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The Stress Field at the Base of a Port in a Cylindrical Pressure Vessel

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ABSTRACT

The stress field at the base of a port hole in a cylindrical pressure vessel is presented. The analysis shows the stresses to be singular in this neighborhood with a singularity strength of the order 0·198. Moreover, locations of possible fatigue crack initiation are identified.

1 INTRODUCTION

It is well known that the majority of fractures that occur in engineering structures are due to fatigue. Fatigue failure is the phenomenon of progressive cracking that, unless detected early, can lead to catastrophic failures. It is essential, therefore, that designers have a complete understanding of this phenomenon and how to deal with it. Pressure vessels are special types of structures and as such not immune to fatigue failures. In one case history, for example, lethal gas was stored into a cylindrical type of pressure vessel via a port hole. Unfortunately, a fatigue crack developed prematurely at the base of the port and the gas began to escape creating, therefore, a hostile environment.

The exact mechanism of the initiation of a fatigue crack is extremely complex and not very well understood. Nevertheless, discussions of some fatigue theories can be found in the existing literature,[1–3] and will not be addressed here. It is more important for us to examine where, when, and under what conditions a fatigue crack is most likely to develop. In general, if

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a repeated load is large enough to cause a fatigue crack, the crack will initiate at a point of maximum stress. This maximum stress is usually due to a stress *concentration* often referred to as a stress riser.

Stress concentrations can occur on the exterior surface of the member in the form of scratches, rust pits or even sharp corners. There exists overwhelming experimental evidence that points to the undesirability of the presence of such flaws and emphasizes how important it is for the designer to take great measures to eliminate all adverse conditions that may lead to the initiation of a fatigue crack. The ultimate strategy, therefore, is to take appropriate measures so that no crack, however small, will manifest itself.

Port holes are quite often necessary components of pressurized vessels. It is not surprising, therefore, that the majority of the service cracks nucleate at the base of such ports. The subject of eventual concern is to derive reliable design criteria that can be used to ensure the structural integrity of the vessel for its entire service life. However, in deriving such criteria, the knowledge of the three-dimensional stress field at such neighborhoods is a prerequisite.

2 FORMULATION OF THE PROBLEM

Let us consider a portion of a thin, shallow† cylindrical vessel, of constant thickness h, which is subjected to a uniform internal pressure q_0 and contains a through-the-thickness cylindrical port of radius a. In this analysis, we shall limit our considerations to elastic, isotropic and homogeneous segments of cylindrical vessels that are subjected to small deformations.

The basic variables in the theory of cylindrical shells are the displacement component $w(x, y)$ in the direction of the z-axis, and a stress function $F(x, y)$, which represents the stress resultants tangent to the middle surface of the shell. Following Marguerre[4] the coupled differential equations governing w and F, with x and y as rectangular coordinates of the base plane (see Fig. 1), are given by:

$$\frac{Eh}{R}\frac{\partial^2 w}{\partial x^2} + \nabla_2^4 F = 0 \tag{1}$$

and

$$\nabla_2^4 w - \frac{1}{RD}\frac{\partial^2}{\partial x^2} = q_0/D \tag{2}$$

where E represents Young's modulus, h the thickness of the vessel, R the

† A shell is called thin if $h/R \leq 0{\cdot}01$ and shallow if $L/R \leq 0{\cdot}1$, where L represents a linear dimension.

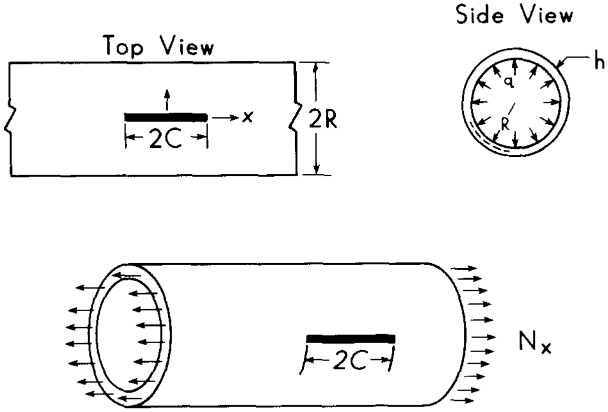

Fig. 1. Geometry and coordinates of a cylindrical pressure vessel with a crack.

radius of curvature, D the flexural rigidity, q_0 the internal pressure and ∇_2^4 the two-dimensional biharmonic operator.

Theoretical investigations of the above equations in the presence of discontinuities such as holes or cracks have been carried out by Van Dyke[5] and Folias,[6] respectively. Moreover, Folias[6] was able to show that there exists a correlation function between a cylindrical pressure vessel and the corresponding case of a flat plate, i.e.

$$\sigma_{\text{shell}} = \{1 + f\}^{-1/2} \sigma_{\text{plate}} \tag{3}$$

where the function f represents a correction factor to account for the geometry of the structure. The value of this geometry correction factor is positive and less than one. For example, in the case where the discontinuity is in an axial crack, the geometry correction factor has been shown to be (see Ref. 6):

$$f = 0{\cdot}317\lambda^2 \tag{4}$$

where

$$\lambda^2 \equiv \{12(1 - \nu^2)\}^{1/2} \frac{c^2}{\sqrt{Rh}} \tag{5}$$

and $2c$ represents the length of the crack.

Although this correlation function was derived on the basis of a homogeneous and isotropic material, the author believes that it reflects the dominant term of a geometrical inherent property that exists between shallow cylindrical shell structures and similarly loaded flat plates of the same material. Such a property is of great practical value for it may be used to predict stresses in pressurized cylinders by knowing only the corresponding stresses in flat plates.

3 THE STRESS FIELD AT THE BASE OF THE PORT

Thus, in the immediate vicinity of the intersection of the two surfaces between the port hole and the cylindrical vessel, one may envision (at least locally) the cylindrical surface being flat. This is because of the shallowness of the pressure vessel relative to the port hole. Moreover, in order to examine the possible singular nature of the stress field in the presence of a 270° material corner, one must consider the fully three-dimensional problem (see Fig. 2).

In particular, we must find three displacement functions u, v and w such that they satisfy the following governing equations

$$\frac{1}{1-2\nu}\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] e + \nabla^2(u, v, w) = 0 \tag{6)–(8}$$

where ∇^2 is the three-dimensional Laplacian operator, ν is Poisson's ratio and

$$e \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{9}$$

in view of which the stress–displacement relations are given by Hooke's law as

$$\sigma_{xx} = 2G\left\{\frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu} e\right\}, \ldots, \tau_{xy} = G\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}, \ldots \tag{10)–(15}$$

with G being the shear modulus.

As to boundary conditions, we require that

$$\text{at } z = h/2: \qquad \sigma_{zz} = \tau_{xz} = \tau_{yz} = 0 \tag{16)–(18}$$

$$\text{at } r = a: \qquad \sigma_{rr} = \tau_{r\theta} = \tau_{rz} = 0 \tag{19)–(21}$$

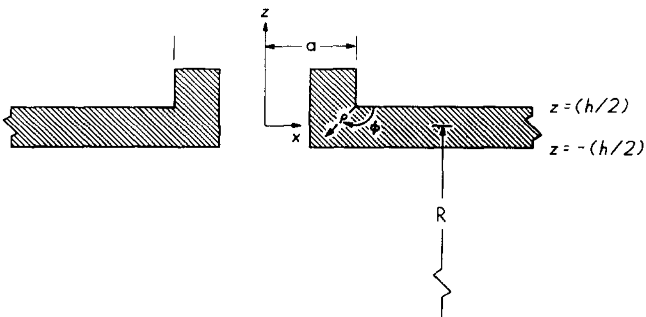

Fig. 2. Geometrical configuration at the base of the port.

In seeking a solution to the problem, we assume the displacement field in the form[7]

$$u = \rho^{\alpha-1} F(\phi)\sin\theta\cos(2\theta) \tag{22}$$

$$v = \rho^{\alpha-1} F(\phi)\cos\theta\cos(2\theta) \tag{23}$$

$$w = \rho^{\alpha-1}\tilde{G}(\phi)\cos(2\theta), \tag{24}$$

where ρ and ϕ represent the local coordinates at the material corner, F and $\tilde{G}$ are arbitary functions to be determined, and α is a constant. Without going into the mathematical details, one finds by substituting eqns (22)–(24) into the governing eqns (6)–(8) and subsequently satisfying the boundary conditions (eqns (16)–(21)) that

(i) *the displacements field*:

$$u = \rho^{\alpha-1}\Psi(\phi)\sin(\theta)\cos(2\theta) + 0(\rho^{\alpha}) \tag{25}$$

$$v = \rho^{\alpha-1}\Psi(\phi)\cos(\theta)\cos(2\theta) + 0(\rho^{\alpha}) \tag{26}$$

$$w = -\rho^{\alpha-1}\Phi(\phi)\cos(2\theta) + 0(\rho^{\alpha}) \tag{27}$$

where

$$\begin{aligned}\Psi(\phi) = \alpha B\left\{2\left(\frac{m-1}{m-2}\right)\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{3\alpha\pi}{2}\right)\cos(\alpha-1)\phi\right.\\ \left.+\frac{m-2}{m}\sin(\alpha-1)\phi\right\} + \alpha(\alpha-1)B\sin\phi\\ \times\left\{\frac{1-\alpha}{\alpha}\tan\left(\frac{3\alpha\pi}{2}\right)\sin(2-\alpha)\phi + \cos(2-\alpha)\phi\right\}\end{aligned} \tag{28}$$

$$\begin{aligned}\Phi(\phi) = \alpha B\left\{\frac{3m-2}{m}\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{3\alpha\pi}{2}\right)\sin(\alpha-1)\phi - 2\left(\frac{m-1}{m}\right)\cos(\alpha-1)\phi\right\}\\ + \alpha(1-\alpha)B\sin\phi\left\{\frac{1-\alpha}{\alpha}\tan\left(\frac{3\alpha\pi}{2}\right)\cos(\alpha-2)\phi + \sin(\alpha-2)\phi\right\}\end{aligned} \tag{29}$$

(ii) *the stress field*:

$$\begin{aligned}\sigma_{rr} = 2G\alpha(\alpha-1)\rho^{\alpha-2}B\left\{2\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{3\alpha\pi}{2}\right)\cos(\alpha-2)\phi + \sin(\alpha-2)\phi\right.\\ \left.-(\alpha-2)\sin\phi\left[\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{3\alpha\pi}{2}\right)\sin(\alpha-3)\phi - \cos(\alpha-3)\phi\right]\right\}\\ \times\cos(2\theta) + 0(\rho^{\alpha-1})\end{aligned} \tag{30}$$

$$\sigma_{\theta\theta} = -2G\alpha(\alpha-1)\rho^{\alpha-2}B\left\{2\left(\frac{m+1}{m}\right)\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{3\alpha\pi}{2}\right)\cos(\alpha-2)\phi + \left(\frac{m+2}{m}\right)\sin(\alpha-2)\phi - (\alpha-2)\sin\phi \times\left[\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{3\alpha\pi}{2}\right)\sin(\alpha-3)\phi - \cos(\alpha-3)\phi\right]\right\}\cos(2\theta) + 0(\rho^{\alpha-1}) \quad (31)$$

$$\sigma_{zz} = 2G\alpha(\alpha-1)\rho^{\alpha-2}B\left\{\sin(\alpha-2)\phi + (\alpha-2)\sin\phi \times\left[\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{3\alpha\pi}{2}\right)\sin(\alpha-3)\phi - \cos(\alpha-3)\phi\right]\right\}\cos(2\theta) + 0(\rho^{\alpha-1}) \quad (32)$$

$$\tau_{rz} = 2G\alpha(\alpha-1)\rho^{\alpha-2}B\left\{\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{3\alpha\pi}{2}\right)\sin(\alpha-2)\phi + (\alpha-2)\sin\phi \times\left[\left(\frac{1-\alpha}{\alpha}\right)\tan\left(\frac{3\alpha\pi}{2}\right)\cos(\alpha-3)\phi + \sin(\alpha-3)\phi\right]\right\}\cos(2\theta) + 0(\rho^{\alpha-1}) \quad (33)$$

$$\tau_{r\theta} = 0(\rho^{\alpha-1}) \quad (34)$$

where $m \equiv 1/v$ and α satisfies the characteristic equation

$$\sin\left[(\alpha-1)\frac{3\pi}{2}\right] = \pm(\alpha-1) \quad (35)$$

The above equation has an infinite number of complex roots. However, of practical interest is only the value of $\mathrm{Re}(\alpha) = 1{\cdot}802$, which implies that the stresses in the vicinity of the base are singular. Interestingly enough, this is exactly the same result as that obtained by Williams,[8] based on plane strain analysis. Finally, it should be noted that B is a constant that is to be determined from the boundary conditions far away from the port. In particular,

$$B \sim \frac{v}{1+v}\sigma_h\{1+f\}^{1/2} \quad (36)$$

where σ_h is the hoop stress and where we have included an appropriate geometry correction factor in order to account for the curvature of the pressure vessel.

4 CONCLUSIONS

The analysis clearly shows that at the base of the port hole, the stresses are singular in nature with a singularity strength of 0·198. This suggests, therefore, that, all things being equal, a fatigue crack is most likely to develop at the 12 or 6 o'clock positions of the port. Designers must exercise caution at such locations, particularly if welding is to be carried out. Another safety precaution may be to eliminate the sharp corner by introducing a small radius at the base of the port. Such preventive measures may ultimately increase the life span of the vessel.

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