Homework 8
Math 5520 - Spring 2020

Let X and Y be a path connected spaces and $B \subset Y$ a path-connected subspace. Let

$$f: B \to X$$

be a continuous map. Let $\sim_f$ be the smallest equivalence relation on the union of X and Y with $y \sim_f f(y)$ for all $y \in B$ and let $X \sqcup_f Y$ be the quotient space for this equivalence relation.

1. Let X be path-connected and B^n the closed n-dimensional map. Let

   $$f: S^{n-1} = \partial B^n \to X$$

   be a continuous map. If $n \geq 3$ show that if

   $$\iota: X \to X \sqcup_f B^n$$

   is the inclusion map then ι_* is an isomorphism.

2. Let

   $$F: B \times [0,1] \to X$$

   be a continuous map. Define

   $$f: B \to X$$

   by $f(x) = F(x,0)$ and

   $$\tilde{F}: B \times [0,1] \to X \times [0,1]$$

   by $\tilde{F}(x,t) = (F(x,t),t)$. Show that $(X \times [0,1]) \sqcup_{\tilde{F}} (Y \times [0,1])$ and $X \sqcup_f Y$ are homotopy equivalent. Let

   $$g: B \to X$$

   be another continuous map with f and g homotopic. Show that $X \sqcup_f Y$ and $X \sqcup_g Y$ are homotopy equivalent.

3. Let $X = U \cup V$ where U and V are open in X. Let $x_0 \in U \cap V$ be a basepoint. Let

   $$\iota_U: U \hookrightarrow X, \iota_V: V \hookrightarrow X \text{ and } \iota_{U \cap V}: U \cap V \hookrightarrow X$$

   be the inclusion maps. If

   $$(\iota_{U \cap V})_*: \pi_1(U \cap V, x_0) \to \pi_1(X, x_0)$$

   is injective show that

   $$(\iota_U)_*: \pi_1(U, x_0) \to \pi_1(X, x_0) \text{ and } (\iota_V)_*: \pi_1(V, x_0) \to \pi_1(X, x_0)$$

   are both injective.

Comments on equivalence relations: A *relation* on a set E is a a subset of the product $E \times E$. Every relation can be extended to a unique smallest equivalence relation. This is a (not difficult) exercise that you should do (but you do not need to turn it in).