MATH 6720
Homework # 6
Due Date: 04/07/2017

1. **Problem 7.2.1** Find the spectral representation of the delta function for the following operators
   (a) $Lu = -u''$, $x \in [0,1]$, $u(0) = u'(1) = 0$,
   (b) $Lu = -u''$, $x \in [0,\infty)$, $u'(0) = 0$.

2. **Problems 7.2.3, 7.2.6, 7.2.7** Find the Fourier transforms of
   (a) $f(t) = t^{1/2} e^{(i+c)t} H(t)$,
   (b) $f(t) = e^{-a|t|}$, $a > 0$,
   (c) $f(t) = e^{-t} H(t)$.
   (d) $f(t) = \sin(\pi t)/(\pi t)$.

3. **Problem 7.2.11** Evaluate $E_x E_\mu$ as defined in the uncertainty principle for $f(x) = e^{-ax^2}$.

4. **Problem 7.5.1** Find two linearly independent solutions on $-\infty < x < \infty$ of $u''(x) + q(x)u = 0$ where
$$q(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases} .$$

5. **Problem 7.5.3** Find the reflection coefficients, bound states, and, if appropriate, transmission coefficients for the operator $Lu = -u'' + q(x)u$ for $q(x) = \beta\delta(x-1)$, $x \in [0,\infty)$, $u(0) = 0$.

6. The following theorem is attributed to Macdonald and Whittaker. It deals with zeroes of analytical functions. A proof has been recently suggested by Ross McPhedran (available on arXiv: `https://arxiv.org/abs/1702.03458v1`).

   Thm. The number of zeros N_z of a non-constant function $f(z)$ in the region bounded by a contour at each point of which $|f(z)| = C$, C being a constant, exceeds the number of zeros of the derived function $f'(z)$ in the same region by unity, the function $f(z)$ being supposed analytic in the region.

   The paper by McPhedran also gives a nice way of checking the theorem using Mathematica (code provided). The code uses a version of Weierstrass theorem to construct an analytical function given random zeros.

   McPhedran's argument assumes that the zeros are simple. What happens for zeros with multiplicity ? This is an open question.

   (a) Download and read McPhedran's paper.
   (b) Reproduce the numerical examples for simple zeros in Matematica. Include the plots.
   (c) Perform numerical experiments for zeros of multiplicity higher than one. Include the plots. Do your experiments suggest that the theorem is still true ?
   (d) (Optional) Modify McPhedran's proof to deal with multiplicity.