LEMMA If $f, g : (X, x_0) \to (Y, y_0)$ are homotopic as pairs then $f_* = g_*$.

PROOF Let $[h] \in \pi_1(X, x_0)$ then

$f_*[h] = [f \circ h]$ & $g_*([h]) = [g \circ h]$

As f & g are homotopic as pairs so are $f \circ h$ & $g \circ h$ $\Rightarrow$ $[f \circ h] = [g \circ h]$. ∎

LEMMA Let $f_t : X \to Y$ be a homotopy & let $x_0 \in X$ be a basepoint. Let $\alpha(t) = f_t(x_0)$ be a path in Y from $y_0 = f_0(x_0)$ & $y_1 = f_1(x_1)$.

Then $\hat{\alpha} \circ (f_0)_* = (f_1)_*$ where

$(f_0)_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$

& $(f_1)_* : \pi_1(X, x_0) \to \pi_1(Y, y_1)$

PROOF Define $\alpha_t : [t, 1] \to Y$ by

$\alpha_t(s) = f_s(x_0)$ for $t \leq s \leq 1$

Given $h \in \pi_1(X, x_0)$ define

$$h_t = \alpha_t * (f_t \circ h) * \bar{\alpha}_t.$$

h_t is a homotopy from

$\bar{\alpha}_0 * (f_0 \circ h) * \alpha_0 \simeq_p [f_0 \circ h] = \hat{\alpha} \circ (f_0)_*([h])$ to

$\bar{\alpha}_1 * (f_1 \circ h) * \alpha_1 = \bar{\alpha} * (f_1 \circ h) * \alpha = (f_1)_*([h])$. ∎

COROLLARY Let $f, g : X \to Y$ be homotopic. (X, Y path connected)

If f_* is : 1) injective

2) surjective

3) trivial

then so is g_*.

DEFORMATION RETRACTS Let $A \subset X$ with A path connected. Then X deformation retracts to A if there is a homotopy

$$F : X \times [0,1] \to X$$

with
1) $F(x,t) = x \quad \forall x \in A$
2) $f_0(x) = F(x,0) = id$
3) $f_1(x) = F(x,1)$ is a retract to A.

THEOREM If X deformation retracts to A then the inclusion of A in X is an isomorphism of fundamental groups.

PROOF Let $\iota_A : A \to X$ be the inclusion. As A is a retract of X, $(\iota_A)_*$ is injective.
Let f_t realize the deformation retract.
Then $\iota_A \circ f_1 : X \to X$ is homotopic to $f_0 = id$ since $\iota_A \circ f_1 = f_1$.
Therefore $(id)_* = (\iota_A \circ f_1)_* = (\iota_A)_* \circ (f_1)_*$
$\Rightarrow (\iota_A)_*$ is surjective. $\blacksquare$

EXAMPLES $\mathbb{R}^n$ & $B^n = \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| \leq 1\}$ deformation retracts to $\vec{0} \in \mathbb{R}^n$:

$$F(\vec{x},t) = (1-t)\cdot\vec{x} + t\cdot\vec{0}.$$

In fact, this holds for any convex subset $A \subset \mathbb{R}^n$ that contains $\vec{0}$. More generally if $\vec{x}_0 \in A$ & the segment between $\vec{x} \in A$ and $\vec{x}_0$ is contained in A then A def. retracts to $\vec{x}_0$.

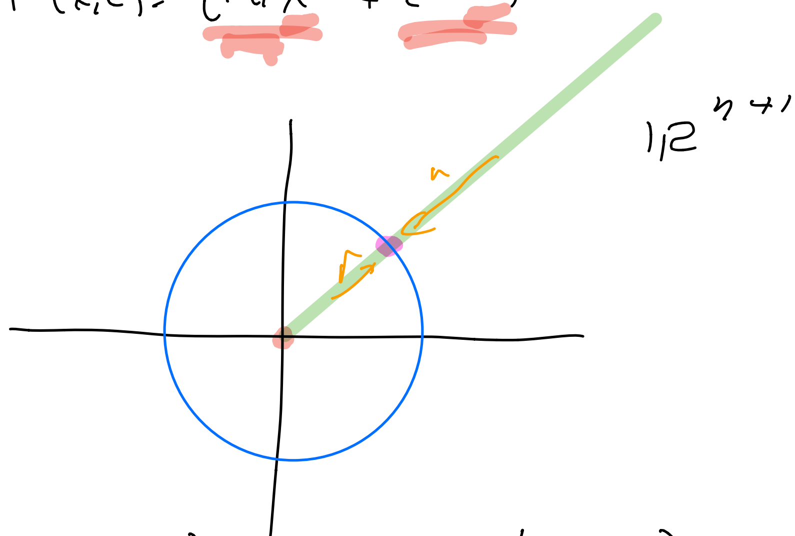
$$S^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid |\vec{x}| = 1\}.$$

$\mathbb{R}^{n+1} \setminus \{0\}$ deformation retracts to S^n:

The retraction is given by $r: \mathbb{R}^n \setminus \{0\} \to S^n$

with $r(\vec{x}) = \frac{\vec{x}}{|\vec{x}|}$. Define

$$F(\vec{x}, t) = (1-t)\cdot\vec{x} + t\cdot r(\vec{x})$$



$$\pi_1(S^n) \cong \pi_1(\mathbb{R}^{n+1} \setminus \{0\})$$

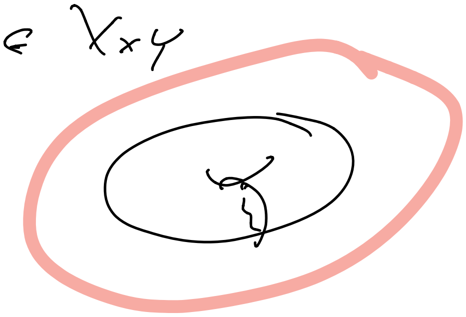
If X deformation retracts to a point $x_0 \in X$ then $X \times Y$ deformation retracts to $\{x_0\} \times Y$:

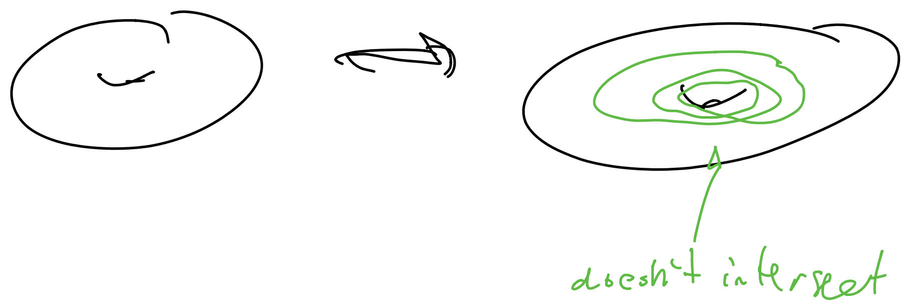
Let $F: X \times [0,1] \to X$ be the deformation retract of X to $\{x_0\}$. Define

$H: (X \times Y) \times [0,1] \to X \times Y$ by

$$H((x,y), t) = (F(x,t), y) \in X \times Y$$

$B^2 \times S^1$ = solid torus





$\boxed{\text{HOMOTOPY EQUIVALENCE}}$ X & Y are *homotopy equivalent* if $\exists$ $f: X \to Y$ & $g: Y \to X$ with

$f \circ g \simeq id_Y$ & $g \circ f \simeq id_X$.

A map $f: X \to Y$ is a *homotopy equivalence* if $\exists$ $g: Y \to X$ with $f \circ g \simeq id_Y$ & $g \circ f \simeq id_X$.

g is the *homotopy inverse* of f.

$\boxed{\text{THEOREM}}$ If $f: X \to Y$ is a homotopy equivalence f_* is an isomorphism.

$\boxed{\text{PROOF}}$ Let $g: Y \to X$ be the homotopy inverse.

Then $f \circ g \simeq id_Y$ & $g \circ f \simeq id_X$.

As $(id_Y)_*$ is an isomorphism so is $(f \circ g)_*$.

Similarly $(g \circ f)_*$ is an isomorphism.

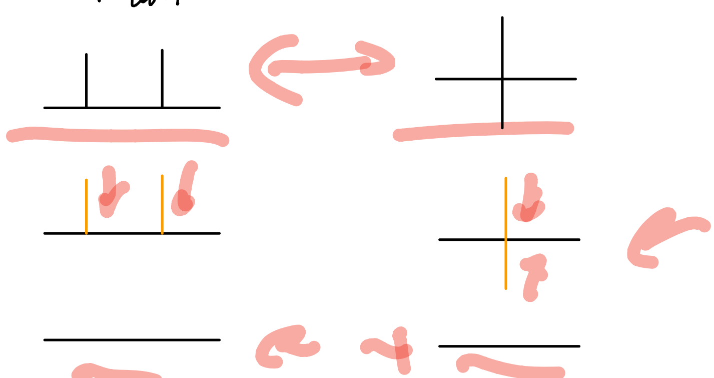
$(f \circ g)_* = f_* \circ g_* \Rightarrow f_*$ is surjective

$(g \circ f)_* = g_* \circ f_* \Rightarrow f_*$ is injective.

$\Rightarrow f_*$ is an isomorphism. ■

Comments:

- homotopy equivalence is an equivalence relation. If X is h.e. to Y and Y is h.e to Z then X is h.e. to Z.
- If X deformation retracts to A then X and A are homotopy equivalent. However there are spaces that are homotopy equivalent but neither one deformation retracts to the other.



- There are spaces with the same fundamental group that are not homotopy equivalent. A point and S^2 are an example. But this is much harder to show. We need more invariants.