Math 6010, Fall 2004: Homework

Homework 3

#2, page 23: Recall that $\boldsymbol{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ iff for all $\boldsymbol{t} \in \mathbf{R}^n$, $\boldsymbol{t}'\boldsymbol{Y} \sim N(\boldsymbol{t}'\boldsymbol{\mu}, \boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t})$. Choose $\boldsymbol{t}$ such that $t_i = 1$ and $t_j = 0$ for $j \neq i$. Then $\boldsymbol{t}'\boldsymbol{Y} = Y_i$, $\boldsymbol{t}'\boldsymbol{\mu} = \mu_i$, and $\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t} = \sigma_{i,i}$.

#3, page 23: Of course, $\boldsymbol{Z} = \boldsymbol{AY}$, where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Therefore, $\boldsymbol{Z} \sim N_2(\boldsymbol{A\mu}, \boldsymbol{A\Sigma A}')$. This is a bivariate normal;

$$\boldsymbol{A\mu} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \qquad \boldsymbol{A\mu A}' = \begin{pmatrix} 10 & -1 \\ -1 & 3 \end{pmatrix}.$$

#5, page 24: Each (X_i, Y_i) is obtained from linear combination of two i.i.d. standard normals. That is, $X_i = a_{i,1}Z_{i,1} + a_{i,2}Z_{i,2}$ and $Y_i = b_{i,1}Z_{i,1} + b_{i,2}Z_{i,2}$, where $Z_{1,1}, Z_{1,2}, Z_{2,1}, Z_{2,2}, \ldots, Z_{n,1}, Z_{n,2}$ are i.i.d. standard normals, and $a_{i,j}$'s and $b_{i,j}$'s are constants. Therefore,

$$\begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ \vdots \\ X_n \\ Y_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{A}_1 & & & \\ & \boldsymbol{A}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{A}_n \end{pmatrix} \begin{pmatrix} Z_{1,1} \\ Z_{1,2} \\ Z_{2,1} \\ Z_{2,2} \\ \vdots \\ Z_{n,1} \\ Z_{n,1} \end{pmatrix},$$

where the empty parts of the matrix with $\boldsymbol{A}$'s in it are are zero, and

$$\boldsymbol{A}_j = \begin{pmatrix} a_{j,1} & a_{j,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}.$$

This proves that $(X_1, Y_1, \ldots, X_n, Y_n)'$ is multivariate normal. Therefore, so is

$$\begin{pmatrix} \overline{X} \\ \overline{Y} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ \vdots \\ X_n \\ Y_n \end{pmatrix}.$$

It is easiest to compute the mean and variance matrix directly though. Suppose $EX_1 = \mu_X$, $EY_1 = \mu_Y$, $\mathrm{Var}X_1 = \sigma_X^2$, $\mathrm{Var}Y_1 = \sigma_Y^2$, and $\mathrm{Cor}(X_1, Y_1) = \rho$. Then, $E\overline{X} = EX_1 = \mu_X$, $E\overline{Y} = EY_1 = \mu_Y$, $\mathrm{Var}\overline{X} = \sigma_X^2/n$, $\mathrm{Var}\overline{Y} = \sigma_Y^2/n$. Finally,

$$\begin{aligned} \mathrm{Cov}(\overline{X}, \overline{Y}) &= \mathrm{Cov}\left(\frac{1}{n}\sum_{i=1}^n X_i \ , \ \frac{1}{n}\sum_{j=1}^n Y_j \right) \\ &= \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \mathrm{Cov}(X_i, Y_j) = \frac{1}{n^2}\sum_{i=1}^n \mathrm{Cov}(X_i, Y_i) \\ &= \frac{\rho\sigma_X\sigma_Y}{n}. \end{aligned}$$

Therefore, $\mathrm{Cor}(\overline{X}, \overline{Y}) = \mathrm{Cov}(\overline{X}, \overline{Y})/\mathrm{SD}(\overline{X})\mathrm{SD}(\overline{Y}) = \rho$. Thus, $(\overline{X}, \overline{Y}) \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_X^2/n & \rho \\ \rho & \sigma_Y^2/n \end{pmatrix}.$$

#6, page 24: Let $\mu_i = EY_2$ and $\sigma_i^2 = \mathrm{Var}Y_i$. Also define $\rho = \mathrm{Cor}(Y_1, Y_2)$.

Define $Z_1 = Y_1 + Y_2$ and $Z_2 = Y_1 - Y_2$. Then we are told that Z_1 and Z_2 are independent $N(0, 1)$'s. Note that

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \text{ where } \boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Therefore, $(Y_1, Y_2)'$ is bivariate normal with

$$E\boldsymbol{Y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \mathrm{Var}\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{A}' = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

#5, page 32: Define

$$\boldsymbol{W} = \left(\mathrm{I}_n - \frac{\boldsymbol{a}\boldsymbol{a}'}{\|\boldsymbol{a}\|^2}\right)\boldsymbol{Y} := \boldsymbol{A}\boldsymbol{Y}.$$

[NB: $\boldsymbol{a}\boldsymbol{a}'$ is an $n \times n$ matrix.] We compute directly to find that

$$\begin{aligned}\mathrm{Cov}(\boldsymbol{W}, \boldsymbol{a}'\boldsymbol{Y}) &= \boldsymbol{A}\mathrm{Cov}(\boldsymbol{Y}, \boldsymbol{Y})\boldsymbol{a} = \boldsymbol{A}\boldsymbol{a} \\ &= \boldsymbol{a} - \frac{\boldsymbol{a}\boldsymbol{a}'}{\|\boldsymbol{a}\|^2}\boldsymbol{a} = \mathbf{0}.\end{aligned}$$

This proves that $\boldsymbol{W}$ and $\boldsymbol{a}'\boldsymbol{Y}$ are independent (Theorem 2.5). Note that $\boldsymbol{A}$ is symmetric and idempotent (i.e., $\boldsymbol{A}^2 = \boldsymbol{A}$). Therefore,

$$\begin{aligned}\|\boldsymbol{W}\|^2 = \boldsymbol{W}'\boldsymbol{W} = \boldsymbol{Y}'\boldsymbol{A}^2\boldsymbol{Y} &= \boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'\frac{\boldsymbol{a}\boldsymbol{a}'}{\|\boldsymbol{a}\|^2}\boldsymbol{Y} \\ &= \boldsymbol{Y}'\boldsymbol{Y} - \frac{\|\boldsymbol{a}'\boldsymbol{Y}\|^2}{\|\boldsymbol{a}\|^2}.\end{aligned}$$

Turn this around to see that $\|\boldsymbol{Y}\|^2 = \|\boldsymbol{W}\|^2 + \|\boldsymbol{a}'\boldsymbol{Y}\|^2/\|\boldsymbol{a}\|^2$. Because $\boldsymbol{W}$ is independent of $\boldsymbol{a}'\boldsymbol{Y}$, the conditional distribution of $\|\boldsymbol{Y}\|^2$ given $\boldsymbol{a}'\boldsymbol{Y} = 0$ is the same as the (unconditional) distribution of $\|\boldsymbol{W}\|^2 = \|\boldsymbol{A}\boldsymbol{Y}\|^2$. Thanks to Theorem 2.8, the said distribution is χ^2_r where r denotes the number of eigenvalues of $\boldsymbol{A}$ that are one; whence, $n - r$ eigenvalues are zero. It remains to prove that $r = n - 1$. This follows immediately from the fact that the only non-zero solution to $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = \boldsymbol{a}$. To see this note that $\boldsymbol{A}\boldsymbol{a} = \mathbf{0}$, so $\boldsymbol{a}$ is a solution to $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$. Suppose there were another non-zero solution to $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$. We can use Gramm–Schmitt to obtain a non-zero solution $\boldsymbol{v}$ to $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$ with the property that $\boldsymbol{v}$ is orthogonal to $\boldsymbol{a}$; i.e., $\boldsymbol{v}'\boldsymbol{a} = 0$. Note that $\boldsymbol{a}\boldsymbol{a}'\boldsymbol{v} = 0$ so that $\mathbf{0} = \boldsymbol{A}\boldsymbol{v} = \boldsymbol{v}$. Therefore, there is exactly one non-zero solution to $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$, and that is $\boldsymbol{x} = \boldsymbol{a}$. Equivalently, the column rank of $\boldsymbol{A}$ is $r = n - 1$.

#6, page 32: Let $X_i = (Y_i - \mu_i)/\sqrt{1-\rho}$ to find that $\boldsymbol{X} \sim N_n(\mathbf{0}, (1-\rho)^{-1}\mathbf{\Sigma})$. Because $(Y_i - \overline{Y})/\sqrt{1-\rho} = X_i - \overline{X}$,

$$\begin{pmatrix} \frac{Y_1 - \overline{Y}}{\sqrt{1-\rho}} \\ \vdots \\ \frac{Y_n - \overline{Y}}{\sqrt{1-\rho}} \end{pmatrix} = \boldsymbol{A}\boldsymbol{X}, \text{ where } \boldsymbol{A} = \frac{1}{\sqrt{1-\rho}}\left(\mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'\right),$$

since $\mathbf{1}_n\mathbf{1}_n'$ is an $n \times n$ matrix of all ones. The first thing to notice is that $\boldsymbol{A}\mathbf{1}_n\mathbf{1}_n' = \mathbf{0}$. This follows from the fact that

$(\mathbf{1}_n\mathbf{1}_n')^2 = n\mathbf{1}_n\mathbf{1}_n'$. In particular, $\boldsymbol{A}^2 = (1-\rho)^{-1}$. In addition,

$$\begin{aligned}\boldsymbol{A}\mathrm{Var}\boldsymbol{X} &= \frac{1}{\sqrt{1-\rho}}\boldsymbol{A}\boldsymbol{\Sigma} \\ &= \sqrt{1-\rho}\boldsymbol{A} + \frac{\rho}{\sqrt{1-\rho}}\boldsymbol{A}\mathbf{1}_n\mathbf{1}_n' = \sqrt{1-\rho}\boldsymbol{A}.\end{aligned}$$

Therefore, $\boldsymbol{A}\mathrm{Var}\boldsymbol{X}$ is idempotent. The corollary on page 30 tells us then that $\|\boldsymbol{AX}\|^2 \sim \chi^2_r$ where $r = \mathrm{rank}(\boldsymbol{A}\mathrm{Var}\boldsymbol{X})$. Note that

$$\|\boldsymbol{AX}\|^2 = \frac{1}{1-\rho}\sum_{i=1}^n (Y_i - \overline{Y})^2.$$

Therefore, it suffices to prove that $r = n-1$. That is, we wish to prove that there is exactly one solution to $\boldsymbol{A}\mathrm{Var}(\boldsymbol{X})\boldsymbol{x} = \mathbf{0}$. This was proved in #5, page 32; simply set $\boldsymbol{a} = \mathbf{1}_n$ there.

#11, page 32: One can check that $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{a}$, where $\boldsymbol{A}$ ($n+1$ columns and n rows) as follows:

$$\boldsymbol{A} = \begin{pmatrix} \phi & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \phi & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \phi & 1 \end{pmatrix}.$$

So $\boldsymbol{Y} \sim N_n(\mathbf{0}, \sigma^2\boldsymbol{AA}')$. To finish, we compute the $n\times n$ matrix,

$$\boldsymbol{AA}' = \begin{pmatrix} \phi^2+1 & \phi & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \phi & \phi^2+1 & \phi & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \phi & \phi^2+1 & \phi & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \phi & \phi^2+1 & \phi & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \phi & \phi^2+1 & \phi \end{pmatrix}.$$

That is, $(\boldsymbol{AA}')_{i,i} = \phi^2+1$, $(\boldsymbol{AA}')_{i,i+1} = (\boldsymbol{AA}')_{i,i-1} = \phi$, and for all $j \notin \{i, i\pm 1\}$, $(\boldsymbol{AA}')_{i,j} = 0$.