International Journal of Fracture Mechanics, Vol. 6, No. 3, September 1970
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

On a Plate Supported by an Elastic Foundation and Containing a Finite Crack

E. S. FOLIAS, PH. D.,

Civil Engineering and Mathematics, University of Utah (U.S.A.)

(Received June 30, 1969)

ABSTRACT
Using an integral formulation, the equation for a plate resting on an elastic foundation, of a spring constant k and containing a crack of length $2c$, is solved for the Kirchhoff bending stresses. The inverse square root singular behavior of the stresses peculiar to crack problems is obtained. Furthermore, this singularity may be related to that found in an unsupported plate by

$$\frac{\sigma_{y_{\text{sup.}}}}{\sigma_{y_{\text{unsup.}}}} \approx \frac{1}{1+a\lambda^2}$$

where a is for small values of the parameter λ is a positive constant.

Nomenclature

$2c$ = crack length
$D = Eh^3/[12(1-\nu^2)]$ = flexural rigidity of a plate
E = Young's modulus of elasticity
G = Shear modulus of elasticity
h = Thickness of a plate
k = Foundation modulus of a plate
$M_y^{(c)}, M_y^{(P)}$ = Bending moments as defined in text
m_0 = Constant as defined in text
$q(x, y)$ = Lateral load
$V_y^{(c)}, V_y^{(P)}$ = Shear force as defined in text
$W(x, y)$ = Transverse deflection of a plate in bending
$W^{(P)}(x, y)$ = Transverse deflection of a continuous plate which has not been weakened by the crack
$W^{(c)}(x, y)$ = Transverse deflection of a plate involving perturbations induced by the presence of the crack
$W_+^{(c)} = \lim_{y\to 0^+} W^{(c)}(x, y)$, $W_-^{(c)} = \lim_{y\to 0^-} W^{(c)}(x, y)$
X, Y, Z = Rectangular coordinates in middle plane of a plate
$x \equiv \frac{X}{c}, \quad y \equiv \frac{Y}{c}, \quad z \equiv \frac{Z}{c}$
$\alpha \equiv (\mathrm{i})^{\frac{1}{2}}$
$\beta \equiv (-\mathrm{i})^{\frac{1}{2}}$
$\gamma = 0.578$ Euler's constant
$\varepsilon, \theta;\ \varepsilon e^{i\theta} = x \pm 1 + iy$
$\zeta \equiv x - \xi$
$\lambda \equiv (k/D)^{\frac{1}{4}} c$
ν = Poisson's ratio
$\nu_0 \equiv 1-\nu$
$\sigma_x, \sigma_y, \tau_{xy}$ = Stress components
$\sigma_x^{(c)}, \sigma_y^{(c)}, \sigma_{xy}^{(c)}$ = Perturbation stress components due to the presence of the crack
$\bar{\sigma}_b \equiv \frac{6Dm_0}{h^2 c^2}$

Introduction

In a previous treatment of this problem, the stress field in the vicinity of a semi-infinite line crack of a plate resting on an elastic foundation was determined [1].

It was recognized at the time that a semi-infinite crack was not of much practical value and that a finite crack would be by far more desirable. Unfortunately, the latter presented numerous mathematical complexities and consequently it was put aside. Recently, however, the author, using techniques which he had developed earlier [2], was able to investigate the case of a finite crack and the results are reported herein.

The results have direct value in civil engineering applications such as roadways, airports and indirect value in the fracture of initially curved vessels [3].

Formulation of the Problem

Consider a homogeneous and isotropic thin plate of uniform thickness h. The plate is resting on an elastic foundation of a spring constant k and contains a line crack of length $2c$ (see fig. 1).

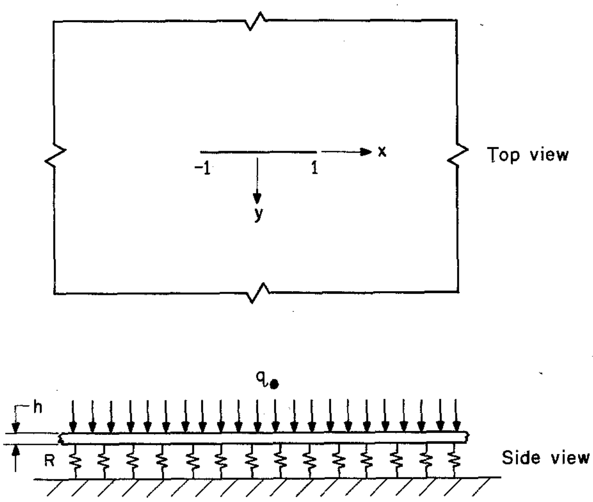

Figure 1

The differential equation governing the displacement function $W(x, y)$, with x and y as dimensionless rectangular coordinates, is given in the classical theory by

$$(\nabla^4+\lambda^4)\,W(x, y) = \frac{q(x, y)c^4}{D}. \tag{1}$$

The boundary conditions at the crack are those of free edges. Inasmuch as classical bending theory is used, only two boundary conditions along the crack may be satisfied. In particular, one must require that the normal moment and equivalent vertical shear to vanish, i.e.

$$\lim_{|y|\to 0}\begin{pmatrix} M_y \\ V_y \end{pmatrix} = 0 \quad \text{for} \quad -1<x<1\,. \tag{2}$$

It is required, in addition, that the function W and all its partial derivatives be continuous for all x and y, except for points on the segment $-1<x<1$, $y=0$. In order not to loose any generality, one may assume that at infinity the plate is loaded in some general manner.

Thus if one seeks the solution to the crack plate problem in the form

$$W(x, y) = W^{(P)}(x, y) + W^{(c)}(x, y)\,, \tag{3}$$

then the moments and stress resultants may be correspondingly written as $M_y = M_y^{(P)} + M_y^{(c)}$, etc.

Suppose, however, that one has already found a particular solution* satisfying equation (1) but there is a residual normal moment M_y and equivalent vertical shear V_y along the crack $|x|<1$ of the form

$$M_y^{(P)} = -\frac{Dm_0}{c^2} \tag{4}$$

$$V_y^{(P)} = 0 \tag{5}$$

where, for simplicity, m_0 will be taken to be a constant.

Assuming, therefore, that a particular solution has been found, we need to find a function $W^{(c)}(x, y)$ such that it satisfies the homogeneous part of the partial differential equation (1) and the following boundary conditions:

at $y=0$ and $|x|<1$

$$M_y^{(c)}(x, 0) = -\frac{D}{c^2}\left[\frac{\partial^2 W^{(c)}}{\partial y^2} + \nu\frac{\partial^2 W^{(c)}}{\partial x^2}\right] = \frac{Dm_0}{c^2} \tag{6}$$

$$V_y^{(c)}(x, 0) = -\frac{D}{c^3}\left[\frac{\partial^3 W^{(c)}}{\partial y^3} + (2-\nu)\frac{\partial^3 W^{(c)}}{\partial x^2 \partial y}\right] = 0 \tag{7}$$

at $y=0$ and $|x|>1$

$$\lim_{|y|\to 0}\left[\frac{\partial^n}{\partial y^n}(W_+^{(c)}) - \frac{\partial^n}{\partial y^n}(W_-^{(c)})\right] = 0 \qquad (n=0, 1, 2, 3). \tag{8}$$

To complete the formulation of the problem, we require that the displacement function $W^{(c)}(x, y)$ together with its first partial derivatives be finite at infinity.

Method of Solution

We construct the following integral representation which has the proper behavior at infinity

$$W^{(c)}(x, y^{\pm}) = \int_0^\infty \{P_1 \exp[-(s^2+\alpha^2\lambda^2)^{\frac{1}{2}}|y|] + P_2 \exp[-(s^2+\beta^2\lambda^2)^{\frac{1}{2}}|y|\} \cos xs\,ds\,, \tag{9}$$

where P_1, P_2 are arbitrary functions of s to be determined from the boundary conditions and the $\pm$ signs refer to $y>0$ and $y<0$ respectively.

Assuming that one can differentiate under the integral sign, formally substituting equation (9) into (7) one has

$$\mp\frac{D}{c^3}\int_0^\infty \{(s^2+\alpha^2\lambda^2)^{\frac{1}{2}}(\nu_0 s^2-\alpha^2\lambda^2)P_1 + (s^2+\beta^2\lambda^2)^{\frac{1}{2}}(\nu_0 s^2-\beta^2\lambda^2)P_2\} \cos xs\,ds = 0 \tag{10}$$

which may be satisfied, for all values of x, if one chooses

$$P_1 = (s^2+\beta^2\lambda^2)^{\frac{1}{2}}(\nu_0 s^2-\beta^2\lambda^2)P(s) \tag{11}$$

$$P_2 = -(s^2+\alpha^2\lambda^2)^{\frac{1}{2}}(\nu_0 s^2-\alpha^2\lambda^2)P(s)\,. \tag{12}$$

Similarly, substituting (9) into (6) and utilizing equations (11) and (12) one obtains

$$\lim_{|y|=0} -\int_0^\infty \{(\nu_0 s^2+\alpha^2\lambda^2)^2(s^2+\beta^2\lambda^2)^{\frac{1}{2}} \exp[-(s^2+\alpha^2\lambda^2)^{\frac{1}{2}}|y|] + \\ -(\nu_0 s^2+\beta^2\lambda^2)^2(s^2+\alpha^2\lambda^2)^{\frac{1}{2}} \exp[-(s^2+\beta^2\lambda^2)^{\frac{1}{2}}|y|]P(s) \cos xs\,ds = m_0\,; \quad \text{for } |x|<1. \tag{13}$$

Next, it can easily be seen that the continuity conditions may be satisfied if one considers the following expression to vanish

* As an illustration of how the local solution may be combined in a particular case, see reference [1].

$$\int_0^\infty (s^4+\lambda^4)^{\frac{1}{2}} P(s) \cos xs\,ds = 0 \; ; \quad \text{for} \quad |x|>1 \,. \tag{14}$$

We have, therefore, reduced our problem to solving the dual integral equation (13), (14) for the unknown function $P(s)$. However, because we are unable to solve dual integral equations of this type, we will reduce the problem to a singular integral equation. If one lets

$$u(x) = \int_0^\infty (s^4+\lambda^4)^{\frac{1}{2}} P(s) \cos xs\,ds \; ; \quad \text{for} \quad |x|<1 \tag{15}$$

Then by Fourier inversion:

$$(s^4+\lambda^4)^{\frac{1}{2}} P(s) = \frac{2}{\pi}\int_0^1 u(\xi)\cos\xi s\,d\xi \,, \tag{16}$$

where the function $u(\xi)$, due to the symmetry of the problem, is even. Thus, formally, substituting (16) into (13) one, after changing the order of integration and rearranging, has

$$\int_{-1}^1 L(\lambda|x-\xi|)\,u(\xi)\,d\xi = -m_0\pi x \quad \text{for} \quad |x|<1 \tag{17}$$

where the kernel L is given by the expression

$$L(\lambda|x-\xi|) \equiv \lim_{|y|\to 0}\int_0^\infty \left\{ \frac{(v_0 s^2+\alpha^2\lambda^2)^2 \exp[-(s^2+\alpha^2\lambda^2)^{\frac{1}{2}}|y|]}{s(s^2+\alpha^2\lambda^2)^{\frac{1}{2}}} + \right.$$

$$\left. - \frac{(v_0 s^2+\beta^2\lambda^2)^2 \exp[-(s^2+\beta^2\lambda^2)^{\frac{1}{2}}|y|]}{s(s^2+\beta^2\lambda^2)^{\frac{1}{2}}} \right\} \sin(x-\xi)s\,ds \tag{18a}$$

and its asymptotic form for small λ's is:

$$L(\lambda|\zeta|) = \frac{(4-v_0)v_0\alpha^2\lambda^2}{\zeta} + \left(\frac{3v_0^2-8v_0+8}{8}\right)\frac{i\pi\lambda^4\zeta}{2} + \left(\frac{73v_0^2-180v_0+128}{9}\right)\frac{\alpha^6\lambda^6\zeta^3}{64} +$$

$$-\left(\frac{5v_0^2-12v_0+8}{3}\right)\frac{\alpha^6\lambda^6\zeta^3}{16}\left(\gamma+\ln\frac{\lambda|\zeta|}{2}\right) + 0\left((\lambda^8\zeta^6 \ln\lambda|\zeta|)\right). \tag{18b}$$

We require that the solution $u(\xi)$ be Hölder continuous for some positive Hölder index μ for all x in the closed interval $[-1, 1]$. In particular $u(\xi)$ is to be bounded near the ends of the crack. A method for constructing such a solution is given in reference (2). Without going into the details, one may find a series expansion of $u(\xi)$ for small values of the parameter λ, i.e.

$$u(\xi) = (1-\xi^2)^{\frac{1}{2}} \sum_{n=0}^{\infty} A_{n+1}\alpha^{2n}\lambda^{2n}(1-\xi^2)^n \tag{19}$$

where

$$(4-v_0)v_0 A_1 = -\frac{m_0}{\alpha^2\lambda^2}\times$$

$$\times\left\{1+\frac{3v_0^2-8v_0+8}{(4-v_0)v_0}\,\frac{\pi\lambda^2}{32}+\frac{\lambda^4}{128}\left[\frac{15v_0^2-36v_0+24}{(4-v_0)v_0}\left(\lambda+\ln\frac{\lambda}{4}\right)+\frac{3v_0^2-6v_0+2}{(4-v_0)v_0}\right]\right\}^{-1} \tag{20a}$$

$$(4-v_0)v_0 A_2 = -\frac{\alpha^2\lambda^2}{64}\left[\frac{11v_0^2-24v_0+12}{9}+\frac{10v_0^2-24v_0+16}{3}\left(\gamma+\ln\frac{\lambda}{4}\right)\right]A_1 \tag{20b}$$

$$(4-v_0)v_0 A_3 = \frac{5v_0^2-12v_0+8}{16\cdot 60}\,A_1 \tag{20c}$$

Finally, using equations (19), (16), (12), (11) and (9) one may derive the following expression for the complementary displacement function $W^{(c)}(x, y)$, i.e.,

$$W^{(c)}(x, y^{\pm}) = \int_0^{\infty} \left\{ \frac{(\nu_0 s^2 - \beta^2\lambda^2)\exp\left[-(s^2+\alpha^2\lambda^2)^{\frac{1}{2}}|y|\right]}{(s^2+\alpha^2\lambda^2)^{\frac{1}{2}}} - \frac{(\nu_0 s^2 - \alpha^2\lambda^2)\exp\left[-(s^2+\beta^2\lambda^2)^{\frac{1}{2}}|y|\right]}{(s^2+\beta^2\lambda^2)^{\frac{1}{2}}} \right\} \times$$

$$\times \left\{ A_1 \frac{J_1(s)}{s} + A_2 \alpha^2\lambda^2 \frac{J_2(s)}{s^2} + \ldots \right\} \cos xs\,ds \tag{21}$$

from which*, the stresses at the surface $z = h/2c$ may be computed as:

$$\sigma_x^{(c)} = \frac{P_b}{(2\varepsilon)^{\frac{1}{2}}} \left(-\frac{3-3\nu}{4} \cos\frac{\theta}{2} - \frac{1-\nu}{4} \cos\frac{5\theta}{2} \right) + 0(\varepsilon^0) \tag{22}$$

$$\sigma_y^{(c)} = \frac{P_b}{(2\varepsilon)^{\frac{1}{2}}} \left(\frac{11+5\nu}{4} \cos\frac{\theta}{2} + \frac{1-\nu}{4} \cos\frac{5\theta}{2} \right) + 0(\varepsilon^0) \tag{23}$$

$$\tau_{xy}^{(c)} = \frac{P_b}{(2\varepsilon)^{\frac{1}{2}}} \left(-\frac{7+\nu}{4} \sin\frac{\theta}{2} - \frac{1-\nu}{4} \sin\frac{5\theta}{2} \right) + 0(\varepsilon^0) \tag{24}$$

where

$$P_b = \frac{\bar{\sigma}_b}{(4-\nu_0)} \left\{ 1 + \frac{3\nu_0^2 - 8\nu_0 + 8}{(4-\nu_0)\nu_0} \frac{\pi\lambda^2}{32} + \frac{\lambda^4}{128} \left[\frac{15\nu_0^2 - 36\nu_0 + 24}{(4-\nu_0)\nu_0} \left(\gamma + \ln\frac{\lambda}{4} \right) + \frac{3\nu_0^2 - 6\nu_0 + 2}{(4-\nu_0)\nu_0} \right] \right\}^{-1} \tag{25}$$

which for $\nu = \frac{1}{3}$ reduces to:

$$P_b = \bar{\sigma}_b \frac{3}{10} \left\{ 1 + \frac{9}{5} \frac{\pi\lambda^2}{32} + \frac{3\lambda^4}{1280} [9.1 + 10 \ln \lambda] \right\}^{-1}. \tag{26}$$

As a consequence of the Kirchhoff boundary condition, the bending shear stress $\tau_{xy}^{(c)}$ does not vanish along the free edge. For the flat sheet this difficulty was discussed by Knowles and Wang who considered Reissner bending theory [4].

Returning to the stresses along the crack prolongation, for example the normal stress $\sigma_y^{(c)}(x, 0)$, one finds using equations (23) and (25) that for small values of the parameter λ

$$\frac{\sigma_y^{(c)}(\varepsilon, 0)}{\bar{\sigma}_b} \approx (2\varepsilon)^{-\frac{1}{2}} \left[1 + \frac{9}{5} \frac{\pi}{32} \lambda^2 \right]^{-1}. \tag{27a}$$

On the other hand, for large values of the parameter λ, reference [1] gives

$$\frac{\sigma_y^{(c)}(\varepsilon, 0)}{\bar{\sigma}_b} \approx (2\varepsilon)^{-\frac{1}{2}} \ \lambda^{-\frac{1}{2}}. \tag{27b}$$

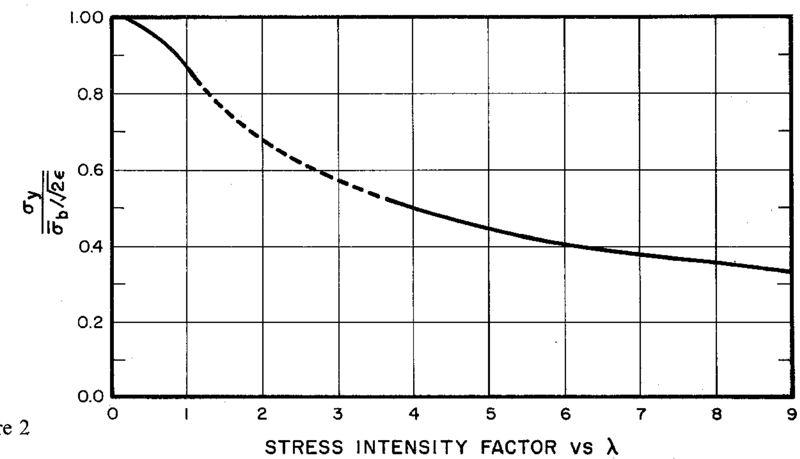

Figure 2

* It can easily be shown that the imitted terms lead to non-singular stress contributions, and therefore, are omitted.

Since the behavior of the stress intensity factor at the two extremes is known, one may construct, as an engineering approximation, a curve with the proper asymptotes. Such a plot is given in figure 2.

A Particular Solution

As an illustration of how the local solution may be combined in a particular case, consider a rectangular strip, infinitely long in the χ-direction and of finite width y^* in the y-direction. Furthermore, let the plate be subjected to a constant moment M^* and zero shear at $y = \pm y^*$, and simultaneously subjected to a uniform normal loading q_0. Reference [1] gives the solution of this problem as

$$w^{(p)}(y) = \frac{q_0}{D\lambda^4} + A \cos\frac{\lambda y}{2^{\frac{1}{2}}} \cosh\frac{\lambda y}{2^{\frac{1}{2}}} + B \sin\frac{\lambda y}{2^{\frac{1}{2}}} \sinh\frac{\lambda y}{2^{\frac{1}{2}}} \tag{28}$$

where the coefficients A and B are given by

$$A = \frac{M^*}{D\lambda^4} \frac{\sin\left(\frac{\partial\lambda^*}{2^{\frac{1}{2}}}\right)\cosh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right) - \cos\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\sinh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)}{\sinh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\cosh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right) + \cos\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\sin\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)}$$

$$B = -\frac{M^*}{D\lambda^2} \frac{\cos\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\sinh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right) + \sin\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\cosh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)}{\cosh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\sinh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right) + \sin\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\cos\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)}$$

from which one may easily deduce that

$$\bar{\sigma}_b \equiv \frac{6Dm_0}{h^2c^2} = -\frac{6D}{h^2}\frac{M^*}{c^2} \frac{\cos\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\sinh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right) + \sin\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\cosh\left(\lambda\frac{y^*}{2^{\frac{1}{2}}}\right)}{\cosh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\sinh\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right) + \sin\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)\cos\left(\frac{\lambda y^*}{2^{\frac{1}{2}}}\right)} \tag{29}$$

Conclusions

The following conclusions may be deduced from the foregoing analysis:

(i) the stresses are proportional to $\varepsilon^{-\frac{1}{2}}$.

(ii) the stresses have the same angular distribution as that of a flat plate.

(iii) the stress intensity factors are functions of the spring constant k and in the limit as $k \to 0$ we recover the unsupported plate. A typical term for small values of the parameter $(k/D)^{\frac{1}{2}} c^2$ is of the form:

$$\frac{\sigma_{\text{sup.}}}{\sigma_{\text{unsup.}}} \approx \left\{1 + a\left(\frac{k}{D}\right)^{\frac{1}{2}} c^2\right\}^{-1} \tag{30}$$

where a is a positive constant. This result indicates that the stresses, and consequently the strains, appear to decrease in magnitude by a factor which depends on the spring constant, the crack length, and the material properties.

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RÉSUMÉ

A l'aide d'une formule d'intégration, on a résolu l'équation d'une plaque reposant sur un support de constante élastique k, comportant une fissure de longueur $2c$, et soumise à des contraintes de flexion de Kirchhoff. On aboutit à une distribution des contraintes dont l'amplitude varie en raison inverse du carré de la distance, comportement singulier propre aux problèmes de fissuration.

En outre, cette singularité peut être mise en balance avec celle que l'on trouve dans une plaque non supportée, par l'expression:

$$\frac{\sigma_{y_{\text{supportée}}}}{\sigma_{y_{\text{non supportée}}}} \approx \frac{1}{1+a\lambda^2}$$

où a est une constante positive pour de petites valeurs du paramètre λ.

ZUSAMMENFASSUNG

Für eine auf einem elastischen Fundament (Elastizitätskonstante k) ruhende Platte, welche einen Riß der Länge $2c$ aufweist und Kirchhoff Biegebeanspruchungen ausgesetzt ist, konnte die Gleichung mit Hilfe einer Integralformel gelöst werden.

Es ergibt sich das den Rißproblemen eigene singulare Gesetz des umgekehrten Verhältnisses zur Quadratwurzle. Außerdem kann diese Singularität mit der für eine nicht gestützte Platte ermittelten über die Gleichung:

$$\frac{\sigma_{y_{\text{gestützt}}}}{\sigma_{y_{\text{ungestützt}}}} = \frac{1}{1+a\lambda^2}$$

in Zusammenhang gebracht werden, wobei a für kleine Werte des Parameters λ eine positive Konstante ist.

Int. Journ. of Fracture Mech., 6 (1970) 257–263