + \rho \Phi^T \Phi \right\} \vec{c} dz - \int_0^{2\pi} \Phi^T \frac{1}{(c_6 + d_6 p(z)) (\Phi\vec{c})^3} dz = \vec{0}.$$

Introducing some notation, let the following $N \times N$ matrices be:

$$\begin{aligned} B_1 &\equiv \int_0^{2\pi} (\Phi')^T \Phi' dz = \int_0^{2\pi} \begin{pmatrix} \varphi'_1 \\ \vdots \\ \varphi'_N \end{pmatrix} (\varphi'_1 \cdots \varphi'_N) dz \\ &= \int_0^{2\pi} \begin{pmatrix} \varphi'_1 \varphi'_1 & \cdots & \varphi'_1 \varphi'_N \\ \vdots & & \vdots \\ \varphi'_N \varphi'_1 & \cdots & \varphi'_N \varphi'_N \end{pmatrix} dz, \\ pB_1 &\equiv \int_0^{2\pi} p(z) (\Phi')^T \Phi' dz, \\ B_4 &\equiv \int_0^{2\pi} \Phi^T \Phi dz, \\ pB_4 &\equiv \int_0^{2\pi} p(z) \Phi^T \Phi dz. \end{aligned}$$

Finally, letting the $N \times N$ matrix A be

$$A = -\frac{\alpha^2}{m^2} (c_6 B_1 + d_6 p B_1) + \rho B_4$$

$$(pB_4)_{ij} = \int_0^\pi \varphi_i \varphi_j dz - \int_\pi^{2\pi} \varphi_i \varphi_j dz.$$

and using

$$\int \sin(iz) \cos(jz) dz = -\frac{\cos((i-j)z)}{2(i-j)} - \frac{\cos((i+j)z)}{2(i+j)}, \quad i^2 \neq j^2$$

one has in the upper left mixed block

$$\begin{aligned} \pi \int_0^\pi \varphi_{i+1} \varphi_{j+n+1} dz &= \int_0^\pi \sin(iz) \cos(jz) dz \\ &= \begin{cases} \frac{2i}{i^2-j^2}, & i+j \text{ odd} \\ 0, & i+j \text{ even} \end{cases} \\ \pi \int_\pi^{2\pi} \varphi_{i+1} \varphi_{j+n+1} dz &= \int_\pi^{2\pi} \sin(iz) \cos(jz) dz \\ &= \begin{cases} -\frac{2i}{i^2-j^2}, & i+j \text{ odd} \\ 0, & i+j \text{ even.} \end{cases} \end{aligned}$$

Now define an $n \times n$ matrix P to be

$$\frac{1}{2}(P)_{ij} = \begin{cases} \frac{2i}{i^2-j^2}, & i+j \text{ odd} \\ 0, & i+j \text{ even} \end{cases}$$

and an n -vector \vec{u} to be

$$\frac{1}{2}(\vec{u})_i = \begin{cases} \sqrt{2}/i, & i \text{ odd} \\ 0, & i \text{ even} \end{cases}$$

as

$$\frac{1}{2}\vec{u} = \int_0^\pi \frac{1}{\sqrt{2}} \begin{pmatrix} \sin(z) \\ \sin(2z) \\ \vdots \\ \sin((n-1)z) \\ \sin(nz) \end{pmatrix} dz = - \int_\pi^{2\pi} \frac{1}{\sqrt{2}} \begin{pmatrix} \sin(z) \\ \sin(2z) \\ \vdots \\ \sin((n-1)z) \\ \sin(nz) \end{pmatrix} dz.$$

The cosine terms on the upper border and left border of the $\Phi^T \Phi$ matrix all give zeros upon integration. With these pB_4 can be written

$$pB_4 = \int_0^\pi \Phi^T \Phi dz - \int_\pi^{2\pi} \Phi^T \Phi dz$$

$$\begin{aligned}
&= \frac{1}{\pi} \left(\begin{array}{c|c|c} 0 & \frac{1}{2}\vec{u}^T & 0 \\ \hline \frac{1}{2}\vec{u} & \frac{1}{2}I & \frac{1}{2}P \\ \hline 0 & \frac{1}{2}P^T & \frac{1}{2}I \end{array} \right) - \frac{1}{\pi} \left(\begin{array}{c|c|c} 0 & -\frac{1}{2}\vec{u}^T & 0 \\ \hline -\frac{1}{2}\vec{u} & \frac{1}{2}I & -\frac{1}{2}P \\ \hline 0 & -\frac{1}{2}P^T & \frac{1}{2}I \end{array} \right) \\
&= \frac{1}{\pi} \left(\begin{array}{c|c|c} 0 & \vec{u}^T & 0 \\ \hline \vec{u} & 0 & P \\ \hline 0 & P^T & 0 \end{array} \right).
\end{aligned}$$

The above gives all the matrices need for the computations. With this periodic basis the periodic boundary conditions are automatically satisfied. If one has a different p , all that need be done differently is a calculation of the pB_4 matrix of integrals.

5.5 A Specific Numerical Result

Using the above basis, the nonlinear equations were solved using the Hybrd1 path in double precision MINPACK. This takes a user supplied initial guess and makes its first step by way of a Jacobian it calculates by forward differences. After this, the path uses a modification of the Powell hybrid method (see [Cowell 1984]). As to specifics, the integration on the right hand side of Equation 5.2 was done with the trapezoidal rule, with $10N$ point evaluations. With $N = 19$ convergence took about one minute of CPU time on a VAX 8600. The units of the constants were pounds (force) per square inch, so

$$c_6 = .606 \times 10^6, \quad d_6 = .104 \times 10^6,$$

To make the units of ρ consistent with these, use

$$1 \text{ pound(force)} = 32.2 \frac{\text{pounds(mass) feet}}{\text{second}^2}$$

to yield

$$\rho = 1.48 \times 10^{-4} \frac{\text{pounds(force) sec}^2}{\text{inch}^4}.$$

The initial guess was η^0 , or

$$\vec{c}_0 = (\rho c_6)^{-1/4} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The L_2 norm of the residuals \vec{r} is used as a measure of accuracy after convergence, where

$$\vec{r} = A\vec{c} - \int_0^{2\pi} \Phi^T \frac{1}{(c_6 + d_6 p(z))(\Phi\vec{c})^3} dz.$$

In Table 5.1, the results are listed for a specific case, namely for $\alpha/m = .0001$.

Shown is the comparison of the computed solution with the perturbation expansion truncated at two terms, η^0 and one term of η^1 ,

$$\eta \approx \eta^0 + \eta^1 \approx (\rho c_6)^{-1/4} + a_1 \sin(z),$$

and six terms, η^0 and five terms of η^1 ,

$$\eta \approx \eta^0 + \eta^1 \approx (\rho c_6)^{-1/4} + \sum_{n=1}^9 a_n \sin(nz).$$

There are six terms since the Fourier sine expansion of the periodic square function has p_n and therefore a_n equal to zero for even n . The percentage error is simply

$$\% \text{ error} = 100 \times \frac{\eta_{\text{perturbation}} - \eta_{\text{computed}}}{\eta_{\text{computed}}}.$$

The average wave (phase) speed is given by

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\varphi'} dz = \frac{1}{2\pi} \int_0^{2\pi} \eta_{\text{computed}}^2 (c_6 + d_6 p) dz$$

which was also computed using the trapezoidal rule with $10N$ terms. As an example of how one computes the wave speed, consider the case of $d_6 = 0$. Here $\eta =$

$(\rho c_6)^{-1/4}$, a constant, and so

$$\text{wave speed} = \eta^2(c_6 + d_6 p) = \sqrt{\frac{c_6}{\rho}} = \sqrt{\frac{0.606 \times 10^6 \frac{\text{pounds}(\text{force})}{\text{inch}^2}}{0.000148 \frac{\text{pounds}(\text{force})\text{sec}^2}{\text{inch}^4}}}$$

and the result is

$$\text{wave speed} = 64000 \frac{\text{inch}}{\text{sec}} = 5330 \frac{\text{feet}}{\text{sec}},$$

or a little over a mile a second.

5.6 Some General Numerical Results

Next, Table 5.2 shows some results for a range of α/m . The numerics are the same as above, and the average percentage error and maximum percentage error refer to the absolute values of the percentage error.

For $\log_{10}(\alpha/m) > -4$ the wave speed is constant, and the errors stay at the limiting values indicated in the table. This is also true for the region $\log_{10}(\alpha/m) < -6$. For values of $\log_{10}(\alpha/m)$ the accuracy of the perturbation depends on how close one is to an eigenvalue of the homogeneous problem, as is demonstrated in Figure 5.1. The relationship between the homogeneous problem and the numerical solution will be explored more fully in the next chapter.

And finally, Figure 5.2 is a wave (phase) speed plot. The particular interest is in the region $-6 < \log_{10}(\alpha/m) < -3$. A break in the lines means that the numerical method did not converge for an interior value of $\log_{10}(\alpha/m)$. Presumably, if more computations were done, each of the peaks would continue on to $+\infty$.

To give some idea of the wavelengths of the waves, recall the $y_{\text{peak to peak}}$ argument of the previous chapter. In the first approximation the wavelength of the wave is

$$\text{wavelength} = y_{\text{peak to peak}} = \sqrt{\frac{c_6}{\rho}} \frac{2\pi}{m} = \sqrt{\frac{c_6}{\rho}} \frac{h}{\alpha},$$

where h is the thickness of two layers. For a single layer thickness of .01 inch, the wavelength for $\alpha/m = 1$ is about 33 meters, and for $\alpha/m = 10^{-6}$, the wavelength is

Table 5.1. The numerical results for $\alpha/m = .0001$ ($\log_{10}(\frac{\alpha}{m}) = -4$).

z	Two Terms			Six Terms	
	$\eta_{comp}(z)$	$\eta_{pert}(z)$	% Error	$\eta_{pert}(z)$	% Error
0.3307	0.328356	0.325574	-0.8	0.325646	-0.8
0.6614	0.328817	0.326130	-0.8	0.326182	-0.8
0.9921	0.329152	0.326559	-0.8	0.326560	-0.8
1.3228	0.329335	0.326813	-0.8	0.326769	-0.8
1.6535	0.329384	0.326865	-0.8	0.326813	-0.8
1.9842	0.329279	0.326710	-0.8	0.326685	-0.8
2.3149	0.329043	0.326364	-0.8	0.326392	-0.8
2.6456	0.328652	0.325865	-0.8	0.325934	-0.8
2.9762	0.328142	0.325266	-0.9	0.325314	-0.9
3.3069	0.327396	0.324634	-0.8	0.324585	-0.9
3.6376	0.326614	0.324035	-0.8	0.323966	-0.8
3.9683	0.326053	0.323536	-0.8	0.323508	-0.8
4.2990	0.325675	0.323190	-0.8	0.323215	-0.8
4.6297	0.325516	0.323035	-0.8	0.323087	-0.7
4.9604	0.325554	0.323087	-0.8	0.323131	-0.7
5.2911	0.325810	0.323341	-0.8	0.323340	-0.8
5.6218	0.326261	0.323769	-0.8	0.323718	-0.8
5.9525	0.326920	0.324326	-0.8	0.324254	-0.8
6.2832	0.327724	0.324950	-0.8	0.324950	-0.8
L_2 norm of residuals = 0.16156447E-14					
Average phase speed = 5426 $\frac{feet}{sec}$					

Table 5.2. The perturbation solution compared with the computed solution for various α/m .

$\log_{10}(\alpha/m)$	Two Terms		Six Terms		Wave (phase) speed $\frac{feet}{sec}$	L_2 norm of Residuals
	% Error		% Error			
	Ave.	Max.	Ave.	Max.	Ave.	
0.000	0.7	0.7	0.7	0.7	5413	0.8432D-19
-1.000	0.7	0.7	0.7	0.7	5413	0.8438D-19
-2.000	0.7	0.7	0.7	0.7	5413	0.7107D-19
-3.000	0.7	0.7	0.7	0.7	5413	0.7936D-19
-4.000	0.8	0.9	0.8	0.9	5426	0.1616D-14
-4.540	7.7	10.2	7.7	10.2	6130	0.2097D-13
-5.000	11.5	22.1	5.1	12.6	5557	0.1267D-13
-5.500	2.8	7.7	0.7	1.8	5323	0.6172D-13
-6.000	1.4	4.1	0.5	1.0	5316	0.1299D-13
-7.000	1.3	4.0	0.5	1.0	5317	0.1101D-13
-8.000	1.3	4.0	0.5	1.0	5317	0.1098D-13
-9.000	1.3	4.0	0.5	1.0	5317	0.1098D-13
-10.000	1.3	4.0	0.5	1.0	5317	0.1098D-13

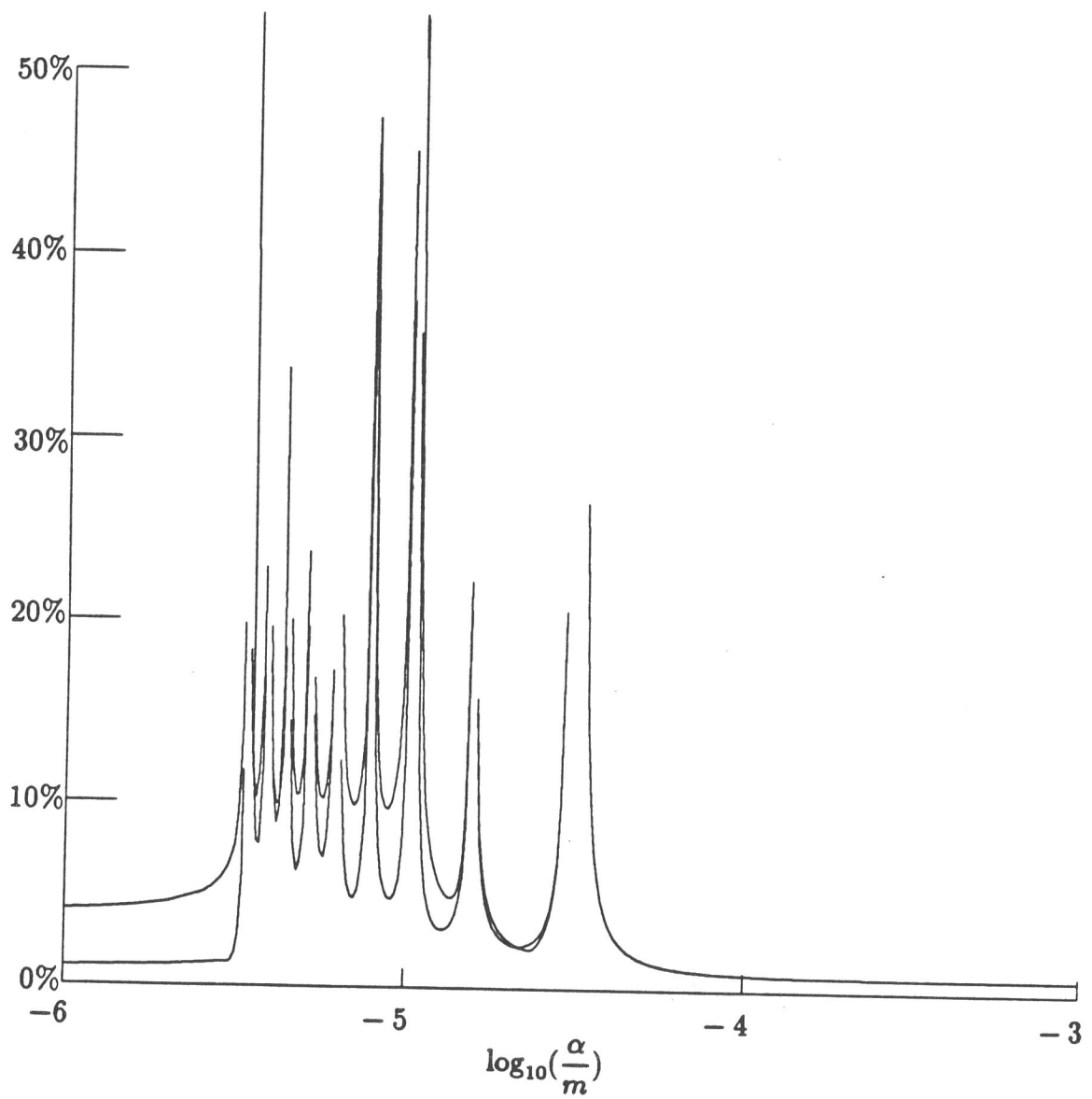


Figure 5.1. A plot of the maximum percentage difference between the two perturbation solutions and the numerical solution. The upper curve is the two term error, and the lower curve is the six term error.

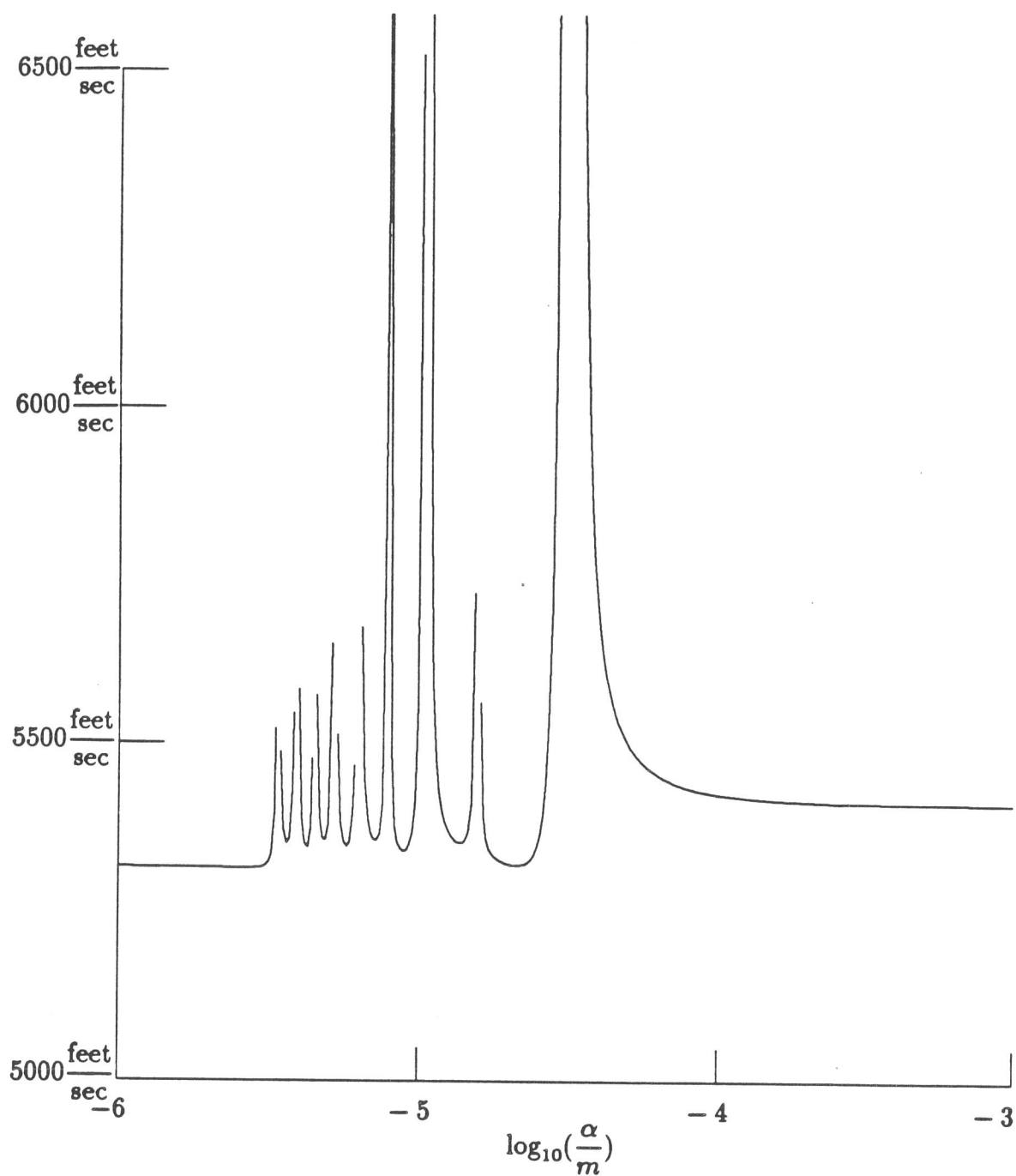


Figure 5.2. The phase speed plot for the transverse waves travelling normal to the laminae. The speed is in feet per second.

about .0033 centimeters. For higher frequencies than this, though some are listed in the table, a model should be used which takes into account the microstructure of the layer. The wavelengths in the region in which there are peaks in the phase speed are approximately given by the formula $2h/n$, for n an integer.

CHAPTER 6

A DISCUSSION OF EXISTENCE

This chapter discusses the existence of the nonlinear ordinary differential equation dealt with in the previous chapter. The existence of a solution is not proved, but a relationship is demonstrated between the failure of the numerical method and the eigenvalues of the linear portion of the equation.

6.1 The Equation

Equation 5.1 is, after dividing through by $c_6\alpha^2/m^2$,

$$((1 + rp(z))\eta')' - \frac{m^2/\alpha^2}{(1 + rp(z))\eta^3} + \left(\frac{\rho}{c_6} \frac{m^2}{\alpha^2}\right)\eta = 0, \quad r = \frac{d_6}{c_6}, \quad (6.1)$$

with periodic boundary conditions of period 2π , and where the prime denotes differentiation with respect to z .

In a paper by Lazer and Solimini ([Lazer and Solimini 1987]), the existence of a periodic solution to the equation

$$u'' - \frac{1}{u^\alpha} = h$$

is proven for $\alpha \geq 1$ and

$$\int_{\text{period}} h < 0.$$

Unfortunately, the proof depends upon the existence of a lower bound on u obtained by the existence of an upper bound on h (which exists because h is piecewise continuous on a closed interval). So the method of proof does not apply to the

equation considered. However, if a solution to Equation 6.1 did exist it would presumably be positive and so

$$\int_0^{2\pi} -\left(\frac{\rho}{c_6} \frac{m^2}{\alpha^2}\right) \eta \, dz < 0$$

and the corresponding $\alpha = 3 > 1$. So it seems reasonable for a solution to exist, by comparing forms.

6.2 A Green's Function

Though used earlier in this monograph, the Green's function will be formally developed here. Let

$$(fu')' + \lambda u = g, \quad f \geq C > 0$$

with C a constant. As the homogeneous part,

$$(fu')' + \lambda u = 0, \quad f \geq C > 0,$$

is a Sturm-Liouville problem, there exist a complete set of eigenvalues and eigenfunctions. Call these $\{\lambda_i\}$ and $\{\varphi_i\}$. If g is square integrable, it can be expressed in terms of the $\{\varphi_i\}$,

$$g = \sum_i g_i \varphi_i.$$

Assuming the solution u to be square integrable, one has

$$u = \sum_i u_i \varphi_i.$$

Using orthonormality of the eigenfunctions, the original equation yields

$$(\lambda - \lambda_i)u_i = g_i,$$

or that

$$u = \sum_i u_i \varphi_i = \sum_i \frac{g_i}{\lambda - \lambda_i} \varphi_i = \sum_i \frac{\int g(\zeta) \overline{\varphi_i(\zeta)} \, d\zeta}{\lambda - \lambda_i} \varphi_i(z).$$

In this, the integration is over the domain of u . If one exchanges the integration and the summation and lets

$$G(z, \zeta, \lambda) = \sum_i \frac{\varphi_i(z)\overline{\varphi_i(\zeta)}}{\lambda - \lambda_i}$$

then u can be written

$$u = \int G(z, \zeta, \lambda)g(\zeta) d\zeta$$

where G is called the Green's function.

If the nonlinearity in Equation 6.1 is ignored, the remainder is a periodic Sturm-Liouville problem as described above. So a Green's function exists and the equation could be written as

$$\eta(z) = \int_0^{2\pi} G(z, \zeta, \lambda) \frac{m^2/\alpha^2}{(1 + rp(\zeta))\eta^3(\zeta)} d\zeta.$$

This does not help a great deal in demonstrating existence, but it does indicate that when

$$\left(\frac{\rho}{c_6} \frac{m^2}{\alpha^2}\right)$$

equals an eigenvalue of

$$((1 + rp(z))u')' + \lambda u = 0 \tag{6.2}$$

a solution would not exist. The reasoning is that the Green's function does not exist there because of the $\lambda - \lambda_i$ term in the denominator, and so one should not expect a solution. However, as will be seen, there are twice as many values of α/m for which a solution does not exist than those indicated by this argument.

6.3 Finding the Eigenvalues

Following the above argument, the job becomes obtaining the eigenvalues of Equation 6.2. These are expected to be very close to the eigenvalues of periodic problem

$$u'' + \lambda u = 0,$$

period 2π , which has eigenvalues $0, 1, 1, 4, 4, \dots$

Using the notation of the last chapter, it is simple to set up a numerical scheme to approximately find them. With

$$u = \Phi \vec{c}$$

the Galerkin method gives for Equation 6.1 a corresponding algebraic eigenvalue problem of

$$(B_1 + rpB_1)\vec{c} = \lambda B_4\vec{c}.$$

This form of the eigenvalue problem can be solved by EISPACK, and was, with $N = 51$. EISPACK will be considered in a bit more detail in the next chapter.

6.4 The Results

As was mentioned in the last chapter, the method did not converge for some regions of α/m . In Table 6.1 are displayed some values in each region for which convergence did not occur, the value

$$2\sqrt{\frac{\rho}{c_6} \frac{m}{\alpha}},$$

the squareroots of the numerically computed eigenvalues of Equation 6.2, and finally the squareroots of the eigenvalues of

$$u'' + \lambda u = 0.$$

The correspondence is clear. In fact, the nonconvergence corresponding to $\lambda \approx 4$ was not found during the phase plot calculation, where the method converged for $\log_{10}(\alpha/m) = -5.11$ and -5.12 . Rather it was found by examining

$$\frac{\alpha}{m} = \frac{2}{\lambda} \sqrt{\frac{\rho}{c_6}}$$

for $\lambda \approx 4$, which gives $\log_{10}(\alpha/m) \approx -5.107$.

Table 6.1. A comparison of the eigenvalues of the homogeneous problem with some α/m where the numerical routine did not converge.

$\log_{10}(\frac{\alpha}{m})$	$2\sqrt{\frac{\rho}{c_6} \frac{m}{\alpha}}$	Squareroot of Eigenvalues	Squareroot of eigenvalues
		of $((1 + rp)u')' + \lambda u = 0$	of $u'' + \lambda u = 0$
		0.0000	0
-4.480	0.9439	0.9821	1
-4.500	0.9884	0.9962	1
-4.520	1.0350		
-4.800	1.9721	1.9646	2
		1.9919	2
-4.980	2.9849	2.9480	3
		2.9867	3
-5.105	3.9804	3.9327	4
		3.9801	4
-5.200	4.9537	4.9192	5
-5.210	5.0690	4.9718	5
-5.280	5.9556	5.9073	6
		5.9618	6
-5.350	6.9972	6.8974	7
		6.9499	7
-5.410	8.0339	7.8893	8
		7.9364	8
-5.470	9.2241	8.8828	9
		8.9211	9
		9.8777	10
		9.9050	10

A surprising item is the factor 2 which appeared in the perturbation and now again in the numerical solution. There is more here than

$$\left(\frac{\rho}{c_6} \frac{m^2}{\alpha^2}\right)$$

being an eigenvalue of the homogeneous problem. So a conjecture:

CONJECTURE 6.1: *The nonlinear equation, Equation 6.1, has solutions except when*

$$4\left(\frac{\rho}{c_6} \frac{m^2}{\alpha^2}\right)$$

equals an eigenvalue of the corresponding homogeneous equation, Equation 6.2.

It is the authors opinion that the numerical method gives solutions at the larger eigenvalues as a computational artifact, though (in the same breath) it is computational difficulties that produce regions of nonconvergence rather than a single point. A reminder, "nonconvergence" of the numerical method means MINPACK ceases iterations feeling it is not making sufficient progress.

CHAPTER 7

A NUMERICAL SOLUTION FOR THE PARALLEL WAVES

In this chapter the parallel waves are solved numerically. It is assumed that

$$u = \eta(y) \sin(m(\lambda x - t))$$

$$v = \mu(y) \cos(m(\lambda x - t))$$

so this would correspond to finding $H_3(y)$, $H_4(y)$, and also λ which is unknown. The Galerkin method will be used, as explained in a previous chapter, and it will give rise to a generalized eigenvalue problem of the form

$$(\lambda^2 A + \lambda B + C)\vec{c} = \vec{0},$$

where A , B and C are square symmetric matrices. Also, as in the previous chapter, the change of variables $z = \alpha y$ has been introduced.

7.1 The Equation

Let \vec{u} be a 2-vector of periodic functions of period 2π . Then the differential operator L associated with the parallel wave system is (see Equation 4.9)

$$L(\vec{u}) \equiv L_1(\vec{u}) + \lambda \frac{m}{\alpha} (L_2(\vec{u}) + L_3(\vec{u})) + \lambda^2 \frac{m^2}{\alpha^2} L_4(\vec{u}) + \rho \frac{m^2}{\alpha^2} \vec{u} = \vec{0},$$

where with D being the differential operator with respect to y ,

$$D \equiv \frac{d}{dy}$$

the L_i are

$$\begin{aligned} L_1(\vec{u}) &= \begin{pmatrix} D\{(c_6 + d_6p)D\} & 0 \\ 0 & c_2D^2 \end{pmatrix} \vec{u} \\ L_2(\vec{u}) &= \begin{pmatrix} 0 & -D(c_6 + d_6p) \\ D(c_3 + d_3p) & 0 \end{pmatrix} \vec{u} \\ L_3(\vec{u}) &= \begin{pmatrix} 0 & -(c_3 + d_3p)D \\ (c_6 + d_6p)D & 0 \end{pmatrix} \vec{u} \\ L_4(\vec{u}) &= \begin{pmatrix} -(c_1 + d_1p) & 0 \\ 0 & -(c_6 + d_6p) \end{pmatrix} \vec{u}. \end{aligned}$$

In L_1 , L_2 and L_3 the order of the differentiation is important.

Next, if \vec{v} is also a 2-vector of periodic functions of period 2π then the equations are

$$\int_0^{2\pi} \vec{v}^T L(\vec{u}) dz = 0.$$

Carrying out an integration by parts on the L_1 and the L_2 term gives

$$\int_0^{2\pi} [(\vec{v}')^T \{\hat{L}_1(\vec{u}) + \lambda \frac{m}{\alpha} \hat{L}_2(\vec{u})\} + \vec{v} \{\lambda \frac{m}{\alpha} L_3(\vec{u}) + \lambda^2 \frac{m^2}{\alpha^2} L_4(\vec{u}) + \rho \frac{m^2}{\alpha^2} \vec{u}\}] dz = 0, \quad (7.1)$$

where

$$\begin{aligned} \hat{L}_1(\vec{u}) &= \begin{pmatrix} -(c_6 + d_6p)D & 0 \\ 0 & -c_2D \end{pmatrix} \vec{u} \\ \hat{L}_2(\vec{u}) &= \begin{pmatrix} 0 & (c_6 + d_6p) \\ -(c_3 + d_3p) & 0 \end{pmatrix} \vec{u}. \end{aligned}$$

Nowhere is p differentiated! So p need be only piecewise continuous.

7.2 Galerkin's Equation

With the differential equation set up in a weak fashion as above, the stage is set for the Galerkin method. Let Φ be a $2 \times 2N$ matrix of 2π periodic basis functions, and \vec{c} a $2N$ -vector:

$$\tilde{\Phi} = \begin{pmatrix} \varphi_1 & \cdots & \varphi_N & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \varphi_1 & \cdots & \varphi_N \end{pmatrix}; \quad \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_{2N} \end{pmatrix}.$$

Then

$$\begin{pmatrix} \eta \\ \mu \end{pmatrix} = \tilde{\Phi} \tilde{c}$$

is the approximate solution. Placing this in Equation 7.1 gives

$$\int_0^{2\pi} [(\tilde{\Phi}')^T \{ \hat{L}_1(\tilde{\Phi}) + \lambda \frac{m}{\alpha} \hat{L}_2(\tilde{\Phi}) \} + \tilde{\Phi} \{ \lambda \frac{m}{\alpha} L_3(\tilde{\Phi}) + \lambda^2 \frac{m^2}{\alpha^2} L_4(\tilde{\Phi}) + \rho \frac{m^2}{\alpha^2} \tilde{\Phi} \}] \tilde{c} dz = \vec{0}, \quad (7.2)$$

as a $2N$ dimensional generalized eigenvalue problem to find λ and \tilde{c} . This is the Galerkin equation for the problem.

Now the problem becomes finding the various matrices defined above. The two \hat{L}_i 's will be done explicitly, as the other terms are similar.

$$\begin{aligned} \int_0^{2\pi} (\tilde{\Phi}')^T \hat{L}_1(\tilde{\Phi}) dz &= \int_0^{2\pi} (\tilde{\Phi}')^T \begin{pmatrix} -(c_6 + d_6 p)D & 0 \\ 0 & -c_2 D \end{pmatrix} \tilde{\Phi} dz \\ &= \int_0^{2\pi} \begin{pmatrix} \varphi_1 & 0 \\ \vdots & \vdots \\ \varphi_N & 0 \\ 0 & \varphi_1 \\ \vdots & \vdots \\ 0 & \varphi_N \end{pmatrix} \begin{pmatrix} -(c_6 + d_6 p)D & 0 \\ 0 & -c_2 D \end{pmatrix} \begin{pmatrix} \varphi_1 & \cdots & \varphi_N & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \varphi_1 & \cdots & \varphi_N \end{pmatrix} dz \\ &= \int_0^{2\pi} \begin{pmatrix} \varphi_1 & 0 \\ \vdots & \vdots \\ \varphi_N & 0 \\ 0 & \varphi_1 \\ \vdots & \vdots \\ 0 & \varphi_N \end{pmatrix} \begin{pmatrix} -(c_6 + d_6 p)\varphi'_1 & \cdots & -(c_6 + d_6 p)\varphi'_N & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -c_2\varphi'_1 & \cdots & -c_2\varphi'_N \end{pmatrix} dz \\ &= \int_0^{2\pi} \begin{pmatrix} -(c_6 + d_6 p)\varphi'_1\varphi'_1 & \cdots & -(c_6 + d_6 p)\varphi'_1\varphi'_N & & & \\ \vdots & & \vdots & & 0 & \\ -(c_6 + d_6 p)\varphi'_N\varphi'_1 & \cdots & -(c_6 + d_6 p)\varphi'_N\varphi'_N & & & \\ & & & -c_2\varphi'_1\varphi'_1 & \cdots & -c_2\varphi'_1\varphi'_N \\ & & & \vdots & & \vdots \\ & & & -c_2\varphi'_N\varphi'_1 & \cdots & -c_2\varphi'_N\varphi'_N \end{pmatrix} dz. \end{aligned}$$

Now there is a more compact way to do this, namely by working with blocks of the matrices. First recall a definition from a previous chapter:

$$\Phi = (\varphi_1(z) \cdots \varphi_N(z)).$$

The $\tilde{\Phi}$ can be written

$$\tilde{\Phi} = \begin{pmatrix} \varphi_1 & \cdots & \varphi_N & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \varphi_1 & \cdots & \varphi_N \end{pmatrix} = \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix}.$$

Also the same definitions for B_i and pB_i will be applicable. Then, with this notation, the same argument as above goes

$$\begin{aligned} & \int_0^{2\pi} (\tilde{\Phi}')^T \hat{L}_1(\tilde{\Phi}) dz \\ &= \int_0^{2\pi} (\tilde{\Phi}')^T \begin{pmatrix} -(c_6 + d_6 p)D & 0 \\ 0 & -c_2 D \end{pmatrix} \tilde{\Phi} dz \\ &= \int_0^{2\pi} \begin{pmatrix} (\Phi')^T & 0 \\ 0 & (\Phi')^T \end{pmatrix} \begin{pmatrix} -(c_6 + d_6 p)D & 0 \\ 0 & -c_2 D \end{pmatrix} \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix} dz \\ &= \int_0^{2\pi} \begin{pmatrix} -(c_6 + d_6 p)(\Phi')^T \Phi' & 0 \\ 0 & -c_2 (\Phi')^T \Phi' \end{pmatrix} dz \\ &= \int_0^{2\pi} \begin{pmatrix} -(c_6 B_1 + d_6 p B_1) & 0 \\ 0 & -c_2 B_1 \end{pmatrix} dz. \end{aligned}$$

Similarly,

$$\begin{aligned} & \int_0^{2\pi} (\tilde{\Phi}')^T \hat{L}_2(\tilde{\Phi}) dz \\ &= \int_0^{2\pi} (\tilde{\Phi}')^T \begin{pmatrix} 0 & (c_6 + d_6 p) \\ -(c_3 + d_3 p) & 0 \end{pmatrix} \tilde{\Phi} dz \\ &= \int_0^{2\pi} \begin{pmatrix} (\Phi')^T & 0 \\ 0 & (\Phi')^T \end{pmatrix} \begin{pmatrix} 0 & (c_6 + d_6 p) \\ -(c_3 + d_3 p) & 0 \end{pmatrix} \begin{pmatrix} \Phi & 0 \\ 0 & \Phi \end{pmatrix} dz \\ &= \int_0^{2\pi} \begin{pmatrix} 0 & (c_6 + d_6 p)(\Phi')^T \Phi \\ -(c_3 + d_3 p)(\Phi')^T \Phi & 0 \end{pmatrix} dz. \end{aligned}$$

Defining

$$\begin{aligned} B_2 &\equiv \int_0^{2\pi} (\Phi')^T \Phi dz \\ pB_2 &\equiv \int_0^{2\pi} p(\Phi')^T \Phi dz \end{aligned}$$

gives

$$\int_0^{2\pi} (\tilde{\Phi}')^T \hat{L}_2(\tilde{\Phi}) dz = \begin{pmatrix} 0 & c_6 B_2 + d_6 p B_2 \\ -(c_3 B_2 + d_3 p B_2) & 0 \end{pmatrix},$$

where (as above) both the left and right hand sides are $2N \times 2N$ matrices.

Using

$$\int_0^{2\pi} \Phi^T(\Phi') dz = \left\{ \int_0^{2\pi} (\Phi')^T \Phi dz \right\}^T = B_2^T$$

the following results for the three other terms in Equation 7.2 are stated:

$$\begin{aligned} \int_0^{2\pi} \tilde{\Phi}^T L_3(\tilde{\Phi}) dz &= \begin{pmatrix} 0 & -(c_3 B_2^T + d_3(pB_2)^T) \\ c_6 B_2^T + d_6(pB_2)^T & 0 \end{pmatrix}; \\ \int_0^{2\pi} \tilde{\Phi}^T L_4(\tilde{\Phi}) dz &= \begin{pmatrix} -(c_1 B_4 + d_1 p B_4) & 0 \\ 0 & -(c_6 B_4 + d_6 p B_4) \end{pmatrix}; \\ \int_0^{2\pi} \tilde{\Phi}^T \tilde{\Phi} dz &= \begin{pmatrix} B_4 & 0 \\ 0 & B_4 \end{pmatrix}. \end{aligned}$$

Putting all these things together with a few more definitions,

$$\begin{aligned} \tilde{B}_4 &= \frac{m^2}{\alpha^2} \int_0^{2\pi} \tilde{\Phi}^T L_4(\tilde{\Phi}) dz = \frac{m^2}{\alpha^2} \begin{pmatrix} -(c_1 B_4 + d_1 p B_4) & 0 \\ 0 & -(c_6 B_4 + d_6 p B_4) \end{pmatrix} \\ \tilde{B}_2 &= \frac{m}{\alpha} \int_0^{2\pi} [(\tilde{\Phi}')^T \hat{L}_2(\tilde{\Phi}) + \tilde{\Phi}^T L_3(\tilde{\Phi})] dz \\ &= \frac{m}{\alpha} \begin{pmatrix} 0 & c_6 B_2 + d_6 p B_2 - (c_3 B_2^T + d_3(pB_2)^T) \\ -(c_3 B_2 + d_3 p B_2) + c_6 B_2^T + d_6(pB_2)^T & 0 \end{pmatrix} \\ \tilde{B}_1 &= \int_0^{2\pi} [(\tilde{\Phi}')^T \hat{L}_1(\tilde{\Phi}) + \rho \frac{m^2}{\alpha^2} \tilde{\Phi}^T \tilde{\Phi}] dz \\ &= \begin{pmatrix} -(c_6 B_1 + d_6 p B_1) + \rho \frac{m^2}{\alpha^2} B_4 & 0 \\ 0 & -c_2 B_1 + \rho \frac{m^2}{\alpha^2} B_4 \end{pmatrix}. \end{aligned}$$

Finally, with all these definitions the Galerkin equation (Equation 7.2) becomes

$$(\lambda^2 \tilde{B}_4 + \lambda \tilde{B}_2 + \tilde{B}_1) \vec{c} = \vec{0} \quad (7.3)$$

where all the \tilde{B}_i 's are $2N \times 2N$ matrices, λ is a generalized eigenvalue, and \vec{c} is a corresponding eigenvector.

If the original form had been

$$\begin{aligned} u &= \eta(y) \cos(m(\lambda x - t)) \\ v &= \mu(y) \sin(m(\lambda x - t)), \end{aligned}$$

then letting $\tilde{B}_2 \mapsto -\tilde{B}_2$ is the only change required. As is shown below, this corresponds to changing the sign of the resulting eigenvalue and changing the signs of the entries on the lower half of the vector \vec{c} . Thus, solving the case originally considered also solves this case.

7.3 Some Properties of the System

Solutions exist when

$$\det(\lambda^2 \tilde{B}_4 + \lambda \tilde{B}_2 + \tilde{B}_1) = 0.$$

This is a $4N$ th degree polynomial in λ . Here are a few propositions which help one understand the problem a bit better.

PROPOSITION 7.1: *If λ is an eigenvalue of Equation 7.3, then so is $-\lambda$.*

PROOF: The proof is done by constructing the eigenvector. If λ is an eigenvalue, then it has a corresponding eigenvector, call it \vec{c}_+ . This vector is of dimension $2N$, and can be broken in the middle into two N -vectors, \vec{c}_η and \vec{c}_μ . With

$$\vec{c}_+ = \begin{pmatrix} \vec{c}_\eta \\ \vec{c}_\mu \end{pmatrix}, \quad \text{let } \vec{c}_- = \begin{pmatrix} \vec{c}_\eta \\ -\vec{c}_\mu \end{pmatrix}.$$

In block form, Equation 7.3 looks like

$$(\lambda^2 \begin{pmatrix} \bullet & 0 \\ 0 & \bullet \end{pmatrix} + \lambda \begin{pmatrix} 0 & \bullet \\ \bullet & 0 \end{pmatrix} + \begin{pmatrix} \bullet & 0 \\ 0 & \bullet \end{pmatrix}) \vec{c} = \vec{0}$$

where \bullet means a nonzero block. It will be seen that if λ and \vec{c}_+ solve this equation, then so do $-\lambda$ and \vec{c}_- . \square

If λ is real, this is actually a physically intuitive result. Since the wave speed is given by $1/\lambda$ and there is no reason to prefer one direction of propagation over the other, both the plus and the minus λ should occur. The same argument for the original ordinary differential equation yields that if λ, η, μ is a solution, then so is $-\lambda, \eta, -\mu$.

If λ is complex and not purely imaginary, this result says the roots come in fours, $\lambda, \bar{\lambda}, -\lambda,$ and $-\bar{\lambda}$, since the polynomial in λ has only real coefficients.

PROPOSITION 7.2: *All the matrices in Equation 7.3 are symmetric.*

PROOF: For the cases of \tilde{B}_1 and \tilde{B}_4 this is obvious since B_1, pB_1, B_4 and pB_4 are all symmetric. For the case of \tilde{B}_2 , let the matrix A be

$$A = c_6 B_2 + d_6 p B_2 - (c_3 B_2^T + d_3 (p B_2)^T).$$

Then

$$\begin{aligned} A^T &= (c_6 B_2 + d_6 p B_2 - (c_3 B_2^T + d_3 (p B_2)^T))^T \\ &= c_6 B_2^T + d_6 (p B_2)^T - (c_3 B_2 + d_3 p B_2). \end{aligned}$$

Thus,

$$\tilde{B}_2 = \frac{m}{\alpha} \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$$

and

$$\tilde{B}_2^T = \frac{m}{\alpha} \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}^T = \frac{m}{\alpha} \begin{pmatrix} 0 & (A^T)^T \\ A^T & 0 \end{pmatrix} = \frac{m}{\alpha} \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} = \tilde{B}_2$$

and so \tilde{B}_2 is also symmetric. \square

With this result, the following applies:

PROPOSITION 7.3: *If $\lambda, \tilde{c}_\lambda$ and μ, \tilde{c}_μ eigenvalues and eigenvectors respectively for*

$$(\lambda^2 A + \lambda B + C)\tilde{c} = 0$$

with A, B and C Hermitian (meaning $A^H = A$, where the superscript H means conjugate transpose), then

$$(\bar{\lambda} - \mu)\tilde{c}_\mu^H \{(\bar{\lambda} + \mu)A + B\}\tilde{c}_\mu = 0.$$

PROOF: This follows by manipulation. First, as μ is an eigenvalue,

$$(\mu^2 A + \mu B + C)\tilde{c}_\mu = 0,$$

and as λ is an eigenvalue,

$$\vec{c}_\lambda^H (\bar{\lambda}^2 A^H + \bar{\lambda} B^H + C^H) = 0.$$

Since all the matrices are Hermitian, $A^H = A$, $B^H = B$ and $C^H = C$ and subtraction of the two equations gives

$$\vec{c}_\lambda^H \{(\bar{\lambda}^2 - \mu^2)A + (\bar{\lambda} - \mu)B\} \vec{c}_\mu = 0.$$

Factoring the $\bar{\lambda} - \mu$ term out gives the desired result. \square

In the case considered here, the \tilde{B}_i are real symmetric matrices, and so are Hermitian. But as the interest is in real eigenvalues, this does not yield any direct information, though it does say that for $\mu = -\lambda$, the resulting eigenvectors are orthogonal with respect to \tilde{B}_2 .

7.4 Rewriting the Generalized Eigenvalue Problem

The way the equation is solved is by converting it to the following form:

$$\begin{pmatrix} -\tilde{B}_1 & 0 \\ -\tilde{B}_2 & I \end{pmatrix} \begin{pmatrix} \vec{c} \\ \vec{y} \end{pmatrix} = \lambda \begin{pmatrix} 0 & I \\ \tilde{B}_4 & 0 \end{pmatrix} \begin{pmatrix} \vec{c} \\ \vec{y} \end{pmatrix} \quad (7.4)$$

where I is an $2N \times 2N$ identity matrix, and \vec{y} is a dummy variable. This gives

$$-\tilde{B}_1 \vec{c} = \lambda \vec{y}$$

and

$$-\tilde{B}_2 \vec{c} + \vec{y} = \lambda \tilde{B}_4 \vec{c}$$

and solving the second equation for \vec{y} , and placing this in the first gives

$$-\tilde{B}_1 \vec{c} = \lambda(\lambda \tilde{B}_4 \vec{c} + \tilde{B}_2 \vec{c})$$

which is the generalized eigenvalue problem to be solved.

7.5 The Fourier Basis

Again the periodic Fourier basis will be used. With this, the approximate solutions are assured to satisfy the periodic boundary conditions. Also, the above equations were derived assuming the basis elements were periodic. In fact, with this choice, all the work is already done. Only two terms in the newly defined matrices have not yet been calculated, and each of these is easy. First, with D the “differentiation” matrix,

$$B_2 = \int_0^{2\pi} (\Phi')^T \Phi dz = \int_0^{2\pi} (\Phi D)^T \Phi dz = D^T B_4$$

and second,

$$pB_2 = \int_0^{2\pi} p(\Phi')^T \Phi dz = \int_0^{2\pi} p(\Phi D)^T \Phi dz = D^T pB_4.$$

It will also be assumed that p is the periodic square function.

7.6 Some Numerical Results

Equation 7.4 is in the form of an eigenvalue problem that EISPACK can solve (see [Cowell 1984]). So this is how the problem was solved. The double precision (single precision yielded grossly inaccurate results) RGG path was used, which provides both the eigenvalues and eigenvectors of the more general eigenvalue problem of the form $A\vec{x} = \lambda B\vec{x}$. This is done by reducing the matrices to Hessenberg form by nonorthogonal elimination methods, and then carrying out an “implicit, double-step, Hessenberg QR iteration”. The units were again in pounds (force) per square inch, so

$$\begin{array}{ll} c_1 = 11.1900 \times 10^6 & d_1 = 9.6350 \times 10^6 \\ c_2 = 1.7180 \times 10^6 & d_2 = 0.0000 \times 10^6 \\ c_3 = 0.6276 \times 10^6 & d_3 = 0.0426 \times 10^6 \\ c_6 = 0.6060 \times 10^6 & d_6 = 0.1040 \times 10^6 \end{array}$$

with $\rho = 0.0571$ pounds (mass) per cubic inch. In Table 7.1 are recorded the results for the longitudinal wave with $\alpha/m = 1$. The routine returned the \vec{c} which

was then normalized by its first entry – the first entry was set equal to $\sqrt{2\pi}$ to compare with $\eta^0 = 1$. There was no difference between the η values, so the zero percentage error is not recorded. The table of μ values and comparisons has the same explanation as the corresponding Table 5.1 for the transverse wave travelling normal to the laminae. As can be seen, the method compares well with six terms of the perturbation.

Next is a table for a range of values. Table 7.2 is similar to Table 5.2 listing values of maximal percentage differences between the computed and the perturbation μ . The longitudinal wave was nondispersive, varying less than fifty feet per second over a wide range of $\log_{10}(\alpha/m)$. For $\log_{10}(\alpha/m)$ below -4 EISPACK had errors, and for $\log_{10}(\alpha/m)$ around and above 3 the “longitudinal” eigenvalues lost all relation to the wave speed, as their corresponding wave speed rapidly increased.

Unfortunately, for small frequencies, the eigenvalues for the transverse wave were very numerically sensitive, and EISPACK was unable to obtain them accurately, even in double precision.

With the wavelengths given by

$$\text{wavelength} = \sqrt{\frac{c_1}{\rho} \frac{h}{\alpha}},$$

for $h = .02$ inch, the wavelength for $\alpha/m = 10^{-3}$ is about 14 centimeters, and the wavelength for $\alpha/m = 10^2$ is about 14 kilometers.

In summary, six terms of the perturbation were seen to compare well with the numerical solution. Further, over a wide range of frequencies the longitudinal waves were nondispersive.

7.7 An Iteration Method for A Specific Wave

As a closing section, another method will be presented for solving the system. If one is interested in the longitudinal wave, it is faster and slightly more accurate

Table 7.1. The numerical results for $\alpha/m = 1$ ($\log_{10}(\frac{\alpha}{m}) = 0$).

z	Two Terms			Six Terms	
	$\mu_{comp}(z)$	$\mu_{pert}(z)$	% Error	$\mu_{pert}(z)$	% Error
0.3307	0.1122E-06	0.1086E-06	-3.231	0.1122E-06	-0.011
0.6614	0.8210E-07	0.9061E-07	10.361	0.8210E-07	-0.001
0.9921	0.5194E-07	0.6280E-07	20.897	0.5195E-07	0.002
1.3228	0.2270E-07	0.2819E-07	24.182	0.2270E-07	-0.002
1.6535	-0.7873E-08	-0.9482E-08	20.425	-0.7873E-08	-0.004
1.9842	-0.3674E-07	-0.4612E-07	25.522	-0.3674E-07	0.000
2.3149	-0.6781E-07	-0.7776E-07	14.673	-0.6782E-07	0.002
2.6456	-0.9585E-07	-0.1010E-06	5.351	-0.9584E-07	-0.007
2.9762	-0.1287E-06	-0.1133E-06	-11.974	-0.1286E-06	-0.013
3.3069	-0.1287E-06	-0.1133E-06	-11.973	-0.1286E-06	-0.012
3.6376	-0.9585E-07	-0.1010E-06	5.350	-0.9584E-07	-0.007
3.9683	-0.6782E-07	-0.7776E-07	14.667	-0.6782E-07	-0.004
4.2990	-0.3674E-07	-0.4612E-07	25.519	-0.3674E-07	-0.002
4.6297	-0.7873E-08	-0.9482E-08	20.430	-0.7873E-08	0.000
4.9604	0.2270E-07	0.2819E-07	24.184	0.2270E-07	-0.001
5.2911	0.5195E-07	0.6280E-07	20.891	0.5195E-07	-0.003
5.6218	0.8210E-07	0.9061E-07	10.357	0.8210E-07	-0.005
5.9525	0.1122E-06	0.1086E-06	-3.229	0.1122E-06	-0.010
6.2832	0.1359E-06	0.1148E-06	-15.542	0.1359E-06	-0.013

Computed Wave Speed = 22913 $\frac{feet}{sec}$
Perturbation Wave Speed = 22914 $\frac{feet}{sec}$

Table 7.2. The perturbation solution compared with the computed solution for various α/m . The perturbation wave speed is 22914 $\frac{feet}{sec}$ and 5332 $\frac{feet}{sec}$.

$\log_{10}(\alpha/m)$	Wave speed $\frac{feet}{sec}$		μ	
	Longitudinal	Transverse	Two Terms Max % Error	Six Terms Max % Error
-3.0	22911	5256	25.72	-0.43
-2.0	22938	5254	25.66	0.12
-1.0	22919	5251	25.59	0.06
0.0	22913		25.52	-0.01
1.0	22913		25.51	-0.02
2.0	22914		25.54	0.03

to let

$$\lambda = \sqrt{\frac{\rho}{c_1}}$$

$$A = \lambda^2 \tilde{B}_4 + \lambda \tilde{B}_2 + \tilde{B}_1$$

and then do an iteration

$$A\vec{x}_{i+1} = \vec{c}_i,$$

$$\vec{c}_{i+1} = \frac{\vec{x}_{i+1}}{|\vec{x}_{i+1}|}.$$

To start, let

$$\vec{c}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

This is finding an eigenvector by inverse iteration, and as A is nearly singular it converges very rapidly. To mention that the same matrix is used throughout the iteration and so it need only be factored once is somewhat meaningless, since one iteration is usually sufficient.

As to the "slightly more accurate" statement, as the entries in the eigenvector span ten orders of magnitude, this approach picks up the very small entries better. However, these entries do not affect (to four significant figures) the values of η and μ .

If one is interested in the transverse wave, let

$$\lambda = \sqrt{\frac{\rho}{c_6}}$$

$$\vec{c}_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

With $\alpha/m = 1$, using double precision LINPACK (DSIFA followed by DSISL), one iteration had a two term maximum percentage error of 26.41%, and a six term maximum percentage error of -1.07% when the numerical result was compared with the respective transverse wave perturbation.

CHAPTER 8

PULSES, OR WAVE PACKETS

It is thought that a material is often damaged when it receives a sharp blow or impact. This blow can be modelled as a Dirac delta function behavior in the velocity of the material at a given time. This chapter briefly considers pulses propagating in one direction.

8.1 Normal Longitudinal Pulse

From Chapter 3 the form of the solution is

$$\begin{aligned}u &= 0 \\v &= f\left(\sqrt{\frac{\rho}{c_2}}y - t\right).\end{aligned}$$

Take an initial condition for a longitudinal wave travelling normal to the laminae,

$$\frac{\partial v}{\partial t}\Big|_{y=0} = V(t),$$

which implies that at $y = 0$ the wave material has been given a specific velocity. A solution is

$$f(s) = \left\{ \int V(t) dt \right\}_{t=-s}.$$

As the phase speed is independent of m , the wave packet does not spread out, and the pulse keeps its original shape. For example, for a Dirac delta function blow,

$$V(t) = \delta(t),$$

one has

$$v = H\left(-\left(\sqrt{\frac{\rho}{c_2}}y - t\right)\right),$$

where $H(t)$ is the Heavyside step function. The Dirac delta function and the Heavyside step function are defined as

$$\int_{-\infty}^{\infty} f(t)\delta(t) dt = f(0), \quad \text{all } f \in C(R)$$

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0. \end{cases}$$

$C(R)$ is the set of continuous functions on the real line.

8.2 The Normal Transverse Case

Equation 4.3 gives

$$u = \eta(y)f(\varphi(y) - t)$$

$$v = 0$$

where

$$\varphi'(y) = \frac{1}{(c_6 + d_6 p)\eta^2}.$$

This case is much more complicated, as the waves are dispersive and there are values of α/m where η does not exist. So only the dispersion will be discussed.

As the wave (phase) speed is dependent on the frequency, a wave packet composed of these waves will spread out. To see how this will occur, the modulation, or group, speed will be calculated. Ignore the $\eta(y)$ amplitude modulation and add two waves of equal amplitude with a given frequency m_1 and m_2 . Then

$$u = \sin(m_1(\varphi_1(y) - t)) + \sin(m_2(\varphi_2(y) - t))$$

which can be written

$$u = 2 \sin\left(\frac{1}{2}(m_1\varphi_1 - m_2\varphi_2 - (m_1 - m_2)t)\right) \cos\left(\frac{1}{2}(m_1\varphi_1 + m_2\varphi_2 - (m_1 + m_2)t)\right).$$

The cosine term is the “carrier” wave, while the sine term is a modulation of the carrier wave. The modulation (group) speed is the speed at which this modulation travels. This is

$$\text{modulation speed} = \frac{m_1 - m_2}{m_1 \varphi'_1 - m_2 \varphi'_2}.$$

In a limit this is (see [Crawford 1968])

$$\text{group speed} = \frac{1}{\frac{\partial(m\varphi')}{\partial m}} = \frac{1}{\varphi' + m \frac{\partial \varphi'}{\partial m}}$$

which is the customary group speed.

In particular, where the phase speed is nearly constant, that is for low and high frequencies, the group speed is roughly equal to the phase speed. In the locale where the phase speed had its peaks tending to infinity, the group speed does the same, only here some of the peaks are tending to plus infinity, while some are tending to minus infinity. This is because a zero is obtained in the denominator of the group speed expression when

$$\frac{\varphi'}{m} = -\frac{\partial \varphi'}{\partial m}.$$

8.3 The Parallel Longitudinal Case

Only the case of the longitudinal pulse will be discussed. Recall for the parallel wave there were four basis elements,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{i=1}^4 a_i H_i(y) \begin{pmatrix} \cos(m(\lambda_i x - t)) \\ \sin(m(\lambda_i x - t)) \end{pmatrix}.$$

The η_i 's and μ_i 's, which make up the H_i 's, depend on m . A general initial boundary traction cannot be accommodated by this approach, since it is very restrictive. Since the longitudinal waves were nondispersive, terms can be combined to build a travelling pulse. However, if one chose to write this as a Fourier integral, an explicit expression for the pulse would be difficult to come by because the terms in the perturbation have a very complicated m dependence.

CHAPTER 9

THE BOUNDARY CONDITIONS AT THE CRACK

Now the interest shifts to the crack. A crack will be introduced into the material by specifying a boundary condition of no tractions on a line segment in the material. Actually, this will be accomplished in two steps. One solves the linear elasticity problem with the specified tractions on the crack (namely those produced by the waves) with the further boundary condition that the solution vanishes at x and y at infinity. This is called the “complementary” problem. Then one subtracts this solution from the wave solution. Since linear elasticity has linear partial differential equations and linear boundary conditions, this has the same effect as specifying zero tractions at the crack.

In the first chapter the tractions were given by $t_i = T_{ij}n_j$ where n_j was the normal to the surface. The crack here will be of length $2a$ lying along the x -axis (see Figure 9.1). So the normal to the crack surface is, on the top

$$\vec{n} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and on the bottom of the crack

$$\vec{n} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}.$$

The tractions are

$$\vec{t}_{top} = \begin{pmatrix} T_{21} \\ T_{22} \\ T_{23} \end{pmatrix}, \quad \vec{t}_{bottom} = \begin{pmatrix} -T_{21} \\ -T_{22} \\ -T_{23} \end{pmatrix}.$$

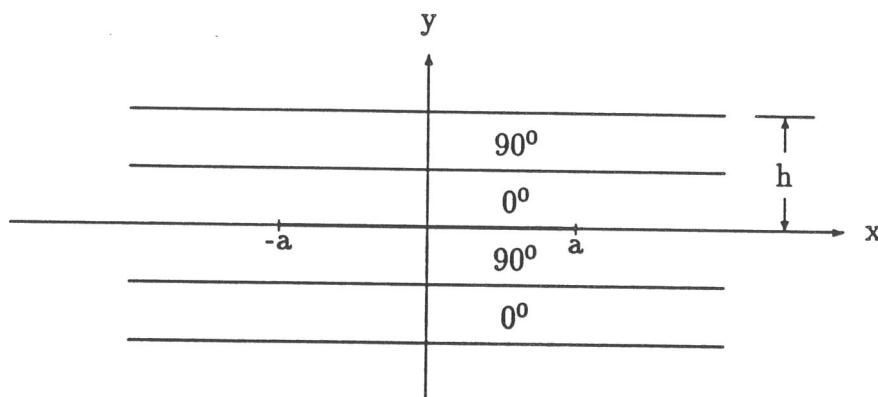


Figure 9.1. The crack.

As the problem has been reduced to a two-dimensional one, in particular a plane strain problem,

$$T_{32} = T_{23} = 2c_{44}\epsilon_{23} = 0.$$

For each of the waves it will be necessary to calculate T_{12} and T_{22} .

To recall some definitions for the composite material considered,

$$\begin{aligned} T_{12} &= 2c_{66}\epsilon_{12} = (c_6 + d_6p)\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ T_{22} &= c_{21}\epsilon_{11} + c_{22}\epsilon_{22} = (c_3 + d_3p)\frac{\partial u}{\partial x} + c_2\frac{\partial v}{\partial y}. \end{aligned} \quad (9.1)$$

On the x -axis $y = 0$, so $p(\alpha y) = p(0)$ is a constant and will be taken to be zero. If it is not zero, one can easily adjust the c_i 's at this point.

9.1 Normal Longitudinal Wave

Recalling Equation 4.2

$$\begin{aligned} u &= 0 \\ v &= f\left(\sqrt{\frac{\rho}{c_2}}y - t\right) \end{aligned}$$

one has

$$T_{12} = 0$$

$$T_{22} = \sqrt{\rho c_2} f' \left(\sqrt{\frac{\rho}{c_2}} y - t \right) \Big|_{y=0} = \sqrt{\rho c_2} f'(-t).$$

If $f(\cdot)$ is $C \sin(m(\cdot))$, C a constant, then

$$T_{22} = C \sqrt{\rho c_2} m \cos(mt).$$

9.2 Normal Transverse Wave

Recalling Equation 4.3

$$u = \eta(y) f(\varphi(y) - t)$$

$$v = 0$$

where

$$\varphi'(y) = \frac{1}{(c_6 + d_6 p) \eta^2}$$

one has

$$\begin{aligned} T_{12} &= c_6 \{ \eta' f + \eta f' \varphi' \} \Big|_{y=0} = c_6 \eta'(0) f(-t) + \frac{1}{\eta(0)} f'(-t) \\ T_{22} &= 0. \end{aligned}$$

These are a bit more complicated but still a periodic time dependence occurs. It has been assumed that $\varphi(0) = 0$. Again let $f(\cdot) = C \sin(m(\cdot))$. Then

$$\frac{T_{12}}{C} = c_6 \eta'(0) \sin(-mt) + \frac{1}{\eta(0)} m \cos(mt) = \frac{m}{\eta(0)} \left(-\frac{c_6 \eta(0) \eta'(0)}{m} \sin(mt) + \cos(mt) \right).$$

Letting

$$\tan(\phi) = \frac{c_6 \eta(0) \eta'(0)}{m}$$

one has

$$T_{12} = \frac{Cm}{\eta(0) \cos(\phi)} \cos(mt + \phi),$$

where ϕ is a function of m .

9.3 Parallel Waves

Recall Equation 4.8

$$u = \eta(y) \cos(m(\lambda x - t))$$

$$v = \mu(y) \sin(m(\lambda x - t)).$$

This form leads to both the “longitudinal” and “transverse” waves. This gives

$$T_{12} = c_6\{\eta'(0) \cos(\cdot) + mA\mu(0) \cos(\cdot)\} = c_6\{\eta'(0) + mA\mu(0)\} \cos(\cdot)$$

$$T_{22} = c_3(-mA)\eta(0) \sin(\cdot) + c_2\mu'(0) \sin(\cdot) = \{-mA c_3\eta(0) + c_2\mu'(0)\} \sin(\cdot)$$

where \cdot means $m(\lambda x - t)$. These boundary tractions are quite complicated, and are included for completeness, as this case will not be dealt with.

CHAPTER 10

THE COMPLEMENTARY PROBLEM

In this chapter, assumptions on the forms of solution will be made which yield the equations to be solved. To outline the approach, first a time dependence will be assumed. Then, the problem will be split into two half planes, an upper plane and a lower plane, and it will be assumed that in each region the solution may be written in terms of a Fourier integral. Continuity conditions off the crack and prescribed tractions on the crack will give rise to a dual integral equation.

10.1 The Form of Solution

Recall from Chapter 3 the equations

$$(c_1 + d_1 p) \partial_1^2 u + \partial_2((c_6 + d_6 p) \partial_2 u) + (c_3 + d_3 p) \partial_1 \partial_2 v + \partial_2((c_6 + d_6 p) \partial_1 v) = \rho u_{tt}$$

$$(c_6 + d_6 p) \partial_1 \partial_2 u + \partial_2((c_3 + d_3 p) \partial_1 u) + (c_6 + d_6 p) \partial_1^2 v + c_2 \partial_2^2 v = \rho v_{tt}.$$

It is assumed the solution can be written

$$u = \eta(x, y) \cos(mt + \phi)$$

$$v = \mu(x, y) \cos(mt + \phi)$$

with

$$\eta(x, y) = \int_{-\infty}^{\infty} A(w, y) e^{iwx} dw$$

$$\mu(x, y) = \int_{-\infty}^{\infty} B(w, y) e^{iwx} dw.$$

The A and the B will be different for the upper plane and the lower plane. Putting this form into the solid equations gives equations for A and B . These are

$$(c_1 + d_1 p)(-w^2 A) + ((c_6 + d_6 p)A_y)_y + (c_3 + d_3 p)(iwB_y) + ((c_6 + d_6 p)(iwB))_y \\ = -\rho m^2 A$$

$$(c_6 + d_6 p)(iwA_y) + ((c_3 + d_3 p)(iwA))_y + (c_6 + d_6 p)(-w^2 B) + c_2 B_{yy} \\ = -\rho m^2 B,$$

where the subscripted y means partial differentiation with respect to y . The boundary conditions are some fixed value at $y = 0$ and A and B vanish at $y = \pm\infty$ for the upper half plane and lower half plane, respectively. If A and B solve this system of ordinary differential equations, then the form of the solution will solve the solid equations.

10.2 The Ordinary Differential Equation

It will be assumed that there exists a basis for the solution of this ordinary differential equation. For the upper half plane let it be

$$\vec{H}_1(w, y), \quad \text{and} \quad \vec{H}_2(w, y).$$

For the lower half plane let it be

$$\vec{H}_3(w, y), \quad \text{and} \quad \vec{H}_4(w, y).$$

Let

$$\vec{H}_i \equiv \vec{H}_i(w, 0) \\ \vec{H}'_i \equiv \left. \frac{\partial \vec{H}_i(w, y)}{\partial y} \right|_{y=0}.$$

Let

$$\vec{a} = \begin{pmatrix} a_1(w) \\ a_2(w) \\ a_3(w) \\ a_4(w) \end{pmatrix}$$

be a 4-vector of w dependent coefficients. Let

$$C = \begin{pmatrix} \vec{H}_1 & \vec{H}_2 & \vec{0} & \vec{0} \\ \vec{0} & \vec{0} & \vec{H}_3 & \vec{H}_4 \end{pmatrix}$$

and

$$C_y = \frac{1}{iw} \begin{pmatrix} \vec{H}'_1 & \vec{H}'_2 & \vec{0} & \vec{0} \\ \vec{0} & \vec{0} & \vec{H}'_3 & \vec{H}'_4 \end{pmatrix}$$

be 4×4 matrices, possibly dependent on w , where the subscript y is to be reminiscent of C_y having a relationship to the y partials of the solution of the ordinary differential equation on the x -axis. The purpose for the $1/iw$ term will be clear later.

With

$$\vec{\eta}(w, y) = \begin{pmatrix} \eta_{upper} \\ \mu_{upper} \\ \eta_{lower} \\ \mu_{lower} \end{pmatrix}$$

one can write

$$\vec{\eta} = \int_{-\infty}^{\infty} C \vec{a} e^{iwx} dw = \int_{-\infty}^{\infty} \begin{pmatrix} a_1(w) \vec{H}_1(w, y) + a_2(w) \vec{H}_2(w, y) \\ a_3(w) \vec{H}_3(w, y) + a_4(w) \vec{H}_4(w, y) \end{pmatrix} e^{iwx} dw.$$

For any \vec{a} the resulting $\vec{\eta}$ written in this form satisfies the solid equations. However, the work comes in trying to fit the boundary conditions, a job that is very difficult for a crack. Needed will be the partial derivatives, as they appear in the tractions:

$$\begin{aligned} \frac{\partial}{\partial x} \vec{\eta} |_{y=0} &= \int_{-\infty}^{\infty} iw C \vec{a} e^{iwx} dw, \\ \frac{\partial}{\partial y} \vec{\eta} |_{y=0} &= \int_{-\infty}^{\infty} iw C_y \vec{a} e^{iwx} dw. \end{aligned}$$

Writing these terms as matrices and vectors, it is hoped that the resulting equations will be a bit more clear.

10.3 The Dual Integral Equation

All the above leads to an equation in which the time dependence has been factored out. It is assumed that the solution is continuous off the crack, or that

the displacements match.

$$\begin{pmatrix} \eta \\ \mu \end{pmatrix}_{\text{upper}} = \begin{pmatrix} \eta \\ \mu \end{pmatrix}_{\text{lower}} ; \quad y = 0, |x| > a.$$

It might appear that one should also set some of the tractions equal in this region off the crack. However, the approach here will be to find a solution satisfying continuity, and then look at the resulting tractions off the crack.

It is assumed that on the crack the solution meets the specified tractions, or with

$$T_{12} = \hat{T}_{12} \cos(mt + \phi)$$

$$T_{22} = \hat{T}_{22} \cos(mt + \phi)$$

that (from the tractions in the previous chapter)

$$\begin{aligned} c_6 \left(\frac{\partial \eta_{\text{upper}}}{\partial y} + \frac{\partial \mu_{\text{upper}}}{\partial x} \right) &= \hat{T}_{12} \\ c_3 \frac{\partial \eta_{\text{upper}}}{\partial x} + c_2 \frac{\partial \mu_{\text{upper}}}{\partial y} &= \hat{T}_{22} \end{aligned}$$

and

$$\begin{aligned} -c_6 \left(\frac{\partial \eta_{\text{lower}}}{\partial y} + \frac{\partial \mu_{\text{lower}}}{\partial x} \right) &= -\hat{T}_{12} \\ -(c_3 \frac{\partial \eta_{\text{lower}}}{\partial x} + c_2 \frac{\partial \mu_{\text{lower}}}{\partial y}) &= -\hat{T}_{22} \end{aligned}$$

for $y = 0, |x| < a$. The minus signs on the left come from the normal on the lower face of the crack pointing down. These signs cancel and will not appear again.

These terms are somewhat tedious. With

$$\vec{H}_i = \begin{pmatrix} (\vec{H}_i)_1 \\ (\vec{H}_i)_2 \end{pmatrix}$$

the upper half plane term can be written out as

$$\int_{-\infty}^{\infty} \begin{pmatrix} c_6 [a_1 (\vec{H}_1)'_1 + a_2 (\vec{H}_2)'_1 + iw \{a_1 (\vec{H}_1)_2 + a_2 (\vec{H}_2)_2\}] \\ c_3 iw \{a_1 (\vec{H}_1)_1 + a_2 (\vec{H}_2)_1\} + c_2 \{a_1 (\vec{H}_1)'_2 + a_2 (\vec{H}_2)'_2\} \end{pmatrix} e^{iw x} dw$$

$$= \begin{pmatrix} \hat{T}_{12} \\ \hat{T}_{22} \end{pmatrix}, \quad y = 0, \quad |x| < a,$$

with the prime terms defined previously. Putting all this together,

$$\begin{aligned} \int_{-\infty}^{\infty} [iw \begin{pmatrix} 0 & c_6 \\ c_3 & 0 \end{pmatrix} (a_1 \vec{H}_1 + a_2 \vec{H}_2) + \begin{pmatrix} c_6 & 0 \\ 0 & c_2 \end{pmatrix} (a_1 \vec{H}'_1 + a_2 \vec{H}'_2)] e^{iw x} dw \\ = \begin{pmatrix} \hat{T}_{12} \\ \hat{T}_{22} \end{pmatrix}, \quad y = 0, \quad |x| < a, \end{aligned}$$

or as

$$a_1 \vec{H}_1 + a_2 \vec{H}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} C \vec{a}$$

one can write this as

$$\begin{aligned} \int_{-\infty}^{\infty} [iw \begin{pmatrix} 0 & c_6 \\ c_3 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} C \vec{a} + \begin{pmatrix} c_6 & 0 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} iw C_y \vec{a}] e^{iw x} dw \\ = \begin{pmatrix} \hat{T}_{12} \\ \hat{T}_{22} \end{pmatrix}, \quad y = 0, \quad |x| < a. \end{aligned}$$

Using this, and similar arguments, the following equation results for the $a_i(w)$:

$$\begin{aligned} \int_{-\infty}^{\infty} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} C \vec{a} e^{iw x} dx = \vec{0}, \quad y = 0, \quad |x| > a; \\ \int_{-\infty}^{\infty} iw \left[\begin{pmatrix} 0 & c_6 \\ c_3 & 0 \\ & 0 & c_6 \\ & c_3 & 0 \end{pmatrix} C + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} C_y \right] \vec{a} e^{iw x} dw \\ = \begin{pmatrix} \hat{T}_{12} \\ \hat{T}_{22} \\ \hat{T}_{12} \\ \hat{T}_{22} \end{pmatrix}, \quad y = 0, \quad |x| < a. \end{aligned} \quad (10.1)$$

The spaces in the matrices are zero entries. In the next chapter these will be solved for the homogeneous case.

CHAPTER 11

THE SOLUTION OF THE CRACK PROBLEM

The ε^0 case will be dealt with. The ε^0 case means the ε^0 consideration of the ordinary differential equation which arose in the process of solving the crack problem. This will deal with the anisotropy of the material, but not the inhomogeneity near the crack. This would be the approximate solution if the crack were at the center of a layer instead of between two layers. For the interlaminar crack problem, this is a first approximation to the solution.

11.1 The Simplifications

The ε^0 problem is being solved, which means using a perturbation for the ordinary differential equation satisfied by the transforms of η and μ . Only the first term of the perturbation will be examined and used.

The ε^0 equation is, with $a = c_3 + c_6$,

$$-w^2 c_1 A^0 + c_6 A_{yy}^0 + i w a B_y^0 = -\rho m^2 A^0$$

$$i w a A_y^0 - w^2 c_6 B^0 + c_2 B_{yy}^0 = -\rho m^2 B^0.$$

Since this system has constant coefficients, a solution of the form

$$\begin{pmatrix} A^0 \\ B^0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{qy}$$

where a , b , and q depend on w but not on y , is expected to solve the equations.

Placing this in the system implies the following:

$$\begin{vmatrix} -w^2c_1 + c_6q^2 + \rho m^2 & iwaq \\ iwaq & -w^2c_6 + c_2q^2 + \rho m^2 \end{vmatrix} = 0.$$

This is a quadratic in q^2 . If q is a solution, so is $-q$.

There is a difference of about ten orders of magnitude between the value of ρ and the values of the c_i 's. Because of this, for not too large m at least, the ρm^2 term in the above is negligible, giving the simpler

$$\begin{vmatrix} -w^2c_1 + c_6q^2 & iwaq \\ iwaq & -w^2c_6 + c_2q^2 \end{vmatrix} \approx 0$$

to solve. The resulting equation is

$$c_2c_6q^4 + (a^2 - c_1c_2 - c_6^2)w^2q^2 + c_1c_6w^4 = 0.$$

One can use the quadratic formula to solve for q^2 . Letting k_+ be the plus and k_- be the minus, then with

$$k_{\pm}^2 = \frac{c_1c_2 + c_6^2 - a^2 \pm \sqrt{(c_1c_2 + c_6^2 - a^2)^2 - 4c_1c_2c_6^2}}{2c_2c_6}$$

the solution is

$$q^2 = k^2w^2.$$

In the example, both k_{\pm}^2 will be real and positive and distinct (for the isotropic case the roots are *not* distinct - anisotropy is being assumed), so let the k_{\pm} be the positive squareroots.

To satisfy the boundary conditions the roots for the upper half plane, namely that the solution vanish at $y = +\infty$, take

$$q_1 = -k_+|w|, \quad q_2 = -k_-|w|$$

For the solution to vanish at $-\infty$ in lower half plane, let

$$q_3 = k_+|w|, \quad q_4 = k_-|w|.$$

The basis of solutions is

$$\begin{aligned}\vec{H}_1(w, y) &= \begin{pmatrix} 1 \\ -b_+ \end{pmatrix} e^{-k_+|w|y}, & \vec{H}_2(w, y) &= \begin{pmatrix} 1 \\ -b_- \end{pmatrix} e^{-k_-|w|y}, \\ \vec{H}_3(w, y) &= \begin{pmatrix} 1 \\ b_+ \end{pmatrix} e^{k_+|w|y}, & \vec{H}_4(w, y) &= \begin{pmatrix} 1 \\ b_- \end{pmatrix} e^{k_-|w|y},\end{aligned}$$

with the \vec{H}_i as defined in the last chapter. Here

$$b_{\pm} = -\frac{-w^2 c_1 + c_6 q_i^2}{i w a q_i} = \frac{c_1 - c_6 k_{\pm}^2}{i a k_{\pm}} \frac{w}{|w|}.$$

The b_{\pm} depend on w . The b_{\pm} and the k_{\pm} are dimensionless. For the specific material considered previously, they are

$$\begin{aligned}k_+ &= 4.120 & k_- &= 0.619 \\ b_+ &= -0.178i & b_- &= -14.339i.\end{aligned}$$

The y -derivatives are

$$\begin{aligned}\vec{H}'_1(w, y) &= \begin{pmatrix} -k_+|w| \\ k_+ b_+ |w| \end{pmatrix} e^{-k_+|w|y}, & \vec{H}'_2(w, y) &= \begin{pmatrix} -k_-|w| \\ k_- b_- |w| \end{pmatrix} e^{-k_-|w|y}, \\ \vec{H}'_3(w, y) &= \begin{pmatrix} k_+|w| \\ k_+ b_+ |w| \end{pmatrix} e^{k_+|w|y}, & \vec{H}'_4(w, y) &= \begin{pmatrix} k_-|w| \\ k_- b_- |w| \end{pmatrix} e^{k_-|w|y}.\end{aligned}$$

11.2 How to Write the Integral Equation

The next step is to find the C and C_y matrices. These are

$$\begin{aligned}C &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ -b_+ & -b_- & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & b_+ & b_- \end{pmatrix}, \\ C_y &= \frac{1}{iw} \begin{pmatrix} -k_+|w| & -k_-|w| & 0 & 0 \\ k_+ b_+ |w| & k_- b_- |w| & 0 & 0 \\ 0 & 0 & k_+|w| & k_-|w| \\ 0 & 0 & k_+ b_+ |w| & k_- b_- |w| \end{pmatrix} \\ &= \frac{1}{i} \frac{|w|}{w} \begin{pmatrix} -k_+ & -k_- & 0 & 0 \\ k_+ b_+ & k_- b_- & 0 & 0 \\ 0 & 0 & k_+ & k_- \\ 0 & 0 & k_+ b_+ & k_- b_- \end{pmatrix}.\end{aligned}$$

C is invertible since $b_+ \neq b_-$, and the part of the dual integral equation, Equation 10.1, specifically within the crack ($|x| < a$) can be written

$$\int_{-\infty}^{\infty} iw \left[\begin{pmatrix} 0 & c_6 \\ c_3 & 0 \\ & 0 & c_6 \\ & c_3 & 0 \end{pmatrix} + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} C_y C^{-1} \right] C \vec{a} e^{iw x} dw$$

$$= \begin{pmatrix} \hat{T}_{12} \\ \hat{T}_{22} \\ \hat{T}_{12} \\ \hat{T}_{22} \end{pmatrix}, \quad y = 0, \quad |x| < a.$$

To see what $C_y C^{-1}$ looks like it will be written out.

$$C_y C^{-1} = \frac{1}{i} \frac{|w|}{w} \begin{pmatrix} -k_+ & -k_- & 0 & 0 \\ k_+ b_+ & k_- b_- & 0 & 0 \\ 0 & 0 & k_+ & k_- \\ 0 & 0 & k_+ b_+ & k_- b_- \end{pmatrix} \frac{1}{b_+ - b_-} \begin{pmatrix} -b_- & -1 & 0 & 0 \\ b_+ & 1 & 0 & 0 \\ 0 & 0 & -b_- & 1 \\ 0 & 0 & b_+ & -1 \end{pmatrix}$$

$$= \frac{-i}{b_+ - b_-} \frac{|w|}{w} \begin{pmatrix} k_+ b_- - k_- b_+ & k_+ - k_- & & \\ b_+ b_- (k_- - k_+) & k_- b_- - k_+ b_+ & & \\ & & k_- b_+ - k_+ b_- & k_+ - k_- \\ & & b_+ b_- (k_- - k_+) & k_+ b_+ - k_- b_- \end{pmatrix}.$$

Now comes a rather large insight which will lead to an approach to solve this problem. The b_{\pm} change sign at $w = 0$, and the k_{\pm} are constant. So this $C_y C^{-1}$ matrix factors into two pieces, one that flips, or changes sign at $w = 0$, and one that is constant for all w . Consider the first column of $C_y C^{-1}$. The term

$$\frac{-i}{b_+ - b_-} \frac{|w|}{w} (k_+ b_- - k_- b_+)$$

flips because each b_{\pm} has a $|w|/w$ in it, and so the $|w|/w$ term is taken to an odd power. On the other hand,

$$\frac{-i}{b_+ - b_-} \frac{|w|}{w} b_+ b_- (k_- - k_+)$$

has an even power of $|w|/w$'s in it, so it is constant for all w and does not flip.

Choosing to write these with superscript f for flip and n for no flip, one has

$$C_y C^{-1} = A^f + A^n$$

where

$$A^f = \frac{-i}{b_+ - b_-} \frac{|w|}{w} \begin{pmatrix} k_+ b_- - k_- b_+ & & & \\ & k_- b_- - k_+ b_+ & & \\ & & k_- b_+ - k_+ b_- & \\ & & & k_+ b_+ - k_- b_- \end{pmatrix},$$

$$A^n = \frac{-i}{b_+ - b_-} \frac{|w|}{w} \begin{pmatrix} 0 & k_+ - k_- & & \\ b_+ b_- (k_- - k_+) & 0 & & \\ & & 0 & k_+ - k_- \\ & & b_+ b_- (k_- - k_+) & 0 \end{pmatrix}.$$

Each of these matrices is invertible.

11.3 How the Flip Affects the Equation

The left hand side of the integral equation can now be written

$$\int_{-\infty}^{\infty} iw \left[\begin{pmatrix} 0 & c_6 \\ c_3 & 0 \\ & 0 & c_6 \\ & c_3 & 0 \end{pmatrix} + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} \{A^f + A^n\} \right] C \vec{a} e^{iw x} dw.$$

But some of these matrices do not depend on w , such as the matrices with the c_i 's, and A^n , and these can be pulled out of the integrals to leave

$$\left[\begin{pmatrix} 0 & c_6 \\ c_3 & 0 \\ & 0 & c_6 \\ & c_3 & 0 \end{pmatrix} + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} A^n \right] \int_{-\infty}^{\infty} iw C \vec{a} e^{iw x} dw$$

$$+ \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} \int_{-\infty}^{\infty} iw A^f C \vec{a} e^{iw x} dw.$$

Now to identify terms. First

$$\int_{-\infty}^{\infty} iw C \vec{a} e^{iw x} dw = \vec{\eta}',$$

where prime denotes differentiation with respect to x . But next is the intriguing item. In the latter integral above, because of the A^f , everything inside is flipped.

To see what this means,

$$\begin{aligned}\vec{\eta} &= \int_{-\infty}^{\infty} C \vec{a} e^{iwx} dw \\ &= \int_{-\infty}^0 C \vec{a} e^{iwx} dw + \int_0^{\infty} C \vec{a} e^{iwx} dw \\ &= \vec{\eta}_- + \vec{\eta}_+.\end{aligned}$$

This last line is a definition. But to flip this means changing the sign of the interior terms for $w < 0$, so one has

$$\int_{-\infty}^{\infty} \frac{w}{|w|} C \vec{a} e^{iwx} dw = \vec{\eta}_+ - \vec{\eta}_-,$$

but this is merely the Hilbert transform! (See [Stenger 1973].) If the Hilbert transform is denoted by \mathcal{H} then

$$\vec{\eta}_+ - \vec{\eta}_- = \mathcal{H}(\vec{\eta}) = \frac{\text{PV}}{\pi i} \int_{-\infty}^{\infty} \frac{\vec{\eta}(y)}{y-x} dy$$

where PV means Cauchy principal value for the integral.

The integral equation can now be written

$$\begin{aligned}\left[\begin{pmatrix} 0 & c_6 \\ c_3 & 0 \\ & 0 & c_6 \\ & c_3 & 0 \end{pmatrix} + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} A^n \right] \vec{\eta}' + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} A^{f+} \mathcal{H}(\vec{\eta}') \\ = \begin{pmatrix} \hat{T}_{12} \\ \hat{T}_{22} \\ \hat{T}_{12} \\ \hat{T}_{22} \end{pmatrix}, \quad y = 0, \quad |x| < a.\end{aligned}\tag{11.1}$$

In this, A^{f+} is A^f evaluated at some positive w . It is well-defined since the terms are constant for positive w .

The dual integral equation has become a Cauchy singular integral equation for the $\vec{\eta}$. The actual form of the matrices A^{f+} and A^n give a beautiful separation of the problem. The entries of the matrix coefficient of $\vec{\eta}'$ are zero when there are nonzero entries in the matrix coefficient of $\mathcal{H}(\vec{\eta}')$, and similarly the entries of the matrix coefficient of $\mathcal{H}(\vec{\eta}')$ are zero when there are nonzero entries in the matrix coefficient of $\vec{\eta}'$.

11.4 A Particular Function and Its Hilbert Transform

Consider the following function:

$$\psi(x) = \begin{cases} 0 & x < -a \\ \sqrt{a^2 - x^2} & |x| < a \\ 0 & a < x. \end{cases}$$

The Hilbert transform of this is (see [Bateman 1954] Volume 2, page 246, or Appendix B)

$$i\mathcal{H}(\psi)(x) = \begin{cases} -x - \sqrt{x^2 - a^2} & x < -a \\ -x & |x| < a \\ -x + \sqrt{x^2 - a^2} & a < x. \end{cases}$$

Both the function ψ and its Hilbert transform $\mathcal{H}(\psi)$ are continuous and vanish at $\pm\infty$. This is expected behavior for the solution of the complementary problem. Also ψ is an even function, while $\mathcal{H}(\psi)$ is an odd function.

As the Hilbert transform of a derivative is the derivative of the Hilbert transform,

$$\begin{aligned} \psi'(x) &= \begin{cases} 0 & x < -a \\ \frac{-x}{\sqrt{a^2 - x^2}} & |x| < a \\ 0 & a < x, \end{cases} \\ i\mathcal{H}(\psi')(x) &= \begin{cases} -1 - \frac{x}{\sqrt{x^2 - a^2}} & x < -a \\ -1 & |x| < a \\ -1 + \frac{x}{\sqrt{x^2 - a^2}} & a < x. \end{cases} \end{aligned}$$

It will be assumed that the solution to the Cauchy singular integral equation can be written in the form

$$\vec{\eta} = \begin{pmatrix} r_1\psi + r_5i\mathcal{H}(\psi) \\ r_2\psi + r_6i\mathcal{H}(\psi) \\ r_3\psi + r_7i\mathcal{H}(\psi) \\ r_4\psi + r_8i\mathcal{H}(\psi) \end{pmatrix}, \quad (11.2)$$

where the r_i 's are dimensionless, hopefully real constants. Since $\vec{\eta}$ is real, and $\mathcal{H}(\psi)$ is imaginary, the r_i 's should be real.

To satisfy the continuity condition off the crack gives some restrictions on the r_i 's. As ψ is zero off the crack, only the $\mathcal{H}(\psi)$ terms need be considered:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \vec{\eta} = \begin{pmatrix} r_5 - r_7 \\ r_6 - r_8 \end{pmatrix} \mathcal{H}(\psi) = \vec{0} \quad |x| > a.$$

So continuity requires

$$r_5 = r_7, \quad r_6 = r_8. \quad (11.3)$$

11.5 Solving the Cauchy Singular Integral Equation

Placing these into the equation causes a separation into two parts: one for ψ ; and one for $\mathcal{H}(\psi)$. As

$$\mathcal{H}(\mathcal{H}(\psi)) = \psi,$$

setting the coefficients of ψ' equal to zero inside the crack gives

$$\left[\begin{pmatrix} 0 & c_6 \\ c_3 & 0 \\ & 0 & c_6 \\ & c_3 & 0 \end{pmatrix} + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} A^n \right] \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} A^{j+} \begin{pmatrix} ir_5 \\ ir_6 \\ ir_7 \\ ir_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (11.4)$$

Next, since the \hat{T}_{ij} are constants, the traction conditions can be satisfied since within the crack $\mathcal{H}(\psi') = i$, a constant. The equation is

$$\left[\begin{pmatrix} 0 & c_6 \\ c_3 & 0 \\ & 0 & c_6 \\ & c_3 & 0 \end{pmatrix} + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} A^n \right] \begin{pmatrix} ir_5 \\ ir_6 \\ ir_7 \\ ir_8 \end{pmatrix} (i) + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} A^{j+} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} (i) = \begin{pmatrix} \hat{T}_{12} \\ \hat{T}_{22} \\ \hat{T}_{12} \\ \hat{T}_{22} \end{pmatrix} \quad (11.5)$$

Solution of these will yield the solution to the displacements along the crack.

To facilitate finding solutions, the coefficient matrices will be written out. They are

$$\left[\begin{pmatrix} 0 & c_6 \\ c_3 & 0 \\ & 0 & c_6 \\ & c_3 & 0 \end{pmatrix} + \begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} A^n \right] =$$

$$\begin{pmatrix} 0 & c_6 \\ c_3 & 0 \\ & 0 & c_6 \\ & c_3 & 0 \end{pmatrix} - i \frac{k_+ - k_-}{b_+ - b_-} \begin{pmatrix} 0 & c_6 \\ -c_2 b_+ b_- & 0 \\ & 0 & c_6 \\ & -c_2 b_+ b_- & 0 \end{pmatrix},$$

$$\begin{pmatrix} c_6 & & & \\ & c_2 & & \\ & & c_6 & \\ & & & c_2 \end{pmatrix} A^{f+} = \frac{-i}{b_+ - b_-} \times$$

$$\begin{pmatrix} c_6(k_+ b_- - k_- b_+) & & & \\ & c_2(k_- b_- - k_+ b_+) & & \\ & & c_6(k_- b_+ - k_+ b_-) & \\ & & & c_2(k_+ b_+ - k_- b_-) \end{pmatrix}.$$

Both these matrices are assumed evaluated at some positive w . As b_{\pm} contains an i , the upper matrix is purely real, while the lower one is purely imaginary.

11.6 The Longitudinal Case

Here $\hat{T}_{12} = 0$ and \hat{T}_{22} is a given constant. Let

$$r_1 = r_3 = r_6 = r_8 = 0.$$

Equations 11.4 and 11.5 reduce to

$$\begin{pmatrix} c_6(1 - i \frac{k_+ - k_-}{b_+ - b_-}) & c_6 \frac{k_+ b_- - k_- b_+}{b_+ - b_-} \\ c_2 \frac{k_- b_- - k_+ b_+}{b_+ - b_-} & -c_3 - c_2 i b_+ b_- \frac{k_+ - k_-}{b_+ - b_-} \end{pmatrix} \begin{pmatrix} r_2 \\ r_5 \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{T}_{22} \end{pmatrix}$$

$$\begin{pmatrix} c_6(1 - i \frac{k_+ - k_-}{b_+ - b_-}) & -c_6 \frac{k_+ b_- - k_- b_+}{b_+ - b_-} \\ -c_2 \frac{k_- b_- - k_+ b_+}{b_+ - b_-} & -c_3 - c_2 i b_+ b_- \frac{k_+ - k_-}{b_+ - b_-} \end{pmatrix} \begin{pmatrix} r_4 \\ r_7 \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{T}_{22} \end{pmatrix}. \quad (11.6)$$

The rest of the equation is automatically satisfied. If r_2 and r_5 satisfy the upper set of equations, then

$$r_4 = -r_2, \quad r_7 = r_5$$

satisfy the lower set of equations, because the lower system differs from the upper only in two signs off the diagonal.

This solves the crack problem for the longitudinal case. The continuity requirement, Equation 11.3, is satisfied since $r_5 = r_7$ and $r_6 = r_8 = 0$.

To discuss the solution a little,

$$\begin{aligned}
 \frac{X_1^2}{(1-r_5)^2} + \frac{X_2^2}{r_2^2} &= \frac{(x+u)^2}{(1-r_5)^2} + \frac{(0+v)^2}{r_2^2} \\
 &= \frac{(x+r_5i\mathcal{H}(\psi))^2}{(1-r_5)^2} + \frac{(r_2\psi)^2}{r_2^2} \\
 &= \frac{(x+r_5(-x))^2}{(1-r_5)^2} + \frac{(r_2\sqrt{a^2-x^2})^2}{r_2^2} \\
 &= x^2 + a^2 - x^2 \\
 &= a^2
 \end{aligned}$$

and so the displacements for the complementary problem form an ellipse. This is not quite the case for the solution of the original problem, which is the wave solution plus the complementary solution. For this case the crack opening is still an ellipse, but it is displaced by an amount v , where v is the displacement of the wave solution.

Since \hat{T}_{22} is directly proportional to both the amplitude of the original wave and the frequency of the original wave,

$$\hat{T}_{22} = C\sqrt{\rho c_2}m$$

where C was a constant, one concludes there is a larger crack opening for a higher frequency wave (since increasing \hat{T}_{22} causes a proportional linear increase in the r_i 's). The crack is more likely to propagate when the crack opening displacement is large than when it is small, and so high frequencies can be expected to be more damaging to the material.

If one does a numerical solution for the dimensionless r_i 's, with the constants used previously, one has

$$r_2 = 11.04 \times 10^{-7} \times (-\hat{T}_{22})$$

$$r_5 = 2.00 \times 10^{-7} \times (-\hat{T}_{22})$$

where \hat{T}_{22} is in the pounds (force) per square inch. A note on the signs. Recall that the complementary solution is subtracted from the wave solution to produce a cancellation of the tractions. So a positive traction gives a positive crack opening, since the minus signs cancel and ψ is positive for $|x| < a$. The maximal crack opening occurs at the center of the crack and is given by

$$\text{crack opening} = (r_2 - r_4)\psi(0) = 2r_2a.$$

The larger the crack opening for this brittle material, the more likely it is for a crack to propagate.

If \hat{T}_{22} is negative, the wave is in its compressive cycle, and the above solution implies that the upper and lower displacements overlap. This clearly does not occur. What actually happens is that the crack closes and during the compression remains closed, so the above complementary solution does not apply.

11.7 The Transverse Case

Here let

$$r_2 = r_4 = r_5 = r_7 = 0$$

and the equations reduce to

$$\begin{pmatrix} c_3 + c_2 i b_+ b_- \frac{k_+ - k_-}{b_+ - b_-} & c_2 \frac{k_- b_- - k_+ b_+}{b_+ - b_-} \\ c_6 \frac{k_+ b_- - k_- b_+}{b_+ - b_-} & -c_6 (1 - i \frac{k_+ - k_-}{b_+ - b_-}) \end{pmatrix} \begin{pmatrix} r_1 \\ r_6 \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{T}_{12} \end{pmatrix}$$

$$\begin{pmatrix} c_3 + c_2 i b_+ b_- \frac{k_+ - k_-}{b_+ - b_-} & -c_2 \frac{k_- b_- - k_+ b_+}{b_+ - b_-} \\ -c_6 \frac{k_+ b_- - k_- b_+}{b_+ - b_-} & -c_6 (1 - i \frac{k_+ - k_-}{b_+ - b_-}) \end{pmatrix} \begin{pmatrix} r_3 \\ r_8 \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{T}_{12} \end{pmatrix}. \quad (11.7)$$

If r_1 and r_6 are a solution to the upper system of equations, then

$$r_3 = -r_1, \quad r_8 = r_6$$

constitute a solution to the lower system. Again the continuity conditions are satisfied as $r_6 = r_8$ and $r_5 = r_7 = 0$.

In this case there is not a nice elliptical crack opening. In fact, the crack does not open at all, since $\mu_{upper} = \mu_{lower}$. The crack faces do displace, but in a slipping fashion – the x -displacements do not match up. Apparently the crack rotates a bit. To obtain some measure of the rotation, at the $x = a$ end of the crack the v displacement is

$$v = \mu_{upper} = \mu_{lower} = r_6 i \mathcal{H}(\psi)(a) = r_6(-a).$$

If θ is the angle the crack makes with the x -axis, then

$$\tan(\theta) = \frac{v(a)}{a} = -r_6.$$

Since r_6 is proportional to the magnitude of the load and frequency of the wave,

$$\hat{T}_{12} = \frac{Cm}{\eta(0) \cos(\phi)},$$

the crack rotates more for large loads or high frequency. When the time dependence is included in T_{12} , the crack rotates back and forth at the wave frequency.

Doing a numerical calculation for the specific material herein,

$$\begin{aligned} r_1 &= 4.32 \times 10^{-7} \times (-\hat{T}_{12}) \\ r_6 &= -2.00 \times 10^{-7} \times (-\hat{T}_{12}) \end{aligned}$$

with \hat{T}_{12} in pounds (force) per square inch.

11.8 The Traction Off the Crack

Taking a last glimpse at the crack, the question is what are the tractions of the complementary solution off the crack for $y = 0$? Are they continuous?

The only thing needed is an explicit expression for T_{11} . But as

$$T_{11} = (c_1 + d_1 p) \frac{\partial u}{\partial x} + c_3 \frac{\partial v}{\partial y}$$

by comparison with Equation 9.1 the value of T_{11} is obtained by simply replacing c_3 by c_1 and c_2 by c_3 in the expression for T_{22} .

Thus all the tractions can be written in terms of $\vec{\eta}$, both on and off the crack (still on the line $y = 0$). With the $\hat{\cdot}$'s meaning time has been factored out,

$$\begin{pmatrix} \hat{T}_{11} \\ \hat{T}_{22} \\ \hat{T}_{12} \end{pmatrix}_{upper} = \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 \\ c_3 & 0 & 0 & 0 \\ 0 & c_6 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c_3 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ c_6 & 0 & 0 & 0 \end{pmatrix} A^n \right\} \vec{\eta}' \\ + \begin{pmatrix} 0 & c_3 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ c_6 & 0 & 0 & 0 \end{pmatrix} A^{f+} \mathcal{H}(\vec{\eta}').$$

One gets the lower tractions by reversing the order of the 3×2 block of zeros and the nonzero block.

Specifically, the longitudinal with $x > a$ has

$$\begin{pmatrix} \hat{T}_{11} \\ \hat{T}_{22} \\ \hat{T}_{12} \end{pmatrix}_{upper} = \left\{ \begin{pmatrix} c_1 & 0 & 0 & 0 \\ c_3 & 0 & 0 & 0 \\ 0 & c_6 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & c_3 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ c_6 & 0 & 0 & 0 \end{pmatrix} A^n \right\} \begin{pmatrix} r_5 i \mathcal{H}(\psi') \\ 0 \\ r_7 i \mathcal{H}(\psi') \\ 0 \end{pmatrix} \\ + \begin{pmatrix} 0 & c_3 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ c_6 & 0 & 0 & 0 \end{pmatrix} A^{f+} \begin{pmatrix} 0 \\ r_2 \mathcal{H}(\psi') \\ 0 \\ r_4 \mathcal{H}(\psi') \end{pmatrix}.$$

Written out,

$$\begin{aligned} \hat{T}_{11} &= \left\{ (c_1 + i c_3 b_+ b_- \frac{k_+ - k_-}{b_+ - b_-}) r_5 - c_3 \frac{k_- b_- - k_+ b_+}{b_+ - b_-} r_2 \right\} \left(-1 + \frac{x}{\sqrt{x^2 - a^2}} \right) \\ \hat{T}_{22} &= \left\{ (c_3 + i c_2 b_+ b_- \frac{k_+ - k_-}{b_+ - b_-}) r_5 - c_2 \frac{k_- b_- - k_+ b_+}{b_+ - b_-} r_2 \right\} \left(-1 + \frac{x}{\sqrt{x^2 - a^2}} \right) \\ \hat{T}_{12} &= 0 \quad x > a. \end{aligned} \quad (11.8)$$

In these, the b_i are evaluated for positive w . The shears (T_{12}) are zero for all time, both on and off the crack. For the lower half plane, since $r_4 = -r_2$ and the lower half of the A^{f+} matrix is minus the upper half of the A^{f+} matrix, the minus signs cancel and the tractions for the upper half plane equal those for the lower half plane. The tractions are continuous off the crack.

To discuss the behavior of these stresses, the wave solution which is to be added to the complementary solution for the complete solution can be ignored as, near the crack tip, the complementary solution's stresses have a singular behavior and dominate everything. Both \hat{T}_{11} and \hat{T}_{22} behave as

$$\hat{T}_{ii} \approx \frac{K_i}{\sqrt{x^2 - a^2}},$$

K_i a constant. Linear elasticity predicts infinite stresses at the crack tip.

To relate this result to the usual stress intensity factor, first one changes the variable $z = x - a$ to look right at the crack tip. The stress intensity factor is defined as (see [Broek 1978])

$$T_{11} = \frac{K}{\sqrt{2\pi z}}$$

for small $z > 0$. For the longitudinal wave this is (from Equation 11.8)

$$\hat{T}_{11} = \{\dots\} \left(-1 + \frac{z + a}{\sqrt{(z + a)^2 - a^2}}\right).$$

This gives

$$\hat{T}_{11} = \{\dots\} \sqrt{\frac{a}{2z}}$$

for z near zero, which gives ¹

$$K = \sqrt{\pi} \left\{ (c_1 + ic_3 b_+ b_- \frac{k_+ - k_-}{b_+ - b_-}) r_5 - c_3 \frac{k_- b_- - k_+ b_+}{b_+ - b_-} r_2 \right\} \sqrt{a}.$$

For the specific material this is

$$K = 4.52 \times \sqrt{a} \times (-\hat{T}_{22}).$$

For an isotropic material the stress intensity factor is the same when defined by

$$T_{22} = \frac{K}{\sqrt{2\pi z}}$$

but in the anisotropic case, this latter definition would give

$$K = \sqrt{\pi} \left\{ (c_3 + ic_2 b_+ b_- \frac{k_+ - k_-}{b_+ - b_-}) r_5 - c_2 \frac{k_- b_- - k_+ b_+}{b_+ - b_-} r_2 \right\} \sqrt{a}.$$

¹If the crack is located at the center of the layer, then the numerical value should be replaced by 6.17 for a 0°-ply and by 1.69 for a 90°-ply respectively.

For the specific material this is

$$K = 1.77 \times \sqrt{a} \times (-\hat{T}_{22}).$$

The K 's are not the same. Because of the anisotropy, the often stated isotropic results (see [Broek 1978]) using the stress intensity factor do not quite apply to this case. But there is something to be learned by this. The coefficient of the singularity depends on the squareroot of half the crack length, as the terms in the brackets do not depend on a . The larger the crack, the higher the resulting stresses in the neighborhood of the crack.

For the transverse case, the stresses are

$$\begin{aligned} \hat{T}_{11} &= 0 \\ \hat{T}_{22} &= 0 \\ \hat{T}_{12} &= c_6 \left\{ \left(1 - i \frac{k_+ - k_-}{b_+ - b_-} \right) r_6 - \frac{k_+ b_- - k_- b_+}{b_+ - b_-} r_1 \right\} \left(-1 + \frac{x}{\sqrt{x^2 - a^2}} \right). \end{aligned} \quad (11.9)$$

As $r_3 = -r_1$ and (as above) the corresponding entry in A^{j+} also changes sign for the lower half plane, the tractions here are also continuous. In this case the shears are singular at the crack tip.

This completes the discussion of the crack in a homogeneous material.

CHAPTER 12

THE NONHOMOGENEOUS CRACK

Finally comes the nonhomogeneous crack discussion. To actually carry out the ε^1 perturbation of the ordinary differential equation is very tedious and leads to no insight (the authors did this). Rather, a qualitative discussion seems more appropriate, with some numbers thrown in.

The question is how the inhomogeneity right in the vicinity of the crack, the crack lies on the interface of two layers, affects the solution. The ε^0 case treats the plane as a homogeneous material. The worst case of the inhomogeneity is when one considers the upper half plane to be all of one material, and the lower half plane to be all of another. The solution approach of the last chapter, using the specific ψ , will not quite work here. However, such an approach is a means to get an approximate solution, meaning a function satisfying the solid equation in the respective half plane, but not quite satisfying all the boundary conditions.

The arguments of the last chapter leading up to Equations 11.4 and 11.5 show that for the tractions on the crack face the equations for the upper half plane and the lower half plane are separate. One can solve the equations for the upper half plane with one set of constants, and the lower half plane for another set of constants, to yield zero tractions on the crack face. Then one can examine the continuity conditions off the crack to get a feel for how wrong the approximate solution is.

Using the constants used throughout the monograph, letting the upper half plane be material 1 (the + in the perturbation, $c_i + d_i$) and the lower half plane be material 2 ($c_i - d_i$) one has for the longitudinal case from Equation 11.6

$$\begin{aligned} r_2 &= 10.04 \times 10^{-7} \times (-\hat{T}_{22}) \\ r_5 &= 1.50 \times 10^{-7} \times (-\hat{T}_{22}) \\ r_4 &= -13.90 \times 10^{-7} \times (-\hat{T}_{22}) \\ r_7 &= 4.51 \times 10^{-7} \times (-\hat{T}_{22}) \end{aligned}$$

where \hat{T}_{22} is in the pounds (force) per square inch.

It is clear the continuity conditions are not satisfied - there is a slip in the u displacements. To see how large this is, compare the u discontinuity at $x = a$ with the maximal crack opening:

$$\frac{u \text{ discontinuity}}{\text{crack opening}} = \frac{(r_5 - r_7)i\mathcal{H}(\psi)(a)}{(r_2 - r_4)\psi(0)} = \frac{(r_5 - r_7)(-a)}{(r_2 - r_4)a} = -0.126.$$

This slip, or discontinuity, is 13% of the crack opening at the center of the crack, and it seems reasonable to view this solution as an approximate solution to the nonhomogeneous problem. Notice that, in this approximation, $v = 0$ off the crack for both the upper and lower half planes. The approximate displacements are given by the expression for $\bar{\eta}$ in terms of the r_i 's, ψ and $\mathcal{H}(\psi)$ in Equation 11.2. To see how much beyond the crack one must go before this discontinuity is half its value at the crack tip, consider

$$\mathcal{H}(\psi)(x) = \frac{a}{2}, \quad x > a.$$

This is

$$-x + \sqrt{x^2 - a^2} = \frac{a}{2}$$

yielding

$$x = \frac{5}{4}a.$$

The discontinuity, which is directly proportional to $\mathcal{H}(\psi)$, is half its value at $a/4$ beyond the crack, or at one-eighth the length of the crack beyond the crack.

The tractions also no longer match up; the greatest difficulty is in the shear \hat{T}_{12} . For the homogeneous case the shears are zero, but in the nonhomogeneous case as

$$T_{12} = c_6 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

the shears are singular since the u displacements are no longer continuous across the line $y = 0$. It seems reasonable to suppose that an exact solution would have large shears also. For a layered material this is of great concern, since interlaminar cracking, or delamination, is the main cause of failure.

This last observation gives some information on the form of an exact solution to the nonhomogeneous crack problem, namely that the v displacements must be nonzero off the crack, or that the Hilbert transform of the u displacements must be nonzero off the crack. This is because it is nonzero v displacements off the crack or nonzero Hilbert transform of the u displacements off the crack which give rise to nonzero shears.

The conclusions are that the crack on the boundary layer will experience very large shears, but that the crack opening is nearly the same as in the homogeneous case (half ellipses).

CHAPTER 13

CONCLUSIONS AND EXTENSIONS

In this final chapter conclusions will be drawn, and a few extensions and directions for further study will be suggested.

13.1 Conclusions

A perturbation approach yielded good results. For a transverse wave travelling normal to the laminae a rather interesting dispersion curve (phase speeds) resulted. The waves travelling parallel to the laminae were found to be nearly purely longitudinal or purely transverse, and so Mr. Ben-Amoz's statement regarding homogeneous behavior was borne out.

On interaction with a crack, the case of waves coming down on the crack, or travelling normal to the laminae, was considered. It was found that crack opening displacement for the longitudinal wave, and rotation of the crack for the transverse wave, was directly proportional to the magnitude of the displacements in the wave and the frequency of the wave. So higher frequency waves will be more destructive, though it is pointed out that the solution process assumed the frequency was not too large in order to ignore a term in the equation. For a crack in the layered material, one should expect high shears for the longitudinal wave travelling normal to the laminae.

13.2 The Layer Not at Ninety Degrees

Most of the time the layers are not placed in a $0^\circ/90^\circ/0^\circ/90^\circ \dots$, but rather at various angles such as $0^\circ/45^\circ/90^\circ/45^\circ \dots$. To deal with this, it is necessary to calculate the constants in Hooke's law after a change of basis. Hooke's law is a relation between two tensors, as the linearized strains form a tensor in the infinitesimal case. This being the case, it is possible to write a change of basis formula for the constants in Hooke's law. Hooke's law can be written

$$T_{ij} = c_{ijkl} E_{kl}$$

where E_{kl} is the matrix of the ϵ_{kl} . In the way Hooke's law is presented in the text, one has

$$\begin{pmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{pmatrix} = \begin{pmatrix} c_{1111} & c_{1122} & c_{1133} & \frac{1}{2}c_{1123} & \frac{1}{2}c_{1131} & \frac{1}{2}c_{1112} \\ c_{2211} & c_{2222} & c_{2233} & \frac{1}{2}c_{2223} & \frac{1}{2}c_{2231} & \frac{1}{2}c_{2212} \\ c_{3311} & c_{3322} & c_{3333} & \frac{1}{2}c_{3323} & \frac{1}{2}c_{3331} & \frac{1}{2}c_{3312} \\ c_{2311} & c_{2322} & c_{2333} & \frac{1}{2}c_{2323} & \frac{1}{2}c_{2331} & \frac{1}{2}c_{2312} \\ c_{3111} & c_{3122} & c_{3133} & \frac{1}{2}c_{3123} & \frac{1}{2}c_{3131} & \frac{1}{2}c_{3112} \\ c_{1211} & c_{1222} & c_{1233} & \frac{1}{2}c_{1223} & \frac{1}{2}c_{1231} & \frac{1}{2}c_{1212} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{pmatrix}. \quad (13.1)$$

Now suppose the $x - y$ plane is rotated an angle θ about the z -axis. Then if \vec{x} is a vector, in the new coordinate system the components of the vector are

$$\vec{x}' = A\vec{x}$$

where

$$A = (a_{ij}) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In terms of this A matrix, the new values for the constants in Hooke's law become

$$c'_{abcd} = a_{ai} a_{bj} a_{ck} a_{dl} c_{ijkl}$$

where, of course, the summation notation is employed.

13.3 A Simpler Approach

The above is a rather tedious way to find the new material constants. An easier approach is to use a similar one to that in [Vinson and Sierakowski 1986].

As the stress tensor is a rank 2 tensor, it transforms as

$$T'_{ab} = a_{ai}a_{bj}T_{ij}.$$

This can be written out as a matrix

$$\begin{pmatrix} T'_{11} \\ T'_{22} \\ T'_{33} \\ T'_{23} \\ T'_{31} \\ T'_{12} \end{pmatrix} = \begin{pmatrix} \cos^2 \theta & \sin^2 \theta & 0 & 0 & 0 & 2 \cos \theta \sin \theta \\ \sin^2 \theta & \cos^2 \theta & 0 & 0 & 0 & -2 \sin \theta \cos \theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos \theta & -\sin \theta & 0 \\ 0 & 0 & 0 & \sin \theta & \cos \theta & 0 \\ -\cos \theta \sin \theta & \sin \theta \cos \theta & 0 & 0 & 0 & \cos^2 \theta - \sin^2 \theta \end{pmatrix} \begin{pmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{pmatrix}.$$

Particular care must be used in deriving this. For example, as T_{ij} is symmetric, $T_{12} = T_{21}$ gives rise to the factors of two in the last column. Call this transformation matrix \tilde{A} . \tilde{A}^{-1} is given by letting $\theta \mapsto -\theta$.

Next, one can write Equation 13.1 in a better fashion for the purpose of finding the new material constants:

$$\begin{pmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{pmatrix} = \begin{pmatrix} c_{1111} & c_{1122} & c_{1133} & \frac{1}{2}c_{1123} & \frac{1}{2}c_{1131} & \frac{1}{2}c_{1112} \\ c_{2211} & c_{2222} & c_{2233} & \frac{1}{2}c_{2223} & \frac{1}{2}c_{2231} & \frac{1}{2}c_{2212} \\ c_{3311} & c_{3322} & c_{3333} & \frac{1}{2}c_{3323} & \frac{1}{2}c_{3331} & \frac{1}{2}c_{3312} \\ c_{2311} & c_{2322} & c_{2333} & \frac{1}{2}c_{2323} & \frac{1}{2}c_{2331} & \frac{1}{2}c_{2312} \\ c_{3111} & c_{3122} & c_{3133} & \frac{1}{2}c_{3123} & \frac{1}{2}c_{3131} & \frac{1}{2}c_{3112} \\ c_{1211} & c_{1222} & c_{1233} & \frac{1}{2}c_{1223} & \frac{1}{2}c_{1231} & \frac{1}{2}c_{1212} \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{23} \\ \epsilon_{31} \\ \epsilon_{12} \end{pmatrix}.$$

To simplify a bit, let \vec{T} be the above vector of stresses, C be the matrix of constants in Hooke's law, and \vec{E} be the above vector of strains. Then under the rotation one has

$$\vec{T}' = C' \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix} \vec{E}'$$

which is

$$\tilde{A}\vec{T} = C' \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix} \tilde{A}\vec{E}.$$

This gives

$$C = \tilde{A}^{-1}C' \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix} \tilde{A} \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix}^{-1}$$

or

$$C' = \tilde{A}C \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 2 & & \\ & & & & 2 & \\ & & & & & 2 \end{pmatrix} \tilde{A}^{-1} \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \frac{1}{2} & & \\ & & & & \frac{1}{2} & \\ & & & & & \frac{1}{2} \end{pmatrix}.$$

If one multiplies out the last three matrices, one arrives at the surprising final result

$$C' = \tilde{A}(\theta)C\tilde{A}^T(-\theta) \quad (13.2)$$

with the reminder that $\tilde{A}^{-1} \neq \tilde{A}^T$. With this one can find the material constants for a layer at an angle θ by viewing it as a rotation of the coordinate axes by $-\theta$.

13.4 How a Layer at an Angle Affects the Equations

Having a layer not at 0° or 90° changes the equations right from the start. It leads to

$$\begin{pmatrix} T_{11} \\ T_{22} \\ T_{12} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{16} \\ c_{21} & c_{22} & c_{26} \\ c_{61} & c_{62} & c_{66} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{pmatrix}. \quad (13.3)$$

so the simple form of the equation studied is not adequate for the analysis of this case.

However, a great deal of difference is not expected in qualitative behavior. The waves travelling in a direction parallel to the laminae will still have an average propagation speed decided by the average of the two layers. The transverse waves travelling in a direction normal to the laminae will exhibit an amplitude modulation. But the longitudinal waves travelling in a direction normal to the laminae will have an analytical behavior similar to that of the transverse waves, as now all the other terms will not vanish away.

13.5 Directions

An interesting direction to pursue would be the effects of glue between the layers. Many composite materials are held together by an adhesive and so this is an important case. Also, one could study in more detail the local behavior of the material near the boundary of two layers. As a first step in these analyses, one could use more complicated $p(x)$ to model the more complicated Hooke's law in each of the regions: the layers and the glues.

The effect on the crack by a wave travelling parallel to the laminae could be tackled, though this would be tedious. It is expected that a longitudinal wave would have little effect on the crack, the effect being similar to that of the transverse wave travelling normal to the laminae. The transverse wave would cause a crack opening similar to that seen in the longitudinal wave travelling normal to the laminae.

Also, the approach of writing the displacements along the crack directly as a Cauchy singular integral equation looks very promising. The solution of the equation directly yields the displacements, and there are no extraneous variables or functions introduced. Numerical methods are improving for the solving of these types of equations (and this is a one-dimensional one). For an exact solution to the crack in the layered solid, a simple analytic solution is not expected, but a numerical method based on this approach might work.

APPENDIX A

THE EXPERIMENTAL DATA

The material constants given in the text were arrived at by the following experimental data supplied to the author.

$$\begin{array}{lll} E_1 = 20.450 & G_{12} = 0.710 & \nu_{12} = 0.291 \\ E_2 = 1.350 & G_{23} = 0.502 & \nu_{23} = 0.333 \\ E_3 = 1.489 & G_{13} = 0.710 & \nu_{13} = 0.291 \end{array}$$

where the E_i and the G_{ij} are in million pounds (force) per square inch, and the ν_{ij} are dimensionless.

The matrix of coefficients C for the general Hooke's law is given by the inverse of the compliance matrix S , so

$$C^{-1} = S = \begin{pmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{pmatrix}.$$

Inverting this S matrix with the above data gave rise to the material constants used in the text.

The density was supplied as 1.58 grams per cubic centimeter.

For those who do not like English units at all, conversion to cgs units is straightforward. Using

$$1 \frac{\text{pound}(\text{force})}{\text{inch}^2} = 68947 \frac{\text{dynes}}{\text{cm}^2}$$

one gets for the coefficients

$$C = \begin{pmatrix} 1436 & 42.98 & 46.21 & 0 & 0 & 0 \\ 42.98 & 107.4 & 40.33 & 0 & 0 & 0 \\ 46.21 & 40.33 & 118.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 34.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 49.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 49.0 \end{pmatrix}$$

with the units being 10^9 dynes per square centimeter.

APPENDIX B

THE PARTICULAR HILBERT TRANSFORM

In this appendix the Hilbert transform of the particular transform used in the text is found. The approach is to use a change of variable followed by a contour integration.

Recall,

$$\psi(x) = \begin{cases} 0 & x < -a \\ \sqrt{a^2 - x^2} & |x| < a \\ 0 & a < x. \end{cases}$$

and

$$\mathcal{H}(\psi)(y) = \frac{\text{PV}}{\pi i} \int_{-\infty}^{\infty} \frac{\psi(x)}{x - y} dx$$

where PV means Cauchy principal value for the integral.

Introducing the change of variable motivated by [Tricomi 1957], page 174,

$$x = a \frac{1 - t^2}{1 + t^2}, \quad dx = a \frac{-4t}{(1 + t^2)^2} dt$$

gives

$$\begin{aligned} \mathcal{H}(\psi)(y) &= \frac{\text{PV}}{\pi i} \int_{-a}^a \frac{\sqrt{a^2 - x^2}}{x - y} dx \\ &= \frac{\text{PV}}{\pi i} \int_{\infty}^0 \frac{2at}{(1 + t^2)(a \frac{1-t^2}{1+t^2} - y)} a \frac{-4t}{(1 + t^2)^2} dt \\ &= \frac{\text{PV}}{\pi i} 8a^2 \int_0^{\infty} \frac{t^2}{\{(a - y) - (a + y)t^2\}(1 + t^2)^2} dt. \end{aligned}$$

As this integrand is a function of t^2 ,

$$\mathcal{H}(\psi)(y) = \frac{\text{PV}}{\pi i} 4a^2 \int_{-\infty}^{\infty} \frac{t^2}{\{(a - y) - (a + y)t^2\}(1 + t^2)^2} dt.$$

The integrand (referred to as I in the sequel) is holomorphic except for a finite number of poles, and so the residue theorem applies for a given contour in the complex plane (see [Ash 1971]). It vanishes at infinity as $1/t^4$, and so an arc at infinity will contribute zero to the integral. The contour will be closed in the upper half plane. All the cases will have in common a double pole at i , and so the residue there will be calculated first.

$$\begin{aligned}
 \text{res}(I, i) &= 4a^2 \lim_{t \rightarrow i} \frac{d}{dt} \frac{(t-i)^2 t^2}{\{(a-y) - (a+y)t^2\}(1+t^2)^2} \\
 &= 4a^2 \lim_{t \rightarrow i} \frac{d}{dt} \frac{t^2}{\{(a-y) - (a+y)t^2\}(t+i)^2} \\
 &= 4a^2 \lim_{t \rightarrow i} \left[\frac{2t}{\{(a-y) - (a+y)t^2\}(t+i)^2} + \frac{(a+y)t^3}{\{(a-y) - (a+y)t^2\}^2(t+i)^2} \right. \\
 &\quad \left. - \frac{2t^2}{\{(a-y) - (a+y)t^2\}(t+i)^3} \right] \\
 &= 4a^2 \left[\frac{2i}{2a(2i)^2} + \frac{(a+y)i^3}{\{2a\}^2(2i)^2} - \frac{2i^2}{2a(2i)^3} \right] \\
 &= 4a^2 \frac{iy}{8a^2} \\
 &= \frac{iy}{2}.
 \end{aligned}$$

This completes the first residue.

Now to look at various cases. For $-a < y < a$ the other poles lie on the axis at

$$\pm \sqrt{\frac{a-y}{a+y}}.$$

The residues are

$$\begin{aligned}
 \text{res}(I, \pm \sqrt{\frac{a-y}{a+y}}) &= 4a^2 \lim_{t \rightarrow \pm \sqrt{\frac{a-y}{a+y}}} \left[\frac{(t \pm \sqrt{\frac{a-y}{a+y}}) t^2}{\{-(a-y) - (a+y)t^2\}(1+t^2)^2} \right] \\
 &= 4a^2 \frac{(\pm \sqrt{\frac{a-y}{a+y}})^2}{\{-(a+y)(\pm 2\sqrt{\frac{a-y}{a+y}})\}(1 + (\pm \sqrt{\frac{a-y}{a+y}})^2)^2} \\
 &= \pm 4a^2 \frac{(\frac{a-y}{a+y})}{\{(a+y)(2\sqrt{\frac{a-y}{a+y}})\}(1 + (\frac{a-y}{a+y})^2)}.
 \end{aligned}$$

Notice the signs of the residues of the \pm roots are the only things which differ, and so a sum of these residues (actually one half the sum since they lie on the axis) gives zero. The double root at i considered above is the only surviving term for this case.

Next consider $y > a$. The roots

$$\pm\sqrt{\frac{a-y}{a+y}} = \pm i\sqrt{\frac{y-a}{a+y}}$$

are poles of the integrand, and the plus one lies in the upper half plane. Following the last case's argument one has

$$\begin{aligned} \operatorname{res}(I, i\sqrt{\frac{y-a}{a+y}}) &= 4a^2 \frac{(i\sqrt{\frac{y-a}{a+y}})^2}{\{-(a+y)2i\sqrt{\frac{y-a}{a+y}}\}(1+(i\sqrt{\frac{y-a}{a+y}})^2)^2} \\ &= 4a^2 \frac{-\frac{y-a}{a+y}}{-(a+y)2i\sqrt{\frac{y-a}{a+y}}(1-\frac{y-a}{a+y})^2} \\ &= -\frac{2a^2}{i} \frac{\sqrt{\frac{y-a}{a+y}}}{-(a+y)(\frac{2a}{a+y})^2} \\ &= -\frac{i}{2}\sqrt{y^2-a^2}. \end{aligned}$$

This completes the calculation of this residue.

Finally, for $y < -a$, the poles are at

$$\pm\sqrt{\frac{a-y}{a+y}} = \pm i\sqrt{\frac{a-y}{-a-y}}$$

and the same argument as above leads to

$$\begin{aligned} \operatorname{res}(I, i\sqrt{\frac{a-y}{-a-y}}) &= 4a^2 \frac{(i\sqrt{\frac{a-y}{-a-y}})^2}{\{-(a+y)2i\sqrt{\frac{a-y}{-a-y}}\}(1+(i\sqrt{\frac{a-y}{-a-y}})^2)^2} \\ &= 4a^2 \frac{-\frac{a-y}{-a-y}}{-(a+y)2i\sqrt{\frac{a-y}{-a-y}}(1-\frac{a-y}{-a-y})^2} \\ &= -\frac{2a^2}{i} \frac{\sqrt{\frac{a-y}{-a-y}}}{-(a+y)(\frac{2a}{a+y})^2} \\ &= \frac{i}{2}\sqrt{y^2-a^2}. \end{aligned}$$

This is the last residue which needed to be calculated.

The Hilbert transform of ψ is now easy to find. For $y < -a$,

$$\begin{aligned}
 \mathcal{H}(\psi)(y) &= \frac{\text{PV}}{\pi i} 4a^2 \int_{-\infty}^{\infty} \frac{t^2}{\{(a-y) - (a+y)t^2\}(1+t^2)^2} dt \\
 &= 2\left(\frac{1}{2\pi i} \int_{\text{contour}} 4a^2 \frac{t^2}{\{(a-y) - (a+y)t^2\}(1+t^2)^2} dt\right) \\
 &= 2(\text{sum of the residues}) \\
 &= 2\left(\frac{iy}{2} + \frac{i}{2}\sqrt{y^2 - a^2}\right) \\
 &= iy + i\sqrt{y^2 - a^2}.
 \end{aligned}$$

Similarly, for $-a < y < a$,

$$\mathcal{H}(\psi)(y) = 2\left(\frac{iy}{2}\right) = iy,$$

and for $a < y$,

$$\mathcal{H}(\psi)(y) = 2\left(\frac{iy}{2} - \frac{i}{2}\sqrt{y^2 - a^2}\right) = iy - i\sqrt{y^2 - a^2}.$$

This is the Hilbert transform of ψ as given in the text.

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