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Complete characterization and synthesis of the response function of elastodynamic networks

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Abstract The response function of a network of springs and masses, an elastodynamic network, is the matrix valued function $\mathbf{W}(\omega)$, depending on the frequency ω, mapping the displacements of some accessible or terminal nodes to the net forces at the terminals. We give necessary and sufficient conditions for a given function $\mathbf{W}(\omega)$ to be the response function of an elastodynamic network, assuming there is no damping. In particular we construct an elastodynamic network that can mimic a suitable response in the frequency or time domain. Our characterization is valid for networks in three dimensions and also for planar networks, which are networks where all the elements, displacements and forces are in a plane. The network we design can fit within an arbitrarily small neighborhood of the convex hull of the terminal nodes, provided the springs and masses occupy an arbitrarily small volume. Additionally, we prove stability of the network response to small changes in the spring constants and/or addition of springs with small spring constants.

Keywords elastic networks · elastodynamic networks · response function · network synthesis

Mathematics Subject Classification (2000) 74B05, 35R02

1 Introduction

Is it possible to design an elastic material that has a prescribed response? This question is answered by Camar-Eddine and Seppecher [4] for linear elastic materials in three dimensions, assuming the macroscopic response is governed by a single displacement field. Their approach consists of three steps. First it is shown how to design a continuum

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material that behaves like an *elastic network* (a network composed of springs). Then the response of elastic networks is characterized, i.e. it is shown how to construct an elastic network with a suitable response. A limiting argument is then used to answer the question for the continuum. As a first step towards solving the characterization problem when the response depends on time, we show how to design an *elastodynamic network* (a network of springs and masses), that can mimic a prescribed response as a function of time (or frequency). Moreover if the springs and masses occupy an arbitrarily small volume, the network can be designed to fit within an arbitrarily small neighborhood of the convex hull of the terminal nodes, which is a requirement for an argument similar to that of Camar-Eddine and Seppecher [4]. An earlier characterization of elastodynamic networks is that of Milton and Seppecher [9]. However the network elements used in the construction [9] are frequency dependent, so the constructed network can only mimic the response function at a single fixed frequency.

In a different context, the approach of Camar-Eddine and Seppecher was applied earlier by the same authors [3] to characterize all possible responses for the conductivity equation, assuming the macroscopic response is governed by a single voltage field. The problem of finding a network with a given response is often called "network synthesis", and the earliest example is Kirchhoff's $Y - \Delta$ theorem, which characterizes the response of any resistor network in three dimensions. Another characterization for resistor networks is that of Curtis, Ingerman and Morrow [5] who consider planar networks that can be embedded inside a disk and where all terminals lie on its boundary. For electrodynamic networks (with resistances, capacitors and inductances), we are only aware of results dealing with the frequency response or impedance of a circuit with two terminals (see Foster [6, 7] and Bott and Duffin [1]). Milton and Seppecher [9] give a construction for $n-$terminal elastodynamic, electrodynamic and acoustic networks which is valid at a single frequency. The electromagnetic analog of elastodynamic networks is considered by the same authors [10, 11]

In §2 we give the properties of the response function of elastic and elastodynamic networks. The construction of a network that matches a response function with the properties in §2 is given for the static case in §3. Note that the characterization of elastic networks by Camar-Eddine and Seppecher [4] is part of a limiting argument on energy functionals, so only non-degenerate three dimensional elastic networks are needed. The degenerate case corresponds to *planar elastic networks* (the network, forces and displacements lie on a plane) and is a set of measure zero which leaves the energy functionals considered in [4] unaffected. We complete the characterization in [4] to include planar elastic networks. Then in §4 we completely characterize the response of elastodynamic networks (planar or in three dimensions) for all frequencies and assuming there is no dissipation (damping) in the network. We include in the appendices two technical results. Appendix A shows that the network response is stable with respect to small changes in the spring constants and the addition (but not deletion) of springs. Appendix B uses stability to give a systematic method of modifying an elastic network to eliminate floppy modes without changing significantly the response. Floppy modes correspond to nodes that can move with zero forces and they are discussed in more length in §2.2.

1.1 Preliminaries

Consider a network composed of springs and masses, and assume we only have access to n "terminal" or "boundary" nodes $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^d$, where the dimension d is either 2 or 3. The network is said to be *planar* if $d = 2$ and the springs do not cross. The static response matrix or displacement-to-forces map is the $nd \times nd$ matrix $\mathbf{W}$ so that

$$\mathbf{f} = \mathbf{W}\mathbf{u},$$

where $\mathbf{u} = (\mathbf{u}_1^T, \dots, \mathbf{u}_n^T)^T$ is the vector of displacements $\mathbf{u}_i$ of the terminal nodes $\mathbf{x}_i$ and $\mathbf{f} = (\mathbf{f}_1^T, \dots, \mathbf{f}_n^T)^T$ is the vector of net forces $\mathbf{f}_i$ acting on node $\mathbf{x}_i$ at equilibrium.

In the dynamic case the displacements $\mathbf{u}(t)$ and $\mathbf{f}(t)$ depend on time t. Let $\widehat{\mathbf{u}}(\omega)$ be the Fourier transform of $\mathbf{u}(t)$,

$$\widehat{\mathbf{u}}(\omega) = \int_{-\infty}^{\infty} \mathbf{u}(t) e^{-i\omega t} dt,$$

and similarly for $\widehat{\mathbf{f}}(\omega)$, where ω is the frequency. Then if ω is not a resonance frequency of the network (a precise definition of resonance is given later in §2.2.2), the response matrix of the network is the possibly complex $nd \times nd$ matrix valued function $\widehat{\mathbf{W}}(\omega)$ such that

$$\widehat{\mathbf{f}}(\omega) = \widehat{\mathbf{W}}(\omega)\widehat{\mathbf{u}}(\omega).$$

For convenience we have chosen to work in the frequency domain. However when $\mathbf{u}(t) = 0$ for $t < 0$, our results can be reformulated for the *transfer function* of the network since $\mathcal{L}[\mathbf{u}(t)](s) = \widehat{\mathbf{u}}(-is)$, where $\mathcal{L}$ denotes the Laplace transform, i.e.

$$\mathcal{L}[\mathbf{u}(t)](s) = \int_0^{\infty} \mathbf{u}(t) e^{-st} dt.$$

In this case the transfer function of the network is $\widehat{\mathbf{W}}(-is)$. As we work only in the frequency domain, we drop the hats in the Fourier transform notation for the sake of clarity (i.e. $\mathbf{u}(\omega) \equiv \widehat{\mathbf{u}}(\omega)$ etc...). Also as there is no dissipation, it suffices to assume that $\mathbf{u}(\omega)$ and $\mathbf{f}(\omega)$ are real to determine the real valued function $\mathbf{W}(\omega)$.

2 The response function of an elastodynamic network

In this section we establish the properties that the response of an elastodynamic network satisfies. We start with the response of networks (static or dynamic) where all the nodes are terminals (§2.1) and then study the case where interior nodes are present (§2.2). We also include some transformations in §2.3 that do not affect the response function.

2.1 Response function for networks without interior nodes

Consider the simple network consisting of two nodes $\mathbf{x}_1$ and $\mathbf{x}_2$ with masses m_1 and m_2, linked with a spring with spring constant $k_{1,2}$. Let $\mathbf{a}_i$ be the force exerted by the spring on node $\mathbf{x}_i$, $i = 1, 2$. By Hooke's law

$$\mathbf{a}_2 = -k_{1,2} \frac{(\mathbf{x}_2 - \mathbf{x}_1)(\mathbf{x}_2 - \mathbf{x}_1)^T}{\|\mathbf{x}_2 - \mathbf{x}_1\|^2} (\mathbf{u}_2 - \mathbf{u}_1) = -\mathbf{a}_1.$$

The laws of motion can be written in matrix form as $-\omega^2\mathbf{M}\mathbf{u} = \mathbf{f} - \mathbf{K}\mathbf{u}$, where

$$\mathbf{K} = k_{1,2}\begin{bmatrix} \mathbf{n}_{1,2}\mathbf{n}_{1,2}^T & -\mathbf{n}_{1,2}\mathbf{n}_{1,2}^T \\ -\mathbf{n}_{1,2}\mathbf{n}_{1,2}^T & \mathbf{n}_{1,2}\mathbf{n}_{1,2}^T \end{bmatrix}, \quad \mathbf{M} = \mathrm{diag}\,(m_1\mathbf{e}, m_2\mathbf{e}),$$
$$\mathbf{n}_{1,2} = \frac{\mathbf{x}_2 - \mathbf{x}_1}{\|\mathbf{x}_2 - \mathbf{x}_1\|},$$

and the vector $\mathbf{e} = (1, \ldots, 1) \in \mathbb{R}^d$ for $d = 2, 3$. Thus the response function of a single spring is given by

$$\mathbf{W}(\omega) = \mathbf{K} - \omega^2\mathbf{M}. \tag{1}$$

When all nodes are terminal nodes (i.e. there are no interior nodes) the response function can also be written in the form (1), but now $\mathbf{K}$ is the stiffness matrix of the network and the mass matrix

$$\mathbf{M} = \mathrm{diag}\,(m_1\mathbf{e}, \ldots, m_n\mathbf{e}), \tag{2}$$

where $m_i \geq 0$ is the mass of the $i-$th node and n is the number of nodes. The stiffness matrix $\mathbf{K}$ of the network is a $n \times n$ matrix of $d \times d$ blocks (i.e. a $nd \times nd$ matrix). The $d \times d$ block $[\mathbf{K}]_{i,j}$ in the $i-$th row and $j-$th column of the stiffness matrix is

$$[\mathbf{K}]_{i,j} = \begin{cases} -k_{i,j}\mathbf{n}_{i,j}\mathbf{n}_{i,j}^T & \text{when } i \neq j, \\ \sum_{l=1, l\neq j}^{n} k_{l,j}\mathbf{n}_{l,j}\mathbf{n}_{l,j}^T & \text{when } i = j. \end{cases} \tag{3}$$

Here $k_{i,j}$ is the stiffness constant of the spring between nodes $\mathbf{x}_i$ and $\mathbf{x}_j$, or zero if there is no spring joining these nodes. The vector $\mathbf{n}_{i,j}$ is a unit vector with direction $\mathbf{x}_i - \mathbf{x}_j$. We only consider non-negative spring stiffnesses $k_{i,j}$. Stiffnesses with a non-zero imaginary part model damping or dissipation of energy in the network and are left for future studies.

2.2 Response function for networks with interior nodes

2.2.1 The static case

The response matrix $\mathbf{W}$ can be obtained from the response matrix $\mathbf{A}$ of the network where all nodes are considered as terminal nodes. The partitioning of the nodes into interior nodes I and terminal (boundary) nodes B induces the following partitioning of $\mathbf{A}$,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{BB} & \mathbf{A}_{BI} \\ \mathbf{A}_{IB} & \mathbf{A}_{II} \end{bmatrix}. \tag{4}$$

Instead of dealing directly with the response matrix $\mathbf{A}$, it is convenient to introduce the quadratic form

$$q_{\mathbf{A}}(\mathbf{u}) = \mathbf{u}^T\mathbf{A}\mathbf{u},$$

which represents twice the total elastic energy stored in the network. In the simple case of a single spring between nodes $\mathbf{x}_1$ and $\mathbf{x}_2$ with spring constant k, the quadratic form is

$$s_{(\mathbf{x}_1,\mathbf{x}_2)}(\mathbf{u}_1, \mathbf{u}_2) = k\left((\mathbf{u}_1 - \mathbf{u}_2) \cdot \frac{\mathbf{x}_1 - \mathbf{x}_2}{\|\mathbf{x}_1 - \mathbf{x}_2\|}\right)^2.$$

We omit the spring constant indices for clarity. When there are more springs $q_{\mathbf{A}}$ is the sum of the quadratic forms for all springs, thus $q_{\mathbf{A}}(\mathbf{u}) \geq 0$.

For general static networks the response matrix is defined indirectly by its quadratic form $q_{\mathbf{W}}$:

$$q_{\mathbf{W}}(\mathbf{u}_B) = \inf_{\mathbf{u}_I} q_{\mathbf{A}}(\mathbf{u}_B, \mathbf{u}_I). \tag{5}$$

By the partitioning (4) we may rewrite

$$q_{\mathbf{A}}(\mathbf{u}_B, \mathbf{u}_I) = \mathbf{u}_B^T \mathbf{A}_{BB} \mathbf{u}_B + 2\mathbf{u}_B^T \mathbf{A}_{BI} \mathbf{u}_I + \mathbf{u}_I^T \mathbf{A}_{II} \mathbf{u}_I.$$

The first order optimality conditions for the minimization (5) are actually the balance of forces at the interior nodes:

$$\mathbf{0} = \nabla_{\mathbf{u}_I} q_{\mathbf{A}}(\mathbf{u}_B, \mathbf{u}_I) = 2\mathbf{A}_{II}\mathbf{u}_I + 2\mathbf{A}_{IB}\mathbf{u}_B.$$

The following lemma shows that for any $\mathbf{u}_B$ it is possible to balance forces at the interior nodes and it implies the minimization (5) has at least a minimizer (since $q_{\mathbf{A}}$ is bounded below). Another way of seeing this lemma is that if there are any "floppy" modes within the interior nodes (i.e. modes that generate displacements with zero forces) then those modes are not coupled to the terminals. Hereinafter $\mathcal{R}(\mathbf{B})$ denotes the range or columnspace of a matrix $\mathbf{B}$.

Lemma 1 *Given the partitioning* (4) *of the response matrix* $\mathbf{A}$ *where all nodes are considered as terminal nodes, we have* $\mathcal{R}(\mathbf{A}_{IB}) \subseteq \mathcal{R}(\mathbf{A}_{II})$.

Proof By reciprocity $\mathbf{A}^T = \mathbf{A}$, thus it is equivalent to prove $\mathcal{N}(\mathbf{A}_{BI}) \supseteq \mathcal{N}(\mathbf{A}_{II})$, where $\mathcal{N}(\mathbf{B})$ denotes the nullspace of a matrix $\mathbf{B}$. Let $\mathbf{u}_I$ be a displacement such that $\mathbf{A}_{II}\mathbf{u}_I = \mathbf{0}$ (i.e. a "floppy" mode). Then

$$\begin{aligned} 0 = \mathbf{u}_I^T \mathbf{A}_{II} \mathbf{u}_I &= \begin{bmatrix} \mathbf{0} \ \mathbf{u}_I^T \end{bmatrix} \mathbf{A} \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_I \end{bmatrix} \\ &= \sum_{\text{springs } i,\, j \in I} s_{(\mathbf{x}_i, \mathbf{x}_j)}(\mathbf{u}_i, \mathbf{u}_j) + \sum_{\text{springs } i \in I, j \in B} k_{i,j} \left(\mathbf{u}_i \cdot \frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|} \right)^2. \end{aligned}$$

Therefore for all nodes $\mathbf{x}_i \in I$ and $\mathbf{x}_j \in B$ that are linked by a spring we must have $\mathbf{u}_i \cdot (\mathbf{x}_i - \mathbf{x}_j) = 0$, which means precisely that $\mathbf{A}_{BI}\mathbf{u}_I = \mathbf{0}$. □

Remark 1 We show later in Appendix B that floppy modes can be eliminated from a network by adding springs with small spring constants. The response of the new network can be made arbitrarily close to that of the original one, provided the new springs have sufficiently small stiffness. Examples of floppy modes are given in Figure 3.

By eliminating the interior nodes, the static response matrix can thus be written in Schur complement form:

$$\mathbf{W} = \mathbf{A}_{BB} - \mathbf{A}_{BI}\mathbf{A}_{II}^{\dagger}\mathbf{A}_{IB}, \tag{6}$$

where † stands for the Moore-Penrose pseudo-inverse, which is is simply the inverse if there are no floppy modes. Recall that one way of defining the pseudo-inverse of a matrix $\mathbf{B}$ is

$$\mathbf{B}^{\dagger} = \lim_{\epsilon \to 0} \mathbf{B}^T (\mathbf{B}\mathbf{B}^T + \epsilon^2 \mathbf{I})^{-1}.$$

We denote by $\mathbf{u} \wedge \mathbf{v}$ the cross product of the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$. For $d = 2$ we have $\mathbf{u} \wedge \mathbf{v} = \det[\mathbf{u}, \mathbf{v}]$ and for $d = 3$, $\mathbf{u} \wedge \mathbf{v} = (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1)^T$. Before reviewing some properties of the static response matrix we need the following definition.

Definition 1 A *balanced system of forces* $\mathbf{f}_i$, $i = 1, \ldots, n$ supported at nodes $\mathbf{x}_i$, $i = 1, \ldots, n$ in $\mathbb{R}^d$ $(d = 2, 3)$ satisfies:

(a) $\sum_{i=1}^{n} \mathbf{f}_i = \mathbf{0}$ (balance of forces)

(b) $\sum_{i=1}^{n} \mathbf{x}_i \wedge \mathbf{f}_i = \mathbf{0}$ (balance of torques)

Lemma 2 *For any stiffness matrix* $\mathbf{A}$ *(of the form* (3)*), the static response matrix* $\mathbf{W}$ *at the terminals (see* (6)*) satisfies the following properties.*

(a) $\mathbf{W} \in \mathbb{R}^{nd \times nd}$.
(b) $\mathbf{W} = \mathbf{W}^T$ *(reciprocity)*
(c) $\mathbf{W}$ *is positive semidefinite (energy is not produced by the network)*
(d) Every column $\mathbf{f} = (\mathbf{f}_1^T, \ldots, \mathbf{f}_n^T)^T$ *of* $\mathbf{W}$ *is a* balanced system of forces *when supported at the nodes* $\mathbf{x}_i$ *in* $\mathbb{R}^d$.

Proof Properties (a), (b) and (d) follow from the construction of the response matrix. We now prove Property (c). Let $\widetilde{\mathbf{W}}$ be the response matrix of a network if all the nodes are considered as terminal nodes. Then for all displacements $\mathbf{u} \in \mathbb{R}^{nd}$ we have

$$\widetilde{q}(\mathbf{u}) = \mathbf{u}^T \widetilde{\mathbf{W}} \mathbf{u} = \sum_{\text{springs } i,j} s_{(\mathbf{x}_i, \mathbf{x}_j)}(\mathbf{u}_i, \mathbf{u}_j) \geq 0,$$

where $s_{(\mathbf{x}_i, \mathbf{x}_j)}(\mathbf{u}, \mathbf{v})$ is the quadratic form associated with the spring between nodes $\mathbf{x}_i$ and $\mathbf{x}_j$. Thus (c) holds for networks where all the nodes are terminals. Using (5) we see that (c) holds for general networks as well. □

2.2.2 The dynamic case

The response function in the dynamic case can be obtained in a similar way as in the static case. First if all the nodes are terminal nodes, the response function $\mathbf{A}(\omega)$ of the network is given by (1). The partitioning of $\mathbf{A}$ induced by the partitioning of the nodes into boundary B and interior I nodes is,

$$\mathbf{A}(\omega) = \begin{bmatrix} \mathbf{K}_{BB} & \mathbf{K}_{BI} \\ \mathbf{K}_{IB} & \mathbf{K}_{II} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M}_{BB} & \\ & \mathbf{M}_{II} \end{bmatrix}.$$

As in the static case we can introduce the quadratic form

$$q_{\mathbf{A}}(\mathbf{u}_B, \mathbf{u}_I; \omega) = \mathbf{u}_B^T (\mathbf{K}_{BB} - \omega^2 \mathbf{M}_{BB}) \mathbf{u}_B + 2 \mathbf{u}_B^T \mathbf{K}_{BI} \mathbf{u}_I + \mathbf{u}_I^T (\mathbf{K}_{II} - \omega^2 \mathbf{M}_{II}) \mathbf{u}_I. \quad (7)$$

Remark 2 Unlike in the static case the quadratic form $q_{\mathbf{A}}(\mathbf{u}_B, \mathbf{u}_I; \omega)$ could be unbounded from below for $\mathbf{u}_B$ fixed. This happens for example if there is a $\mathbf{u}_I$ so that $\mathbf{K}_{II} \mathbf{u}_I \neq \mathbf{0}$ and $\mathbf{M}_{II} \mathbf{u}_I \neq \mathbf{0}$. Then for ω large enough the matrix in the last term of $q_{\mathbf{A}}$ becomes indefinite. Thus we cannot define the response function at the terminals through a minimization principle similar to (5).

The dynamic response function at the terminals is the displacement-to-forces map at the critical point $\nabla_{\mathbf{u}_I} q_{\mathbf{A}}(\mathbf{u}_B, \mathbf{u}_I; \omega) = \mathbf{0}$, if such critical point exists. Because $q_{\mathbf{A}}$ may be unbounded below, this critical point could be a saddle point for the quadratic $q_{\mathbf{A}}$ with $\mathbf{u}_B$ fixed. The frequencies ω for which there is no critical point (i.e. there is some $\mathbf{u}_B$ so that $\nabla_{\mathbf{u}_I} q_{\mathbf{A}}(\mathbf{u}_B, \mathbf{u}_I; \omega) \neq \mathbf{0}$ for all $\mathbf{u}_I$) are important physically and in our derivation and correspond to the *resonance frequencies* of the network.

To give an expression for the dynamic response function we partition the interior nodes into nodes J with positive mass and massless nodes L, so that $I = J \cup L$. Therefore $\mathbf{M}_{JJ}$ is positive definite but $\mathbf{M}_{LL} = \mathbf{0}$.

Lemma 3 *For any $nd \times nd$ stiffness matrix $\mathbf{K}$ (see (3)) and $nd \times nd$ mass matrix $\mathbf{M}$ (see (2)), the response function at the terminals is*

$$\mathbf{W}(\omega) = \widetilde{\mathbf{K}}_{BB} - \omega^2 \mathbf{M}_{BB} - \widetilde{\mathbf{K}}_{BJ}(\widetilde{\mathbf{K}}_{JJ} - \omega^2 \mathbf{M}_{JJ})^{-1} \widetilde{\mathbf{K}}_{JB}, \tag{8}$$

provided that ω^2 is not an eigenvalue of $\mathbf{M}_{JJ}^{-1/2} \widetilde{\mathbf{K}}_{JJ} \mathbf{M}_{JJ}^{-1/2}$. Here we have used the submatrices of the matrix

$$\widetilde{\mathbf{K}} = \begin{bmatrix} \widetilde{\mathbf{K}}_{BB} & \widetilde{\mathbf{K}}_{BJ} \\ \widetilde{\mathbf{K}}_{JB} & \widetilde{\mathbf{K}}_{JJ} \end{bmatrix} = \begin{bmatrix} \mathbf{K}_{BB} & \mathbf{K}_{BJ} \\ \mathbf{K}_{JB} & \mathbf{K}_{JJ} \end{bmatrix} - \begin{bmatrix} \mathbf{K}_{BL} \\ \mathbf{K}_{JL} \end{bmatrix} \mathbf{K}_{LL}^{\dagger} \begin{bmatrix} \mathbf{K}_{LB} & \mathbf{K}_{LJ} \end{bmatrix}. \tag{9}$$

Proof The matrix $\widetilde{\mathbf{K}}$ is the response matrix for the network with terminals $B \cup J$ and interior nodes L and can be obtained from (6). Since the nodes L are massless the dynamic response at the nodes $B \cup J$ is $\widetilde{\mathbf{K}} - \omega^2 \mathrm{diag}\,(\mathbf{M}_{BB}, \mathbf{M}_{JJ})$. Since $\mathbf{M}_{JJ}$ is non-singular, the matrix $\widetilde{\mathbf{K}}_{JJ} - \omega^2 \mathbf{M}_{JJ}$ is singular if and only if ω^2 is an eigenvalue of $\mathbf{M}_{JJ}^{-1/2} \widetilde{\mathbf{K}}_{JJ} \mathbf{M}_{JJ}^{-1/2}$. Thus when ω^2 is not an eigenvalue of $\mathbf{M}_{JJ}^{-1/2} \widetilde{\mathbf{K}}_{JJ} \mathbf{M}_{JJ}^{-1/2}$, we can equilibrate forces at the nodes J and get the expression for the response function. □

A corollary of Lemma 3 is that if ω is a resonance frequency of the network then ω^2 must be an eigenvalue of the matrix $\mathbf{M}_{JJ}^{-1/2} \widetilde{\mathbf{K}}_{JJ} \mathbf{M}_{JJ}^{-1/2}$. The expression for the response function in Lemma 3 leads to the following properties.

Lemma 4 *The response function $\mathbf{W}(\omega)$ of any network of springs and masses with n terminals is of the form*

$$\mathbf{W}(\omega) = \mathbf{A} - \omega^2 \mathbf{M} + \sum_{i=1}^{p} \frac{\mathbf{C}^{(i)}}{\omega^2 - \omega_i^2} \in \mathbb{R}^{nd \times nd}, \tag{10}$$

where the matrix $\mathbf{M} = \mathrm{diag}\,(m_1 \mathbf{e}, \ldots, m_n \mathbf{e})$ is real diagonal with the masses of the boundary nodes in the diagonal, the vector $\mathbf{e} = [1, \ldots, 1] \in \mathbb{R}^d$, the matrices $\mathbf{C}^{(i)}$ are real symmetric positive semidefinite, and the static response

$$\mathbf{W}(0) = \mathbf{A} - \sum_{i=1}^{p} \omega_i^{-2} \mathbf{C}^{(i)},$$

is real symmetric positive semidefinite and balanced (i.e. it satisfies the conditions (a)–(d) of Lemma 2). The resonant frequencies are distinct, finite and satisfy $\omega_i^2 > 0$.

Proof Let $I = J \cup L$ be a partition of the interior nodes into massless nodes L and nodes with positive mass J, and let $\widetilde{\mathbf{K}}$ be defined as in (9). By Lemma 3 the response function at the terminals can be rewritten as

$$\mathbf{W}(\omega) = \widetilde{\mathbf{K}}_{BB} - \omega^2\mathbf{M}_{BB} - \widetilde{\mathbf{K}}_{BJ}\mathbf{M}_{JJ}^{-1/2}(\mathbf{C} - \omega^2\mathbf{I})^{-1}\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JB},$$

where $\mathbf{C} = \mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JJ}\mathbf{M}_{JJ}^{-1/2}$ and ω^2 is not an eigenvalue of $\mathbf{C}$. The matrix $\mathbf{C}$ is symmetric positive semidefinite because $\widetilde{\mathbf{K}}$ is a symmetric positive semidefinite matrix (Lemma 2). Let $\{\omega_j^2\}_{j=1}^N$ be the (nonnegative) eigenvalues of $\mathbf{C}$ and $\{\mathbf{c}_j\}_{j=1}^N$ be a corresponding orthonormal basis of eigenvectors of $\mathbf{C}$, where $N = |J|$. When $\omega \neq 0$ and $\omega^2 \neq \omega_j^2$, the response function $\mathbf{W}(\omega)$ becomes

$$\mathbf{W}(\omega) = \widetilde{\mathbf{K}}_{BB} - \omega^2\mathbf{M}_{BB} + \sum_{j=1}^{N} \frac{\widetilde{\mathbf{c}}_j\widetilde{\mathbf{c}}_j^T}{\omega^2 - \omega_j^2}, \tag{11}$$

with $\widetilde{\mathbf{c}}_j = \widetilde{\mathbf{K}}_{BJ}\mathbf{M}_{JJ}^{-1/2}\mathbf{c}_j$, $j = 1, \ldots, N$. Let $r = \operatorname{rank}(\mathbf{C}) = \operatorname{rank}(\widetilde{\mathbf{K}}_{JJ})$ and assume the eigenvalues are ordered such that $\omega_j^2 > 0$ for $j = 1, \ldots, r$. Clearly $\widetilde{\mathbf{K}}_{JJ}\mathbf{M}_{JJ}^{-1/2}\mathbf{z} = \mathbf{0}$ if and only if $\mathbf{C}\mathbf{z} = \mathbf{0}$. Thus $\mathbf{M}_{JJ}^{-1/2}\mathbf{c}_j \in \mathcal{N}(\widetilde{\mathbf{K}}_{JJ})$, for $j = r+1, \ldots, N$. By Lemma 1, we have $\mathcal{N}(\widetilde{\mathbf{K}}_{JJ}) \subseteq \mathcal{N}(\widetilde{\mathbf{K}}_{JB})$ which means that

$$\widetilde{\mathbf{c}}_j = \mathbf{0} \text{ for } j = r+1, \ldots, N.$$

In other words, only the first r terms of the sum in (11) are nonzero. We obtain the form (10) of the response function from (11) by setting $\mathbf{A} = \widetilde{\mathbf{K}}_{BB}$ and $\mathbf{M} = \mathbf{M}_{BB}$. The matrices $\mathbf{C}^{(i)}$ are the sum of the matrices $\widetilde{\mathbf{c}}_j\widetilde{\mathbf{c}}_j^T$ that correspond to the same resonance ω_i^2, thus the $\mathbf{C}^{(i)}$ must be real symmetric positive semidefinite.

We now show that $\mathbf{W}(0) = \widetilde{\mathbf{K}}_{BB} - \widetilde{\mathbf{K}}_{BJ}\widetilde{\mathbf{K}}_{JJ}^\dagger\widetilde{\mathbf{K}}_{JB}$, i.e. at $\omega = 0$ the dynamic response function is the static response of the network. Then the properties of $\mathbf{W}(0)$ follow from Lemma 2. First note that from (11),

$$\mathbf{W}(0) = \widetilde{\mathbf{K}}_{BB} - \widetilde{\mathbf{K}}_{BJ}\mathbf{M}_{JJ}^{-1/2}\mathbf{C}^\dagger\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JB}$$

where we used that

$$\mathbf{C}^\dagger = \sum_{j=1}^{r} \omega_j^{-2}\mathbf{c}_j\mathbf{c}_j^T.$$

It is sufficient to show that $\mathbf{u}_J = -\mathbf{M}_{JJ}^{-1/2}\mathbf{C}^\dagger\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JB}\mathbf{u}_B$ equilibrates the forces at the interior nodes for any terminal displacements $\mathbf{u}_B$. Indeed we have

$$\mathbf{C}\mathbf{M}_{JJ}^{1/2}\mathbf{u}_J = (\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JJ}\mathbf{M}_{JJ}^{-1/2})\mathbf{M}_{JJ}^{1/2}\mathbf{u}_J = -\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JB}\mathbf{u}_B, \tag{12}$$

since Lemma 1 and $\mathbf{M}_{JJ}$ invertible imply

$$\mathcal{R}(\mathbf{C}) = \mathcal{R}(\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JJ}) \supseteq \mathcal{R}(\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JB}).$$

The balance of forces at the nodes J (i.e. $\widetilde{\mathbf{K}}_{JJ}\mathbf{u}_J = -\widetilde{\mathbf{K}}_{JB}\mathbf{u}_B$) comes from multiplying (12) by $\mathbf{M}_{JJ}^{1/2}$ on the left. □

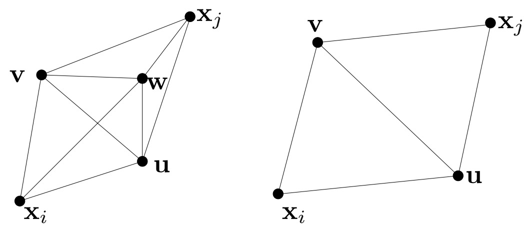

Fig. 1 Truss structure replacing a spring between nodes $\mathbf{x}_i$ and $\mathbf{x}_j$ without changing the response function. This structure can be used to avoid a line for (a) networks in $\mathbb{R}^3$ or (b) planar networks.

2.3 Network transformations not affecting the response function

We need a few elementary transformations that allow more flexibility with the placement of springs in a network. We assume throughout this text that the springs occupy an arbitrarily small volume and that the nodes are points which may or may not have a mass attached to them.

2.3.1 Avoiding a line

It is possible to transform a spring in order to avoid a line or a crossing (for networks in $\mathbb{R}^3$). The construction for networks $\mathbb{R}^3$ is given in [9, Example 3.2] and consists of replacing the spring by a *simple truss* as shown in Figure 1(a). A similar construction can be done for planar networks, see Figure 1(b). If a network in $\mathbb{R}^3$ has springs crossing, these can be eliminated via this transformation since the position of the additional interior nodes $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is not fixed. Moreover, the additional nodes in the truss structures can be chosen to avoid a finite number of points.

2.3.2 Virtual crossings

A network with all springs in $\mathbb{R}^2$ is not necessarily planar because its springs may cross. However [9, Example 3.15] shows how to replace such a crossing by a planar network with exactly the same response function. This transformation involves adding a node at the crossing point of the springs and carefully choosing the spring constants. To avoid a finite number of points one can first replace one of the springs by a simple truss as in Figure 1(b) and use virtual crossings to transform the network into a planar network.

2.4 The superposition principle

A fundamental tool for our construction of a network reproducing the response function is the following result, which is valid for both planar and $\mathbb{R}^3$ networks.

Lemma 5 *Let $\mathbf{W}_1$ and $\mathbf{W}_2$ be the response matrices of two networks (planar or in $\mathbb{R}^3$, static or dynamic) sharing the same terminals but with no interior nodes in common. Then the response function of both networks together is $\mathbf{W}_1 + \mathbf{W}_2$.*

Proof The result follows from the reasoning in [9, Remark 3.9] and the frequency independent transformations in §2.3. For the planar case any crossing can be eliminated using [9, Example 3.15]. Note that the transformations in §2.3 allow one to avoid a finite number of locations (except the terminals). □

3 Characterization of the static response

Building upon the seminal work of Camar-Eddine and Seppecher [4], we give necessary and sufficient conditions for a function to be the response function for either a planar network or a network in $\mathbb{R}^3$. The sufficiency is proved constructively and relies on the existence of networks that have rank one response matrices, as is described in detail in the remaining part of §3.

Recall that the ϵ-neighborhood $\mathbf{C}_\epsilon$ of a set $\mathbf{C}$ is the set,

$$\mathbf{C}_\epsilon = \{\mathbf{x} \in \mathbb{R}^2 \mid \text{dist}(\mathbf{x}, \mathbf{C}) \le \epsilon\}. \tag{13}$$

If the set $\mathbf{C}$ is convex then the set $\mathbf{C}_\epsilon$ is also convex because of the convexity of the function $\text{dist}(\mathbf{x}, \mathbf{C})$ for convex $\mathbf{C}$ (see e.g. [2, §3.2.5]).

Theorem 1 *For any choice of terminal node positions, any real symmetric positive semidefinite and balanced $nd \times nd$ matrix $\mathbf{W}$ (i.e. a matrix with the properties of Lemma 2) is the response matrix of a purely elastic network which is either planar or in $\mathbb{R}^3$. Moreover, any internal nodes in the construction can be chosen within an $\epsilon-$neighborhood of the convex hull of the terminals, and avoiding a finite number of positions.*

Proof By properties (a)–(c) in Lemma 2 the matrix $\mathbf{W}$ can be written as a sum of rank one matrices $\mathbf{W}_i$:

$$\mathbf{W} = \sum_{i=1}^{n} \mathbf{W}_i,$$

where $\mathbf{W}_i = \lambda_i \mathbf{w}_i \mathbf{w}_i^T$ and $(\lambda_i, \mathbf{w}_i)$ is an eigenpair of $\mathbf{W}$, $\lambda_i \ge 0$, for $i = 1, \dots, n$. Each $\mathbf{W}_i$ satisfies properties (a)–(d) in Lemma 2. Properties (a)–(c) are easy to check for $\mathbf{W}$ and (d) follows by linearity, since it holds for each $\mathbf{W}_i$. Owing to Theorem 3 (Theorem 2 in the planar case), it is possible to construct a network with matrix response equal to $\mathbf{W}_i$. By the superposition principle we obtain a general network with response $\mathbf{W}$. If the desired network is planar, then every crossing between springs can be transformed to a planar network through a truss-like structure [9, Examples 3.2, 3.15]. As discussed in §2.3, such transformations can be chosen to avoid a finite number of points in $\mathbb{R}^d$. □

Remark 3 If $\mathbf{W}$ is the static response matrix of a network and α is a positive constant then clearly $\alpha\mathbf{W}$ is the response matrix of the same spring network, but where all the spring constants are multiplied by α.

3.1 Planar networks with rank one static response matrices

The main result in this section is Theorem 2 which is the statement of Theorem 1 for rank one response matrices, i.e. it shows that for any rank one response matrix satisfying Lemma 2 it is possible to find a planar network that realizes it. We first prove Theorem 2 for three terminal networks in §3.1.1, then for four and more terminals in §3.1.2.

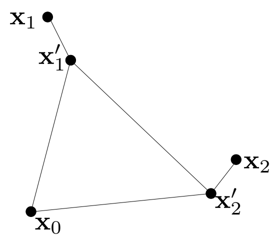

Fig. 2 A planar network with rank one response. Any resulting force at terminals is proportional to $(\mathbf{f}_0^T, \mathbf{f}_1^T, \mathbf{f}_2^T)^T$.

3.1.1 Three terminal rank one static planar networks

We show how to construct a three terminal planar elastic network realizing any valid rank one response matrix. If $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2 \in \mathbb{R}^2$ is the balanced system of forces at the nodes $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$, the construction depends on rank $[\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0]$. More precisely Lemma 7 corresponds to the case when this rank is two and Lemma 8 when this rank is one. Since we are in $\mathbb{R}^2$ these are the only non-trivial cases available, which shows Theorem 2 for planar three terminal networks.

Remark 4 (Pierre Seppecher, private communication) The easiest way to construct a three terminal rank one network is to add a node at the intersection of the force lines (three forces that are balanced meet at a single point in 2D, this can be shown by writing the torque balance equation for the intersection point). The only problem with this construction is that the extra node can end up far away if the force lines are almost parallel.

We start with the following intermediate result. A similar result is shown in three dimensions by Camar-Eddine and Seppecher [4, Lemma 5].

Lemma 6 *Let* $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{f}_2$ *be a set of balanced forces at the nodes* $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ *in* $\mathbb{R}^2$. *Then if* rank $[\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0] = 2$, *there is an* $\epsilon > 0$ *such that the points* $\mathbf{x}_0$, $\mathbf{x}_1' = \mathbf{x}_1 + \epsilon \mathbf{f}_1$ *and* $\mathbf{x}_2' = \mathbf{x}_2 + \epsilon \mathbf{f}_2$ *are not collinear. Moreover* ϵ *can be chosen arbitrarily small and so that* $\mathbf{x}_1'$ *and* $\mathbf{x}_2'$ *do not coincide with a finite number of points.*

Proof If it were true that for all $\epsilon > 0$ the three points $\mathbf{x}_0$, $\mathbf{x}_1'$ and $\mathbf{x}_2'$ are collinear, then the second degree polynomial in ϵ, $p(\epsilon) = \det(\mathbf{x}_1' - \mathbf{x}_0, \mathbf{x}_2' - \mathbf{x}_0)$ is identically zero. That the constant coefficient of $p(\epsilon)$ vanishes means that $\det(\mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0) = 0$, or that the points $\mathbf{x}_0$, $\mathbf{x}_1$ and $\mathbf{x}_2$ are collinear. Since the lemma for $\mathbf{x}_0 = \mathbf{x}_1 = \mathbf{x}_2$ is trivial to prove, we may assume without loss of generality that there is some $\alpha \in \mathbb{R}$ such that

$$\mathbf{x}_2 - \mathbf{x}_0 = \alpha(\mathbf{x}_1 - \mathbf{x}_0), \tag{14}$$

swapping the indices 1 and 2 if necessary. Since the coefficient in ϵ of $p(\epsilon)$ vanishes we get $\det(\mathbf{f}_1, \mathbf{x}_2 - \mathbf{x}_0) + \det(\mathbf{x}_1 - \mathbf{x}_0, \mathbf{f}_2) = 0$ or equivalently $\det(\mathbf{x}_1 - \mathbf{x}_0, -\alpha\mathbf{f}_1 + \mathbf{f}_2) = 0$. Now the torque balance implies that

$$(\mathbf{x}_1 - \mathbf{x}_0) \wedge (\mathbf{f}_1 + \alpha \mathbf{f}_2) = 0 \tag{15}$$

Putting both (14) and (15) in matrix form, there are some real β and γ such that

$$\begin{bmatrix} \mathbf{I} & \alpha\mathbf{I} \\ -\alpha\mathbf{I} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix} = \begin{bmatrix} \beta(\mathbf{x}_1 - \mathbf{x}_0) \\ \gamma(\mathbf{x}_1 - \mathbf{x}_0) \end{bmatrix}.$$

The determinant of the matrix above is $(\alpha^2+1)^2 \neq 0$, thus $\operatorname{rank}[\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0] = 1$ which contradicts the hypothesis of the lemma. Finally since $p(\epsilon)$ is not identically zero, one can choose an arbitrarily small ϵ that avoids a finite number of points.
□

Lemma 7 *Let $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{f}_2$ be a set of balanced forces at the nodes $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbb{R}^2$. If $\operatorname{rank}[\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0] = 2$ then there exists a purely elastic planar network with force response proportional to $\mathbf{f} = (\mathbf{f}_0^T, \mathbf{f}_1^T, \mathbf{f}_2^T)^T$, or in other words a (rank one) response matrix proportional to $\mathbf{f}\mathbf{f}^T$. The internal nodes of such a network can be chosen to avoid a finite number of points and within an $\epsilon-$neighborhood of the convex hull of the terminals.*

Proof First observe the hypothesis that $\operatorname{rank}[\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0] = 2$ implies the existence of at least one permutation σ of $\{0, 1, 2\}$ such that

$$\mathbf{f}_{\sigma(1)} \wedge (\mathbf{x}_{\sigma(1)} - \mathbf{x}_{\sigma(0)}) \neq 0 \tag{16}$$

Indeed, if (16) is false for all permutations σ, i.e.,

$$\begin{aligned} \mathbf{f}_1 \wedge (\mathbf{x}_1 - \mathbf{x}_0) = 0 = \mathbf{f}_2 \wedge (\mathbf{x}_2 - \mathbf{x}_0), \\ \mathbf{f}_2 \wedge (\mathbf{x}_2 - \mathbf{x}_1) = 0 = \mathbf{f}_0 \wedge (\mathbf{x}_0 - \mathbf{x}_1), \\ \mathbf{f}_0 \wedge (\mathbf{x}_0 - \mathbf{x}_2) = 0 = \mathbf{f}_1 \wedge (\mathbf{x}_1 - \mathbf{x}_2) \end{aligned}$$

we have that $\operatorname{rank}[\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0] = 1$ and this contradicts our initial hypothesis. So, without loss of generality we can assume that,

$$\mathbf{f}_1 \wedge (\mathbf{x}_1 - \mathbf{x}_0) \neq 0. \tag{17}$$

Next, let $\mathbf{x}'_i = \mathbf{x}_i + \epsilon\mathbf{f}_i$, for $i = 1, 2$. By Lemma 6, if $\operatorname{rank}[\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0] = 2$, there is $\epsilon > 0$ such that the points $\mathbf{x}_0$, $\mathbf{x}'_1$ and $\mathbf{x}'_2$ are not collinear and do not coincide with a finite number of points. Consider the network in Figure 2 (the spring constants are irrelevant for this proof). Let $\mathbf{A}$ be the response matrix for the network including both terminal ($B = \{0, 1, 2\}$) and interior ($I = \{1', 2'\}$) nodes. The associated quadratic form is

$$q_{\mathbf{A}}(\mathbf{u}_B, \mathbf{u}_I) = \sum_{i=1}^{2} s_{(\mathbf{x}_0, \mathbf{x}'_i)}(\mathbf{u}_0, \mathbf{u}'_i) + \sum_{i=1}^{2} s_{(\mathbf{x}_i, \mathbf{x}'_i)}(\mathbf{u}_i, \mathbf{u}'_i) + s_{(\mathbf{x}'_1, \mathbf{x}'_2)}(\mathbf{u}'_1, \mathbf{u}'_2). \tag{18}$$

It suffices to show that the quadratic form $q_{\mathbf{M}}(\mathbf{u}_B)$ at the terminal nodes has codimension one (since $\mathbf{M}$ is positive semidefinite, we do have $\ker \mathbf{M} = \ker q_{\mathbf{M}}$). Actually $\mathbf{u}_B = (\mathbf{u}_0^T, \mathbf{u}_1^T, \mathbf{u}_2^T)^T \in \ker q_{\mathbf{M}}$ if and only if there exists $\mathbf{u}_I = (\mathbf{u}_1'^T, \mathbf{u}_2'^T)^T$ such that all terms in the sum (18) vanish or equivalently,

$$(\mathbf{u}'_i - \mathbf{u}_0)^T (\mathbf{x}'_i - \mathbf{x}_0) = 0, \text{ for } i = 1, 2, \tag{19}$$

$$(\mathbf{u}'_i - \mathbf{u}_i)^T \mathbf{f}_i = 0, \text{ for } i = 1, 2, \tag{20}$$

$$(\mathbf{u}'_2 - \mathbf{u}'_1)^T (\mathbf{x}'_2 - \mathbf{x}'_1) = 0. \tag{21}$$

Property (19) is equivalent to

$$\mathbf{u}_1' - \mathbf{u}_0 = a\mathbf{R}_\perp(\mathbf{x}_1' - \mathbf{x}_0)$$
$$\mathbf{u}_2' - \mathbf{u}_0 = b\mathbf{R}_\perp(\mathbf{x}_2' - \mathbf{x}_0)$$

for some a, b reals and where

$$\mathbf{R}_\perp = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Property (21) implies that $b = a$, i.e. the only infinitesimal deformation which does not change the side lengths of the triangle in Figure 2 is an infinitesimal rigid motion (translation plus rotation). Since $\mathbf{f}_i \wedge \mathbf{x}_i' = \mathbf{f}_i \wedge \mathbf{x}_i$ and the forces and torques are balanced we conclude from (20) that

$$\sum_{i=1}^{2} \mathbf{f}_i \cdot (\mathbf{u}_i - \mathbf{u}_0) = a \sum_{i=1}^{2} \mathbf{f}_i \wedge (\mathbf{x}_i' - \mathbf{x}_0) = a \sum_{i=0}^{2} \mathbf{f}_i \wedge \mathbf{x}_i = 0.$$

Thus $\ker q_{\mathbf{M}} \subseteq \operatorname{span}\{(\mathbf{f}_0^T, \mathbf{f}_1^T, \mathbf{f}_2^T)^T\}^\perp$.
To prove the other side of the inclusion we start with $\mathbf{u}_i$, $i = 0, 1, 2$ satisfying

$$\sum_{i=1}^{2} \mathbf{f}_i \cdot (\mathbf{u}_i - \mathbf{u}_0) = 0 \tag{22}$$

and we seek $\mathbf{u}_i'$, $i = 1, 2$ such that (19), (20) and (21) hold. For any real a, the choice

$$\mathbf{u}_1' = \mathbf{u}_0 + a\mathbf{R}_\perp(\mathbf{x}_1' - \mathbf{x}_0) \quad \text{and} \quad \mathbf{u}_2' = \mathbf{u}_0 + a\mathbf{R}_\perp(\mathbf{x}_2' - \mathbf{x}_0),$$

satisfies (19) and (21). Next, note that from the definition of the points $\mathbf{x}_1', \mathbf{x}_2'$ we have,

$$(\mathbf{x}_i' - \mathbf{x}_0) \wedge \mathbf{f}_i = (\mathbf{x}_i - \mathbf{x}_0) \wedge \mathbf{f}_i, \text{ for } i = 1, 2.$$

Using the latter, the balance of forces and (17), property (20) follows by taking

$$a = \frac{\mathbf{f}_1 \cdot (\mathbf{u}_1 - \mathbf{u}_0)}{\mathbf{f}_1 \wedge (\mathbf{x}_1 - \mathbf{x}_0)} = \frac{\mathbf{f}_2 \cdot (\mathbf{u}_2 - \mathbf{u}_0)}{\mathbf{f}_2 \wedge (\mathbf{x}_2 - \mathbf{x}_0)}.$$

This proves that $\ker q_{\mathbf{M}} = \operatorname{span}\{(\mathbf{f}_0^T, \mathbf{f}_1^T, \mathbf{f}_2^T)^T\}^\perp$, so $q_{\mathbf{M}}$ can be written for some $c > 0$ as,

$$q_{\mathbf{M}}(\mathbf{u}_B) = c \left(\sum_{i=0}^{2} \mathbf{f}_i \cdot \mathbf{u}_i \right)^2.$$

□

Lemma 8 *Let* $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{f}_2$ *be a set of balanced forces at the nodes* $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$ *in* $\mathbb{R}^2$. *If* $\operatorname{rank}[\mathbf{f}_1, \mathbf{f}_2, \mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0] = 1$ *then there exists a purely elastic planar network with force response proportional to* $(\mathbf{f}_0^T, \mathbf{f}_1^T, \mathbf{f}_2^T)^T$. *The internal nodes of such a network can be chosen to avoid a finite number of points and within an* $\epsilon-$*neighborhood of the convex hull of the terminals.*

Proof We build up on the idea presented in [4, Theorem 5]. It can be easily observed (by choosing a coordinate system with $\mathbf{x}_0$ as origin) that for any point $\mathbf{y}$ in the plane, not collinear with $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2$, there is a force $\mathbf{f}$, such that the following families,

$$((\mathbf{f}, \mathbf{y}), (\mathbf{f}_0 + \mathbf{f}_2 - \mathbf{f}, \mathbf{x}_0), (\mathbf{f}_1, \mathbf{x}_1)) \text{ and } ((-\mathbf{f}, \mathbf{y}), (\mathbf{f} - \mathbf{f}_2, \mathbf{x}_0), (\mathbf{f}_2, \mathbf{x}_2)), \tag{23}$$

form balanced systems of forces. Then, by using Lemma 7 there exists purely elastic networks with response matrices proportional with $(\mathbf{f}^T, \mathbf{f}_0^T + \mathbf{f}_2^T - \mathbf{f}^T, \mathbf{f}_1^T)^T$, and $(-\mathbf{f}^T, \mathbf{f}^T - \mathbf{f}_2^T, \mathbf{f}_2^T)^T$ respectively. It is possible to choose the spring constants so that the associated quadratic forms are (see Remark 3),

$$\begin{aligned} &q'(\mathbf{v}, \mathbf{u}_0, \mathbf{u}_1) = (\mathbf{f} \cdot \mathbf{v} + (\mathbf{f}_2 - \mathbf{f}) \cdot \mathbf{u}_0 + \mathbf{f}_0 \cdot \mathbf{u}_0 + \mathbf{f}_1 \cdot \mathbf{u}_1)^2 \text{ and} \\ &q''(\mathbf{v}, \mathbf{u}_0, \mathbf{u}_2) = (-\mathbf{f} \cdot \mathbf{v} + (\mathbf{f} - \mathbf{f}_2) \cdot \mathbf{u}_0 + \mathbf{f}_2 \cdot \mathbf{u}_2)^2. \end{aligned} \tag{24}$$

Now, let us consider the infimum,

$$\widetilde{q}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2) = \inf_{\mathbf{v} \in \mathbb{R}^2} \{q'(\mathbf{v}, \mathbf{u}_0, \mathbf{u}_1) + q''(\mathbf{v}, \mathbf{u}_0, \mathbf{u}_2)\}. \tag{25}$$

The necessary condition of the infimum,

$$\mathbf{f} \cdot \mathbf{v} = -\frac{1}{2}\left((\mathbf{f}_2 - \mathbf{f}) \cdot \mathbf{u}_0 + \mathbf{f}_0 \cdot \mathbf{u}_0 + \mathbf{f}_1 \cdot \mathbf{u}_1\right) - \frac{1}{2}\left((\mathbf{f}_2 - \mathbf{f}) \cdot \mathbf{u}_0 - \mathbf{f}_2 \cdot \mathbf{u}_2\right), \tag{26}$$

implies the statement of the Lemma, i.e.,

$$\widetilde{q}(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2) = \frac{1}{2}\left(\sum_{i=0}^{2} \mathbf{f}_i \cdot \mathbf{u}_i\right)^2. \tag{27}$$

The point $\mathbf{y}$ and any additional points in Lemma 7 can be chosen to avoid a finite number of points in the plane, and within an $\epsilon-$neighborhood of the convex hull of the terminals. □

3.1.2 General rank one planar networks

We show in Lemma 10 the construction of a network realizing a valid four terminal rank one response and then generalize the result to any number of terminals in Theorem 2. We start our argument with Lemma 9 which is a technical result needed later in this section.

Lemma 9 *Let $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ and $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ be a balanced system of forces in $\mathbb{R}^2$. Then there exists a point $\mathbf{y}_*$ in an ϵ-neighborhood of the convex hull of the set $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ such that $\mathbf{y}_* \notin \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2\}$ and*

$$\mathbf{f}_i \wedge \mathbf{x}_i + \mathbf{f}_j \wedge \mathbf{x}_j - (\mathbf{f}_i + \mathbf{f}_j) \wedge \mathbf{y}_* = 0, \textit{ for some } i, j \in \{0, 1, 2, 3\}, \ i \neq j,$$

or in other words, the three forces $\mathbf{f}_i$, $\mathbf{f}_j$ and $-(\mathbf{f}_i + \mathbf{f}_j)$ supported at the nodes $\mathbf{x}_i$, $\mathbf{x}_j$ and $\mathbf{y}_*$ are balanced. The point $\mathbf{y}_*$ can be chosen to avoid a finite number of positions.

Proof For any pair of indices $\{i, j\}$, with $i \neq j$ and $i, j \in \{0, 1, 2, 3\}$, let $\mathbf{f}_{ij} : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$\mathbf{f}_{ij}(\mathbf{y}) = \mathbf{f}_i \wedge \mathbf{x}_i + \mathbf{f}_j \wedge \mathbf{x}_j - (\mathbf{f}_i + \mathbf{f}_j) \wedge \mathbf{y}. \tag{28}$$

Using the balance of forces relations it can be easily observed that

$$\mathbf{f}_{ij} = \mathbf{f}_{ji} = -\mathbf{f}_{kt} = -\mathbf{f}_{tk}, \text{ for any } \{i, j, k, t\} = \{0, 1, 2, 3\}. \tag{29}$$

Let $\mathbf{C}$ be the convex hull of $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, that is

$$\mathbf{C} = \left\{ \mathbf{z} \in \mathbb{R}^2 \mid \mathbf{z} = \sum_{i=0}^{3} c_i \mathbf{x}_i, \text{ for } c_i \geq 0 \text{ and } \sum_{i=0}^{3} c_i = 1 \right\},$$

and $\mathbf{C}_\epsilon$ be the ϵ-neighborhood of the set $\mathbf{C}$ as defined in (13).

Next we show that there exists a pair of indices $\{i, j\}$ with $i, j \in \{0, 1, 2, 3\}$, with the property that there exists a point $\mathbf{y}_* \in \mathbf{C}_\epsilon$, such that $\mathbf{f}_{ij}(\mathbf{y}_*) = 0$. We reason by contradiction. Assume that the above is not true, i.e.,

For all pairs $\{i, j\}$, with $i \neq j$ and $i, j \in \{0, 1, 2, 3\}$ we have

$$\mathbf{f}_{ij}(\mathbf{y}) \neq 0, \text{ for any } \mathbf{y} \in \mathbf{C}_\epsilon. \tag{30}$$

Using the continuity of the functions $\mathbf{f}_{ij}$ and the convexity of the set $\mathbf{C}_\epsilon$, from (30) we obtain that all the functions $\mathbf{f}_{ij}$ have constant strictly positive or strictly negative sign over $\mathbf{C}_\epsilon$. Using this observation, together with the relations (29), for any partition $\{i, j\} \cup \{m, n\} = \{0, 1, 2, 3\}$ we have

$$\mathbf{f}_{ij}(\mathbf{y})\mathbf{f}_{mn}(\mathbf{y}) < 0 \text{ for } \mathbf{y} \in \mathbf{C}_\epsilon. \tag{31}$$

For simplicity we shall call from now the *complement* of $\mathbf{f}_{ij}$ the function $\mathbf{f}_{kt}$ with $\{k, t\} = \{0, 1, 2, 3\} \setminus \{i, j\}$.

From (29), (30) and (31) we conclude there are six different functions $\mathbf{f}_{ij}$, with $i, j \in \{0, 1, 2, 3\}$, out of which three functions have strictly positive sign on the set $\mathbf{C}_\epsilon$ while their complements have strictly negative sign on $\mathbf{C}_\epsilon$. Using this observation, it can be easily checked that for $\{i, j, k, t\} = \{0, 1, 2, 3\}$, at least one of the following triplets of functions $(\mathbf{f}_{ij}, \mathbf{f}_{ik}, \mathbf{f}_{it})$, $(\mathbf{f}_{ij}, \mathbf{f}_{ik}, \mathbf{f}_{jk})$, $(\mathbf{f}_{ij}, \mathbf{f}_{jt}, \mathbf{f}_{it})$, $(\mathbf{f}_{kt}, \mathbf{f}_{ik}, \mathbf{f}_{it})$, has a constant strict sign over the set $\mathbf{C}_\epsilon$ (the three last triplets are obtained by replacing one of the elements of the first triplet by its complement). Indeed, if $\mathbf{f}_{ij}$, $\mathbf{f}_{ik}$ and $\mathbf{f}_{it}$ do not have the same sign, either two are positive and one is negative or two are negative and one is positive. By replacing one by its complement we get three functions of the same sign. From the balance of forces relations one can immediately observe that the sum of the functions in any of these triplets is of the form

$$\pm 2\mathbf{f}_{i_0} \wedge (\mathbf{y} - \mathbf{x}_{i_0}) \text{ for some } i_0 \in \{0, 1, 2, 3\}. \tag{32}$$

Finally relation (32) leads to a contradiction. Indeed for a triplet with the property that all the functions in the triplet have the same strict sign, the sum of its functions must have the same sign on $\mathbf{C}_\epsilon$. However from (32) the sum cannot have constant sign over the set $\mathbf{C}_\epsilon$ because it equals zero for $\mathbf{y} = \mathbf{x}_{i_0}$. Thus the hypothesis (30) is false and we have that there exists a pair of distinct indices $\{i, j\} \subset \{0, 1, 2, 3\}$ so that $\mathbf{f}_{ij}(\mathbf{y}_*) = 0$ for some $\mathbf{y}_* \in \mathbf{C}_\epsilon$.

From (32) it is possible to choose $\mathbf{y}_* \notin \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_2\}$. Indeed from Lemma 10 there are two indices $\{m, n\}$ such that

$$\mathbf{f}_{mn}(\mathbf{y}_*) = 0. \tag{33}$$

Consider the point $\mathbf{y}_\delta = \mathbf{y}_* + \delta(\mathbf{f}_m + \mathbf{f}_n)$ with $\delta > 0$. For δ small enough it is clear that $\mathbf{y}_\delta \in \mathbf{C}_\epsilon \setminus \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ and from (28) we obtain

$$\begin{aligned}\mathbf{f}_{mn}(\mathbf{y}_\delta) &= \mathbf{f}_m \wedge \mathbf{x}_m + \mathbf{f}_n \wedge \mathbf{x}_n - (\mathbf{f}_m + \mathbf{f}_n) \wedge \mathbf{y}_\delta \\ &= \mathbf{f}_{mn}(\mathbf{y}_*) - \delta(\mathbf{f}_m + \mathbf{f}_n) \wedge (\mathbf{f}_m + \mathbf{f}_n) \\ &= 0,\end{aligned}$$

which implies the statement of the Lemma. □

The next Lemma 10 shows that for any balanced system of four forces in $\mathbb{R}^2$ there exists a purely elastic four terminal planar network with proportional (rank one) force response.

Lemma 10 *Let* $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3$ *and* $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ *be a balanced system of forces in* $\mathbb{R}^2$*. Then, there is a purely elastic four terminal planar network with force response proportional to* $(\mathbf{f}_0^T, \mathbf{f}_1^T, \mathbf{f}_2^T, \mathbf{f}_3^T)^T$*. The internal nodes of such a network can be chosen away from a finite number of points, and within an* $\epsilon-$*neighborhood of the convex hull of the terminals.*

Proof From Lemma 9 we have that there exists a pair of indices $\{i, j\}$, a point $\mathbf{y}_* \in \mathbf{C}_\epsilon \setminus \{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, and a force $\mathbf{f} = \mathbf{f}_i + \mathbf{f}_j$, so that the sets $(\mathbf{y}_*, \mathbf{f}), (\mathbf{x}_i, \mathbf{f}_i), (\mathbf{x}_j, \mathbf{f}_j)$ and $(\mathbf{y}_*, -\mathbf{f}), (\mathbf{x}_k, \mathbf{f}_k), (\mathbf{x}_t, \mathbf{f}_t)$ are balanced sets of forces. From the results of the previous section we have that both sets have a rank one network reproducing the forces. By rescaling the spring constants (see Remark 3) their associated quadratic forms are,

$$\begin{aligned}q_1(\mathbf{v}, \mathbf{u}_i, \mathbf{u}_j) &= \left(\mathbf{f} \cdot \mathbf{v} + \mathbf{f}_i \cdot \mathbf{u}_i + \mathbf{f}_j \cdot \mathbf{u}_j\right)^2 \\ q_2(\mathbf{v}, \mathbf{u}_k, \mathbf{u}_t) &= (-\mathbf{f} \cdot \mathbf{v} + \mathbf{f}_k \cdot \mathbf{u}_k + \mathbf{f}_t \cdot \mathbf{u}_t)^2\end{aligned}$$

where $\{k, t\} = \{0, 1, 2, 3\} \setminus \{i, j\}$. The quadratic form for both networks taken together is

$$q(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \inf_{\mathbf{v}} \{q_1(\mathbf{v}, \mathbf{u}_i, \mathbf{u}_j) + q_2(\mathbf{v}, \mathbf{u}_k, \mathbf{u}_t)\}.$$

The optimality conditions are

$$\mathbf{f} \cdot \mathbf{v} = -\frac{1}{2}(\mathbf{f}_i \cdot \mathbf{u}_i + \mathbf{f}_j \cdot \mathbf{u}_j) + \frac{1}{2}(\mathbf{f}_k \cdot \mathbf{u}_k + \mathbf{f}_t \cdot \mathbf{u}_t)$$

which yield the desired result

$$q(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \frac{1}{2}\left(\sum_{i=0}^{3} \mathbf{f}_i \cdot \mathbf{u}_i\right)^2.$$

□

The following Theorem is the main result of this section. We use the previous results for three and four terminal networks to prove the result in the general case of p-terminal networks by induction, following the approach of Camar-Eddine and Seppecher [4].

Theorem 2 *Let* $\mathbf{f}_i$ *and* $\mathbf{x}_i$*,* $i = 1, \dots, p$ *be a balanced system of forces in* $\mathbb{R}^2$*. There is a purely elastic* p *terminal planar network with a force response proportional to* $(\mathbf{f}_1^T, \dots, \mathbf{f}_p^T)^T$ *and with internal nodes in an* $\epsilon-$*neighborhood of the convex hull of the terminals and avoiding a finite number of points.*

Proof We use an induction argument in the number of terminals as in [4, Theorem 5], which we reproduce here for completeness. The $p = 2$ case corresponds to a single spring with same direction as the forces. The cases $p = 3$ and $p = 4$ are proved in Lemma 7, Lemma 8 and Lemma 10. Assume for the induction argument that the theorem holds for any $t < p$ terminals. Let r be the integer part of $p/2$, and let $\mathbf{y}$ be a node distinct from the terminals $\mathbf{x}_0, \dots, \mathbf{x}_p$. Then there are two forces $\mathbf{f}$ and $\mathbf{f}'$ such that both families

$$((\mathbf{f}, \mathbf{y}), (\mathbf{f}_1 + \mathbf{f}', \mathbf{x}_1), (\mathbf{f}_2, \mathbf{x}_2), \dots, (\mathbf{f}_{r+1}, \mathbf{x}_{r+1})) \text{ and}$$
$$((-\mathbf{f}, \mathbf{y}), (-\mathbf{f}', \mathbf{x}_1), (\mathbf{f}_{r+1}, \mathbf{x}_{r+1}), \dots, (\mathbf{f}_p, \mathbf{x}_p)).$$

are balanced systems of forces, as can easily be seen by taking $\mathbf{f}' = -(\mathbf{f} + \mathbf{f}_1 + \mathbf{f}_2 + \ldots + \mathbf{f}_{r+1})$ and choosing $\mathbf{x}_1$ as the origin of coordinates. These families have $r + 1$ and $p - r + 1$ terminal nodes, and both have less than p terminals when $p > 4$. By the induction hypothesis there are rank one networks with associated quadratic forms

$$q'(\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_{r+1}) = \left(\mathbf{f} \cdot \mathbf{v} + \mathbf{f}' \cdot \mathbf{u}_1 + \sum_{i=1}^{r+1} \mathbf{f}_i \cdot \mathbf{u}_i \right)^2, \text{ and}$$
$$q''(\mathbf{v}, \mathbf{u}_{r+1}, \dots, \mathbf{u}_p) = \left(-\mathbf{f} \cdot \mathbf{v} - \mathbf{f}' \cdot \mathbf{u}_1 + \sum_{i=r+1}^{p} \mathbf{f}_i \cdot \mathbf{u}_1 \right)^2.$$

The quadratic form of both networks together is

$$\widetilde{q}(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_p) = \inf_{\mathbf{v} \in \mathbb{R}^2} q'(\mathbf{v}, \mathbf{u}_1, \dots, \mathbf{u}_{r+1}) + q''(\mathbf{v}, \mathbf{u}_{r+1}, \dots, \mathbf{u}_p).$$

The optimality conditions are:

$$\mathbf{f} \cdot \mathbf{v} = -\frac{1}{2} \left(\mathbf{f}' \cdot \mathbf{u}_1 + \sum_{i=1}^{r+1} \mathbf{f}_i \cdot \mathbf{u}_i \right) + \frac{1}{2} \left(\mathbf{f}' \cdot \mathbf{u}_1 - \sum_{i=r+1}^{p} \mathbf{f}_i \cdot \mathbf{u}_1 \right).$$

Thus the quadratic form of both networks is a rank one with

$$\widetilde{q}(\mathbf{u}_1, \dots, \mathbf{u}_p) = \frac{1}{2} \left(\sum_{i=1}^{p} \mathbf{f}_i \cdot \mathbf{u}_i \right)^2.$$

Notice that the additional point $\mathbf{y}$ can be picked inside an $\epsilon-$neighborhood of the convex hull of the terminal nodes and avoiding a finite number of points. Also we can use virtual crossings and trusses to make the network planar (see §2.3). □

Remark 5 The networks in Lemmas 7, 8, 9, 10, and Theorem 2 may have crossing springs so are not strictly planar. To make them planar is suffices to convert all spring crossings with non-zero angle to a node as is done in [9, Example 3.15]. Zero-angle crossings can be eliminated by replacing springs with simple trusses [9, Example 3.2]. These network transformations are discussed in §2.3.

3.2 Networks in $\mathbb{R}^3$ with rank one static response matrices

The construction of these networks is essentially due to Camar-Eddine and Seppecher [4]. We first complete their construction of rank one four terminal networks to include some degenerate cases which correspond to planar networks (Lemma 11). Then following Camar-Eddine and Seppecher [4] we use induction (Theorem 3) to derive rank one networks with an arbitrary number of terminals.

Lemma 11 *Let $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ be a set of balanced forces at the nodes $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ in $\mathbb{R}^3$. There is a purely elastic rank one network with force response proportional to $(\mathbf{f}_0^T, \mathbf{f}_1^T, \mathbf{f}_2^T, \mathbf{f}_3^T)^T$. Moreover the internal nodes can be chosen within an $\epsilon-$neighborhood of the convex hull of the terminals and avoiding a finite number of points.*

Proof As in the planar case there are two cases depending on the value of

$$r \equiv \operatorname{rank}[\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{x}_1 - \mathbf{x}_0, \mathbf{x}_2 - \mathbf{x}_0, \mathbf{x}_3 - \mathbf{x}_0].$$

The construction for $r = 3$ is given in [4, Lemma 5]. When $r \leq 2$ the network is planar, so the result follows from Theorem 2. □

Theorem 3 *Let $\mathbf{f}_i$ and $\mathbf{x}_i$, $i = 1, \ldots, p$ be a balanced system of forces in $\mathbb{R}^3$. There is a purely elastic p terminal network with a force response proportional to $(\mathbf{f}_1^T, \ldots, \mathbf{f}_p^T)^T$. Moreover the internal nodes can be chosen within an $\epsilon-$neighborhood of the convex hull of the terminals and avoiding a finite number of points.*

Proof The result follows from an induction argument similar to that of Camar-Eddine and Seppecher [4, Theorem 5]. See also the proof of Theorem 2. □

4 Characterization of the dynamic response function

To fully characterize the dynamic response matrices, we take a function $\mathbf{W}(\omega)$ as in Lemma 4 and show that we can construct a network that has $\mathbf{W}(\omega)$ as its frequency response. The construction relies on the static case (Theorem 1) and the existence of a network of springs and masses with rank one response that has exactly one prescribed resonance (Lemma 12). Both networks in $\mathbb{R}^3$ and planar can be constructed.

Theorem 4 *Let $\mathbf{W}(\omega)$ be a real $nd \times nd$ matrix valued function of ω of the form*

$$\mathbf{W}(\omega) = \mathbf{A} - \omega^2 \mathbf{M} + \sum_{i=1}^{p} \frac{\mathbf{C}^{(i)}}{\omega^2 - \omega_i^2},$$

where $\mathbf{M} = \operatorname{diag}(m_1 \mathbf{e}, \ldots, m_n \mathbf{e})$, the vector $\mathbf{e} = [1, \ldots, 1] \in \mathbb{R}^d$, the matrices $\mathbf{C}^{(i)}$ are real symmetric positive semidefinite and $\mathbf{W}(0)$ is real positive semidefinite and balanced (i.e. $\mathbf{W}(\omega)$ satisfies the properties of Lemma 4). Then for any choice of terminal node positions, there is a network (either planar or in $\mathbb{R}^3$) of springs and masses with $\mathbf{W}(\omega)$ as its response function. Moreover the internal nodes of such a network can be chosen to avoid a finite number of positions and within an $\epsilon-$neighborhood of the convex hull of the terminals.

Proof It is convenient to rewrite (10) as,

$$\mathbf{W}(\omega) = \mathbf{W}(0) - \omega^2 \mathbf{M} + \sum_{i=1}^{p} \mathbf{C}^{(i)} \frac{\omega^2}{\omega_i^2(\omega^2 - \omega_i^2)}. \tag{34}$$

Since $\mathbf{W}(0)$ has the properties of Lemma 2, by Theorem 1 there is a (static) network of springs that has $\mathbf{W}(0)$ as its response matrix. Since the $\mathbf{C}^{(i)}$ are positive semidefinite we can use the spectral decomposition to write

$$\mathbf{C}^{(i)} = \sum_{j=1}^{n_i} \lambda_j^{(i)} \mathbf{c}_j^{(i)} (\mathbf{c}_j^{(i)})^T, \text{ for } i = 1, \ldots, p,$$

where the $(\lambda_j^{(i)}, \mathbf{c}_j^{(i)})$ are the eigenpairs of $\mathbf{C}^{(i)}$ with $\lambda_j^{(i)} > 0$ and $n_i = \operatorname{rank}(\mathbf{C}^{(i)})$. Assume that we can construct a network with response function

$$\left(\sqrt{\lambda_j^{(i)}} \omega_i^{-1} \mathbf{c}_j^{(i)}\right) \left(\sqrt{\lambda_j^{(i)}} \omega_i^{-1} \mathbf{c}_j^{(i)}\right)^T \frac{\omega^2}{\omega^2 - \omega_i^2}.$$

The explicit construction of such a network is postponed to the next Section (Lemma 12). By the superposition principle there is a network with response the sum appearing in (34). Finally to obtain the $-\omega^2 \mathbf{M}$ term, simply endow the terminal node $\mathbf{x}_i$ with a mass equal to the $id-$th diagonal element of $\mathbf{M}$ (The mass of node i is repeated d times in the $nd \times nd$ matrix $\mathbf{M}$). To obtain a planar network, simply replace all spring crossings by "virtual crossings" (see §2.3). □

4.1 Rank one response matrices with resonance

The following lemma shows how to design a network with arbitrary rank one response and one single resonance at a prescribed frequency. The network we construct has a purely dynamic response as it has a zero response matrix in the static case $\omega = 0$. The construction is valid for both networks in $\mathbb{R}^3$ and planar networks.

Lemma 12 *Let $\mathbf{x}_i$ and $\mathbf{f}_i$, $i = 1, \ldots, n$, be arbitrary points and forces and $\omega_0 \neq 0$ a given finite resonance frequency. There is a network with terminals $\mathbf{x}_i$ composed of springs and masses with rank one response function*

$$\mathbf{W}(\omega) = \mathbf{f}\mathbf{f}^T \frac{\omega^2}{\omega^2 - \omega_0^2}, \text{ where } \mathbf{f}^T = (\mathbf{f}_1^T, \mathbf{f}_2^T, \ldots, \mathbf{f}_n^T).$$

Moreover the internal nodes of such a network can be chosen to avoid a finite number of positions and within an $\epsilon-$neighborhood of the convex hull of the terminals.

Proof Let us choose two distinct nodes $\mathbf{x}_{n+1}$ and $\mathbf{x}_{n+2}$, in the convex hull of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, and forces $\mathbf{f}_{n+1}$ and $\mathbf{f}_{n+2}$ so that the system $(\mathbf{x}_i, \mathbf{f}_i)$, $i = 1, \ldots, n+2$ is balanced. To do this we take a force $\mathbf{f}_{n+2} \neq \mathbf{0}$ in the line

$$(\mathbf{x}_{n+2} - \mathbf{x}_{n+1}) \wedge \mathbf{f}_{n+2} = -\sum_{i=1}^{n} (\mathbf{x}_i - \mathbf{x}_{n+1}) \wedge \mathbf{f}_i, \tag{35}$$

and choose $\mathbf{f}_{n+1}$ such that $\sum_{i=1}^{n+2} \mathbf{f}_i = \mathbf{0}$. Note that the choice of the nodes $\mathbf{x}_{n+1}$ and $\mathbf{x}_{n+2}$ is unrestricted in dimension $d = 2$ because equation (35) admits a solution $\mathbf{f}_{n+2}$ provided $\mathbf{x}_{n+1} \neq \mathbf{x}_{n+2}$. To guarantee solvability of (35) in dimension $d = 3$, the nodes need to be chosen such that $\mathbf{x}_{n+2} - \mathbf{x}_{n+1}$ is orthogonal to the right hand side of (35). Then by Theorem 2 there is a rank one network with force response proportional to $(\mathbf{f}^T, \mathbf{f}_{n+1}^T, \mathbf{f}_{n+2}^T)^T$. Attach a mass m to nodes $\mathbf{x}_{n+1}$ and $\mathbf{x}_{n+2}$. The spring constants in the network can be rescaled (Remark 3) so that the equations of motion are

$$\begin{bmatrix} \mathbf{w}_B \\ m\omega^2 \mathbf{u}_I \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{a} \end{bmatrix} \begin{bmatrix} \mathbf{f}^T & \mathbf{a}^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_B \\ \mathbf{u}_I \end{bmatrix},$$

where $\mathbf{a}^T = (\mathbf{f}_{n+1}^T, \mathbf{f}_{n+2}^T) \neq \mathbf{0}$, the displacements $\mathbf{u}_B$ and $\mathbf{u}_I$ are respectively at the "boundary" nodes $\mathbf{x}_1, \ldots, \mathbf{x}_n$ and the "interior" nodes $\mathbf{x}_{n+1}, \mathbf{x}_{n+2}$. Then solving the system for the resulting forces $\mathbf{w}_B$ at the "boundary" nodes we get,

$$\mathbf{w}_B = \mathbf{f}\mathbf{f}^T \mathbf{u}_B \left(1 + \frac{\|\mathbf{a}\|^2}{m\omega^2 - \|\mathbf{a}\|^2}\right) = \mathbf{f}\mathbf{f}^T \mathbf{u}_B \frac{\omega^2}{\omega^2 - \|\mathbf{a}\|^2 / m}.$$

Finally choose the mass $m = \|\mathbf{a}\|^2 / \omega_0^2$. The position of the internal nodes $\mathbf{x}_{n+1}$ and $\mathbf{x}_{n+2}$ is flexible and by Theorem 2 so is that of any interior nodes in the rank one network involved in the construction. ◻

A Stability to small perturbations

We show that the response function of an elastodynamic network is stable to changes in the network, which could come from either modifying the spring constants of existing springs or possibly adding springs with small spring constants between any two nodes in the network. However we do not allow springs to be deleted from the network. We first show stability of the response of static networks and then stability for the response function of elastodynamic networks.

A.1 Stability in the static case

Let $\mathbf{A}$ be the response matrix of an elastic network with all nodes considered as terminals and $\mathbf{W}$ be the response matrix at the terminal nodes as given by (6). If we add or modify (but not delete) springs then the new response with all nodes considered as terminals is $\mathbf{A} + \epsilon\mathbf{E}$, $\epsilon > 0$, and its response at the terminals $\mathbf{W}(\epsilon)$. We prove the following result

Lemma 13 *Let $\epsilon > 0$. As $\epsilon \to 0$, we have $\mathbf{W}(\epsilon) \to \mathbf{W}$.*

The stability result for the static case may seem surprising at first because the pseudo-inverse we used to find the response at the terminals (6) is not continuous (see e.g. [8, §5.5.5]). However Lemma 1 guarantees that the instabilities are controlled as they remain (roughly speaking) in $\mathcal{N}(\mathbf{A}_{II})$. Before showing Lemma 13 we need to establish the following relation between the floppy modes of the perturbed and unperturbed stiffness matrices.

Lemma 14 *For all $\epsilon > 0$ sufficiently small, $\mathcal{N}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II}) \subseteq \mathcal{N}(\mathbf{A}_{II})$. Moreover $\mathcal{N}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})$ is* independent *of ϵ, and depends only on the connectivity of the new network. In other words if a network is perturbed by adding springs or modifying existing springs, then a floppy mode of the perturbed network must be a floppy mode of the unperturbed network.*

Proof If $\mathbf{u}_I$ is a floppy mode of the perturbed network then:

$$\begin{aligned} 0 = \mathbf{u}_I^T(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})\mathbf{u}_I &= \begin{bmatrix}\mathbf{0} \ \mathbf{u}_I^T\end{bmatrix}(\mathbf{A} + \epsilon\mathbf{E})\begin{bmatrix}\mathbf{0} \\ \mathbf{u}_I\end{bmatrix} \\ &= \sum_{\substack{\text{old springs} \\ i\in I,\, j\in B\cup I}} (k_{i,j} + \epsilon l_{i,j})\left((\mathbf{u}_i - \mathbf{u}_j)\cdot\frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|}\right)^2 \\ &+ \sum_{\substack{\text{new springs} \\ i\in I,\, j\in B\cup I}} \epsilon l_{i,j}\left((\mathbf{u}_i - \mathbf{u}_j)\cdot\frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|}\right)^2. \end{aligned}$$

Since the perturbed network is a spring network, the new spring constants should be positive and all the terms in the sums above vanish, a condition which is independent of ϵ. Since all the terms in the sums above vanish it follows that $\mathbf{u}_I \in \mathcal{N}(\mathbf{A}_{II})$:

$$\mathbf{u}_I^T\mathbf{A}_{II}\mathbf{u}_I = \sum_{\substack{\text{old springs} \\ i\in I,\, j\in B\cup I}} k_{i,j}\left((\mathbf{u}_i - \mathbf{u}_j)\cdot\frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|}\right)^2 = 0.$$

□

We are now ready to prove stability for the static case.

Proof (of Lemma 13) Let $\epsilon > 0$ be sufficiently small. By Lemma 14 it is possible to find a unitary matrix $[\mathbf{U}, \mathbf{V}, \mathbf{W}]$ *independent* of ϵ such that $\mathcal{R}([\mathbf{V}, \mathbf{W}]) = \mathcal{N}(\mathbf{A}_{II})$ and $\mathcal{R}(\mathbf{W}) = \mathcal{N}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})$. Writing $\mathbf{A}_{II} + \epsilon\mathbf{E}_{II}$ in the new basis gives,

$$\mathbf{A}_{II} + \epsilon\mathbf{E}_{II} = [\mathbf{U}, \mathbf{V}, \mathbf{W}]\begin{bmatrix}\widetilde{\mathbf{A}} + \epsilon\widetilde{\mathbf{E}}_1 & \epsilon\widetilde{\mathbf{E}}_2 & \mathbf{0} \\ \epsilon\widetilde{\mathbf{E}}_2^T & \epsilon\widetilde{\mathbf{E}}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}[\mathbf{U}, \mathbf{V}, \mathbf{W}]^T, \tag{36}$$

where $\widetilde{\mathbf{A}} \equiv \mathbf{U}^T\mathbf{A}_{II}\mathbf{U}$. Because of our choice of basis both $\widetilde{\mathbf{A}}$ and the non-zero block in (36) must be invertible and symmetric positive definite. The inverse of this block is

$$\begin{aligned} &\begin{bmatrix}\widetilde{\mathbf{A}} + \epsilon\widetilde{\mathbf{E}}_1 & \epsilon\widetilde{\mathbf{E}}_2 \\ \epsilon\widetilde{\mathbf{E}}_2^T & \epsilon\widetilde{\mathbf{E}}_3\end{bmatrix}^{-1} \\ &= \begin{bmatrix}\widetilde{\mathbf{A}}(\epsilon)^{-1}(\mathbf{I} + \epsilon\widetilde{\mathbf{E}}_2\widetilde{\mathbf{E}}(\epsilon)^{-1}\widetilde{\mathbf{E}}_3^{-1}\widetilde{\mathbf{E}}_2^T\widetilde{\mathbf{A}}(\epsilon)^{-1}) & -\widetilde{\mathbf{A}}(\epsilon)^{-1}\widetilde{\mathbf{E}}_2\widetilde{\mathbf{E}}(\epsilon)^{-1}\widetilde{\mathbf{E}}_3^{-1} \\ -\widetilde{\mathbf{E}}(\epsilon)^{-1}\widetilde{\mathbf{E}}_3^{-1}\widetilde{\mathbf{E}}_2^T\widetilde{\mathbf{A}}(\epsilon)^{-1} & \epsilon^{-1}\widetilde{\mathbf{E}}(\epsilon)^{-1}\widetilde{\mathbf{E}}_3^{-1}\end{bmatrix} \\ &= \begin{bmatrix}\widetilde{\mathbf{A}}^{-1} + \epsilon\mathbf{G}_1 + o(\epsilon) & -\widetilde{\mathbf{A}}^{-1}\widetilde{\mathbf{E}}_2\widetilde{\mathbf{E}}_3^{-1} + \epsilon\mathbf{G}_2 + o(\epsilon) \\ -\widetilde{\mathbf{E}}_3^{-1}\widetilde{\mathbf{E}}_2^T\widetilde{\mathbf{A}}^{-1} + \epsilon\mathbf{G}_2^T + o(\epsilon) & \epsilon^{-1}\widetilde{\mathbf{E}}_3^{-1} + \mathbf{G}_3 + o(1)\end{bmatrix}, \end{aligned} \tag{37}$$

where $\widetilde{\mathbf{E}}(\epsilon) = (\mathbf{I} - \epsilon\widetilde{\mathbf{E}}_3^{-1}\widetilde{\mathbf{E}}_2^T\widetilde{\mathbf{A}}(\epsilon)^{-1}\widetilde{\mathbf{E}}_2)$ and $\widetilde{\mathbf{A}}(\epsilon) = \widetilde{\mathbf{A}} + \epsilon\mathbf{E}_1$. Notice that $\widetilde{\mathbf{E}}_3$ is a submatrix of a symmetric positive definite matrix and thus must be invertible. The second equality comes from the standard perturbation formula for the inverse (see e.g. [8, §2.3.4]) and the matrices $\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$ are independent of ϵ. We now examine the response of the perturbed matrix. Since the first term in (6) is linear in $\mathbf{E}$, it is stable to perturbations. Now the negative of the second term in (6) can be written as:

$$\begin{aligned} &(\mathbf{A}_{BI} + \epsilon\mathbf{E}_{BI})(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^\dagger(\mathbf{A}_{IB} + \epsilon\mathbf{E}_{IB}) = \mathbf{A}_{BI}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^\dagger\mathbf{A}_{IB} \\ &+ \epsilon\mathbf{E}_{BI}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^\dagger\mathbf{A}_{IB} + \epsilon\mathbf{A}_{BI}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^\dagger\mathbf{E}_{IB} + \epsilon^2\mathbf{E}_{BI}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^\dagger\mathbf{E}_{IB}. \end{aligned} \tag{38}$$

Moreover in the basis $[\mathbf{U}, \mathbf{V}, \mathbf{W}]$ the pseudo-inverse becomes,

$$(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^\dagger = [\mathbf{U}, \mathbf{V}]\begin{bmatrix}\widetilde{\mathbf{A}} + \epsilon\widetilde{\mathbf{E}}_1 & \epsilon\widetilde{\mathbf{E}}_2 \\ \epsilon\widetilde{\mathbf{E}}_2^T & \epsilon\widetilde{\mathbf{E}}_3\end{bmatrix}^{-1}[\mathbf{U}, \mathbf{V}]^T.$$

Using Lemma 1 and Lemma 14, we have $\mathbf{A}_{BI}[\mathbf{V},\mathbf{W}] = [\mathbf{0},\mathbf{0}]$ and $(\mathbf{A}_{BI} + \epsilon\mathbf{E}_{BI})\mathbf{W} = \mathbf{0}$. The leading order asymptotics of each of the terms in (38) are

$$\begin{aligned}
\mathbf{A}_{BI}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^{\dagger}\mathbf{A}_{IB} &= \mathbf{A}_{BI}\mathbf{U}\widetilde{\mathbf{A}}^{-1}\mathbf{U}^T\mathbf{A}_{IB} + \mathcal{O}(\epsilon) = \mathbf{A}_{BI}\mathbf{A}_{II}^{\dagger}\mathbf{A}_{IB} + \mathcal{O}(\epsilon),\\
\epsilon\mathbf{E}_{BI}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^{\dagger}\mathbf{A}_{IB} &= \epsilon\mathbf{E}_{BI}(\mathbf{U}\widetilde{\mathbf{A}}^{-1}\mathbf{U}^T - \mathbf{V}\widetilde{\mathbf{E}}_3^{-1}\widetilde{\mathbf{E}}_2^T\widetilde{\mathbf{A}}^{-1}\mathbf{U}^T)\mathbf{A}_{IB} + \mathcal{O}(\epsilon^2),\\
\epsilon\mathbf{A}_{BI}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^{\dagger}\mathbf{E}_{IB} &= \epsilon\mathbf{A}_{BI}(\mathbf{U}\widetilde{\mathbf{A}}^{-1}\mathbf{U}^T - \mathbf{U}\widetilde{\mathbf{A}}^{-1}\widetilde{\mathbf{E}}_2\widetilde{\mathbf{E}}_3^{-1}\mathbf{V}^T)\mathbf{E}_{IB} + \mathcal{O}(\epsilon^2),\\
\epsilon^2\mathbf{E}_{BI}(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^{\dagger}\mathbf{E}_{IB} &= \epsilon\mathbf{E}_{BI}\mathbf{V}\widetilde{\mathbf{E}}_3^{-1}\mathbf{V}^T\mathbf{E}_{IB} + \mathcal{O}(\epsilon^2).
\end{aligned}$$

which proves the desired result:

$$(\mathbf{A}_{BI} + \epsilon\mathbf{E}_{BI})(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})^{\dagger}(\mathbf{A}_{IB} + \epsilon\mathbf{E}_{IB}) = \mathbf{A}_{BI}\mathbf{A}_{II}^{\dagger}\mathbf{A}_{IB} + \mathcal{O}(\epsilon).$$

□

A.2 Stability in the dynamic case

As in the static case, we deal only with network perturbations that modify existing springs or add new springs, but excluding spring deletions. Denote by $\mathbf{K}$ the response when all the nodes are terminals and let $\epsilon > 0$ be sufficiently small so that $\mathbf{K} + \epsilon\mathbf{E}$ is the response of the perturbed network. We show the following result.

Lemma 15 *Partition the interior nodes I into massless nodes L and nodes with mass J, as in Lemma 3. Let ω be a frequency such that ω^2 is not an eigenvalue of $\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JJ}\mathbf{M}_{JJ}^{-1/2}$, and $\mathbf{K}+\epsilon\mathbf{E}$ be the response of the perturbed network where we allow for new springs or changes in the spring constants, but no spring deletions. Then as $\epsilon \to 0$,*

$$\mathbf{W}(\omega;\epsilon) = \mathbf{W}(\omega) + \mathcal{O}(\epsilon),$$

where $\mathbf{W}(\omega;\epsilon)$ (resp. $\mathbf{W}(\omega)$) is the response at the terminal nodes of the perturbed (resp. unperturbed) network at frequency ω.

Proof By Lemma 13 the matrix $\widetilde{\mathbf{K}}$ (the response matrix of the network with terminals $B \cup J$, see (9)) is stable to such spring perturbations, meaning that the response of the perturbed network at the nodes $B \cup J$ satisfies $\widetilde{\mathbf{K}}(\epsilon) = \widetilde{\mathbf{K}} + \epsilon\widetilde{\mathbf{E}} + o(\epsilon)$, for some matrix $\widetilde{\mathbf{E}}$ independent of ϵ. Since both the perturbed and unperturbed responses are symmetric, the Wielandt-Hoffman theorem (see e.g. [8, §8.1.2]) implies that there is a reordering of the eigenvalues $\omega_i^2(\epsilon)$ of $\widetilde{\mathbf{K}}_{JJ}(\epsilon)$ such that

$$|\omega_i^2(\epsilon) - \omega_i^2| \le \epsilon\|\widetilde{\mathbf{E}}\|_F,$$

with ω_i^2 being the eigenvalues of $\widetilde{\mathbf{K}}_{JJ}$ and $\|\cdot\|_F$ denoting the Frobenius matrix norm. Therefore if ϵ is sufficiently small and ω is not an eigenvalue of $\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JJ}\mathbf{M}_{JJ}^{-1/2}$ then ω is not an eigenvalue of $\mathbf{M}_{JJ}^{-1/2}\widetilde{\mathbf{K}}_{JJ}(\epsilon)\mathbf{M}_{JJ}^{-1/2}$ either and the matrix $\widetilde{\mathbf{K}}(\epsilon) - \omega^2\mathbf{M}_{JJ}$ is invertible. Thus using (8) and the standard perturbation formula for the inverse, it is possible to show that the response at the terminals of the perturbed matrix is $\mathbf{W}(\omega;\epsilon) = \mathbf{W}(\omega) + \mathcal{O}(\epsilon)$. □

B Eliminating floppy modes by adding springs

We use the stability results from the previous Appendix to show that if a network has floppy modes then there is a network with no floppy modes and a response function arbitrarily close to that of the original network. Some examples of floppy modes (for planar networks) are given in Figure 3. Our strategy to remove floppy modes is to connect all nodes (be them terminal or interior nodes) of the network by springs with small spring constants, thus creating a *complete graph* with the nodes $I \cup B$ and springs as edges. By Lemma 13 the response of the new network can be made arbitrarily close to the response of the unperturbed network.

Let $\mathbf{A}$ be the response of the network where all nodes are terminals and let $\mathbf{A} + \epsilon\mathbf{E}$ be response when we have added all these new springs. Then if $\mathbf{u}_I$ is a floppy mode of the new network, proceeding as in Lemma 14 gives

$$\begin{aligned}
0 = \mathbf{u}_I^T(\mathbf{A}_{II} + \epsilon\mathbf{E}_{II})\mathbf{u}_I &= \begin{bmatrix}\mathbf{0} \ \mathbf{u}_I^T\end{bmatrix}(\mathbf{A} + \epsilon\mathbf{E})\begin{bmatrix}\mathbf{0}\\ \mathbf{u}_I\end{bmatrix}\\
&= \sum_{\substack{\text{old springs}\\ i\in I,\ j\in B\cup I}} (k_{i,j} + \epsilon l_{i,j})\left((\mathbf{u}_i - \mathbf{u}_j)\cdot\frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|}\right)^2\\
&+ \sum_{\substack{\text{new springs}\\ i\in I,\ j\in B\cup I}} \epsilon l_{i,j}\left((\mathbf{u}_i - \mathbf{u}_j)\cdot\frac{\mathbf{x}_i - \mathbf{x}_j}{\|\mathbf{x}_i - \mathbf{x}_j\|}\right)^2.
\end{aligned}$$

This is equivalent to saying that

$$\forall i \in I \text{ and } j \in B \cup I, \quad (\mathbf{u}_i - \mathbf{u}_j)\cdot(\mathbf{x}_i - \mathbf{x}_j) = 0. \tag{39}$$

Assume that we are working in d dimensions and that we have $d+1$ nodes $\mathbf{x}_1,\ldots,\mathbf{x}_{d+1}$ that form a non-degenerate triangle ($d = 2$) or tetrahedron ($d = 3$) and for which $\mathbf{u}_1 = \mathbf{u}_2 = \cdots = \mathbf{u}_{d+1} = \mathbf{0}$. Then since every interior node $\mathbf{y}$ is connected to the $\mathbf{x}_1,\ldots,\mathbf{x}_{d+1}$, equation (39) implies that $\mathbf{v}\cdot(\mathbf{y} - \mathbf{x}_i) = 0$, $i = 1,\ldots,d+1$, where $\mathbf{v}$ is the displacement associated with $\mathbf{y}$. Since these special "anchor" nodes are non-degenerate, $\operatorname{rank}[\mathbf{x}_1 - \mathbf{y}, \mathbf{x}_2 - \mathbf{y}, \ldots, \mathbf{x}_{d+1} - \mathbf{y}] = d$ and we must have $\mathbf{v} = \mathbf{0}$. Repeating this for every interior node we get $\mathbf{u}_I = \mathbf{0}$, and so the network does not have any floppy modes.

We now need to show which networks have "anchor" nodes. Clearly if the network has $d+1$ terminal nodes forming a non-degenerate triangle (in $d = 2$) or tetrahedron (in $d = 3$), then the network does not have any floppy modes, since the terminal nodes do not move.

If we are in $d = 2$ dimension and the terminal nodes do not form a non-degenerate triangle, then all terminals must lie on a line. Since the network has at least two terminal nodes (otherwise we cannot balance forces), we can add two interior nodes with a truss as in §2.3 without changing the response. Let $\mathbf{x}_1, \mathbf{x}_2$ be two terminal nodes and $\mathbf{y}$ be one of the interior nodes of the truss, with associated displacements $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{0}$ and $\mathbf{v}$. Then condition (39) implies that $\mathbf{v}\cdot(\mathbf{x}_1 - \mathbf{y}) = \mathbf{v}\cdot(\mathbf{x}_2 - \mathbf{y}) = 0$, i.e. $\mathbf{v} = \mathbf{0}$. Thus the nodes $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}$ form an "anchor" and the network does not have any floppy modes.

If we are in $d = 3$ dimension and we cannot form a non-degenerate tetrahedron from the terminal nodes, then the terminals must lie on a plane. Assume further that the terminals do not lie on a line, we shall deal with this case later. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be three terminal nodes forming a triangle. Then replacing e.g. the spring between $\mathbf{x}_1$ and $\mathbf{x}_2$ by a truss (as in §2.3), we introduce three new interior nodes and at least one of them $\mathbf{y}$ is not in the plane where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ lie. Let $\mathbf{v}$ be the displacement of $\mathbf{y}$ and $\mathbf{u}_1 = \mathbf{u}_2 = \mathbf{u}_3 = \mathbf{0}$ be the displacements of the boundary nodes. Then condition (39) implies that $\mathbf{v}\cdot(\mathbf{x}_i - \mathbf{y}) = 0$ for $i = 1, 2, 3$. Since $\operatorname{rank}[\mathbf{x}_1 - \mathbf{y}, \mathbf{x}_2 - \mathbf{y}, \mathbf{x}_3 - \mathbf{y}] = 3$, we must have $\mathbf{v} = \mathbf{0}$. Therefore the nodes $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}$ form an "anchor" and the network does not have any floppy modes.

If we are in $d = 3$ dimension and all the terminal nodes lie on a line then every interior node $\mathbf{y}$ forms a triangle with two terminal nodes $\mathbf{x}_1, \mathbf{x}_2$. In this case condition (39) means that the displacement $\mathbf{v}$ of node $\mathbf{y}$ is orthogonal to the plane formed by $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}$, and in particular to the axis where all terminals lie. Thus in this case the floppy modes cannot be eliminated by adding springs or interior nodes, as any additional interior node is in this situation as well. This corresponds to rotations of the network around the axis where all the terminals lie.

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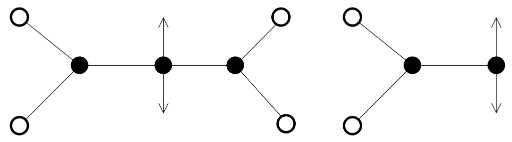

Fig. 3 Two types of floppy modes. The terminals are white circles and the interior nodes are in black. The direction in which the node can move with zero force is given with arrows.

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