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On the Three-Dimensional Theory of Cracked Plates

This paper discusses a method for solving three-dimensional mixed-boundary-value problems which arise in elastostatics. Specifically, the method is applied to a plate of finite thickness which contains a finite, through the thickness, line crack. The analysis shows that (a) in the interior of the plate only the stresses σ_x, σ_y, σ_z, τ_{xy} are singular of order 1/2; (b) in the vicinity of the corner point all the stresses are singular of order [(1/2) + 2ν]; (c) as the thickness $h \to \infty$ the plane strain solution is recovered and; (d) as $\nu \to 0$ the plane stress solution is recovered. Finally, it is found that in the neighborhood of the corner points, even though the displacements are singular for certain values of the Poisson's ratios, the derived stress field satisfies the condition of local finite energy.

Introduction

A major debility in current fracture mechanics work is the ignorance of the effects of thickness on the mechanism of failure. For example, the common experimental observation of a change from ductile failure at the edge to brittle fracture at the center of broken sheet material has so far defied analysis. Yet an orderly theoretical attack on the problem can provide important guidance to this and other phases of fracture research. The most potent mathematical tool for this attack is the linear theory of infinitesimal elasticity as applied to a cracked plate of finite thickness. Although this theory cannot include the nonelastic behavior of the material at the crack tip per se, it can evince many characteristics of the actual behavior of a cracked plate, including those due to thickness. Thus the theory of elasticity is a logical fountainhead for detailed theoretical study.

The mathematical difficulties, however, posed by three-dimensional problems in elasticity are substantially greater than those associated with plane stress or plane strain. Nevertheless, a few particular cases have been solved and can be found in references [1–2].[1]

The stress distribution in a thick plate containing a smooth circular cavity has been discussed by Sternberg and Sadowsky [3], Green [4], and Alblas [5]. Their work has shown that the thickness of the plate can exert appreciable influence on the stress concentrations of the circular hole.

An attempt has been made recently, by Hartranft and Sih [6] to investigate the triaxial characteristics of the crack-edge stress field in a thick plate using a variational principle. In their analysis, they have assumed that the local stress field interior to the plate is in a state of plane strain thus forcing the normal, to the plate faces, stress to vanish in a boundary-layer sense. Under these assumptions, their results show that the stress state depends on the plate thickness to crack length ratio and a dimensionless parameter characterizing the stress distribution across the thickness. Finally, the authors emphasize that although these results are considered to be a refinement over those of generalized plane stress they remain still an approximation to the three-dimensional problems.

This paper describes a method for constructing solutions to some three-dimensional mixed boundary-value problems which arise in elastostatics. In particular, the method is applied to the problem of a uniform extension of an infinite plate containing a through the thickness line crack.

Formulation of the Problem

Consider the equilibrium of a homogeneous, isotropic, elastic plate which occupies the space $|x| < \infty$, $|y| < \infty$, $|z| < h$ and contains a plane crack in the x-z- plane. The crack faces, defined by $|x| < c$, $y = 0^{\pm}$, $|z| \leq h$, and the plate faces $|z| = h$ are free of stress and constraint. Loading is applied on the periphery of the plate $|x|$, $|y| \to \infty$ and is given by

$$\sigma_x = \tau_{xy} = \tau_{yz} = 0, \quad \sigma_y = \bar{\sigma}_0.$$

In the absence of body forces, the coupled differential equations governing the displacement functions u, v, and w are

$$\frac{m}{m-2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\theta + \nabla^2(u, v, w) = 0 \quad (1)\text{–}(3)$$

where ∇^2 is the Laplacian operator, $m \equiv 1/\nu$, ν is Poisson's ratio,

$$\theta \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \quad (4)$$

[1] Numbers in brackets designate References at end of paper.

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and the stress-displacement relations are given by Hook's law as

$$\sigma_x = 2G\left\{\frac{\partial u}{\partial x} + \frac{\theta}{m-2}\right\}, \ldots, \tau_{xy} = G\left\{\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right\}, \ldots \quad (5)\text{–}(10)$$

with G being the shear modulus.

As to boundary conditions, one must require that at

$$|x| < c,\ y = 0^{\pm},\ |z| \leq h: \ \tau_{xy} = \tau_{yz} = \sigma_y = 0 \quad (11)$$

$$|z| = h: \ \tau_{xz} = \tau_{yz} = \sigma_z = 0 \quad (12)$$

$$|y| \longrightarrow \infty \text{ and all } x: \ \tau_{xy} = \tau_{yz} = 0,\ \sigma_y = \bar{\sigma}_0 \quad (13)$$

$$|x| \longrightarrow \infty: \sigma_x = \tau_{xy} = \tau_{zx} = 0. \quad (14)$$

It is found convenient to seek the solution to the crack plate problem in the form

$$u = u^{(P)} + u^{(C)},\ \text{etc.}, \quad (15)$$

where the first component represents the usual "undisturbed" or "particular" solution of a plate without the presence of a crack. Such a particular solution can easily be constructed and for the particular problem at hand is

$$u^{(P)} = -\frac{\bar{\sigma}_0}{2G\Delta}(m-2)^2 x,\ v^{(P)} = -[1-(m-1)^2]\frac{\bar{\sigma}_0(m-2)}{2G\Delta}y,\ w^{(P)} = -(m-2)^2\frac{\bar{\sigma}_0}{2G\Delta}z, \quad (16)$$

where

$$\Delta = (m-1)^3 - 3(m-1) + 2.$$

Mathematical Statement of the Complementary Problem

In view of the particular solution, we need to find three functions $u^{(C)}(x, y, z)$, $v^{(C)}(x, y, z)$, and $w^{(C)}(w, y, z)$, such that they satisfy simultaneously the partial differential equations (1)–(3) and the following boundary conditions:

At

$$|x| < c,\ y = 0^{\pm},\ |z| \leq h:$$

$$\tau_{xy}^{(C)} = \tau_{yz}^{(C)} = 0,\ \sigma_y^{(C)} = -\bar{\sigma}_0 \quad (17)$$

At

$$|z| = h: \ \tau_{xz}^{(C)} = \tau_{yz}^{(C)} = \sigma_z^{(C)} = 0 \quad (18)$$

$$\sqrt{x^2+y^2} \longrightarrow \infty: u^{(C)},\ v^{(C)} \text{ and } w^{(C)} \text{ are to be bounded.} \quad (19)$$

Method of Solution

In constructing a solution to the system (1)–(3) we use the method described in reference [1] to recover the following ordinary differential equations of the independent variable z

$$\frac{d^2u^{(C)}}{dz^2} + \left(D^2 + \frac{m}{m-2}\partial_1^2\right)u^{(C)} + \left(\frac{m}{m-2}\partial_1\partial_2\right)v^{(C)} + \left(\frac{m}{m-2}\partial_1\right)\frac{dw^{(C)}}{dz} = 0 \quad (20)$$

$$\frac{d^2v^{(C)}}{dz^2} + \left(D^2 + \frac{m}{m-2}\partial_2^2\right)v^{(C)} + \left(\frac{m}{m-2}\partial_1\partial_2\right)u^{(C)} + \left(\frac{m}{m-2}\partial_2\right)\frac{dw^{(C)}}{dz} = 0 \quad (21)$$

$$2\left(\frac{m-1}{m-2}\right)\frac{d^2w^{(C)}}{dz^2} + \left(\frac{m}{m-2}\partial_1\right)\frac{du^{(C)}}{dz} + \left(\frac{m}{m-2}\partial_2\right)\frac{dv^{(C)}}{dz} + D^2w^{(C)} = 0, \quad (22)$$

where the symbols of differentiation

$$\partial_1 \equiv \frac{\partial}{\partial x},\ \partial_2 \equiv \frac{\partial}{\partial y},\ D^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

are to be interpreted as numbers.

Upon integrating the foregoing system subject to the initial conditions,

$$u^{(C)} = u_0,\ v^{(C)} = v_0,\ w^{(C)} = 0,\ \frac{du^{(C)}}{dz} = 0,\ \frac{dv^{(C)}}{dz} = 0,\ \frac{dw^{(C)}}{dz} = w_0' \text{ for } z = 0, \quad (23)$$

one has after a few simple calculations[2]

$$u^{(C)} = \cos(zD)u_0 - \frac{m}{2(m-2)}\frac{z\sin(zD)}{D}\partial_1\theta_0 \quad (24)$$

$$v^{(C)} = \cos(zD)v_0 - \frac{m}{2(m-2)}\frac{z\sin(zD)}{D}\partial_2\theta_0 \quad (25)$$

$$w^{(C)} = \frac{\sin(zD)}{D}w_0' + \frac{m}{2(m-2)}\left[\frac{\sin(zD)}{D} - z\cos(zD)\right]\theta_0 \quad (26)$$

$$\theta^{(C)} = \cos(zD)\theta_0 \quad (27)$$

where

$$\theta_0 = \partial_1 u_0 + \partial_2 v_0 + w_0'. \quad (28)$$

The stresses on a plane perpendicular to the z-axis may now be expressed as

$$\frac{1}{G}\tau_{zx}^{(C)} = -\frac{\sin(zD)}{D}(D^2u_0 - \partial_1 w_0') - \frac{m}{m-2}z\cos(zD)\partial_1\theta_0 \quad (29)$$

$$\frac{1}{G}\tau_{yz}^{(C)} = -\frac{\sin(zD)}{D}(D^2v_0 - \partial_2 w_0') - \frac{m}{m-2}z\cos(zD)\partial_2\theta_0 \quad (30)$$

$$\frac{1}{G}\sigma_z^{(C)} = 2\cos(zD)w_0' + \left[\frac{2}{m-2}\cos(zD) + \frac{m}{m-2}(zD)\sin(zD)\right]\theta_0 \quad (31)$$

and boundary condition (18) now takes the form

$$\begin{aligned} d_{11}u_0 + d_{12}v_0 + d_{13}w_0' &= 0 \\ d_{21}u_0 + d_{22}v_0 + d_{23}w_0' &= 0 \\ d_{31}u_0 + d_{32}v_0 + d_{33}w_0' &= 0, \end{aligned} \quad (32)$$

where the differential operators d_{ik} are defined as

$$d_{11} = -\left[D\sin(hD) + \frac{mh}{m-2}\cos(hD)\partial_1^2\right],\ d_{22} = -\left[D\sin(hD) + \frac{mh}{m-2}\cos(hD)\partial_2^2\right],\ d_{12} = d_{21} = -\frac{mh}{m-2}\cos(hD)\partial_1\partial_2,\ d_{33} = \frac{2(m-1)}{m-2}\cos(hD) + \frac{mh}{m-2}D\sin(hD),\ d_{13} = \partial_1 d_3^*,\ d_{23} = \partial_2 d_3^*,\ d_3^* = \frac{\sin(hD)}{D} - \frac{mh}{m-2}\cos(hD),\ d_{31} = \partial_1 d_3^{**},\ d_{32} = \partial_2 d_3^{**},\ d_3^{**} = \frac{2}{m-2}\cos(hD) + \frac{mh}{m-2}D\sin(hD). \quad (33)$$

[2] One must represent cos (zD) and sin $(zD)/D$ in the form of a series in powers of zD (they will, of course, contain even powers of D) and, further, interpret the symbol D^2 as a differential operator acting on the functions $u_0(x, y)$, $v_0(x, y)$, and $w'_0(x, y)$.

Keeping in mind that the differential operators ∂_1, ∂_2, D^2 obey the same formal rules of addition and multiplication as numbers, the solution of system (32) is given by

$$u_0 = \chi_1(x, y), \quad v_0 = \chi_2(x, y), \quad w_0' = \chi_3(x, y), \qquad (34)$$

where the unknown displacement functions χ_1, χ_2, χ_3 satisfy the differential relations

$$Q\chi_i = 0, \quad i = 1, 2, 3 \qquad (35)$$

$$Q = \begin{vmatrix} d_{11}d_{12}d_{13} \\ d_{21}d_{22}d_{23} \\ d_{31}d_{32}d_{33} \end{vmatrix} = \frac{2m}{m-2} h^2 D^4 \frac{\sin (hD)}{hD}\left[1 + \frac{\sin (2hD)}{2hD}\right]. \qquad (36)$$

We construct, next, the following integral representations for u_0, v_0, w_0', which have the proper behavior at infinity

$$u_0(x, y^\pm) = \int_0^\infty \Big\{(P_1 + |y| Q_1) e^{-s|y|} + \sum_{\nu=1}^{\infty} R_\nu^{(1)} e^{-\sqrt{s^2+\beta_\nu^2}|y|} + \sum_{n=1}^{\infty} S_n^{(1)} e^{-\sqrt{s^2+\alpha_n^2}|y|}\Big\} \sin (xs) ds \qquad (37)$$

$$v_0(x, y^\pm) = \mp \int_0^\infty \Big\{(P_2 + |y| Q_2) e^{-s|y|} + \sum_{\nu=1}^{\infty} R_\nu^{(2)} e^{-\sqrt{s^2+\beta_\nu^2}|y|} + \sum_{n=1}^{\infty} S_n^{(2)} e^{-\sqrt{s^2+\alpha_n^2}|y|}\Big\} \cos (xs) ds \qquad (38)$$

$$w_0'(x, y^\pm) = \int_0^\infty \Big\{(P_3 + |y| Q_3) e^{-s|y|} + \sum_{\nu=1}^{\infty} R_\nu^{(3)} e^{-\sqrt{s^2+\beta_\nu^2}|y|} + \sum_{n=1}^{\infty} S_n^{(3)} e^{-\sqrt{s^2+\alpha_n^2}|y|}\Big\} \cos (xs) ds. \qquad (39)$$

The $\pm$ signs refer to $y > 0$ and $y < 0$, respectively, $\alpha_n = (n\pi/h)(n = 1,2,3 \ldots)$, and β_ν are the roots of the equation

$$\sin (2\beta_\nu h) = -(2\beta_\nu h). \qquad (40)$$

This equation has an infinite number of complex roots which appear in groups of four, one in each quadrant of the complex plane and only two of each group of four roots are relevant to the present work. These are chosen to be the complex conjugate pairs with positive real parts. The only real root $\beta_\nu = 0$ must be ignored.[3]

Finally, an examination of the solution shows that the unknown functions P_1, Q_1, $R_\nu^{(1)}$, $S_n^{(1)}$, etc., are not all independent. Assuming, therefore, that one can differentiate under the integral sign and inserting equations (37)–(39) into (32) one finds

$$Q_2 = -Q_1, \; Q_3 = 0, \; P_3 = \frac{2}{m+1} Q_1, \; s(P_1 + P_2) = -\frac{3m-1}{m+1} Q_1, \qquad (41)\text{–}(44)$$

$$R_\nu^{(1)} \equiv R_\nu, \; sR_\nu^{(2)} = -\sqrt{s^2 + \beta_\nu^2}\, R_\nu \qquad (45)\text{–}(46)$$

$$\left[1 + \frac{m}{m-2} \cos^2 (\beta_\nu h)\right] R_\nu^{(3)} = -\beta_\nu^2 \left[1 - \frac{m}{m-2} \cos^2 (\beta_\nu h)\right]\frac{R_\nu}{s} \qquad (47)$$

$$S_n^{(1)} \equiv S_n, \; S_n^{(2)} = -\frac{s}{\sqrt{s^2 + \alpha_n^2}} S_n, \; S_n^{(3)} = 0. \qquad (48)\text{–}(50)$$

In order to facilitate our subsequent discussion, it is found convenient at this stage to summarize our results. Defining

$$\Gamma_\nu \equiv \frac{\sqrt{s^2 + \beta_\nu^2}}{1 + \frac{m}{m-2} \cos^2 (\beta_\nu h)} \frac{\sin (\beta_\nu h)}{\beta_\nu h} R_\nu, \qquad (51)$$

we may now write the complementary displacements and complementary stresses in the form:

(i) Complementary Displacements

$$u^{(c)} = \int_0^\infty \Big\{(P_1 + |y| Q_1 + \frac{1}{m+1} z^2 s Q_1) e^{-s|y|} - \frac{1}{m-2} \sum_{\nu=1}^{\infty} \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2 + \beta_\nu^2}} \cos (\beta_\nu h)[(m - 2 + m \cos^2 (\beta_\nu h)) \cos (\beta_\nu z) - m\beta_\nu z \sin (\beta_\nu z)] + \sum_{n=1}^{\infty} S_n e^{-\sqrt{s^2+\alpha_n^2}|y|} \cos (\alpha_n z)\Big\} \sin (xs) ds \qquad (52)$$

$$v^{(c)} = \mp \int_0^\infty \Big\{\Big(-\frac{3m-1}{m+1} \frac{Q_1}{s} - P_1 - |y| Q_1 - \frac{1}{m+1} s z^2 Q_1\Big) e^{-s|y|} + \frac{1}{m-2} \sum_{\nu=1}^{\infty} \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s} \times \cos (\beta_\nu h) [(m - 2 + m \cos^2 (\beta_\nu h)) \cos (\beta_\nu z) - m\beta_\nu z \sin (\beta_\nu z)] - \sum_{n=1}^{\infty} S_n \frac{s}{\sqrt{s^2 + \alpha_n^2}} e^{-\sqrt{s^2+\alpha_n^2}|y|} \cos (\alpha_n z)\Big\} \cos (xs) ds \qquad (53)$$

$$w^{(c)} = \int_0^\infty \Big\{\Big(\frac{2}{m+1} z Q_1\Big) e^{-s|y|} + \frac{1}{m-2} \times \sum_{\nu=1}^{\infty} \beta_\nu \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s\sqrt{s^2 + \beta_\nu^2}} \cos (\beta_\nu h)[(2m - 2 - m \cos^2(\beta_\nu h)) \sin (\beta_\nu z) - m\beta_\nu z \cos (\beta_\nu z)]\Big\} \cos (xs) ds \qquad (54)$$

$$\theta^{(c)} = \int_0^\infty \Big\{-2 \frac{m-2}{m+1} Q_1 e^{-s|y|} + 2 \sum_{\nu=1}^{\infty} \beta_\nu^2 \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s\sqrt{s^2 + \beta_\nu^2}} \times \cos (\beta_\nu h) \cos (\beta_\nu z)\Big\} \cos (xs) ds. \qquad (55)$$

(ii) Complementary Stresses

$$\frac{\sigma_z^{(c)}}{2G} = \frac{m}{m-2} \int_0^\infty \sum_{\nu=1}^{\infty} \Gamma_\nu \beta_\nu^2 \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s\sqrt{s^2 + \beta_\nu^2}} \cos (\beta_\nu h) \times [\sin^2 (\beta_\nu h) \cos (\beta_\nu z) + \beta_\nu z \sin (\beta_\nu z)] \times \cos (xs) ds \qquad (56)$$

$$\frac{\tau_{xz}^{(c)}}{G} = \int_0^\infty \Big\{\frac{2m}{m-2} \sum_{\nu=1}^{\infty} \beta_\nu \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2 + \beta_\nu^2}} \cos (\beta_\nu h) \times [\cos^2 (\beta_\nu h) \sin (\beta_\nu z) + \beta_\nu z \cos (\beta_\nu z)] - \sum_{n=1}^{\infty} S_n \alpha_n e^{-\sqrt{s^2+\alpha_n^2}|y|} \sin (\alpha_n z)\Big\} \sin (xs) ds. \qquad (57)$$

$$\frac{\tau_{yz}^{(c)}}{G} = \pm \int_0^\infty \Big\{\frac{2m}{m-2} \sum_{\nu=1}^{\infty} \beta_\nu \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s} \cos (\beta_\nu h) \times [\cos^2 (\beta_\nu h) \sin (\beta_\nu z) + \beta_\nu z \cos (\beta_\nu z)] - \sum_{n=1}^{\infty} S_n \frac{\alpha_n s\, e^{-\sqrt{s^2+\alpha_n^2}|y|}}{\sqrt{s^2 + \alpha_n^2}} \sin (\alpha_n z)\Big\} \cos (xs) ds. \qquad (58)$$

$$\frac{\sigma_x^{(c)}}{2G} \int_0^\infty \Big\{\Big(sP_1 + |y| sQ_1 + \frac{1}{m+1} s^2 z^2 Q_1 - \frac{2}{m+1} Q_1\Big) e^{-s|y|} + \frac{2}{m-2} \sum_{\nu=1}^{\infty} \beta_\nu^2 \Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s\sqrt{s^2 + \beta_\nu^2}} \cos (\beta_\nu h) \cos (\beta_\nu z) - \frac{1}{m-2} \sum_{\nu=1}^{\infty} \Gamma_\nu \frac{s e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2 + \beta_\nu^2}} \times \cos (\beta_\nu h) [(m - 2 + m \cos^2 (\beta_\nu h)) \cos (\beta_\nu z) \qquad (59)$$

[3] The roots of equation (40), as given by reference [7], are tabulated in Appendix 1. We shall postpone the ordering of the roots until a later time.

$$-m\,\beta_\nu z \sin(\beta_\nu z)] + \sum_{n=1}^{\infty} S_n\, s e^{-\sqrt{s^2+\alpha_n^2}|y|} \cos(\alpha_n z)\Big\} \times \cos(xs)ds \quad (59)$$

(*Cont.*)

$$\frac{\sigma_y^{(c)}}{2G} = \int_0^\infty \Big\{\Big(-\frac{2m}{m+1}Q_1 - sP_1 - |y|sQ_1 - \frac{1}{m+1}s^2z^2Q_1\Big)e^{-s|y|} + \frac{2}{m-2}\sum_{\nu=1}^{\infty}\beta_\nu^2\Gamma_\nu \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{s\sqrt{s^2+\beta_\nu^2}} \times \cos(\beta_\nu h)\cos(\beta_\nu z) + \frac{1}{m-2} \times \sum_{\nu=1}^{\infty}\Gamma_\nu \frac{\sqrt{s^2+\beta_\nu^2}}{s} e^{-\sqrt{s^2+\beta_\nu^2}|y|}\cos(\beta_\nu h)\,[(m-2 + m\cos^2(\beta_\nu h))\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)] - \sum_{n=1}^{\infty} S_n s e^{-\sqrt{s^2+\alpha_n^2}|y|}\cos(\alpha_n z)\Big\}\cos(xs)ds \quad (60)$$

$$\frac{\tau_{xy}^{(c)}}{G} = \mp\int_0^\infty \Big\{\Big(2\Big(\frac{m-1}{m+1}\Big)Q_1 + 2sP_1 + \frac{2}{m+1}z^2s^2Q_1 + 2|y|sQ_1\Big)e^{-s|y|} - \frac{2}{m-2} \times \sum_{\nu=1}^{\infty}\Gamma_\nu e^{-\sqrt{s^2+\beta_\nu^2}|y|}\cos(\beta_\nu h)[(m-2 + m\cos^2(\beta_\nu h))\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)] + \sum_{n=1}^{\infty} S_n \frac{2s^2+\alpha_n^2}{\sqrt{s^2+\alpha_n^2}} e^{-\sqrt{s^2+\alpha_n^2}|y|}\cos(\alpha_n z)\Big\}\sin(xs)ds \quad (61)$$

By direct substitution, it can easily be ascertained that the foregoing complementary displacements satisfy Navier's equations and furthermore the corresponding stresses $\sigma_z^{(c)}$, $\tau_{zx}^{(c)}$, $\tau_{yz}^{(c)}$ do vanish at the plate faces $z = \pm h$.

Finally, if we consider the following two combinations to vanish:

$$\frac{2m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s}\cos(\beta_\nu h)\,[\sin^2(\beta_\nu h)\cos(\beta_\nu z) + \beta_\nu z\sin(\beta_\nu z)] + \sum_{n=1}^{\infty}\frac{sS_n}{\sqrt{s^2+\alpha_n^2}}\cos(\alpha_n z) + \frac{4m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s} = 0 \quad (62)$$

and

$$\frac{2}{m-2}\sum_{\nu=1}^{\infty}\Gamma_\nu\cos(\beta_\nu h)[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)] - \sum_{n=1}^{\infty}\frac{2s^2+\alpha_n^2}{\sqrt{s^2+\alpha_n^2}}S_n\cos(\alpha_n z) - \frac{2}{1+m}s^2z^2Q_1 - 2sP_1 - 2\frac{m-1}{m+1}Q_1 = 0 \quad (63)$$

for all $|z| \leq h$, then two of the remaining stress boundary conditions are satisfied automatically, i.e.,

$$\tau_{xy}^{(c)} = \tau_{yz}^{(c)} = 0 \text{ for all } x,\ |z| \leq h \text{ and } y = 0.$$

If now one uses the Fourier series expansions given in Appendix 2 and equation (62), he can express the unknown coefficients S_n in terms of the Γ_ν's, in particular

$$\frac{S_n}{\sqrt{s^2+\alpha_n^2}} = (-1)^n\frac{8m}{m-2}\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s^2}\frac{\beta_\nu^4}{(\beta_\nu^2-\alpha_n^2)^2} \quad (64)$$

and furthermore, since Q_1 is arbitrary, we let

$$s^2Q_1 = \frac{m^2-1}{m-2}\sum_{\nu=1}^{\infty}\Gamma_\nu\beta_\nu^2. \quad (65)$$

We may also combine equations (62) and (63) to obtain

$$-\frac{2}{m-2}\sum_{\nu=1}^{\infty}\Gamma_\nu\{(3m-2-m\cos^2(\beta_\nu h))\cos(\beta_\nu h) \times \cos(\beta_\nu z) + m\beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)\} + \frac{2m}{m-2}\sum_{\nu=1}^{\infty} \times \Gamma_\nu\Big(\frac{\beta_\nu^2}{s^2}\Big)\{(1+\cos^2(\beta_\nu h))\cos(\beta_\nu h)\cos(\beta_\nu z) - \beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)\} = -\frac{2}{1+m}s^2z^2Q_1 + \frac{8m}{m-2}\sum_{\nu=1}\Gamma_\nu - 2sP_1 - 2\Big(\frac{m-1}{m+1}\Big)Q_1;\ |z| \leq h. \quad (66)$$

The foregoing equation is valid for all values of z and for $z = 0$ yields

$$-\frac{2}{m-2}\sum_{\nu=1}^{\infty}\Gamma_\nu(3m-2-m\cos^2(\beta_\nu h))\cos(\beta_\nu h) + \frac{2m}{m-2}\sum_{\nu=1}^{\infty}\Gamma_\nu\Big(\frac{\beta_\nu}{s}\Big)^2(1+\cos^2(\beta_\nu h)) \times \cos(\beta_\nu h) = \frac{8m}{m-2}\sum_{\nu=1}^{\infty}\Gamma_\nu - 2sP_1 - 2\Big(\frac{m-1}{m+1}\Big)Q_1. \quad (67)$$

Equations (65) and (67) enable one to express P_1 in terms of the Γ_ν's. Finally, in view of equation (67), equation (66) may now be written in the form[4]

$$\sum_{\nu=1}^{\infty}\Gamma_\nu\cos(\beta_\nu h)\{(3m-2-m\cos^2(\beta_\nu h))(\cos(\beta_\nu z)-1) + m\beta_\nu z\sin(\beta_\nu z)\} - m\sum_{\nu=1}^{\infty}\Gamma_\nu\cos(\beta_\nu h)\Big(\frac{\beta_\nu}{s}\Big)^2 \times\{1+\cos^2(\beta_\nu h))(\cos(\beta_\nu z)-1) - \beta_\nu z\sin(\beta_\nu z)\} = (m-1)\sum_{\nu=1}^{\infty}\Gamma_\nu\beta_\nu^2z^2;\ |z| \leq h \quad (68)$$

which contains the Γ_ν's as the only unknowns.

Returning now to the last boundary condition we require that

$$\int_0^\infty \sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s^3}\Big\{-\frac{m^2-1}{m-2}\beta_\nu^2 - \frac{1}{m-2}\cos(\beta_\nu h)[(2\beta_\nu^2 + (s^2+\beta_\nu^2)(m-2+m\cos^2(\beta_\nu h)))\cos(\beta_\nu z) - m(s^2+\beta_\nu^2)\beta_\nu z\sin(\beta_\nu z)]\Big[1 - \frac{s}{\sqrt{s^2+\beta_\nu^2}}\Big] + \frac{8m}{m-2}\sum_{n=1}^{\infty}\frac{\beta_\nu^4(-1)^n}{(\beta_\nu^2-\alpha_n^2)^2}[(s^2+\alpha_n^2) - s\sqrt{s^2+\alpha_n^2}] \times\cos(\alpha_n z)\Big\}s\cos(xs)ds = -\frac{\sigma_0}{2G};\ |x| < c,\ |z| \leq h, \quad (69)$$

where we have made use of equations (63)–(65). Furthermore, along $|x| > c$ and $|y| = 0$ we must require the complementary displacements together with their first partial derivatives to be continuous for all $|z| < h$. The latter can be accomplished if one considers the following integral combination to vanish

$$\int_0^\infty \frac{\Gamma_\nu}{s^3}\cos(xs)ds = 0;\ |x| > c. \quad (70)$$

The problem, therefore, is to solve the dual integral equation (69)–(70) for the unknown functions Γ_ν/s^3 subject to the condition (68). But this is plausible for the Γ_ν's are complex.

Solution of the Integral Equation

Seeking a solution of the form

$$\Big(\frac{\Gamma_\nu}{s^3}\Big) = \sum_{k=0}^{\infty}A_\nu^{(k)}\frac{J_{k+1}(sc)}{(sc)^{k+1}} \quad (71)$$

[4] Note that equation (68) represents the vanishing of the integrand of equation (61) on the plane $|y| = 0$.

and employing the method described in Appendix 3 one finds after some straightforward computations that

$$\sum_{\nu=1}^{\infty}\sum_{k=0}^{\infty} A_\nu^{(k)}\left\{-\frac{m^2-1}{m-2}\beta_\nu^2 I_1^{kj} - \frac{2\beta_\nu^2}{m-2}\cos(\beta_\nu h)\cos(\beta_\nu z)\right.$$
$$\times I_2^{kj}(\beta_\nu) - \frac{1}{m-2}\cos(\beta_\nu h)[(m-2+m\cos^2(\beta_\nu h))$$
$$\times\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)]\, I_3^{kj}(\beta_\nu)$$
$$\left. + \frac{8m}{m-2}\sum_{n=1}^{\infty}\frac{\beta_\nu^4(-1)^n}{(\beta_\nu^2-\alpha_n^2)^2} I_3^{kj}(\alpha_n)\cos(\alpha_n z)\right\} =$$
$$-\left(\frac{\bar{\sigma}_0}{2G}\right)\frac{1}{2^{j+1}}\frac{1}{(j+1)!};\ |z| \le h \quad (j = 0, 1, 2, \ldots), \tag{72}$$

where the integrals I_n^{kj} are defined as

$$I_1^{kj} = \int_0^\infty s\frac{J_{k+1}(sc)}{(sc)^{k+1}}\frac{J_{j+1}(sc)}{(sc)^{j+1}}ds$$
$$= \frac{1}{2^{k+j+1}k!j!(k+j+1)c^2},\ I_2^{kj}(\beta_\nu)$$
$$= \int_0^\infty s\left\{1 - \frac{s}{\sqrt{s^2+\beta_\nu^2}}\right\}\frac{J_{k+1}(sc)}{(sc)^{k+1}}\frac{J_{j+1}(sc)}{(sc)^{j+1}}ds,\ I_3^{kj}(\beta_\nu)$$
$$= \int_0^\infty s\{(s^2+\beta_\nu^2) - s\sqrt{s^2+\beta_\nu^2}\}\frac{J_{k+1}(sc)}{(sc)^{k+1}}\frac{J_{j+1}(sc)}{(sc)^{j+1}}ds. \tag{73}$$

Similarly equation (68), taking into account the identity

$$z^2\frac{J_{n+1}(z)}{(z)^{n+1}} = (2n)\frac{J_n(z)}{(z)^n} - \frac{J_{n-1}(z)}{(z)^{n-1}}, \tag{74}$$

yields

$$\sum_{\nu=1}^{\infty} A_\nu^{(0)}a_\nu(z) = 0,\ \sum_{\nu=1}^{\infty} A_\nu^{(1)}a_\nu(z) = 0 \tag{75a, b}$$

$$\sum_{\nu=1}^{\infty}\{2kA_\nu^{(k)}a_\nu(z) - A_\nu^{(k+1)}a_\nu(z) + A_\nu^{(k-1)}(\beta_\nu c)^2 b_\nu(z)\} = 0$$
$$(k = 1, 2, 3, \ldots), \tag{75c}$$

where for convenience we have defined

$$a_\nu(z) \equiv -(m-1)\beta_\nu^2 z^2 + \cos(\beta_\nu h)\{(3m-2$$
$$- m\cos^2(\beta_\nu h))(\cos(\beta_\nu z) - 1) + m\beta_\nu z\sin(\beta_\nu z)\} \tag{76}$$

and

$$b_\nu(z) \equiv -m\cos(\beta_\nu h)\{(1+\cos^2(\beta_\nu h))(\cos(\beta_\nu z) - 1)$$
$$- \beta_\nu z\sin(\beta_\nu z)\}. \tag{77}$$

It is clear now that the solution of equation (72) and (75) will determine the value of the coefficients $A_\nu^{(k)}$, which are functions of the crack to thickness and Poisson's ratios. However, as we will see later, the stresses ahead of the crack tip are proportional to $A_\nu^{(0)}$. Consequently, one needs only to compute that coefficient. Unfortunately, the exact solution is not only a difficult but also a very expensive numerical task. Therefore, pending further numerical study, the author in this paper must settle for the evaluation of $A_\nu^{(0)}$ from a truncated system.

The Approximate Coefficients $A_\nu^{(0)}$

For the numerical calculations, we employ the approximation discussed in Appendix 3 and furthermore, restrict ourselves to the main contribution of equation (72) which comes primarily from $k = 0$ and $j = 0$, i.e.,

$$\sum_{\nu=1}^{\infty}\{A_\nu^{(0)}\beta_\nu^2\}\left\{-\frac{m^2-1}{m-2} - \frac{2}{m-2}\left[1 - 2I_1\left(\frac{\beta_\nu c}{\sqrt{2}}\right)K_1\left(\frac{\beta_\nu c}{\sqrt{2}}\right)\right]\right.$$
$$\times\cos(\beta_\nu h)\cos(\beta_\nu z) - \frac{\cos(\beta_\nu h)}{2(m-2)}[(m-2+m\cos^2(\beta_\nu h)) \tag{78}$$

Table 1 Coefficients $A_\nu^{(0)}$ for $\nu = 1/3$ and $(c/h) = 4$

ν	$\mathrm{Re}\left\{\dfrac{A_\nu^{(0)}}{\left(\frac{\bar{\sigma}_0}{2G}\right)}\right\}$	$\mathrm{Im}\left\{\dfrac{A_\nu^{(0)}}{\left(\frac{\bar{\sigma}_0}{2G}\right)}\right\}$
1	2.65720184-04	−9.03913332-03
2	2.65719806-04	9.03913239-03
3	1.61273938-04	−2.22927507-04
4	1.61273985-04	2.22927492-04
5	3.47861974-05	−2.97662441-05
6	3.47861928-05	2.97662514-05
7	1.12677765-05	−7.29232636-06
8	1.12677776-05	7.29232192-06
9	4.65317754-06	−2.46500667-06
10	4.65317822-06	2.46500579-06
	etc.	

$$\times\cos(\beta_\nu z) - m\beta_\nu z\sin(\beta_\nu z)]\left[2 - 2I_1\left(\frac{\beta_\nu c}{\sqrt{2}}\right)K_1\left(\frac{\beta_\nu c}{\sqrt{2}}\right)\right]$$
$$+ \frac{8m}{m-2}\sum_{n=1}^{\infty}(-1)^n\frac{\beta_\nu^2}{(\beta_\nu^2-\alpha_n^2)^2}\left(\frac{\alpha_n^2}{2}\right)\left[2 - 2I_1\left(\frac{\alpha_n c}{\sqrt{2}}\right)\right.$$
$$\left.\left.\times K_1\left(\frac{\alpha_n c}{\sqrt{2}}\right)\right]\cos(\alpha_n z)\right\} = -\frac{\bar{\sigma}_0}{2G}c^2;\ |z| \le h \tag{78}$$

(Cont.)

and[5]

$$\sum_{\nu=1}^{\infty} A_\nu^{(0)}\{-(m-1)\beta_\nu^2 z^2 + \cos(\beta_\nu h)[(3m-2$$
$$- m\cos^2(\beta_\nu h))(\cos(\beta_\nu z) - 1) + m\beta_\nu z\sin(\beta_\nu z)]\} = 0;$$
$$|z| \le h. \tag{79}$$

Defining now the roots $\beta_2, \beta_4, \beta_6, \ldots$ to be the complex conjugates of $\beta_1, \beta_3, \beta_5, \ldots$, we conclude that the unknown coefficients $A_2^{(0)}, A_4^{(0)}, A_6^{(0)}, \ldots$ must also be complex conjugates of $A_1^{(0)}$, $A_3^{(0)}, A_5^{(0)}, \ldots$ and are to be determined from equations (78) and (79).

For the solution of the foregoing system, we use the method discussed in reference [8, pp. 54–56]. Specifically, we expand the functions $\cos(\beta_\nu z)$ and $\beta_\nu z\sin(\beta_\nu z)$ in terms of $\cos(\alpha_n z)$ and equating terms we recover the following infinite system of equations:

$$\sum_{\nu=1}^{\infty}\{\beta_\nu^2 A_\nu^{(0)}\}\left\{-\frac{m^2-1}{m-2} - \frac{1}{m-2}\left[2I_1\left(\frac{\beta_\nu c}{\sqrt{2}}\right)K_1\left(\frac{\beta_\nu c}{\sqrt{2}}\right)\right]\right\}$$
$$= -\frac{\bar{\sigma}_0}{2G}c^2,\ \sum_{\nu=1}^{\infty}\{\beta_\nu^2 A_\nu^{(0)}\}\frac{\beta_\nu^2}{(\beta_\nu^2-\alpha_n^2)^2}\left\{[\beta_\nu^2\right.$$
$$- (m+1)\alpha_n^2]\left[-2I_1\left(\frac{\beta_\nu c}{\sqrt{2}}\right)K_1\left(\frac{\beta_\nu c}{\sqrt{2}}\right)\right]$$
$$\left. + 2m\alpha_n^2\left[1 - 2I_1\left(\frac{\alpha_n c}{\sqrt{2}}\right)K_1\left(\frac{\alpha_n c}{\sqrt{2}}\right)\right]\right\} = 0,$$

$$\sum_{\nu=1}^{\infty} A_\nu^{(0)}\left\{(m-1)\frac{(\beta_\nu h)^2}{3} + 2(2m-1)\right.$$
$$\left. + (3m-2-m\cos^2(\beta_\nu h))\cos(\beta_\nu h)\right\} = 0,$$

$$\sum_{\nu=1}^{\infty}(A_\nu^{(0)}\beta_\nu^2)\left\{\frac{1}{\alpha_n^2} + \frac{1}{\beta_\nu^2-\alpha_n^2} + \frac{m}{m-1}\frac{\beta_\nu^2}{(\beta_\nu^2-\alpha_n^2)^2}\right\} = 0. \tag{80}$$

It should be pointed out that the foregoing system is extremely sensitive to even small changes of the coefficients. As a result, the methods of "collocation" and "least squares" led us to a nonconvergent solution. The aforementioned method, however, furnishes us with coefficients that do converge as the number of roots used increases, Fig. 1.

[5] The reader should be cautioned that the second derivative of equation (79) may not necessarily be zero at $z = \pm h$. In fact, we will show later that at those points the second derivative has a weak singularity.

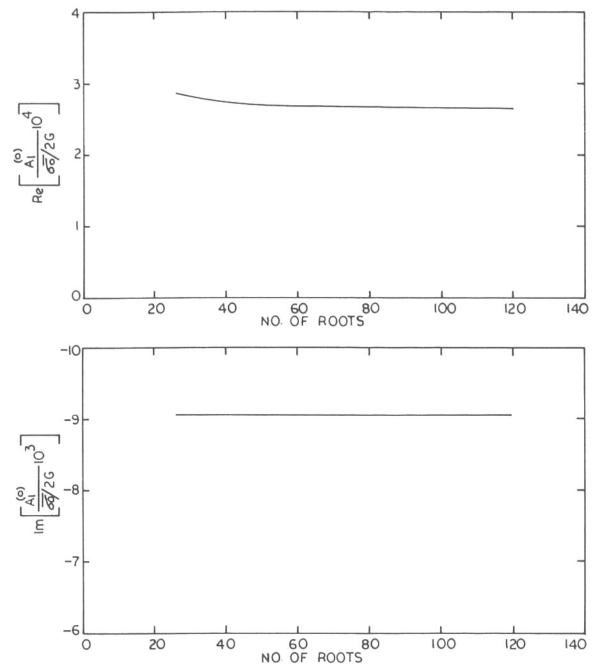

Fig. 1 Typical convergence of coefficients $A_\nu^{(0)}$ for $\nu = 1/3$ and $c/h = 4$

To satisfy equations (78) and (79) exactly, one needs to know the coefficients $A_\nu^{(0)}$ accurately to many significant figures. This can be seen in Figs. 2 and 3 where we compare numerical results corresponding to 120 and 250 roots. As the reader can see, the difference, is noticeable even though the coefficients have changed only slightly. In Table 1, we show the first 10 coefficients $A_\nu^{(0)}$ corresponding to 250 roots and $c/h = 4$.

It is now time for us to examine the behavior of equation (79) at the two end points $z = \pm h$. To accomplish this, we define

$$f(\xi) = \sum_{\nu=1}^{\infty} A_\nu^{(0)} \cos [\beta_\nu h \xi] \tag{81}$$

and rewrite the equation in the form

$$(m-1)[f(1+\zeta) + f(1-\zeta)] + \frac{m}{2}[(1-\zeta)f'(1+\zeta) + (1+\zeta)f'(1-\zeta)] + A\zeta^2 + B = 0, \tag{82}$$

where

$$\zeta \equiv \frac{z}{h}, \quad A \equiv -(m-1)\sum_{\nu=1}^{\infty} A_\nu^{(0)}\beta_\nu^2 h^2$$

$$B \equiv -\sum_{\nu=1}^{\infty} A_\nu^{(0)} \cos(\beta_\nu h)[3m - 2 - m\cos^2(\beta_\nu h)]. \tag{83}$$

This is a difference-differential equation the solution of which is easily found (see Appendix 4) to be

$$f(\xi) = C_0(2-\xi)^{2-\frac{2}{m}} - \frac{(m-1)^2}{2^{2/m}m^2} C_0\xi^2 + \tilde{B}, \tag{84}$$

with C_0 and $\tilde{B}$ as arbitrary constants.[6] Or, in view of our original notation, equation (84) becomes

$$2\sum_{\nu=1}^{\infty} A_\nu^{(0)} \cos(\beta_\nu h)\cos(\beta_\nu z) = C_0\left[\left(1 - \frac{z}{h}\right)^{2-2/m} + \left(1 + \frac{z}{h}\right)^{2-2/m}\right] - \frac{2(m-1)^2}{2^{2/m}m^2} C_0\left(1 + \frac{z^2}{h^2}\right) + 2\tilde{B}. \tag{85}$$

It is now clear that equation (79) may not be differentiated twice with respect to z in the neighborhood of the two end points.

[6] The constant C_0 is to be determined from equation (78) while $\tilde{B}$ is indeterminate.

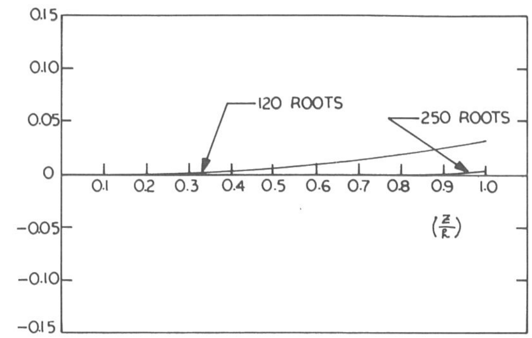

Fig. 2 Numerical results for equation (79) with $\nu = 1/3$ and $c/h = 4$

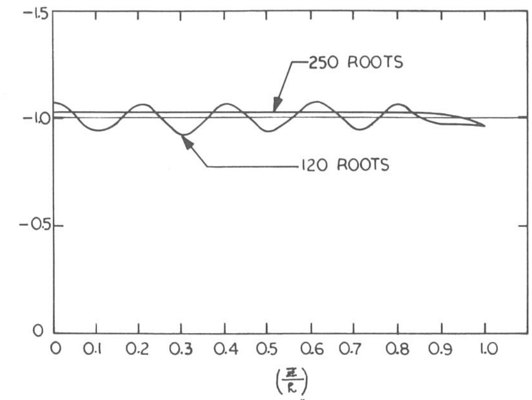

Fig. 3 Numerical results for equation (78) with $\nu = 1/3$ and $c/h = 4$

Stress Field Ahead of the Crack Tip

The stresses ahead of the crack tip may now be expressed in terms of the following four types of integrals[7]

$$M_1 = \int_0^\infty J_1(sc)\left\{e^{-s|y|} - \frac{s}{\sqrt{s^2+\beta_\nu^2}} e^{-\sqrt{s^2+\beta_\nu^2}|y|}\right\}\cos(xs)ds \tag{86}$$

$$M_2 = \int_0^\infty J_1(sc)\left\{\left(s^2 + \frac{\beta_\nu^2}{2} - \frac{\beta_\nu^2}{2}s|y|\right)e^{-s|y|} - s\sqrt{s^2+\beta_\nu^2}\, e^{-\sqrt{s^2+\beta_\nu^2}|y|}\right\}\cos(xs)ds \tag{87}$$

$$M_3 = \int_0^\infty J_1(sc)e^{-s|y|-isx}ds = -\sqrt{\frac{c}{2\epsilon}}\, e^{i\left(\frac{\phi}{2}\right)} + O(\epsilon^0) \tag{88}$$

$$M_4 = |y|\int_0^\infty sJ_1(sc)e^{-s|y|-isx}ds = -\frac{1}{4}\sqrt{\frac{c}{2\epsilon}}\left[e^{i\left(\frac{\phi}{2}\right)} - e^{i\left(\frac{5\phi}{2}\right)}\right] + O(\epsilon^0) \tag{89}$$

Unfortunately, the first two integrals we have not as yet been able to evaluate in closed form. However, inasmuch as we are primarily interested in determining the order of the singularity which prevails in the neighborhood of the point where the crack front intersects the free surface, one may consider h to be large enough so that an asymptotic expansion for small β_ν is justifiable.[8] In effect, this is analogous to perturbing the solution about the well-known plane-strain solution. In view of the foregoing and by virtue of equation (85) the stresses and displacements are found to be

(i) Stresses

[7] The higher-order Bessel terms of equation (71) contribute to stress terms of $O(\epsilon^{1/2})$.

[8] By analytic continuation, this may now be extended to be applicable for all β_ν's. Alternatively, one can easily show by contour integration that M_1 and M_2 are of the $O(\epsilon^0)$ for all $y \to \epsilon \sin\phi$ and $x \to c + \epsilon\cos\phi$. Thus equations (90)–(98) are valid for all (c/h) ratios.

$$\sigma_x^{(c)} = \bar{\sigma}_0 \Lambda \left\{ \frac{1}{\left(1 - \frac{z}{h}\right)^{2\nu}} + \frac{1}{\left(1 + \frac{z}{h}\right)^{2\nu}} \right\} \sqrt{\frac{c}{2\epsilon}} \left\{ \frac{1}{2} \cos\left(\frac{\phi}{2}\right) - \frac{1}{4} \sin\phi \sin\left(\frac{3\phi}{2}\right) \right\} + O(\epsilon^0) \quad (90)$$

$$\sigma_y^{(c)} = \bar{\sigma}_0 \Lambda \left\{ \frac{1}{\left(1 - \frac{z}{h}\right)^{2\nu}} + \frac{1}{\left(1 + \frac{z}{h}\right)^{2\nu}} \right\} \sqrt{\frac{c}{2\epsilon}} \left\{ \frac{1}{2} \cos\left(\frac{\phi}{2}\right) + \frac{1}{4} \sin\phi \sin\left(\frac{3\phi}{2}\right) \right\} + O(\epsilon^0) \quad (91)$$

$$\sigma_z^{(c)} = \nu\bar{\sigma}_0 \Lambda \left\{ \frac{1}{\left(1 - \frac{z}{h}\right)^{2\nu}} + \frac{1}{\left(1 + \frac{z}{h}\right)^{2\nu}} \right\} \times \sqrt{\frac{c}{2\epsilon}} \cos\left(\frac{\phi}{2}\right) + O(\epsilon^0) \quad (92)$$

$$\tau_{xy}^{(c)} = \bar{\sigma}_0 \Lambda \left\{ \frac{1}{\left(1 - \frac{z}{h}\right)^{2\nu}} + \frac{1}{\left(1 + \frac{z}{h}\right)^{2\nu}} \right\} \times \sqrt{\frac{c}{2\epsilon}} \left\{ \frac{1}{4} \sin\phi \cos\left(\frac{3\phi}{2}\right) \right\} + O(\epsilon^0) \quad (93)$$

$$\tau_{yz}^{(c)} = -\nu\bar{\sigma}_0 \frac{\Lambda}{h} \left\{ \frac{1}{\left(1 - \frac{z}{h}\right)^{2\nu+1}} - \frac{1}{\left(1 + \frac{z}{h}\right)^{2\nu+1}} \right\} \times \sqrt{\frac{c\epsilon}{2}} \left\{ \frac{1}{2} \sin\phi \cos\left(\frac{\phi}{2}\right) \right\} + O(\epsilon') \quad (94)$$

$$\tau_{xz}^{(c)} = \nu\bar{\sigma}_0 \frac{\Lambda}{h} \left\{ \frac{1}{\left(1 - \frac{z}{h}\right)^{2\nu+1}} - \frac{1}{\left(1 + \frac{z}{h}\right)^{2\nu+1}} \right\} \times \sqrt{\frac{c\epsilon}{2}} \left\{ (1 - 2\nu) \cos\left(\frac{\phi}{2}\right) + \frac{1}{2} \sin\phi \sin\left(\frac{\phi}{2}\right) \right\} + O(\epsilon') \quad (-h + \epsilon < z < h - \epsilon) \quad (95)$$

(ii) Displacements

$$u^{(c)} = \bar{\sigma}_0 \frac{\Lambda}{2G} \left\{ \frac{1}{\left(1 - \frac{z}{h}\right)^{2\nu}} + \frac{1}{\left(1 + \frac{z}{h}\right)^{2\nu}} \right\} \times \sqrt{\frac{c\epsilon}{2}} \left\{ \frac{m-2}{m} \cos\left(\frac{\phi}{2}\right) + \frac{1}{2} \sin\phi \sin\left(\frac{\phi}{2}\right) \right\} + O(\epsilon') \quad (96)$$

$$v^{(c)} = \bar{\sigma}_0 \frac{\Lambda}{2G} \left\{ \frac{1}{\left(1 - \frac{z}{h}\right)^{2\nu}} + \frac{1}{\left(1 + \frac{z}{h}\right)^{2\nu}} \right\} \times \sqrt{\frac{c\epsilon}{2}} \left\{ 2\left(\frac{m-1}{m}\right) \sin\left(\frac{\phi}{2}\right) - \frac{1}{2} \sin\phi \cos\left(\frac{\phi}{2}\right) \right\} + O(\epsilon') \quad (97)$$

$$w^{(c)} = 0 + 0(\epsilon') \quad (-h + \epsilon < z < h - \epsilon) \quad (98)$$

where we have defined

$$\Lambda \equiv \frac{C_0}{\lim_{h \to \infty} C_0}. \quad (99)$$

The constant Λ is a function of the Poisson's ratio ν and c/h. Its behavior[9] for $\nu = 1/3$ and various crack size to thickness ratios is given in Fig. 4.

It is interesting to note that as $h \to \infty$ one recovers precisely the plane strain solution. Also, as $\nu \to 0$ the usual $1/\sqrt{\epsilon}$ singularity is recovered and furthermore the stresses $\sigma_z^{(c)}$, $\tau_{yz}^{(c)}$, $\tau_{xz}^{(c)}$ do vanish. In fact, this is the solution of plane stress.

Finally, the author would like to emphasize that the relations (90)–(98) are exact for all $-h + \epsilon < \epsilon < h - \epsilon$, ϵ being the usual cylindrical radius around the crack tip. This point will be discussed further in the following section.

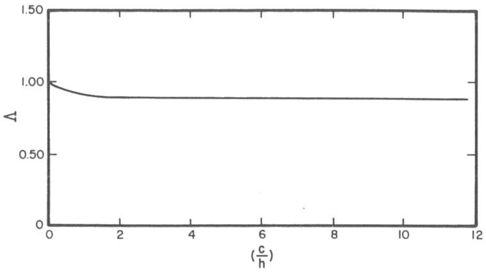

Fig. 4 Stress coefficient Λ versus *(c/h)* with $\nu = 1/3$

The Stress Field at the Corner

One of the academic questions that has defied researchers for a long time is the order of the singularity that prevails at the corner point. Thus, in order for us to study the stress distribution in the vicinity of that point, we introduce the local spherical coordinate system defined by the transformation

$$x - c = \rho \sin\theta \cos\phi = \epsilon \cos\phi, \quad y = \rho \sin\theta \sin\phi = \epsilon \sin\phi, \quad h - z = \rho \cos\theta, \quad (100)$$

Initially, one is tempted to substitute the foregoing transformation into equations (90)–(95) and thus recover the desired stress field. For example, the stress $\sigma_z^{(c)}$ may read

$$\sigma_z^{(c)} \sim \nu\bar{\sigma}\Lambda \frac{h^{2\nu}\sqrt{c}}{\sqrt{2}\rho^{1/2+2\nu}} \frac{1}{\cos^{2\nu}\theta\sqrt{\sin\theta}} \cos\left(\frac{\phi}{2}\right) + \ldots \quad (101)$$

Furthermore, one may deduce that all stresses, including $\tau_{xz}^{(c)}$ and $\tau_{yz}^{(c)}$ which are nonsingular in the interior portion of the plate, are proportional to $\rho^{-(1/2+2\nu)}$.

A closer inspection, however, reveals that the stress $\sigma_z^{(c)}$, for example, becomes infinite as $\theta \to \pi/2$. This of course contradicts one of the boundary conditions and immediately suggests that something is wrong with the solution. But this is not quite true. A careful investigation shows that terms of the order $O(\epsilon^{-1/2+2n}\beta_\nu^{2n})$, $n = 0,1,2,\ldots$, also contribute to the same order of singularity and consequently must be accounted for.

To establish, therefore, the complete θ-angular dependence one must return to the exact expression; i.e.,

$$\sigma_z^{(c)} = -\frac{m}{m-2} \sum_{\nu=1}^{\infty} \beta_\nu^2 [(\beta_\nu h) \sin(\beta_\nu h) \cos(\beta_\nu z) - (\beta_\nu z) \cos(\beta_\nu h) \sin(\beta_\nu z)] \times \int_0^\infty \frac{\Gamma_\nu}{s^2} \frac{s}{\sqrt{s^2 + \beta_\nu^2}} e^{-\sqrt{s^2+\beta_\nu^2}|y|} \cos(xs)\,ds, \quad (102)$$

which in view of the definition

$$I_\mp = \sum_{\nu=1}^{\infty} \cos[\beta_\nu(h \pm z)] \times \int_0^\infty \frac{\Gamma_\nu}{s^2} \frac{s e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2 + \beta_\nu^2}} \cos(xs)\,ds, \quad (103)$$

may also be written in the simple form

$$\frac{\sigma_z^{(c)}}{2G} = -\frac{1}{2(1-2\nu)} \left\{ (h - z) \frac{\partial^3 I_-}{\partial z^3} - (h + z) \frac{\partial^3 I_+}{\partial z^3} \right\}. \quad (104)$$

It remains, therefore, for us to evaluate the function $I_\pm$. With this in mind, we substitute equation (71) into equation (103) to obtain

$$I_\mp = \sum_{\nu=1}^{\infty} \cos[\beta_\nu(h \pm z)] \int_0^\infty \left\{ A_\nu^{(0)} J_1(sc) + A_\nu^{(1)} \frac{J_2(sc)}{(sc)} + A_\nu^{(2)} \frac{J_3(sc)}{(sc)^2} + \ldots \right\} \frac{s e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2 + \beta_\nu^2}} \cos(xs)\,ds. \quad (105)$$

[9] The reader should realize that Λ is proportional to $A_\nu^{(0)}$ and as a result Fig. 4 reflects the value as computed from the truncated equation (78). The exact behavior, of course, should be determined from equations (72) and (75). The author intends to do this as soon as research funds are secured.

The function $I_\pm$ may now be expressed in terms of a new function ${}^{(0)}I_\pm$ defined by

$$ {}^{(0)}I_{\mp} = \sum_{\nu=1}^{\infty} A_\nu{}^{(0)} \cos [\beta_\nu(h \pm z)] \times \int_0^\infty J_1(sc) \frac{se^{-\sqrt{s^2+\beta_\nu{}^2}|y|}}{\sqrt{s^2+\beta_\nu{}^2}} \cos (xs) ds. \quad (106) $$

It is now possible to evaluate explicitly this new function ${}^{(0)}I_\pm$ (see Appendix 5) to be[10]

$$ {}^{(0)}I_{\mp} = -\frac{C_0}{2}\sqrt{\frac{c}{2\epsilon}}\, h^{2\nu-2} \cos\left(\frac{\phi}{2}\right)\{[h \mp z - i\epsilon]^{2-2\nu} + [h \mp z + i\epsilon]^{2-2\nu}\} + R_n, \quad (107) $$

where R_n, here and in the following, will represent the remaining terms leading to singularities of a lesser strength.

Returning next to equation (105) and utilizing property (74) of the Bessel functions one finds after some manipulations[11] that

$$ I_{\mp} = \sum_{\nu=1}^{\infty} A_\nu{}^{(0)} \cos [\beta_\nu(h \pm z)] \int_0^\infty J_1(sc) \times \left\{1 - \left(\frac{m-1}{m}\right)\left(\frac{\beta_\nu}{s}\right)^2 \left[1 + \frac{1}{m2!}\left(\frac{\beta_\nu}{s}\right)^2 + \frac{(m+1)}{m^2 3!}\left(\frac{\beta_\nu}{s}\right)^4 + \ldots\right]\right\} \times \frac{se^{-\sqrt{s^2+\beta_\nu{}^2}|y|}}{\sqrt{s^2+\beta_\nu{}^2}} \cos (xs) ds + R_n \quad (108a) $$

$$ = \sum_{\nu=1}^{\infty} A_\nu{}^{(0)} \cos [\beta_\nu(h \pm z)] \int_0^\infty \times J_1(sc)\left[\frac{s^2-\beta_\nu{}^2}{s^2}\right]^{\frac{m-1}{m}} \frac{se^{-\sqrt{s^2+\beta_\nu{}^2}|y|}}{\sqrt{s^2+\beta_\nu{}^2}} \cos (xs) ds + R_n \quad (108b) $$

The author now believes that due to the convergent character of the series (108a) one may also write

$$ I_{\mp} = {}^{(0)}I_{\mp} + (1-\nu)\frac{\partial^2}{\partial z^2}\iint {}^{(0)}I_{\mp} dx dx - \left(\frac{\nu}{2}\right)(1-\nu)\frac{\partial^4}{\partial z^4}\iiiint {}^{(0)}I_{\mp} dx dx dx dx + \ldots + R_n, \quad (108) $$

in view of which the stress $\sigma_z{}^{(c)}$ now reads

$$ \sigma_z{}^{(c)} = -\frac{1}{2(1-2\nu)}(h-z)\left\{\frac{\partial^{3\,(0)}I_-}{\partial z^3} + (1-\nu)\frac{\partial^5}{\partial z^5}\iint {}^{(0)}I_- dx dx + \ldots\right\} + \frac{1}{2(1-2\nu)}(h+z) \times \left\{\frac{\partial^{3\,(0)}I_+}{\partial z^3} + (1-\nu)\frac{\partial^5}{\partial z^5}\iint {}^{(0)}I_+ dx dx + \ldots\right\} + R_n. \quad (109) $$

[10] The reader should also notice that in terms of the spherical coordinates around the corner point $z = h$,

$$ {}^{(0)}I_- = -C_0\sqrt{\frac{c}{2}}\,\rho^{3/2-2\nu}h^{2\nu-2}\frac{\cos\left(\frac{\phi}{2}\right)}{\sqrt{\sin\theta}} \cos [(2-2\nu)\theta] + R_n, \quad {}^{(0)}I_+ = 0 + R_n. $$

[11] At first glance, it seems that only the first term gives a contribution of the order $\rho^{3/2-2\nu}$. However, a closer inspection reveals that the terms $A_\nu{}^{(2)}$, $A_\nu{}^{(4)}$, etc., also give contributions of the same order. This can be seen from equation (75c). For example, when $k = 1$

$$ \sum_{\nu=1}^{\infty} A_\nu{}^{(2)} a_\nu(z) = \sum_{\nu=1}^{\infty} A_\nu{}^{(0)}(\beta_\nu c)^2 b_\nu(z). $$

Thus, if one lets

$$ A_\nu{}^{(2)} = {}_hA_\nu{}^{(2)} + {}_pA_\nu{}^{(2)}, $$

where ${}_pA_\nu{}^{(2)}$ corresponds to the particular solution, he finds that

$$ {}_pA_\nu{}^{(2)} = -\left(\frac{m-1}{m}\right)(\beta_\nu c)^2 A_\nu{}^{(0)}, \text{ etc.} $$

It is appropriate at this time to investigate the asymptotic behavior of the stress $\sigma_z{}^{(c)}$ *(i)* in the interior portions of the plate and *(ii)* in the neighborhood of the corner points.

(i) Inner Layers. That is the region where $-h + \epsilon < z < h - \epsilon$. Expanding the first terms in powers of $\epsilon/(h-z)$ and the last terms in powers of $\epsilon/(h+z)$ one notices that only the first term of the expression inside the braces contributes to the singular portion of the stress. In fact, one recovers precisely the result of equation (92). We now have, therefore, a better understanding of the type of expansion that equations (90)–(98) represent.

(ii) Outer Layers. That is the neighborhood of the corner point $z = h$. In this case, all the terms inside the first braces contribute to the same order of singularity $\rho^{-1/2-2\nu}$. The terms inside the second braces, however, contribute to less order singularities and consequently may be neglected. Thus, using the spherical coordinates (100), one may write

$$ \sigma_z{}^{(c)} = \nu\bar{\sigma}_0\Lambda\sqrt{\frac{c}{2}}\frac{h^{2\nu}}{\rho^{1/2+2\nu}}\frac{\cos\theta\cos[(2\nu+1)\theta]}{\sqrt{\sin\theta}}\cos\left(\frac{\phi}{2}\right) - \frac{(1-\nu)}{2(1-2\nu)}(h-z)\frac{\partial^5}{\partial z^5}\iint {}^{(0)}I_- dx dx + \ldots + R_n. \quad (110) $$

Although the remaining integrals cannot be evaluated explicitly, it is rather obvious that as $z \to h$ the terms of the order $\rho^{-1/2-2\nu}$ do vanish.

The reader should furthermore notice that as $\nu \to 0$ $\sigma_z{}^{(c)} \to 0$, for the third derivative with respect to z kicks out a factor of ν.

In a similar manner, the complimentary displacement function $w^{(c)}$ becomes

$$ w^{(c)} = -\frac{1}{2}\frac{\partial I_-}{\partial z} - \frac{(h-z)}{2(1-2\nu)}\frac{\partial^2 I_-}{\partial z^2} - \frac{1}{2}\frac{\partial I_+}{\partial z} + \frac{(h+z)}{2(1-2\nu)}\frac{\partial^2 I_+}{\partial z^2} + R_n. \quad (111) $$

It's asymptotic expansion in the inner layers gives precisely the result of equation (98), while in the vicinity of the corner point $z = h$

$$ w^{(c)} = \left[\frac{\bar{\sigma}_0}{4G}\right]\Lambda\sqrt{\frac{c}{2}}\,h^{2\nu}\rho^{1/2-2\nu}\frac{\cos\left(\frac{\phi}{2}\right)}{\sqrt{\sin\theta}}\{\cos\theta\cos(2\nu\theta) - \cos[(2\nu-1)\theta]\} - \left(\frac{1-\nu}{2}\right)\left\{\frac{\partial^3}{\partial z^3}\iint {}^{(0)}I_- dx dx + \frac{(h-z)}{2(1-2\nu)}\frac{\partial^4}{\partial z^4}\iint {}^{(0)}I_- dx dx + \ldots\right\} + R_n. \quad (112) $$

In summary, it is evident from the foregoing that in the vicinity of the corner point the displacements are proportional to $\rho^{1/2-2\nu}$ and the stresses proportional to $\rho^{-1/2-2\nu}$. This result, however, seems to be somewhat unorthodox inasmuch as for certain Poisson's ratios the displacements become singular at that point. Physically, what the solution really shows, is that linear theory is inadequate in predicting the actual behavior of the material at such corner points. Be that as it may, we observe that the strength of the singularity is such that the local strain energy is finite for all $\nu \leq ½$. This of course is not coincidental. In fact, singularities of such strength have also been observed in diffraction theory. In this case, the questions of existence and uniqueness of such type of solutions have been discussed by Bouwkamp [9], Heins and Silver [10], and Wilcox [11, 12].

Following reference [11] one can show [13] that, in the field of elastostatics, solutions that do satisfy the condition of locally finite energy are unique.

Finally, it remains for us to show that the solution satisfies the remaining boundary conditions on the plate faces $z = \pm h$. To show this, we define the following functions:

$$ T^* = \sum_{n=1}^{\infty}\frac{8m}{m-2}(-1)^n\sum_{\nu=1}^{\infty}\frac{\beta_\nu{}^4}{(\beta_\nu{}^2-\alpha_n{}^2)^2}\cos(\alpha_n z) \times \int_0^\infty \frac{\Gamma_\nu}{s^2}\frac{se^{\sqrt{s^2+\alpha_n{}^2}|y|}}{\sqrt{s^2+\alpha_n{}^2}}\cos(xs)ds \quad (113) $$

$$\Phi = \frac{1-\nu}{1-2\nu}(I_- + I_+),$$

$$\Psi = \frac{1}{2(1-2\nu)}\left\{(h-z)\frac{\partial I_-}{\partial z} - (h+z)\frac{\partial I_+}{\partial z}\right\}, \quad (114)$$

in view of which the complementary displacements may now be written in the simple form

$$u^{(c)} = \frac{\partial}{\partial x}(\Phi - \Psi) + \int_0^x \frac{\partial^2 T^*}{\partial y^2} dx + R_n \quad (115)$$

$$v^{(c)} = \frac{\partial}{\partial y}(\Phi - \Psi) - \frac{\partial T^*}{\partial y} + R_n \quad (116)$$

$$w^{(c)} = -\frac{\partial}{\partial z}(\Phi + \Psi) + R_n. \quad (117)$$

The stresses $\tau_{yz}^{(c)}$ and $\tau_{xz}^{(c)}$ can now be computed as

$$\frac{\tau_{yz}^{(c)}}{G} = -\frac{\partial^2 T^*}{\partial y \partial z} - \frac{\partial^2(2\Psi)}{\partial y \partial z}, \quad \frac{\tau_{xz}^{(c)}}{G} = \int_0^x \frac{\partial^3 T^*}{\partial y^2 \partial z} dx - \frac{\partial^2(2\Psi)}{\partial x \partial z}. \quad (118)\text{–}(119)$$

Now in order to examine the behavior of the above stresses as $z \to h$, one must either determine explicitly the harmonic function T^* or express it in terms of the function $I_\pm$. The latter can be accomplished with the use of the relation

$$\frac{e^{-\sqrt{s^2+\alpha_n^2}|y|}}{\sqrt{s^2+\alpha_n^2}} = \int_{|y|}^\infty J_0(s\sqrt{t^2-|y|^2})e^{-\alpha_n t}dt, \quad (120)$$

and the residue theorem. In particular, one can show that[12]

$$4\sum_{n=1}^\infty \frac{(-1)^n \beta_\nu^4}{(\beta_\nu^2-\alpha_n^2)^2}\frac{e^{-\sqrt{s^2+\alpha_n^2}|y|}}{\sqrt{s^2+\alpha_n^2}}\cos(\alpha_n z) = [\beta_\nu h \sin(\beta_\nu h) \times \cos(\beta_\nu z) - \beta_\nu z\cos(\beta_\nu h)\sin(\beta_\nu z)]\frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}}$$
$$-\frac{2e^{-s|y|}}{s} - \beta_\nu^2\cos(\beta_\nu h)\cos(\beta_\nu z)\frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}}\left[\frac{|y|}{\sqrt{s^2+\beta_\nu^2}} + \frac{1}{s^2+\beta_\nu^2}\right] + 4\beta_\nu^4\frac{h}{\pi}\int_{|y|}^\infty J_0(s\sqrt{t^2-|y|^2})$$
$$\times \int_0^\infty \frac{\sin(Rt)\cosh(Rz)}{(\beta_\nu^2+R^2)^2\sinh(Rh)}dRdt, \quad (121)$$

and consequently that

$$\frac{\partial^2 T^*}{\partial y\partial z} = \frac{1}{1-2\nu}\frac{\partial}{\partial z}\left\{(h-z)\frac{\partial^2 I_-}{\partial y\partial z} - (h+z)\frac{\partial^2 I_+}{\partial y \partial z}\right\} + \frac{1}{1-2\nu}|y|\left\{\frac{\partial^3 I_-}{\partial z^3} + \frac{\partial^3 I_+}{\partial z^3}\right\} - \frac{8}{1-2\nu}\frac{h}{\pi}\sum_{\nu=1}^\infty \beta_\nu^4$$
$$\times \int_0^\infty \frac{\Gamma_\nu}{s^2} s\cos(xs)\frac{\partial^2}{\partial y^2}\left\{\int_{|y|}^\infty J_0(s\sqrt{t^2-|y|^2}) \times \int_0^\infty \frac{R\sin(Rt)}{(\beta_\nu^2+R^2)^2}\frac{\sinh(Rz)}{\sinh(Rh)}dRdt\right\}. \quad (122)$$

Unfortunately, the author has not as yet been able to determine the last harmonic function explicitly. Consequently, he is unable to compute the remaining stresses and displacements. However, the derivative of the function with respect to z, at $z = \pm h$, can be evaluated since

$$\int_0^\infty \frac{R\sin(Rt)}{(\beta_\nu^2+R^2)^2}dR = \left(\frac{\pi}{4}\right)\left(\frac{t}{\beta_\nu}\right)e^{-\beta_\nu t}, \quad (123)$$

that is,

$$\frac{\partial^2 T^*}{\partial y\partial z} \longrightarrow -\frac{1}{1-2\nu}\left(\frac{\partial^2 I_-}{\partial y\partial z}\right)_{z\to h} + \frac{1}{1-2\nu}|y|\left[\frac{\partial^3 I_-}{\partial z^3} + \frac{\partial^3 I_+}{\partial z^3}\right]_{z\to h} + \frac{2}{1-2\nu}|y|h\sum_{\nu=1}^\infty \beta_\nu^4$$
$$\times \int_0^\infty \frac{\Gamma_\nu}{s^2}\frac{se^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}}\cos(xs)ds + R_n. \quad (124)$$

But from equation (103)

$$\frac{\partial^3}{\partial z^3}[I_- + I_+] = 2\sum_{\nu=1}^\infty \beta_\nu^3\cos(\beta_\nu h)\sin(\beta_\nu z) \times \int_0^\infty \frac{\Gamma_\nu}{s^2}\frac{se^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}}\cos(xs)ds \quad (125)$$

and therefore

$$\lim_{z\to h}\frac{\partial^3}{\partial z^3}[I_- + I_+] = -2\sum_{\nu=1}^\infty \beta_\nu^4 h\int_0^\infty \frac{\Gamma_\nu}{s^2}\frac{se^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}}\cos(xs)ds. \quad (126)$$

Substituting this result into equation (124) and subsequently into equation (118) one clearly sees that

$$\left(\frac{\tau_{yz}^{(c)}}{G}\right)_{z\to h} \longrightarrow 0.$$

It remains finally to show that the other stress $\tau_{xz}^{(c)}$ also vanishes at the plate faces $z = \pm h$. To do this, one must determine the behavior of the function

$$\int_0^x \frac{\partial^3 T^*}{\partial y^2 \partial z}dx$$

as $z \to h$. Unfortunately, here one may not use the same steps as those of the stress $\tau_{yz}^{(c)}$ because the integrand now corresponds to a third derivative with respect to z. Consequently, the evaluation of the integrand at $z = h$ is not permissible. What this really means is that one must integrate first with respect to x and then take the limit as $z \to h$. The actual details of this, however, are very difficult—at least as far as the author is concerned. But our solution was constructed in such a way that this should be the case. This can be seen from equation (57) for as $z \to h$, $\tau_{xz}^{(c)} \to 0$ in view of equation (40).

Finally, one may compute[13] the total strain energy stored in the system to be

$$W \approx \frac{\pi(1-\nu^2)\bar{\sigma}_0^2 c^2(2h)}{E}\left\{1 + \frac{2\nu}{1-2\nu}\frac{1}{\left(\frac{\sigma_0}{2G}c^2\right)} \times \sum_{\nu=1}^\infty\sum_{k=0}^\infty A_\nu^{(k)}\beta_\nu^2\left[\frac{1}{2^{k+1}(k+1)!} - \frac{I_{k+1}\left(\frac{\beta_\nu c}{\sqrt{2}}\right)K_1\left(\frac{\beta_\nu c}{\sqrt{2}}\right)}{\left(\frac{\beta_\nu c}{\sqrt{2}}\right)^k}\right]\right\}, \quad (127)$$

from which the two well known limits may now be recovered,[14] i.e.,

$$W_{h\to 0^+} \longrightarrow \frac{\bar{\sigma}_0^2\pi c^2(2h)}{E}; \text{ generalized plane stress}$$

and

$$W_{h\to\infty} \longrightarrow \frac{\bar{\sigma}_0^2\pi c^2(1-\nu^2)}{E}(2h); \text{ plane strain.}$$

[12] The reader should notice that equation (121), together with its first and second partial derivatives, is valid for all $|z| \le h$. However, its third derivative with respect to z—or with respect to y for that matter—is only valid for $|z| < h$.

[13] Use equation (53), the approximation of Appendix 3, and the relation

$$W = -\frac{1}{2}\int_{-h}^{h}\int_{-c}^{c}\{(v^+ - v^-)\cdot\sigma_y\}_{y=0}dxdz.$$

[14] Use $\int_{-h}^{h}$ equation (72) dz for $j = 0$.

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APPENDIX 1

The roots of the equation $\sin(2\beta_\nu h) = -(2\beta_\nu h)$.

The equation has an infinite number of complex roots which appear in groups of four. However, as it was pointed out in the text, for this analysis only the roots with positive real parts are pertinent and furthermore, the only real root $\beta_\nu = 0$ must be discarded. Thus, if we define the roots $\beta_2, \beta_4, \beta_6, \ldots$ to be the complex conjugates of the roots $\beta_1, \beta_3, \beta_5, \ldots$, then by setting

$$2\beta_\nu h = x_\nu + iy_\nu;\ \nu = 1,3,5\ldots$$

and using a Newtom-Rampson numerical method one finds

ν	x_ν	y_ν
1	4.21239	2.25073
3	10.71254	3.10315
5	17.07337	3.55109
7	23.39836	3.85881
	etc.	

Furthermore, the asymptotic behavior of the roots for large ν, i.e., for $\nu = 15, 17, 19, \ldots$, is given by the following simple relations:

$$x_\nu \simeq \left(\nu + \frac{1}{2}\right)\pi,\quad y_\nu \simeq \cos h^{-1}\left[\left(\nu + \frac{1}{2}\right)\pi\right].$$

APPENDIX 2

We list the Fourier series expansions with their corresponding range of validity as follows:

$$z^2 = \frac{h^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{\alpha_n^2}\cos(\alpha_n z);\ -h \le z \le h$$

$$\cos(\beta_\nu z) = \frac{\sin(\beta_\nu h)}{\beta_\nu h} + \frac{2\beta_\nu^2\sin(\beta_\nu h)}{\beta_\nu h}\sum_{n=1}^{\infty}\frac{(-1)^n}{\beta_\nu^2-\alpha_n^2}\times\cos(\alpha_n z);\ -h \le z \le h$$

$$z\sin(\beta_\nu z) = \frac{\sin(\beta_\nu h) - h\beta_\nu\cos(\beta_\nu h)}{h\beta_\nu^2} + 2\sum_{n=1}^{\infty}(-1)^n\left\{\frac{\sin(\beta_\nu h)}{h}\frac{\beta_\nu^2+\alpha_n^2}{(\beta_\nu^2-\alpha_n^2)^2} - \cos(\beta_\nu h)\frac{\beta_\nu}{\beta_\nu^2-\alpha_n^2}\right\}\cos(\alpha_n z);\ -h \le z \le h$$

$$\sin(\beta_\nu z) = \frac{2\sin(\beta_\nu h)}{h}\sum_{n=1}^{\infty}\frac{(-1)^n\alpha_n}{\beta_\nu^2-\alpha_n^2}\sin(\alpha_n z);\ -h < z < h$$

$$z\cos(\beta_\nu z) = 2\sum_{n=1}^{\infty}\left\{\frac{\cos(\beta_\nu h)}{\beta_\nu^2-\alpha_n^2} - \frac{2\beta_\nu h\sin(\beta_\nu h)}{h^2(\beta_\nu^2-\alpha_n^2)^2}\right\}(-1)^n\alpha_n\sin(\alpha_n z);\ -h < z < h$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(\beta_\nu^2-\alpha_n^2)}\cos(\alpha_n z) = \frac{\beta_\nu h\cos(\beta_\nu z) - \sin(\beta_\nu h)}{2\beta_\nu^2\sin(\beta_\nu h)};\ |z| \le h$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\beta_\nu^2}{(\beta_\nu^2-\alpha_n^2)^2}\cos(\alpha_n z) = \frac{\beta_\nu^2 hz\sin(\beta_\nu z) - 2\sin(\beta_\nu h) + \beta_\nu h\sin^2(\beta_\nu h)\cos(\beta_\nu z)}{4\beta_\nu^2\sin(\beta_\nu h)};\ |z| \le h$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n\alpha_n^2}{(\beta_\nu^2-\alpha_n^2)^2}\cos(\alpha_n z) = \frac{\beta_\nu^2 hz\sin(\beta_\nu z) + \beta_\nu h(\sin^2(\beta_\nu h) - 2)\cos(\beta_\nu z)}{4\beta_\nu^2\sin(\beta_\nu h)};\ |z| \le h$$

APPENDIX 3

Consider the solution of the following dual set

$$\int_0^\infty sg(s;\beta)f(s)\cos(xs)ds = -1;\ |x| < c$$

$$\int_0^\infty f(s)\cos(xs)ds = 0;\ |x| > c$$

for the unknown function $f(s)$. The function $g(s;\beta)$ is assumed to be continuous for all values of s and β and furthermore for simplicity β to be real. We now assume that the solution can be written in the form

$$f(s) = \sum_{n=0}^{\infty} A^{(n)}\frac{J_{n+1}(sc)}{(sc)^{n+1}}.$$

The advantage of such a form is that it satisfies the second equation automatically and furthermore, if one defines

$$v(x) = \int_0^\infty f(s)\cos(xs)dx;\ |x| < c,$$

it shows that $v(x)$ can be written as

$$v(x) = \sum_{n=0}^{\infty} A^{(n)}\frac{\sqrt{\pi}}{2^{n+1}\Gamma\left(n+\frac{3}{2}\right)}\left(1 - \frac{x^2}{c^2}\right)^{n+1/2}$$

In the field of fracture mechanics, for example, the physical significance of the function $v(x)$ is usually the vertical displacement along the crack face and consequently such an expansion is plausible.

Returning next to the first equation one finds

$$\sum_{n=0}^{\infty} A^{(n)} \int_0^{\infty} sg(s;\ \beta)\frac{J_{n+1}(sc)}{(sc)^{n+1}} \cos(xs)ds = -1;\ |x| < c.$$

and upon multiplying both sides with

$$\frac{1}{2^k\Gamma(k + 3/2)\sqrt{\pi}c^{2k+2}}(c^2 - x^2)^{k+1/2}$$

and integrating with respect to x from 0 to c

$$\sum_{n=0}^{\infty} A^{(n)} \int_0^{\infty} sg(s;\ \beta)\frac{J_{n+1}(sc)}{(sc)^{n+1}}\frac{J_{k+1}(sc)}{(sc)^{k+1}} ds = -\frac{1}{2^{k+1}(k+1)!};\ k = 0, 1, 2\ldots$$

If we now define

$$h_{n,k}(\beta) = \int_0^{\infty} sg(s;\ \beta)\frac{J_{n+1}(sc)}{(sc)^{n+1}}\frac{J_{k+1}(sc)}{(sc)^{k+1}} ds,$$

then the unknown coefficients $A^{(n)}$ are to be determined from the following infinite system of algebraic equations:

$$\sum_{n=0}^{\infty} A^{(n)} h_{n,k}(\beta) = -\frac{1}{2^{k+1}(k+1)!};\ k = 0, 1, 2\ldots.$$

The questions of existence and uniqueness of solutions of infinite systems of equations have been studied and the results are discussed in reference [8, pp. 20–44]. The conditions, therefore, that one must impose on the function $g(s, \beta)$ are the following:

1 The integrals $h_{n,k}(\beta)$ must exist.

2 The coefficients $h_{n,k}(\beta)$ must be such that the resulting infinite system of equations has a unique solution.

As a first example, we let $g(s, \beta) = 1$ the solution of which is well known. In this case, the integrals

$$h_{n,k}{}^0 = \int_0^{\infty} s \cdot \frac{J_{n+1}(sc)}{(sc)^{n+1}}\frac{J_{k+1}(sc)}{(sc)^{k+1}} ds = \left(\frac{1}{2}\right)^{n+k+1}\frac{1}{c^2 k!\,n!\,(n+k+1)}$$

and our infinite system therefore reads

$$\sum_{n=0}^{\infty}\frac{A^{(n)}}{c^2}\left(\frac{1}{2}\right)^{n+k+1}\frac{1}{k!\,n!\,(n+k+1)} = -\frac{1}{2^{k+1}(k+1)!};\ k = 0, 1, 2, \ldots$$

The solution of the foregoing system is

$$A^{(0)} = -c^2 \text{ and } A^{(n)} = 0 \text{ for } n \geq 1,$$

which of course leads to the well-known solution

$$f(s) = -c^2\frac{J_1(sc)}{(sc)}.$$

As a second example, we let

$$g(s, \beta) = 1 - \frac{s}{\sqrt{s^2 + \beta^2}}$$

which gives

$$h_{n,k}(\beta) = \int_0^{\infty} s\left\{1 - \frac{s}{\sqrt{s^2+\beta^2}}\right\}\frac{J_{n+1}(sc)}{(sc)^{n+1}}\frac{J_{k+1}(sc)}{(sc)^{k+1}} ds.$$

But $h_{n,k}(\beta) \leq h^0{}_{nk}$ (numerically) therefore from reference [8] we deduce that there exists a unique solution which converges as the number of terms increases.

The aforementioned integrals may be evaluated approximately if one uses the following identity:

$$1 - \frac{s}{\sqrt{s^2+\beta^2}} = \frac{\beta^2}{s^2+\beta^2}\frac{1}{2 - \left\{\dfrac{\beta^2}{s^2+\beta^2}\dfrac{1}{\left[1 + \dfrac{s}{\sqrt{s^2+\beta^2}}\right]}\right\}}.$$

Furthermore notice that for large β one may replace the expression inside the brackets by 1, hence

$$1 - \frac{s}{\sqrt{s^2+\beta^2}} \approx \frac{\left(\frac{\beta}{\sqrt{2}}\right)^2}{s^2 + \left(\frac{\beta}{\sqrt{2}}\right)^2}$$

The reader should furthermore notice that this is a well-known numerical approximation which is fairly good for all values of β. It is an easy matter now to show that $h_{0,0}$, for example, becomes

$$h_{0,0}(\beta) \approx \left\{\frac{1}{2} - I_1\left(\frac{\beta c}{\sqrt{2}}\right)K_1\left(\frac{\beta c}{\sqrt{2}}\right)\right\}.$$

APPENDIX 4

In order to solve the difference-differential equation (82) we use the following relation

$$\left[\frac{f(u)}{u}\right]' = \frac{f'(u)}{u} - \frac{f(u)}{u^2}.$$

Thus the homogeneous equation may now be written as

$$(m-1)[f(1+\zeta) + f(1-\zeta)] + \frac{m}{2}(1-\zeta^2)\left\{\frac{f(1+\zeta)}{(1+\zeta)^2} + \frac{f(1-\zeta)}{(1-\zeta)^2} + \left[\frac{f(1+\zeta)}{(1+\zeta)}\right]' - \left[\frac{f(1-\zeta)}{(1-\zeta)}\right]'\right\} = 0,$$

where the prime represents differentiation with respect to the arguments. Finally, simplifying and integrating with respect to ζ from 0 to ζ one finds after some very simple manipulations that

$$\frac{f1+\zeta)}{(1+\zeta)} - \frac{f(1-\zeta)}{(1-\zeta)} = -\int_{1-\zeta}^{1+\zeta}\frac{f(\xi)}{\xi}\left\{\frac{1}{\xi} + \frac{2\left(1 - \frac{1}{m}\right)}{2-\xi}\right\}d\xi.$$

It is now evident that

$$-\frac{f(\xi)}{\xi}\left\{\frac{1}{\xi} + \frac{2\left(1-\frac{1}{m}\right)}{2-\xi}\right\} = \left\{\frac{f(\xi)}{\xi}\right\}',$$

which consequently leads to

$$f(\xi) = C_0(2-\xi)^{2-2\nu}$$

To this, of course, one must also attach a particular solution.

APPENDIX 5

In order to prove equation (107), one must first find an asymptotic expansion for the integral

$$M(\epsilon;\beta) = \int_0^{\infty} J_1(sc)\frac{s}{\sqrt{s^2+\beta_\nu{}^2}}e^{-\sqrt{s^2+\beta_\nu{}^2}|y|}\cos(xs)ds$$

when $|y| = \epsilon \sin\phi$ and $x = c + \epsilon\cos\phi$, ϵ being small and positive quantity. Recalling next that

$$M(\epsilon;0) = -\sqrt{\frac{c}{2\epsilon}}\cos\left(\frac{\phi}{2}\right) + O(\epsilon^\circ)$$

and noticing that the function M satisfies the equation

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} - \beta_\nu^2 M = 0,$$

one looks for an asymptotic expansion of the form

$$M = \sum_{n=0}^{\infty} A_n \epsilon^{n-1/2} f_n(\phi;\beta_\nu^2).$$

Substituting into the differential equation and equating powers of ϵ one has

$$M = -\sqrt{\frac{C}{2\epsilon}} \cos\left(\frac{\phi}{2}\right)\left\{1 + \frac{\epsilon^2\beta_\nu^2}{2!} + \frac{\epsilon^4\beta_\nu^4}{4!} + \ldots\right\} + O(\epsilon^0).$$

Thus equation (106), after the summation over ν is carried out, leads to the result of (107). Notice that the $O(\sqrt{\epsilon})$ terms will lead to singularities of a lesser strength for it represents a series with terms such as

$$\sqrt{\epsilon},\ \sqrt{\epsilon}\epsilon^2\beta_\nu^2,\ \sqrt{\epsilon}\epsilon^4\beta_\nu^4,\ \text{etc.}$$

$$M = \sqrt{\frac{C}{2\epsilon}} \cosh(\epsilon\beta) \cos\left(\frac{\phi}{2}\right) \cos[\beta(h-z)]$$