MATH 6720
Homework # 4
Due Date: 02/22/2017

1. Show that, if $a > 0$, then
$$\int_0^\infty \frac{\cos(ax)}{(1+x^2)^2}dx = \frac{\pi(a+1)e^{-a}}{4}.$$

2. Let $f(z) = \exp(-cl(z))$, where $l(re^{i\theta}) = \ln r + i\theta$ for $\theta \in (0, 2\pi)$, be a branch of z^{-c} if $0 < c < 1$. Let L_1 be the line segment $[r + \delta i, R + \delta i]$ for $0 < r < 1 < R$ and $\delta > 0$. Show that
$$\int_r^R \frac{t^{-c}}{1+t}dt = \lim_{\delta \to 0^+} \int_{L_1} \frac{f(z)}{1+z}dz.$$
HINT: Define $g(t, \delta)$ ont he compact set $[r, R] \times [0, \pi/2]$ by
$$g(t, \delta) = \left| \frac{f(t+i\delta)}{1+t+i\delta} - \frac{t^{-c}}{1+t} \right|$$
when $\delta > 0$ and $g(t, 0) \equiv 0$. Show that g is continuous and hence uniformly continuous. Show that if $\varepsilon > 0$, then there exists $\delta_0 > 0$ such that $\int_r^R g(t, \delta)dt \leq \varepsilon$ for $\delta < \delta_0$.

3. Let S be the unit sphere and $\mathbb{C}_\infty$ the extended plane. Let $\mathbb{C}_\infty$ be represented by S using the stereographic projection.
   (a) Find the point in S corresponding to $0, 1+i, 1+2i$.
   (b) Which subsets of S corresponds to the real and imaginary axes in $\mathbb{C}$?

4. An irrotational 2D flow has stream function $\Psi = A(x-c)y$, where A, c are constants. A circular cylinder of radius a is introduced, its centre being at the origin. Find the complex potential, and hence the stream function, of the resulting flow. Use Blasius's theorem to calculate the force exerted on the cylinder.

5. Show that the problem of irrotational flow past a circular cylinder may be formulated in terms of the potential $\Phi(r, \theta)$ as follows
$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0,$$
with $\Phi \sim Ur\cos\theta$ as $r \to \infty$ and $\frac{\partial \Phi}{\partial r} = 0$ on $r = a$. Obtain the potential using the method of separation of variables.

6. **Problem 6.3.6**
   (a) Describe the flow associated with the function $f(z) = Az^2$. Make a contour plot of $\Psi(x, y)$.
   (b) Modify the flow so that it flows around a circle of radius 1 centered at $z = i$. What is the lift on this circle ?
   (c) Add circulation to this flow by adding the term $i\gamma \ln(z - i)$. What is the lift on the circle in this flow ?