**MATH 6750**
**Homework # 1**
**Due Date: September 12, 2019 at the beginning of class.**

1. Derive the following useful identity:

$$\mathbf{u}\cdot\nabla\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla(\tfrac{1}{2}|\mathbf{u}|^2).$$

2. Evaluate the following expressions:

$$\delta_{ij}\delta_{ij} \qquad \text{and} \qquad \epsilon_{ijk}\frac{\partial^2\phi}{\partial x_i\partial x_j}.$$

3. Let $\mathbf{x}$ stand for the position vector in 3 dimensions. Given that $r^2 = x_i x_i$, show that

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}.$$

4. A two-dimensional flow field has the following velocity components:

$$u = x(1+t) \quad v = 1 \quad w = 0.$$

   Determine the following quantities for this flow field:

   (a) The equation of the streamline that passes through the point (1,1) at $t = 0$. Note that the equation of the streamline is $\frac{dx_i}{ds} = u_i(x_i, t)$ for $t$ fixed with I.C at $s = 0$.

   (b) The equation of the pathline for a particle relased at the point (1,1) at time $t = 0$.

   (c) The equation of the streakline that passes through the point (1,1) as seen at $t = 0$. Note that this means that the particle was released at the point (1,1) at a time $t = \tau$.

   (d) Sketch all three paths.

5. The motion of a certain continuous medium is defined by the equations

$$x_1 = \frac{1}{2}(\alpha_1+\alpha_2)\,e^t + \frac{1}{2}(\alpha_1-\alpha_2)\,e^{-t} \quad x_2 = \frac{1}{2}(\alpha_1+\alpha_2)\,e^t - \frac{1}{2}(\alpha_1-\alpha_2)\,e^{-t} \quad x_3 = \alpha_3.$$

   (a) Express the velocity components in terms of the material coordinates and time.

   (b) Express the velocity components in terms of spatial coordinates and time.

6. Show that

$$\frac{d}{dt}\int_{\Omega_t}\mathbf{x}\times(\rho\mathbf{u})d\mathbf{x} = \int_{\Omega_t}\mathbf{x}\times\left(\rho\frac{D\mathbf{u}}{Dt}\right)d\mathbf{x}.$$

7. An antisymmetric second rank tensor $\mathbf{T}$ has only 3 independent components; consequently, it may be represented by using the components of a vector, say $\mathbf{u}$. Show that the relationship between $\mathbf{T}$ and $\mathbf{u}$ may be expressed as

$$\mathbf{T} = \frac{1}{2}\epsilon\cdot\mathbf{u} \qquad \text{and} \qquad \mathbf{u} = -\epsilon : \mathbf{T}.$$

8. In fluid statics, the governing equation is $\nabla P = \rho\mathbf{g}$, where $\rho$ is the density (assumed constant). This equation can be integrated to give $P(\mathbf{x}) = P_0 + \rho\mathbf{g}\cdot\mathbf{x}$, where $P_0$ is a constant.

(a) The hydrodynamic force acting on a body moving through a fluid may be calculated by integrating the normal component of the stress tensor $\mathbf{T}$ over the body surface $S$: $F_h = \int_S \hat{\mathbf{n}} \cdot \mathbf{T}\, dS$. For a static, submerged body of arbitrary shape, show that the force $\mathbf{F}$ and torque $\mathbf{L}_O$ (taken about some point $O$) exerted *on* the body by the fluid are given by

$$\mathbf{F} = -\int_S P\, \hat{\mathbf{n}}\, dS \qquad \mathbf{L}_O = \int_S \mathbf{x} \times \hat{\mathbf{n}}\, P\, dS,$$

where $\mathbf{x}$ is measured relative to the origin $O$, and $\hat{\mathbf{n}}$ is the unit normal directed from the body into the fluid.

(b) Use the Divergence Theorem to determine explicit expressions for the force and torque on the body.

9. Consider an iceberg floating in seawater. Find the fraction of the volume of the iceberg that shows above the sea surface.

10. Water flows through a duct of height $2h$ and width $W$. The velocity varies across the duct according to

$$\frac{u(y)}{U} = 1 - \left(\frac{y}{h}\right)^2.$$

$U$ is some given fluid velocity at infinity. Find the volume, mass, and momentum fluxes over the cross-sectional area of the duct. Note that $\mathbf{u} = u(y)\mathbf{e}_1$ and $\hat{\mathbf{n}} = \mathbf{e}_1$. Assume that the density is constant.