LIFTING LEMMA Let $f:([0,1],\{0\}) \to (S^1,\{0\})$.

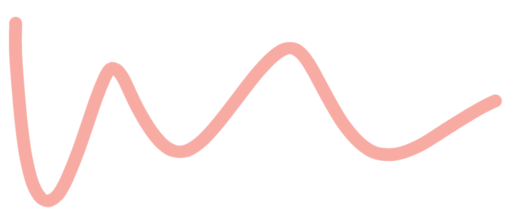
$\exists!\ \tilde{f} : ([0,1],\{0\}) \to (\mathbb{R},\{0\})$ with $f = \pi \circ \tilde{f}$.

$$\begin{tikzcd}
 & (\mathbb{R}, 0) \arrow[d, "\pi"] \\
([0,1],\{0\}) \arrow[ur, "\tilde{f}"] \arrow[r, "f"] & (S^1, \{0\})
\end{tikzcd}$$

Pf We need to find a partition

$t_0 = 0 < t_1 < \dots < t_n = 1$

of $[0,1]$ s.t. that for each interval $[t_i, t_{i+1}]$ $\exists$ an evenly covered nbd. $U_i \subset S^1$ with $f([t_i, t_{i+1}]) \subset U_i$.



Now assume $\tilde{f}$ is defined on $[0, t_i]$. As $f(t_i) \subset U_i$, we have $\tilde{f}(t_i) \in \pi^{-1}(U_i)$.

Let $\tilde{U}_i$ be the component of $\pi^{-1}(U_i)$ that contains $\tilde{f}(t_i)$.

π_i^{-1} is the inverse of $\pi|_{\tilde{U}_i}$.

Extend $\tilde{f}$ to $[t_i, t_{i+1}]$ by

$$\tilde{f}|_{[t_i, t_{i+1}]} = f \circ \pi_i^{-1}.$$

PF OF EXISTENCE OF PARTITION

$$f: ([0,1], \{0,1\}) \to (S^1, [0])$$

- for all $s \in [0,1]$ $\exists\ t_0 < s < t_1$

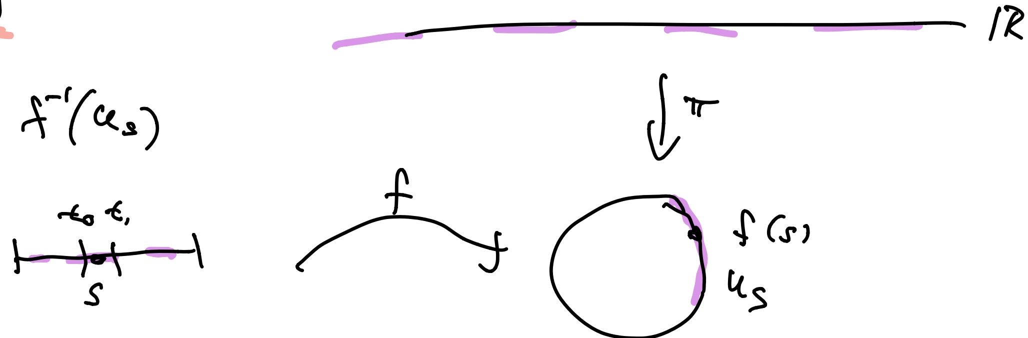
  with $f([t_0,t_1]) \subset U_s$, U_s evenly covered.

  SMALL FIX

  If $s=0$ then $t_0=0$, $s=1$ the $t_1=1$.

$[0, t_1]$

$f^{-1}(U_s)$



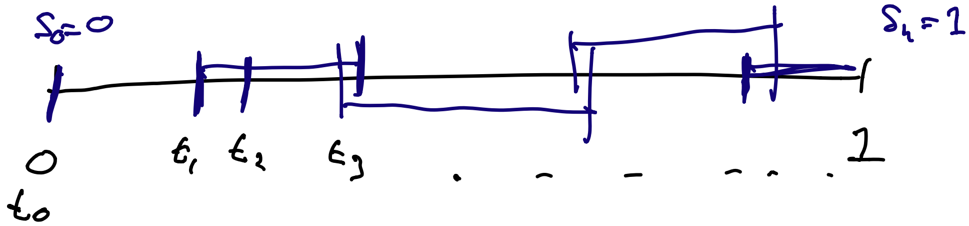
Choosing $[t_0,t_1] \subset U_s$ with $s \in (t_0, t_1]$

- The interiors of the U_s cover $[0,1]$.

  As $[0,1]$ is compact there exists a

  finite sub-cover associated to the intervals for $s_0=0, s_1, \dots, s_n=1$.

The end points of the associated intervals are the partition.



Each $f([t_i, t_{i+1}]) \subset U_{s_k}$

HOMOTOPY LIFTING LEMMA

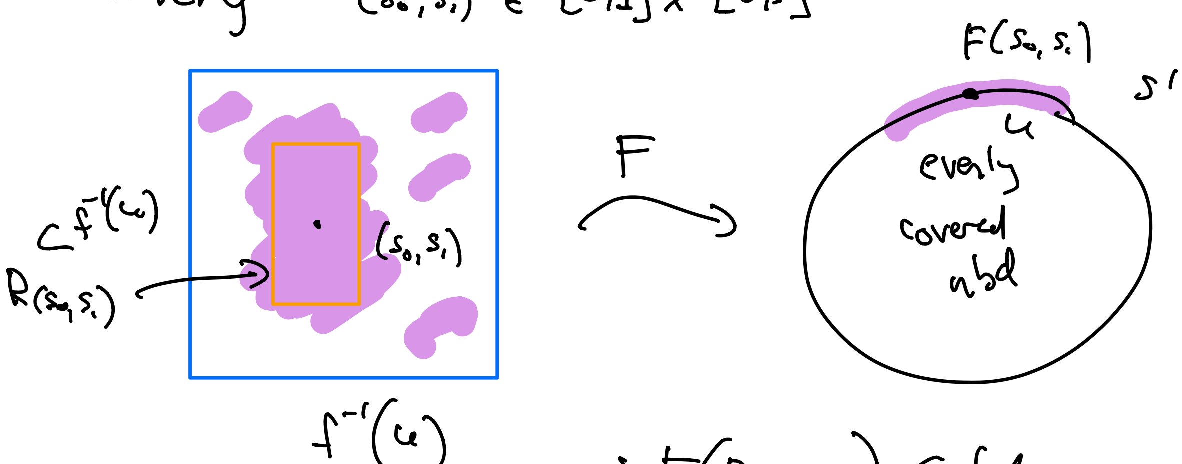
Let $F : ([0,1]\times[0,1], \{(0,0)\}) \to (S^1, \{1\})$.

$\exists !\ \tilde{F} : ([0,1]\times[0,1], \{(0,0)\}) \to (\mathbb{R}, \{0\})$

with $F = \pi \circ \tilde{F}$.

PF The proof has the same strategy. We need to partition the square $[0,1]\times[0,1]$.

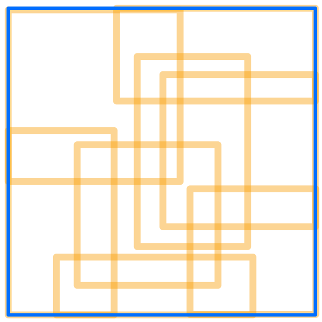
- Every $(s_0, s_1) \in [0,1]\times[0,1]$



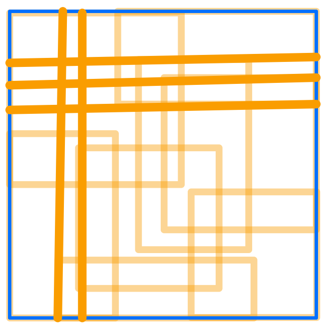
$\Rightarrow F(R(s_0,s_1)) \subset U$

The interior of the $R_{(s_0, s_1)}$ will cover $[0,1] \times [0,1]$

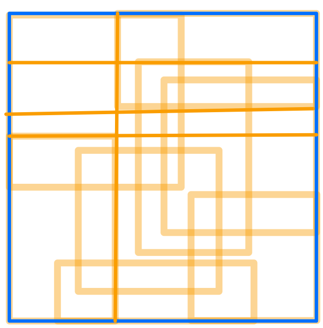
$\Rightarrow \exists$ a finite subcover.

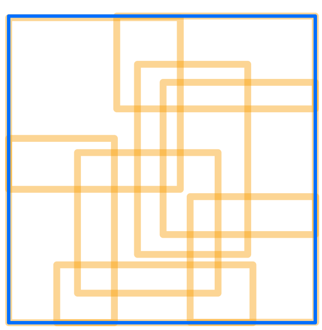


Finite collection of $R(s_0, s_1)$ that cover $[0,1] \times [0,1]$

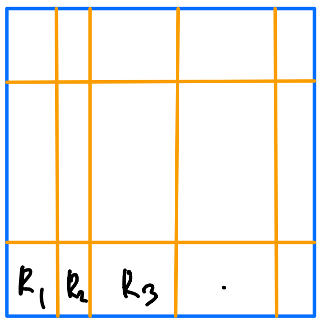


Extend the side of each rectangle



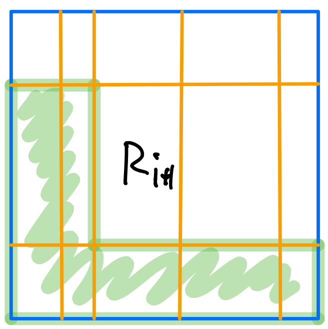


With the partition of the square:

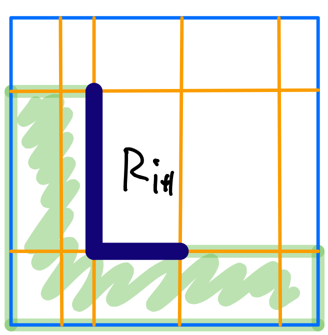


F will map each sub-rectangle to an evenly cover nbd in S'

$F(R_i) \subset U_i$
with U_i evenly covered.



Assume $\tilde{F}$ is defined on the first i rectangles



$\tilde{F}$ is continuous on the dark blue edges.

Since $F(\mathbf{L}) \subset U_{i+1}$ we have $\tilde{F}(\mathbf{L}) \subset \pi^{-1}(U_{i+1})$ and

Let π^{-1}_{i+1} be the inverse of $\pi|_{\tilde{u}_{i+1}}$

$\mathbf{L}$ is connected we must have $\tilde{F}(\mathbf{L})$ is contained in a single component $\tilde{u}_{i+1}$ of $\pi^{-1}(u_{i+1})$.

$$\tilde{F}\big|_{R_{i+1}} = F \circ \pi^{-1}_{i+1}.$$

Uniqueness

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\pi"] \\ X \arrow[ur, "\tilde{f}_0,\, \tilde{f}_1"] \arrow[r, "f"] & S^1 \end{tikzcd}$$

$\tilde{f}_0, \tilde{f}_1$ are continuous lifts of f.

The subspace $A := \{x \in X \mid \tilde{f}_0(x) = \tilde{f}_1(x)\}$ is open & closed.

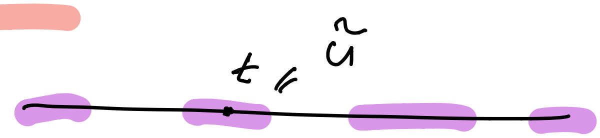
$\boxed{\text{CLOSED}}$ Define $\tilde{f}_0 \times \tilde{f}_1 : X \longrightarrow \mathbb{R} \times \mathbb{R}$ by $\tilde{f}_0 \times \tilde{f}_1(x) = (\tilde{f}_0(x), \tilde{f}_1(x))$.

This map is continuous & the diagonal $\Delta \subset \mathbb{R} \times \mathbb{R}$ is closed.

So $A = (\tilde{f}_0 \times \tilde{f}_1)^{-1}(\Delta)$ is closed.

$\boxed{\text{OPEN}}$ $x \in A$, $\tilde{f}_0(x) = \tilde{f}_1(x) = t$.

U an evenly covered nbd of $\pi(t) = f(x) \in S^1$.



$\tilde{U}$ the component of $\pi^{-1}(U)$ that contains t.

$\pi|_{\tilde{U}} : \tilde{U} \to U \subset S^1$ is a homeomorphism.

V is a nbd. of $x \in X$ st $f(V) \subset \tilde{U}$. since $\tilde{f}_i$ are lifts

On V, $f|_V = \pi|_{\tilde{U}} \circ \tilde{f}_i|_V$

$\pi|_{\tilde{U}}$ is invertible with inverse π_t^{-1} so

$$\tilde{f}_i|_V = \pi_t^{-1} \circ f|_V \Rightarrow \tilde{f}_0|_V = \tilde{f}_1|_V .$$

$\tilde{f}_0 = \tilde{f}_1$ on V

$\Rightarrow$ if $x \in A \Rightarrow \exists$ an open nbd V of x with $V \subset A$

$\Rightarrow A$ is open

$\boxed{\text{uniqueness for L. L.}}$

$$\begin{tikzcd} & (\mathbb{R}, \{0\}) \arrow[d, "\pi"] \\ ([0,1], \{0\}) \arrow[ur, "\tilde{f}_0, \tilde{f}_1"] \arrow[r, "f"] & (S^1, \{0\}) \end{tikzcd}$$

$A = \{t \in [0,1] \mid \tilde{f}_0(t) = \tilde{f}_1(t)\}$

A is open & closed.

A is also non-empty since $0 \in A$. $\Rightarrow A = [0,1]$ & $\tilde{f}_0 \equiv \tilde{f}_1$.

The fundamental group.

(X, x_0) $\quad X$ is a topological space

$x_0 \in X$ $\quad$ basepoint

We will give a group structure to the set of homotopy classes

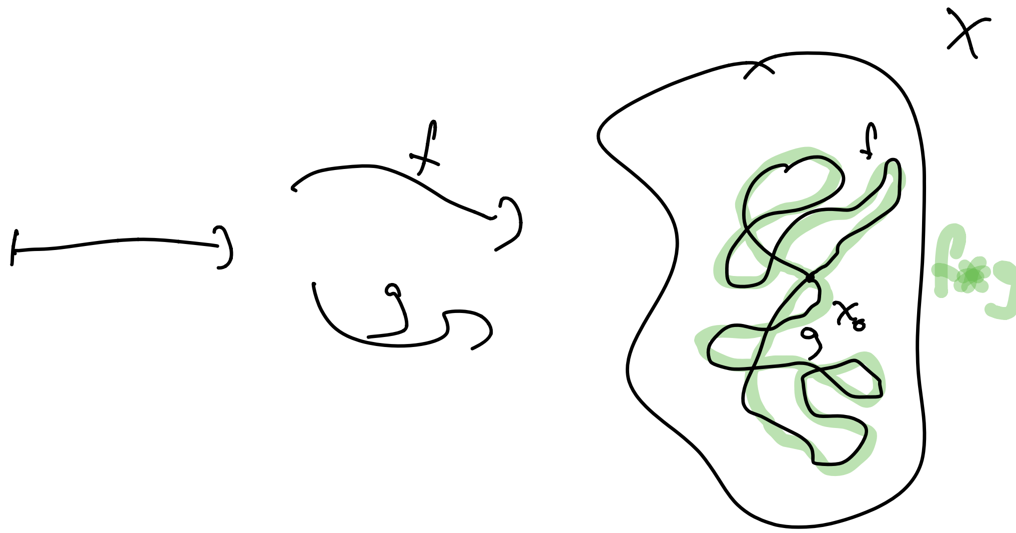
$([0,1], \{0,1\}) \to (X, \{x_0\})$.

First we need to define the operation:

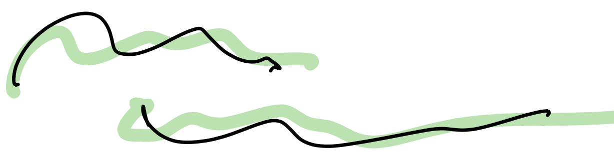
$f, g : ([0,1], \{0,1\}) \to (X, \{x_0\})$

$$f * g(t) = \begin{cases} g(2t) & t \in [0, \frac{1}{2}] \\ f(2t-1) & t \in (\frac{1}{2}, 1] \end{cases}$$

$f * g : ([0,1], \{0,1\}) \to (X, \{x_0\})$



f
g
X
f
g
x_0



$\boxed{\text{GROUPS}}$ A group is a pair $(G, \cdot)$ where G is a set and $\cdot$ is a binary operation satisfying:

$\boxed{\text{Closure}}$ If $a, b \in G \Rightarrow a \cdot b \in G$.

$\boxed{\text{identity}}$ There exists an $e \in G$ s.t

$$a \cdot e = e \cdot a = a \quad \forall a \in G.$$

$\boxed{\text{inverse}}$ $\forall a \in G$, $\exists a^{-1} \in G$ such that

$$a \cdot a^{-1} = a^{-1} \cdot a = e.$$

$\boxed{\text{associativity}}$ $\forall a, b, c \in G$ we have

$$(a \cdot b) \cdot c = a \cdot (b \cdot c).$$